

CHARACTERIZATIONS OF NORM-PARALLELISM IN SPACES OF CONTINUOUS FUNCTIONS

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ABSTRACT. In this paper, we consider the characterization of norm-parallelism problem in some classical Banach spaces. In particular, for two continuous functions f, g on a compact Hausdorff space K , we show that f is norm-parallel to g if and only if there exists a probability measure (i.e. positive and of full measure equal to 1) μ with its support contained in the norm attaining set $\{x \in K : |f(x)| = \|f\|\}$ such that $|\int_K \overline{f(x)}g(x)d\mu(x)| = \|f\| \|g\|$.

1. INTRODUCTION AND PRELIMINARIES

Let $(X, \|\cdot\|)$ be a normed space and denote, as usual, by \mathbb{B}_X and \mathbb{S}_X its closed unit ball and unit sphere, respectively, and denote the topological dual of X by X^* . Let $\mathcal{C}_b(\Omega)$ and $\mathcal{C}(K)$ denote the Banach spaces of all bounded continuous functions on a locally compact Hausdorff space Ω , with the usual norm $\|f\| = \sup_{x \in \Omega} |f(x)|$ ($f \in \mathcal{C}_b(\Omega)$) and all continuous functions on a compact Hausdorff space K , with the usual norm $\|f\| = \max_{x \in K} |f(x)|$ ($f \in \mathcal{C}(K)$), respectively. By $\mathcal{C}_u(\mathbb{B}_X, X)$ we denote the space of all uniformly continuous X valued functions on \mathbb{B}_X endowed with the supremum norm. Given a bounded function $f : \mathbb{S}_X \rightarrow X$, its numerical radius is

$$v(f) := \sup\{|x^*(f(x))| : \|x^*\| = x^*(x) = 1\}.$$

Let us comment that for a bounded function $f : \mathbb{B}_X \rightarrow X$, the above definitions apply by just considering $v(f) := v(f|_{\mathbb{S}_X})$.

Recall that an element $x \in X$ is said to be norm-parallel to another element $y \in X$ (see [12, 17]), in short $x \parallel y$, if

$$\|x + \lambda y\| = \|x\| + \|y\| \quad \text{for some } \lambda \in \mathbb{T}.$$

Here, as usual, \mathbb{T} is the unit circle of the complex plane. In the framework of inner product spaces, the norm-parallel relation is exactly the usual vectorial parallel relation, that is, $x \parallel y$ if and only if x and y are linearly dependent. In the setting of normed linear spaces, two linearly dependent vectors are norm-parallel, but the converse is false in general. Notice that the norm-parallelism is symmetric and \mathbb{R} -homogenous, but not transitive (i.e., $x \parallel y$ and $y \parallel z \not\Rightarrow x \parallel z$; see [18, Example 2.7], unless X is smooth at y ; see [15, Theorem 3.1]).

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In the context of continuous functions, the well-known Daugavet equation

$$\|T + Id\| = \|T\| + 1$$

is a particular case of parallelism. Here Id denotes, as usual, the identity function. Applications of this equation arise in solving a variety of problems in approximation theory; see [14] and the references therein.

Some characterizations of the norm–parallelism for operators on various Banach spaces and elements of an arbitrary Hilbert C^* -module were given in [2, 3, 7, 9, 13, 15, 16, 17, 18].

In particular, for bounded linear operators T, S on a Hilbert space $(H, [\cdot, \cdot])$, it was proved in [17, Theorem 3.3] that $T \parallel S$ if and only if there exists a sequence of unit vectors $\{\xi_n\}$ in H such that

$$\lim_{n \rightarrow \infty} |[T\xi_n, S\xi_n]| = \|T\| \|S\|.$$

Further, for compact operators T, S it was obtained in [16, Theorem 2.10] that $T \parallel S$ if and only if there exists a unit vector $\xi \in H$ such that

$$|[T\xi, S\xi]| = \|T\| \|S\|.$$

In [18], the authors also considered the characterization of norm parallelism problem for operators when the operator norm is replaced by the Schatten p -norm ($1 < p < \infty$). More precisely, it was proved in [18, Proposition 2.19] that $T \parallel S$ in the Schatten p -norm if and only if

$$\left| \operatorname{tr}(|T|^{p-1} U^* S) \right| = \|T\|_p^{p-1} \|S\|_p,$$

where $T = U|T|$ is the polar decomposition of T .

Some other related topics can be found in [1, 5, 6, 7, 10, 16], and the references therein.

It is our aim in the next section to give characterizations of the norm–parallelism in $\mathcal{C}_b(\Omega)$ and $\mathcal{C}(K)$. More precisely, for $f, g \in \mathcal{C}_b(\Omega)$ we prove that $f \parallel g$ if and only if there exists a sequence of probability measures μ_n concentrated at the set $\{x \in \Omega : |f(x)| \geq \|f\| - \varepsilon\}$ ($\varepsilon > 0$) such that

$$\left| \lim_{n \rightarrow \infty} \int_{\Omega} \overline{f(x)} g(x) d\mu_n(x) \right| = \|f\|^{-1} \|g\| \lim_{n \rightarrow \infty} \int_{\Omega} |f(x)|^2 d\mu_n(x).$$

Moreover, for $f, g \in \mathcal{C}(K)$ we show that $f \parallel g$ if and only if there exists a probability measure μ with support contained in the norm attaining set $\{x \in K : |f(x)| = \|f\|\}$ such that

$$\left| \int_K \overline{f(x)} g(x) d\mu(x) \right| = \|f\| \|g\|.$$

Finally, in the next section, we state a characterization of the norm-parallelism for uniformly continuous X valued functions on \mathbb{B}_X to the identity function. Actually, we show that if X is a Banach space and $f \in \mathcal{C}_u(\mathbb{B}_X, X)$, then $f \parallel Id$ if and only if $\|f\| = v(f)$.

2. MAIN RESULTS

We begin with the following results, which will be useful in other contexts as well.

Lemma 2.1. [17, Theorem 4.1] *Let X be a normed space. For $x, y \in X$ the following statements are equivalent:*

- (i) $x \parallel y$.
- (ii) *There exists a norm one linear functional φ over X such that $\varphi(x) = \|x\|$ and $|\varphi(y)| = \|y\|$.*

Lemma 2.2. [8, Theorem 3.1] *Let Ω be a locally compact Hausdorff space and let $f, g \in \mathcal{C}_b(\Omega)$. Then*

$$\lim_{t \rightarrow 0^+} \frac{\|f + tg\| - \|f\|}{t} = \inf_{\varepsilon > 0} \sup_{x \in M_f^\varepsilon} \operatorname{Re}(e^{-i \arg(f(x))} g(x)),$$

where $M_f^\varepsilon = \{x \in \Omega : |f(x)| \geq \|f\| - \varepsilon\}$.

We use some techniques of [8] to prove the following theorem. Recall that a probability measure is a positive measure of total mass 1.

Theorem 2.3. *Let Ω be a locally compact Hausdorff space and let $f, g \in \mathcal{C}_b(\Omega)$. Then the following statements are equivalent:*

- (i) *For every $\varepsilon > 0$, there exists a sequence of probability measures μ_n concentrated at M_f^ε such that*

$$\|f\| \left| \lim_{n \rightarrow \infty} \int_{\Omega} \overline{f(x)} g(x) d\mu_n(x) \right| = \|g\| \lim_{n \rightarrow \infty} \int_{\Omega} |f(x)|^2 d\mu_n(x).$$

- (ii) $f \parallel g$.

Proof. (i) \Rightarrow (ii) Suppose (i) holds. Let $\varepsilon > 0$. So, there exist a sequence of probability measures μ_n concentrated at M_f^ε and $\lambda \in \mathbb{T}$ such that

$$\|g\| \lim_{n \rightarrow \infty} \int_{\Omega} |f(x)|^2 d\mu_n(x) = \lambda \|f\| \lim_{n \rightarrow \infty} \int_{\Omega} \overline{f(x)} g(x) d\mu_n(x),$$

and hence

$$\lim_{n \rightarrow \infty} \int_{\Omega} \overline{f(x)} (\|g\| f(x) - \lambda \|f\| g(x)) d\mu_n(x) = 0. \quad (2.1)$$

Let us now define a linear functional $\varphi : \operatorname{span}\{f, (\|g\|f - \lambda\|f\|g)\} \rightarrow \mathbb{C}$ by setting

$$\varphi(\alpha f + \beta(\|g\|f - \lambda\|f\|g)) = \alpha \|f\| \quad (\alpha, \beta \in \mathbb{C}).$$

We have

$$\begin{aligned}
& \left\| \alpha f + \beta (\|g\|f - \lambda \|f\|g) \right\|^2 \\
&= \sup_{x \in \Omega} \left| \alpha f(x) + \beta (\|g\|f(x) - \lambda \|f\|g(x)) \right|^2 \\
&= \sup_{x \in \Omega} \left(|\alpha|^2 |f(x)|^2 + 2\operatorname{Re} [\overline{\alpha\beta} \overline{f(x)} (\|g\|f(x) - \lambda \|f\|g(x))] \right. \\
&\quad \left. + |\beta|^2 |\|g\|f(x) - \lambda \|f\|g(x)|^2 \right) \\
&\geq \left| \int_{\Omega} \left(|\alpha|^2 |f(x)|^2 + 2\operatorname{Re} [\overline{\alpha\beta} \overline{f(x)} (\|g\|f(x) - \lambda \|f\|g(x))] \right. \right. \\
&\quad \left. \left. + |\beta|^2 |\|g\|f(x) - \lambda \|f\|g(x)|^2 \right) d\mu_n(x) \right| \\
&\geq |\alpha|^2 \int_{\Omega} |f(x)|^2 d\mu_n(x) + 2\operatorname{Re} [\overline{\alpha\beta} \int_{\Omega} \overline{f(x)} (\|g\|f(x) - \lambda \|f\|g(x)) d\mu_n(x)] \\
&\quad \left(\text{since } \int_{\Omega} |\beta|^2 |\|g\|f(x) - \lambda \|f\|g(x)|^2 d\mu_n(x) \geq 0 \right) \\
&\geq |\alpha|^2 (\|f\| - \varepsilon)^2 + 2\operatorname{Re} [\overline{\alpha\beta} \int_{\Omega} \overline{f(x)} (\|g\|f(x) - \lambda \|f\|g(x)) d\mu_n(x)]. \\
&\quad \left(\text{since } \mu_n \text{ is a probability measure concentrated at } M_f^\varepsilon \right)
\end{aligned}$$

Thus

$$\begin{aligned}
& \left\| \alpha f + \beta (\|g\|f - \lambda \|f\|g) \right\|^2 \\
&\geq |\alpha|^2 (\|f\| - \varepsilon)^2 + 2\operatorname{Re} [\overline{\alpha\beta} \int_{\Omega} \overline{f(x)} (\|g\|f(x) - \lambda \|f\|g(x)) d\mu_n(x)]. \quad (2.2)
\end{aligned}$$

Letting $\varepsilon \rightarrow 0^+$ and $n \rightarrow \infty$ in (2.2), then by (2.1) we obtain

$$\left\| \alpha f + \beta (\|g\|f - \lambda \|f\|g) \right\| \geq |\alpha| \|f\|.$$

Hence $\left| \varphi \left(\alpha f + \beta (\|g\|f - \lambda \|f\|g) \right) \right| \leq \left\| \alpha f + \beta (\|g\|f - \lambda \|f\|g) \right\|$ and $\varphi(f) = \|f\|$. Thus $\|\varphi\| = 1$. Therefore, the Hahn-Banach theorem extends φ to a linear functional $\tilde{\varphi}$ on $\mathcal{C}_b(\Omega)$, with $\|\tilde{\varphi}\| = 1$. Since $\tilde{\varphi}(f) = \|f\|$ and $\tilde{\varphi}(\|g\|f - \lambda \|f\|g) = 0$, we get $\tilde{\varphi}(\lambda g) = \|g\|$, hence $|\tilde{\varphi}(g)| = \|g\|$. Now, by Lemma 2.1, we conclude that $f \parallel g$.

(ii) \Rightarrow (i) Let $f \parallel g$. By Lemma 2.1, there exists a norm one linear functional φ over $\mathcal{C}_b(\Omega)$ such that $\varphi(f) = \|f\|$ and $|\varphi(g)| = \|g\|$. So, there exists $\lambda \in \mathbb{T}$ such

that $\varphi(\lambda g) = \|g\|$. From $\|\varphi\| = 1$, $\varphi(f) = \|f\|$ and $\varphi(\lambda g) = \|g\|$ it follows that

$$\begin{aligned} \frac{\left\| f + re^{i\theta}(\|g\|f - \lambda\|f\|g) \right\| - \|f\|}{r} &\geq \frac{\left| \varphi\left(f + re^{i\theta}(\|g\|f - \lambda\|f\|g) \right) \right| - \|f\|}{r} \\ &= \frac{\left| \varphi(f) + re^{i\theta}\|g\|\varphi(f) - re^{i\theta}\|f\|\varphi(\lambda g) \right| - \|f\|}{r} \\ &= \frac{\left| \|f\| + re^{i\theta}\|g\|\|f\| - re^{i\theta}\|f\|\|g\| \right| - \|f\|}{r} = 0 \end{aligned}$$

for all $r > 0$ and all $\theta \in [0, 2\pi)$. Hence, by Lemma 2.2, we get

$$\inf_{\varepsilon > 0} \sup_{x \in M_f^\varepsilon} \operatorname{Re} \left(e^{i\theta} e^{-i \arg(f(x))} (\|g\|f(x) - \lambda\|f\|g(x)) \right) \geq 0$$

for all $\theta \in [0, 2\pi)$. Thus for all $\varepsilon > 0$ the set

$$K_\varepsilon := \{ e^{-i \arg(f(x))} (\|g\|f(x) - \lambda\|f\|g(x)) : x \in M_f^\varepsilon \}$$

contains at least one element with nonnegative real part under all rotations around the origin. Hence the values of the function $\|g\|f - \lambda\|f\|g$ on M_f^ε are not contained in an open half plane with boundary that contains the origin. So, for all $\varepsilon > 0$ the closed convex hull of the set K_ε contains the origin. The convex hull of K_ε consists of points of the form $\int_\Omega e^{-i \arg(f(x))} (\|g\|f(x) - \lambda\|f\|g(x)) d\mu_\varepsilon(x)$, where μ_ε is a probability measure supported on a finite subset of M_f^ε (see [11], chap. 3). Let $n_0 \in \mathbb{N}$ and $\frac{1}{n_0} < \varepsilon$. Thus for every $n \geq n_0$, there is a probability measure μ_n concentrated at M_f^ε such that

$$\left| \int_\Omega e^{-i \arg(f(x))} (\|g\|f(x) - \lambda\|f\|g(x)) d\mu_n(x) \right| < \frac{1}{n}. \quad (2.3)$$

We have

$$\begin{aligned} &\left| \int_\Omega \overline{f(x)} (\|g\|f(x) - \lambda\|f\|g(x)) d\mu_n(x) \right| \\ &= \left| \int_\Omega \left(\overline{f(x)} - \|f\| e^{-i \arg(f(x))} \right) (\|g\|f(x) - \lambda\|f\|g(x)) d\mu_n(x) \right. \\ &\quad \left. + \int_\Omega \|f\| e^{-i \arg(f(x))} (\|g\|f(x) - \lambda\|f\|g(x)) d\mu_n(x) \right| \\ &\leq \left| \int_\Omega \left(\overline{f(x)} - \|f\| e^{-i \arg(f(x))} \right) (\|g\|f(x) - \lambda\|f\|g(x)) d\mu_n(x) \right| \\ &\quad + \|f\| \left| \int_\Omega e^{-i \arg(f(x))} (\|g\|f(x) - \lambda\|f\|g(x)) d\mu_n(x) \right|. \end{aligned} \quad (2.4)$$

Furthermore, since $\|f\| - \frac{1}{n} \leq |f(x)| \leq \|f\|$ for all $x \in M_f^\varepsilon$ we have

$$\begin{aligned} & \left| \int_{\Omega} \left(\overline{f(x)} - \|f\| e^{-i \arg(f(x))} \right) \left(\|g\| f(x) - \lambda \|f\| g(x) \right) d\mu_n(x) \right| \\ & \leq \frac{1}{n} \int_{\Omega} \left| \|g\| f(x) - \lambda \|f\| g(x) \right| d\mu_n(x) \\ & \leq \frac{1}{n} \left(\|g\| \int_{\Omega} |f(x)| d\mu_n(x) + \|f\| \int_{\Omega} |g(x)| d\mu_n(x) \right) \\ & \leq \frac{2}{n} \|f\| \|g\|. \end{aligned} \quad (2.5)$$

By (2.3), (2.4) and (2.5) we get

$$\left| \int_{\Omega} \overline{f(x)} \left(\|g\| f(x) - \lambda \|f\| g(x) \right) d\mu_n(x) \right| \leq \frac{2}{n} \|f\| \|g\| + \|f\| \frac{1}{n}. \quad (2.6)$$

Taking $\lim_{n \rightarrow \infty}$ in (2.6), we obtain

$$\lim_{n \rightarrow \infty} \int_{\Omega} \overline{f(x)} \left(\|g\| f(x) - \lambda \|f\| g(x) \right) d\mu_n(x) = 0,$$

and hence

$$\|f\| \left| \lim_{n \rightarrow \infty} \int_{\Omega} \overline{f(x)} g(x) d\mu_n(x) \right| = \|g\| \lim_{n \rightarrow \infty} \int_{\Omega} |f(x)|^2 d\mu_n(x).$$

□

Next, we present a characterization of the norm-parallelism for continuous functions on a compact Hausdorff space K . We will need the following lemma.

Lemma 2.4. [8, Theorem 3.1] *Let K be a compact Hausdorff space and let $f, g \in \mathcal{C}(K)$. Then*

$$\lim_{t \rightarrow 0^+} \frac{\|f + tg\| - \|f\|}{t} = \max_{x \in M_f} \operatorname{Re} \left(e^{-i \arg(f(x))} g(x) \right),$$

where $M_f = \{x \in K : |f(x)| = \|f\|\}$.

Theorem 2.5. *Let K be a compact Hausdorff space and let $f, g \in \mathcal{C}(K)$. Then the following statements are equivalent:*

- (i) *There exists a probability measure μ with support contained in the norm attaining set M_f such that*

$$\left| \int_K \overline{f(x)} g(x) d\mu(x) \right| = \|f\| \|g\|.$$

- (ii) $f \parallel g$.

Proof. We proceed as in the proof of Theorem 2.3.

(i) \Rightarrow (ii) Suppose (i) holds. So, there exists $\lambda \in \mathbb{T}$ such that

$$\int_K \overline{f(x)} \left(\|g\| f(x) - \lambda \|f\| g(x) \right) d\mu(x) = 0.$$

Thus

$$\begin{aligned} & \left\| \alpha f + \beta (\|g\|f - \lambda \|f\|g) \right\|^2 \\ & \geq |\alpha|^2 \|f\|^2 + |\beta|^2 \int_{M_f} \left| \|g\|f(x) - \lambda \|f\|g(x) \right|^2 d\mu(x) \geq |\alpha|^2 \|f\|^2, \end{aligned}$$

for any $\alpha, \beta \in \mathbb{C}$. Let $\varphi : \text{span}\{f, (\|g\|f - \lambda \|f\|g)\} \rightarrow \mathbb{C}$ be the linear functional defined as

$$\varphi\left(\alpha f + \beta (\|g\|f - \lambda \|f\|g)\right) = \alpha \|f\| \quad (\alpha, \beta \in \mathbb{C}).$$

Hence $\varphi(f) = \|f\|$, $\varphi(\lambda g) = \|g\|$ and $\|\varphi\| = 1$. By the Hahn-Banach theorem, φ extends to a linear functional $\tilde{\varphi}$ on $\mathcal{C}(K)$, of the same norm. Since $\tilde{\varphi}(f) = \|f\|$ and $|\tilde{\varphi}(g)| = \|g\|$, Lemma 2.1 yields $f \parallel g$.

(ii) \Rightarrow (i) Let $f \parallel g$. By Lemma 2.1, there exist $\lambda \in \mathbb{T}$ and a norm one linear functional φ over $\mathcal{C}(K)$ such that $\varphi(f) = \|f\|$ and $\varphi(\lambda g) = \|g\|$. Hence by Lemma 2.4, we get

$$\max_{x \in M_f} \text{Re}\left(e^{-i \arg(f(x))} (\|g\|f(x) - \lambda \|f\|g(x))\right) \geq 0.$$

So, the convex hull of the set $\{\overline{f(x)}(\|g\|f(x) - \lambda \|f\|g(x)) : x \in M_f\}$ consists of points of the form $\int_K \overline{f(x)}(\|g\|f(x) - \lambda \|f\|g(x)) d\mu(x)$, where μ is a probability measure supported on a finite subset of M_f . Then there is a sequence μ_n of probability measures such that

$$\lim_{n \rightarrow \infty} \int_K \overline{f(x)}(\|g\|f(x) - \lambda \|f\|g(x)) d\mu_n(x) = 0.$$

By the Banach-Alaoglu compactness theorem in dual space, there is a probability measure μ such that $\lim_{i \rightarrow \infty} \mu_{n_i} = \mu$. Thus the support of μ is contained in M_f and we obtain

$$\left| \int_K \overline{f(x)}g(x) d\mu(x) \right| = \|f\| \|g\|.$$

□

As a consequence of Theorem 2.5 we have the following result.

Corollary 2.6. *Let K be a compact Hausdorff space and let $f, g \in \mathcal{C}(K)$. If $M_f = \{x_0\}$, then the following statements are equivalent:*

- (i) $f \parallel g$.
- (ii) $\{x_0\} \subseteq M_g$.

We closed this paper with the following equivalence theorem. More precisely, we state a characterization of the norm-parallelism for uniformly continuous X valued functions on \mathbb{B}_X to the identity function. Note that since \mathbb{B}_X is convex and bounded, then every function in $\mathcal{C}_u(\mathbb{B}_X, X)$ is also bounded. Before stating our result, let us quote a result from [4].

Lemma 2.7. [4, Corollary 2.4] *Let X be a Banach space. Let $0 < \theta < 2$ and suppose $y \in \mathbb{B}_X$ and $y^* \in \mathbb{B}_{X^*}$ satisfy $\operatorname{Re} y^*(y) > 1 - \theta$. Then, there are $z \in \mathbb{S}_X$ and $z^* \in \mathbb{S}_{X^*}$ such that*

$$z^*(z) = 1, \quad \|y - z\| < \sqrt{2\theta} \quad \text{and} \quad \|y^* - z^*\| < \sqrt{2\theta}.$$

Theorem 2.8. *Let X be a Banach space and let $f \in \mathcal{C}_u(\mathbb{B}_X, X)$. Then the following statements are equivalent:*

- (i) $v(f) = \|f\|$.
- (ii) $f \parallel Id$,

where Id stands for the identity function.

Proof. (i) \Rightarrow (ii) Let $v(f) = \|f\|$. For every $\varepsilon > 0$, we may find $x \in \mathbb{S}_X$ and $x^* \in \mathbb{S}_{X^*}$ such that $x^*(x) = 1$ and $|x^*(f(x))| > \|f\| - \varepsilon$. Let $x^*(f(x)) = \bar{\lambda}|x^*(f(x))|$ with $\lambda \in \mathbb{T}$. We have

$$\begin{aligned} 1 + \|f\| &\geq \|Id + \lambda f\| \geq \|x + \lambda f(x)\| \\ &\geq |x^*(x + \lambda f(x))| = |x^*(x) + \lambda x^*(f(x))| \\ &= |1 + \lambda \bar{\lambda} |x^*(f(x))|| = 1 + |x^*(f(x))| > 1 + \|f\| - \varepsilon. \end{aligned}$$

Thus

$$1 + \|f\| \geq \|Id + \lambda f\| > 1 + \|f\| - \varepsilon.$$

Letting $\varepsilon \rightarrow 0^+$, we obtain $\|Id + \lambda f\| = 1 + \|f\|$, or equivalently, $f \parallel Id$.

(ii) \Rightarrow (i) Let $f \parallel Id$. So, there exists $\lambda \in \mathbb{T}$ such that

$$\|Id + \lambda f\| = 1 + \|f\|.$$

Fix $0 < \varepsilon < 1$. Since $f \in \mathcal{C}_u(\mathbb{B}_X, X)$, there exists $0 < \delta < \varepsilon$ such that

$$\|y - z\| < \delta \implies \|f(y) - f(z)\| < \varepsilon \quad (y, z \in \mathbb{B}_X). \quad (2.7)$$

Since $1 + \|f\| = \sup_{y \in \mathbb{B}_X} \|y + \lambda f(y)\|$, there exists $y \in \mathbb{B}_X$ such that

$$\|y + \lambda f(y)\| > 1 + \|f\| - \frac{\delta^2}{2}.$$

Then we may find $y^* \in \mathbb{S}_{X^*}$ such that

$$\operatorname{Re} y^*(y + \operatorname{Re} y^*(\lambda f(y))) > 1 + \|f\| - \frac{\delta^2}{2},$$

which yields

$$\operatorname{Re} y^*(y) > 1 - \frac{\delta^2}{2} \quad (2.8)$$

and

$$\operatorname{Re} y^*(\lambda f(y)) > \|f\| - \frac{\delta^2}{2}. \quad (2.9)$$

By (2.8) and Lemma 2.7, there are $z \in \mathbb{S}_X$ and $z^* \in \mathbb{S}_{X^*}$ such that

$$z^*(z) = 1, \quad \|y - z\| < \delta \quad \text{and} \quad \|y^* - z^*\| < \delta. \quad (2.10)$$

So, by (2.10) and (2.7) we get

$$\|f(y) - f(z)\| < \varepsilon. \tag{2.11}$$

By (2.10) and (2.11) it follows that

$$\begin{aligned} \left| \operatorname{Re}z^*(\lambda f(z)) - \operatorname{Re}y^*(\lambda f(y)) \right| &\leq \left| \operatorname{Re}z^*(\lambda f(z) - \lambda f(y)) \right| + \left| \operatorname{Re}(z^* - y^*)(\lambda f(y)) \right| \\ &\leq \|\lambda f(z) - \lambda f(y)\| + \|z^* - y^*\| \leq \varepsilon + \delta, \end{aligned}$$

whence

$$\left| \operatorname{Re}z^*(\lambda f(z)) - \operatorname{Re}y^*(\lambda f(y)) \right| < \varepsilon + \delta. \tag{2.12}$$

So, by (2.9) and (2.12) we obtain

$$\operatorname{Re}z^*(\lambda f(z)) > \|f\| - \frac{\delta^2}{2} - (\varepsilon + \delta) > \|f\| - 3\varepsilon.$$

This implies

$$\|f\| \geq v(f) \geq \operatorname{Re}z^*(\lambda f(z)) > \|f\| - 3\varepsilon.$$

Letting $\varepsilon \rightarrow 0^+$, we conclude $v(f) = \|f\|$. □

As an immediate consequence of Theorem 2.8 we have the following result.

Corollary 2.9. *Let X be a Banach space and let $f \in \mathcal{C}_u(\mathbb{B}_X, X)$. If $f \parallel Id$, then $\|f\| = \sup_{x \in \mathbb{S}_X} \|f(x)\|$.*

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