

Loose edges

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Abstract

We consider formal power series in several variables with coefficients in an arbitrary field such that their Newton polyhedron has a loose edge. We show that if the symbolic restriction of the power series f to such an edge is a product of two coprime polynomials, then f factorizes in the ring of power series.

1 Introduction

Notation. We denote by $\mathbb{R}_{\geq 0}$ (respectively $\mathbb{R}_{> 0}$) the set of nonnegative (respectively positive) real numbers. The symbol $\langle \cdot, \cdot \rangle$ denotes the standard scalar product. We use a multi-index notation $\underline{x}^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n)$.

We start from a quick reminder of convex geometry.

Let $f \in \mathbb{K}[[x_1, \dots, x_n]]$, $f = \sum a_\alpha \underline{x}^\alpha$ be a nonzero power series. We define the *Newton polyhedron* $\Delta(f)$ as the convex hull of the set $\{\alpha : a_\alpha \neq 0\} + \mathbb{R}_{\geq 0}^n$.

The *symbolic restriction* of f to $A \subset \Delta(f)$ is defined as the power series

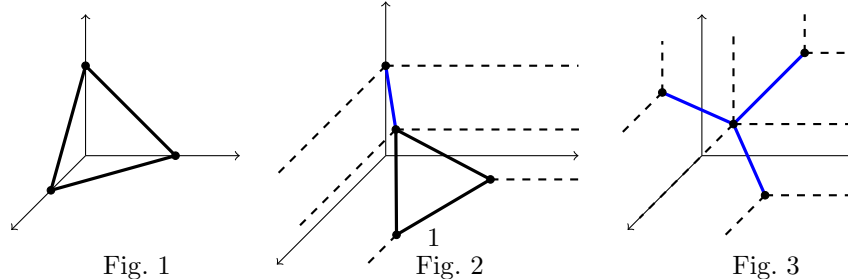
$$f|_A = \sum_{\alpha \in A} a_\alpha \underline{x}^\alpha.$$

Given $\Delta = \Delta(f)$, for any $\xi \in \mathbb{R}_{\geq 0}^n$ we call the set

$$\Delta^\xi := \{a \in \Delta : \langle \xi, a \rangle = \min_{b \in \Delta} \langle \xi, b \rangle\}$$

a *face* of Δ . A Newton polyhedron has a finite number of faces. A face Δ^ξ is compact if and only if $\xi \in \mathbb{R}_{> 0}^n$. A face of dimension 0 (respectively 1) is called a *vertex* (respectively an *edge*). Following [4], we call a compact edge of a Newton polyhedron a *loose edge* if it is not contained in any compact face of dimension ≥ 2 .

Several Newton polyhedra are drawn in the pictures that follow. The segments marked in blue are loose edges



The Newton polyhedron in Figure 1 does not have any loose edge. This is the typical situation.

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The Newton polyhedron in Figure 2 has a loose edge with the end point at $(0, 0, d)$. The Weierstrass polynomials $f \in \mathbb{K}[[x_1, \dots, x_n]][z]$ such that $\Delta(f)$ is of this type were studied in [5] and in [6] where $\Delta(f)$ is called an *orthant associated polyhedron*

In Figure 3 all compact edges are loose. A Newton polyhedron with this property is called in [3] a *polygonal Newton polyhedron*. Notice that the term polygonal Newton polyhedron can be a bit misleading since the union of compact edges in Figure 3 is not homeomorphic to any polygon.

Every compact edge of a plane Newton polyhedron is loose as illustrated in Figure 4.

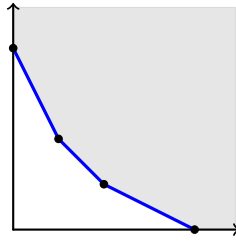


Fig. 4

Below are the main results of the paper.

Theorem 1.1. *Let $f \in \mathbb{K}[[x_1, \dots, x_n]]$ be a formal power series with coefficients in a field \mathbb{K} . Assume that the Newton polyhedron $\Delta(f)$ has a loose edge E . If $f|_E$ is a product of two relatively prime polynomials G and H , where G is not divided by any variable, then there exist powers series g, h such that $f = gh$ and $g|_{E_1} = G, h|_{E_2} = H$ for some E_1, E_2 such that $E = E_1 + E_2$.*

In the above theorem E_1 is a loose edge of $\Delta(g)$ parallel to E and E_2 is a compact face of $\Delta(h)$ which is a loose edge parallel to E or a vertex if H is a monomial.

Corollary 1.2. *Assume that the Newton polyhedron of $f \in \mathbb{K}[[x_1, \dots, x_n]]$ has a loose edge and at least three vertices. Then f is not irreducible.*

Corollary 1.3. *Assume that the Newton polyhedron of $f \in \mathbb{K}[[x_1, \dots, x_n]]$ has a loose edge E . If f is irreducible, then E is the only compact edge of $\Delta(f)$ and $f|_E = cF^k$, where F is an irreducible polynomial. Moreover, if \mathbb{K} is algebraically closed, then $f|_E = (a\underline{x}^\alpha + b\underline{x}^\beta)^k$ with a primitive lattice vector $\alpha - \beta$.*

We will say that a segment $E \subset \mathbb{R}^n$ is *descendant* if E is parallel to some vector $c = (c_1, \dots, c_n)$ such that $c_i \geq 0$ for $i = 1, \dots, n-1$ and $c_n < 0$.

Theorem 1.4. *Let $f \in \mathbb{K}[[x_1, \dots, x_{n-1}]][[x_n]]$. Assume that the Newton polyhedron $\Delta(f)$ has a loose and descendant edge E . If $f|_E$ is a product of two relatively prime polynomials G and H , where G is monic with respect to x_n , then there exist uniquely determined $g, h \in \mathbb{K}[[x_1, \dots, x_{n-1}]][[x_n]]$ such that $f = gh$, g is a monic polynomial with respect to x_n and $g|_{E_1} = G, h|_{E_2} = H$ for some E_1, E_2 such that $E = E_1 + E_2$.*

2 Proofs

At the beginning of this section we establish some results about loose edges.

Lemma 2.1. *Let Δ be a Newton polyhedron with a loose edge E that has ends $a, b \in \mathbb{R}^n$. Then for every $c \in \Delta$ and every $\xi \in \mathbb{R}_{\geq 0}^n$ such that $\langle \xi, a \rangle = \langle \xi, b \rangle$ one has $\langle \xi, c \rangle \geq \langle \xi, a \rangle$.*

Proof. Let V be the set of vertices of Δ . If $V = \{a, b\}$ then Lemma 2.1 follows easily. Hence in the rest of the proof assume that there exists a vertex of Δ different from a or b and consider the function

$$\psi(\xi) = \min_{v \in V \setminus \{a, b\}} \langle \xi, v \rangle - \langle \xi, a \rangle, \quad \xi \in \mathbb{R}_{\geq 0}^n.$$

Since the set of vertices is finite, this function is well defined and continuous.

Since E is a compact face of Δ , there exists $\xi_0 \in \mathbb{R}_{> 0}^n$ such that $\langle \xi_0, a \rangle = \langle \xi_0, b \rangle$ and $\langle \xi_0, c \rangle > \langle \xi_0, a \rangle$ for all $c \in \Delta \setminus E$. This yields

$$\psi(\xi_0) > 0.$$

Suppose that there exist $c \in \Delta$ and $\xi_1 \in \mathbb{R}_{\geq 0}^n$ such that $\langle \xi_1, c \rangle < \langle \xi_1, a \rangle = \langle \xi_1, b \rangle$. Since $c = \lambda_1 v_1 + \dots + \lambda_s v_s + z$, for some $z \in \mathbb{R}_{\geq 0}^n$, $v_1, \dots, v_s \in V$ and $\lambda_i \geq 0$ such that $\lambda_1 + \dots + \lambda_s = 1$, we get $\langle \xi_1, v \rangle < \langle \xi_1, a \rangle$ for at least one vertex $v \in V$. Thus

$$\psi(\xi_1) < 0.$$

It follows from the above that there exist ξ in the segment, joining ξ_0 and ξ_1 , such that $\psi(\xi) = 0$. It means that there exists $v \in V \setminus \{a, b\}$ such that $\langle \xi, v \rangle = \langle \xi, a \rangle = \langle \xi, b \rangle \leq \langle \xi, d \rangle$ for all $d \in \Delta$. This implies that Δ^ξ is a compact (since $\xi \in \mathbb{R}_{> 0}^n$) face of dimension ≥ 2 , which contains E . \square

Lemma 2.2. *Let Δ be a Newton polyhedron with a loose edge E that has end points $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$. If $\min(a_1, b_1) = \dots = \min(a_n, b_n) = 0$, then a and b are the only vertices of Δ .*

Proof. By assumption there exist two nonempty and disjoint subsets A, B of the set of indices $\{1, \dots, n\}$ such that $a_i > 0$ for $i \in A$, $a_i = 0$ for $i \in \{1, \dots, n\} \setminus A$, $b_j > 0$ for $j \in B$ and $b_j = 0$ for $j \in \{1, \dots, n\} \setminus B$.

Let $c = (c_1, \dots, c_n)$ be an arbitrary point of Δ different from a and b . For any $i \in A, j \in B$ consider the vector ξ_{ij} , which has only two nonzero entries: $1/a_i$ at place i and $1/b_j$ at place j ($i \neq j$ since $A \cap B = \emptyset$). Then $\langle \xi_{ij}, a \rangle = \langle \xi_{ij}, b \rangle = 1$ and $\langle \xi_{ij}, c \rangle = c_i/a_i + c_j/b_j$. By Lemma 2.1 we get $c_i/a_i + c_j/b_j \geq 1$. It follows that

$$\min_{i \in A} c_i/a_i + \min_{j \in B} c_j/b_j \geq 1.$$

Choose constants $\lambda \geq 0, \mu \geq 0$ such that $\lambda \leq \min_{i \in A} c_i/a_i, \mu \leq \min_{j \in B} c_j/b_j$ and $\lambda + \mu = 1$. Then $c = \lambda a + \mu b + z$ for some $z \in \mathbb{R}_{\geq 0}^n$, hence c cannot be a vertex of Δ . \square

Lemma 2.3. *Let $c \in \mathbb{Z}^n$ be a point with at least one positive and at least one negative coordinate. Then there exists such a basis ξ_1, \dots, ξ_n of the vector space \mathbb{R}^n that $\xi_i \in \mathbb{N}^n$ for $i = 1, \dots, n$ and $\langle \xi_i, c \rangle = 0$ for $i = 1, \dots, n-1$.*

Proof. For any basis ξ_1, \dots, ξ_n of \mathbb{R}^n set $w = (w_1, \dots, w_n) := (\langle \xi_1, c \rangle, \dots, \langle \xi_n, c \rangle)$. By the hypothesis of the lemma, for the standard basis $\xi_1 = (1, 0, \dots, 0), \dots, \xi_n = (0, \dots, 0, 1)$ of \mathbb{R}^n there exist $w_i, w_j \neq 0$ of opposite signs. We will gradually modify the standard basis until we reach a basis such that only one coordinate of w is nonzero. Below we outline the algorithm.

1. If there are only two nonzero entries w_j, w_k of w and $w_j + w_k = 0$, then replace ξ_k by $\xi_k + \xi_j$ and stop.
2. Let j be the index such that $|w_j| = \min\{|w_i| : w_i \neq 0, i \in \{1, \dots, n\}\}$.
 - (a) If there exists w_k of sign opposite to this of w_j , such that $|w_k| > |w_j|$, then replace ξ_k by $\xi_k + \xi_j$,
 - (b) otherwise if there are at least two entries w_k, w_l of sign opposite to this of w_j , then replace ξ_k by $\xi_k + \xi_j$,
 - (c) otherwise set $j := k$ and perform step (a).
3. Go to step 1.

After every loop of the above algorithm w_k is replaced by $w_k + w_j$. Hence the number $\sum_{i=1}^n |w_i|$ decreases and the algorithm must terminate. \square

We encourage the reader to apply the algorithm from the proof in a simple case, for example for $c = (2, 3, -4)$.

From now on up to the end of this section we fix a loose edge E . Let c be a primitive lattice vector parallel to E . Applying Lemma 2.3 to c we find $n-1$ linearly independent vectors $\xi_1, \dots, \xi_{n-1} \in \mathbb{N}^n$ which are orthogonal to E . For every monomial \underline{x}^α we set $\omega(\underline{x}^\alpha) := (\langle \xi_1, \alpha \rangle, \dots, \langle \xi_{n-1}, \alpha \rangle)$. We call this vector a *weight* of \underline{x}^α . Since $\omega(\underline{x}^\alpha \underline{x}^\beta) = \omega(\underline{x}^\alpha) + \omega(\underline{x}^\beta)$ for any monomials $\underline{x}^\alpha, \underline{x}^\beta$, the ring $\mathbb{K}[x_1, \dots, x_n]$ turns into a graded ring $\bigoplus_{w \in \mathbb{N}^{n-1}} R_w$, where R_w is spanned by monomials of weight w .

All monomials of R_w are of the form \underline{x}^{a+ic} where \underline{x}^a is a fixed monomial of R_w and i is an integer. This shows that R_w is a finite dimensional vector space over \mathbb{K} since there is only a finite number of integers i such that all coordinates of $a+ic$ are nonnegative. One can happen that $\dim R_w = 0$. Take for example the weight $\omega(x_1^{\alpha_1} x_2^{\alpha_2}) = 3\alpha_1 + 2\alpha_2$. Then there is no monomial of weight 1, hence $R_1 \subset \mathbb{K}[x_1, x_2]$ is a zero-dimensional vector space.

Let $M \subset \mathbb{N}^{n-1}$ be the set of weights satisfying the condition: $z \in M$ if and only if there exists $\alpha \in \mathbb{Z}^n$ such that $\omega(\underline{x}^\alpha) = z$ and $\langle \xi, \alpha \rangle \geq 0$ for all $\xi \in \mathbb{R}_{\geq 0}^n$ which are orthogonal to E . Observe that M is closed under addition. Moreover for any $w \in \mathbb{N}^{n-1}$ such that $\dim R_w > 0$ we have $w \in M$.

Lemma 2.4. *Let $w \in \mathbb{N}^{n-1}$ and $z \in M$. Assume that R_w contains two coprime monomials. Then*

$$\dim R_{w+z} = \dim R_w + \dim R_z - 1.$$

Proof. For any $v \in \mathbb{N}^{n-1}$ the dimension of the vector space R_v is equal to the number of monomials \underline{x}^α such that $\omega(\underline{x}^\alpha) = v$, hence is equal to the number of lattice points in the set

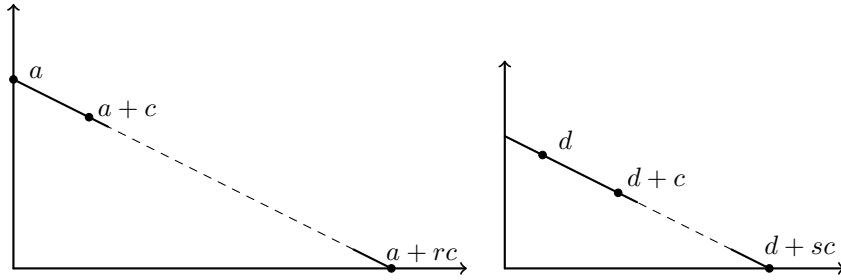
$$l_v = \{ \alpha \in \mathbb{R}_{\geq 0}^n : (\langle \xi_1, \alpha \rangle, \dots, \langle \xi_{n-1}, \alpha \rangle) = v \}.$$

Notice that l_v is the intersection of the straight line $\{a + tc : t \in \mathbb{R}\}$, where $\omega(\underline{x}^a) = v$ and c is a primitive lattice vector parallel to the edge E , with $\mathbb{R}_{\geq 0}^n$.

By the assumption, R_w contains two coprime monomials \underline{x}^a and \underline{x}^b . Hence $\min(a_1, b_1) = \dots = \min(a_n, b_n) = 0$ which yields the partition of $\{1, \dots, n\}$ to three sets $A = \{i \in \{1, \dots, n\} : a_i = 0, b_i > 0\}$, $B = \{i \in \{1, \dots, n\} : a_i > 0, b_i = 0\}$ and $C = \{i \in \{1, \dots, n\} : a_i = 0, b_i = 0\}$. Since A and B are nonempty, a and b are the endpoints of the segment l_w . We may assume without loss of generality that the vector c is pointed in the direction of $b - a$. Then if $\dim R_w = r + 1$, then the lattice points of l_w are $a, a + c, \dots, b = a + rc$.

If the lattice points of l_z are $d, d + c, \dots, d + sc$ then the lattice points of l_{w+z} , are $a + d, a + d + c, \dots, a + d + (r + s)c$. (see Figure 5). This ends the proof in the case $\dim R_z > 0$.

Now suppose that $\dim R_z = 0$. Let \underline{x}^d be any monomial with integer exponents such that $\omega(\underline{x}^d) = z$. Replacing this monomial by \underline{x}^{d+kc} , for suitably chosen integer k , we may assume that $d_{i_0} < 0$ for some $i_0 \in A$ and $d_i + c_i \geq 0$ for all $i \in A$. By the assumption that $z \in M$ we get inequalities $d_{i_0}/c_{i_0} - d_j/c_j \geq 0$ for $j \in B$. Hence $d_j \geq 0$ for all $j \in B$. By the same argument we have $d_j \geq 0$ for $j \in C$. Notice that $d_j + c_j < 0$ for at least one $j \in B$, otherwise all exponents of \underline{x}^{d+c} would be nonnegative. All this information implies that $a + d + c, a + d + 2c, \dots, a + d + rc$ are the only lattice points of l_{w+z} which finishes the proof. \square



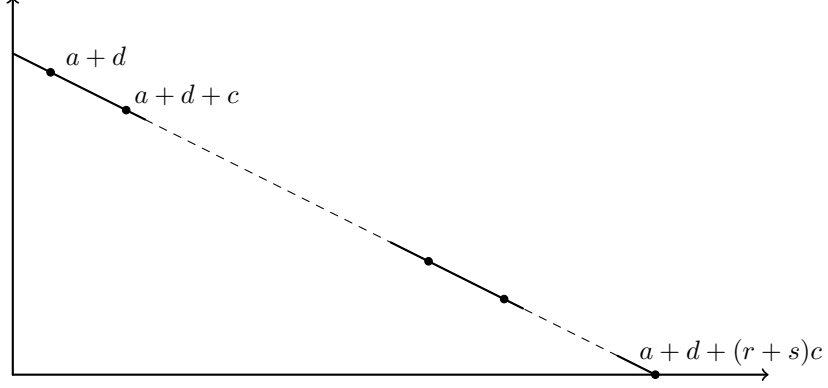


Fig. 5

Lemma 2.5. *Let $G \in R_w$ and $H \in R_z$ be coprime polynomials. If G is not divisible by any monomial then for every $i \in M$*

$$GR_{z+i} + HR_{w+i} = R_{w+z+i}.$$

Proof. Consider the following exact sequence

$$0 \longrightarrow R_i \xrightarrow{\Phi} R_{z+i} \times R_{w+i} \xrightarrow{\Psi} R_{w+z+i},$$

where $\Phi: \eta \mapsto (\eta H, -\eta G)$ and $\Psi: (\psi, \varphi) \mapsto \psi G + \varphi H$. The assumption on G implies that R_w satisfies the hypothesis of Lemma 2.4. Hence we get $\dim \text{Im } \Psi = \dim R_{z+i} + \dim R_{w+i} - \dim R_i = \dim R_{z+i} + (\dim R_w + \dim R_i - 1) - \dim R_i = \dim R_w + \dim R_{z+i} - 1 = \dim R_{w+z+i}$, which implies that Ψ is surjective. \square

Proof of Theorem 1.1. Since all monomials of $f|_E$ have the same weight, it follows from the equality $f|_E = GH$ that $G \in R_w$ and $H \in R_z$ for some $w, z \in \mathbb{N}^{n-1}$. Let \underline{x}^α be any monomial which appears with nonzero coefficient in the power series f and \underline{x}^{α_0} be a fixed monomial of $f|_E$. By Lemma 2.1 we have $\langle \xi, \alpha \rangle \geq \langle \xi, \alpha_0 \rangle$ for every $\xi \in \mathbb{R}_{\geq 0}^n$ which is orthogonal to E . This yields $\omega(\underline{x}^{\alpha - \alpha_0}) \in M$. Since $\omega(\underline{x}^{\alpha_0}) = w + z$, we get $f = \sum_{i \in M} f_{w+z+i}$, where $f_{w+z+i} \in R_{w+z+i}$.

Let $g_w := G$ and $h_z := H$. Then $f_{w+z} = g_w h_z$. Let us set M in degree-lexicographic order. Using Lemma 2.5 we can find recursively $g_{w+i} \in R_{w+i}$ and $h_{z+i} \in R_{z+i}$ such that

$$g_w h_{z+i} + h_z g_{w+i} = f_{w+z+i} - F_i,$$

where

$$F_i = \sum_{\substack{k+l=i, \\ k, l < i}} g_{w+k} h_{z+l}.$$

If $g := \sum_{i \in M} g_{w+i}$ and $h := \sum_{i \in M} h_{z+i}$, then $f = gh$.

Let $\xi = \xi_1 + \dots + \xi_{n-1}$. Then for $E_1 := \Delta(g)^\xi$ and $E_2 := \Delta(h)^\xi$ we have $g|_{E_1} = g_w$, $h|_{E_2} = h_z$ and $E = E_1 + E_2$. \square

Remark 2.6. *The assumption of Theorem 1.1 that G is not divisible by any variable cannot be omitted. Consider the power series $f = (x_3^2 + x_1x_2)(x_3 + x_1x_2)$. Its Newton polyhedron has a loose edge E with endpoints $(1, 1, 1)$, $(2, 2, 0)$ and $f|_E = x_1x_2(x_3 + x_1x_2)$. The irreducible factors of f are $f_1 = x_3^2 + x_1x_2$ and $f_2 = x_3 + x_1x_2$. Hence f cannot be a product of power series g, h such that $g|_{E_1} = x_2(x_3 + x_1x_2)$ and $h|_{E_2} = x_1$.*

Proof of Corollary 1.2. Assume that $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$ are the ends of a loose edge E of the Newton polyhedron $\Delta(f)$. Since $\Delta(f)$ has at least three vertices, Lemma 2.2 implies that $c = (\min(a_1, b_1), \dots, \min(a_n, b_n))$ has at least one nonzero coordinate. The monomials \underline{x}^a and \underline{x}^b appear in the polynomial $f|_E$ with nonzero coefficients and their greatest common divisor equals \underline{x}^c . Thus $\underline{x}^{-c} \cdot f|_E$ is not divisible by any variable, so $\underline{x}^{-c} \cdot f|_E$ and \underline{x}^c are relatively prime. Using Theorem 1.1 we obtain that f is not irreducible. \square

Lemma 2.7. *Let $G \in R_w$ and $H_j \in R_{z_j}$ for $j = 1, 2$. Assume that for every $i \in M$*

$$GR_{z_j+i} + H_jR_{w+i} = R_{w+z_j+i}$$

for $j = 1, 2$. Then for every $i \in M$

$$GR_{z_1+z_2+i} + H_1H_2R_{w+i} = R_{w+z_1+z_2+i}.$$

Proof. By assumptions of the lemma we get

$$\begin{aligned} GR_{z_1+z_2+i} + H_1H_2R_{w+i} &= G(R_{z_1+z_2+i} + H_1R_{z_2+i}) + H_1H_2R_{w+i} \\ &= GR_{z_1+z_2+i} + H_1(GR_{z_2+i} + H_2R_{w+i}) \\ &= GR_{z_1+z_2+i} + H_1R_{w+z_2+i} = R_{w+z_1+z_2+i}. \end{aligned}$$

\square

Assume that E is a descendant loose edge. By definition, there exists a lattice vector $c = (c_1, \dots, c_n)$ parallel to E such that $c_i \geq 0$ for $i = 1, \dots, n-1$ and $c_n < 0$. Let $R'_w := R_w \cap \mathbb{K}[x_1, \dots, x_{n-1}]$ for $w \in \mathbb{N}^{n-1}$. Since every line parallel to E intersects $\mathbb{R}^{n-1} \times \{0\}$ transversely, the dimension of R'_w is 0 or 1.

We claim that for every $w \in M$, $z \in \mathbb{N}^{n-1}$, $0 \neq H \in R'_z$ and $\psi \in R'_{w+z}$ one has $\psi/H \in R'_w$. To prove this claim it is enough to consider $H = \underline{x}^\alpha \in R'_z$ and $\psi = \underline{x}^\beta \in R'_{w+z}$. Then $\omega(\underline{x}^{\beta-\alpha}) = w$. Denote $v_i := e_i - \frac{c_i}{c_n}e_n \in \mathbb{R}_{\geq 0}^n$, where e_1, \dots, e_n is the standard basis of \mathbb{R}^n . Since every vector v_i is orthogonal to E and $w \in M$, we have $0 \leq \langle v_i, \beta - \alpha \rangle = \beta_i - \alpha_i$ for $i = 1, \dots, n-1$. Thus $\psi/H = \underline{x}^{\beta-\alpha}$ is a monomial with nonnegative exponents.

Lemma 2.8. *Let $G \in R_w$ and $H \in R_z$ be coprime polynomials. If G is monic with respect to x_n , then for every $i \in M$*

$$GR_{z+i} + HR_{w+i} = R_{w+z+i}. \quad (1)$$

Proof. First, we prove (1) in the special case $G = x_n$ and $H \in R'_z$.

Every $F \in R_{w+z+i}$ can be written in the form $F = x_n\phi + \psi$ where ϕ, ψ are polynomials and ψ does not depend on x_n . Then $\phi \in R_{z+i}$ and $\psi = H\psi'$ where $\psi' = \psi/H \in R'_{w+i}$. This gives (1) in the special case. All remaining cases follow from Lemma 2.5 and Lemma 2.7. \square

Proof of Theorem 1.4. Proceeding as in the proof of Theorem 1.1, but using Lemma 2.8 instead of Lemma 2.5 we obtain $\bar{g}, \bar{h} \in \mathbb{K}[[x_1, \dots, x_{n-1}, x_n]]$ such that $f = \bar{g}\bar{h}$, $\bar{g}|_{E_1} = G$ and $\bar{h}|_{E_2} = H$ for some segments E_1, E_2 , where $E = E_1 + E_2$. The assumptions that the loose edge E is descendant and the polynomial G is monic with respect to x_n imply that the Newton polyhedron of \bar{g} has a vertex $(0, \dots, 0, d)$ for some positive integer d . Therefore $\bar{g}(0, \dots, 0, x_n) = vx_n^d$ for some $v \in \mathbb{K}[[x_n]]$ such that $v(0) \neq 0$. It means that \bar{g} fulfills assumptions of the Weierstrass Preparation Theorem, which implies that $\bar{g} = u\hat{g}$, where u is a power series such that $u(0) \neq 0$ and $\hat{g} \in \mathbb{K}[[x_1, \dots, x_{n-1}]][[x_n]]$ is a Weierstrass polynomial. Putting $g = \hat{g}$ and $h = u\bar{h}$ we get $f = gh$. Since we can also obtain h using the polynomial division of f by g in the ring $\mathbb{K}[[x_1, \dots, x_{n-1}]][[x_n]]$, we conclude that h is a polynomial with respect to x_n . \square

3 Relation with known results

Corollary 1.3 generalizes to $n > 2$ variables the following well-known fact.

Theorem 3.1. *Assume that a power series $f \in \mathbb{C}[[x, y]]$ written as a sum of homogeneous polynomials $f = f_d + f_{d+1} + \dots$, where the subindex means the degree, is irreducible. Then f_d is a power of a linear form.*

Below we quote some notations of [1] and Lemma A1 of that paper.

Let \mathbb{K} be a field and fix a weight $\omega(x^i y^j) := ni + mj$ for $n, m \in \mathbb{N}$. Given $0 \neq F \in \mathbb{K}[[x, y]]$, we will consider its decomposition in ω -quasihomogeneous forms

$$F(x, y) = F_{a+b}(x, y) + F_{a+b+1}(x, y) + \dots,$$

where the subindex means the ω -weight.

Theorem 3.2. *Assume that $F_{a+b}(x, y) = f_a(x, y)g_b(x, y)$, where $f_a, g_b \in \mathbb{K}[[x, y]]$ are quasihomogeneous and coprime. Then, there exist*

$$f, g \in \mathbb{K}[[X, Y]], \quad f = f_a + f_{a+1} + \dots, \quad g = g_b + g_{b+1} + \dots$$

such that $F = fg$. Moreover if f_a is an irreducible polynomial, then f is an irreducible power series.

Theorem 1.1 can be seen as a generalization of this results.

Corollary 1.3 generalizes the following result of [2]

Theorem 3.3. *If $\phi \in \mathbb{C}\{x_1, \dots, x_n\}$ is irreducible and has a polygonal Newton polyhedron $\Delta(\phi)$, then the polyhedron $\Delta(\phi)$ has only one compact edge E and the polynomial $\phi|_E$ is a power of an irreducible polynomial.*

The term polygonal Newton polyhedron in the statement of the above theorem is used for Newton polyhedra such that all their compact edges are loose.

Theorem 1.4 generalizes the main result of [6] quoted below.

Theorem 3.4. *Let $P(Z) \in k[[x_1, \dots, x_n]][Z]$ be a monic Weierstrass polynomial. Assume that $P(Z)$ has an orthant associated polyhedron and that $P|_\Gamma \in k[x_1, \dots, x_n, Z]$ is the product of two coprime monic polynomials $S_1, S_2 \in k[x_1, \dots, x_n, Z]$, respectively, of degree d_1 and d_2 . Then there exist two monic polynomials $P_1, P_2 \in k[[x_1, \dots, x_n]][Z]$, respectively, of degrees d_1 and d_2 in Z such that*

i) $P = P_1 P_2$,

ii) there is at least one $i \in \{1, 2\}$ such that P_i has an orthant associated polyhedron and if Γ_i denotes the compact face of $\Delta(P_i)$ containing the points $(0, \dots, 0, d_i)$, then $P_i|_{\Gamma_i} = S_i$ and Γ_i is parallel to Γ .

The term *orthant associated polyhedron* in the statement of the above theorem means a Newton polyhedron that has a loose edge with endpoint $(0, \dots, 0, d)$.

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