COMPONENTS OF THE HILBERT SCHEME OF SMOOTH PROJECTIVE CURVES USING RULED SURFACES

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ABSTRACT. Let $\mathcal{I}_{d,g,r}$ be the union of irreducible components of the Hilbert scheme whose general points correspond to smooth irreducible non-degenerate curves of degree d and genus g in \mathbb{P}^r . We use families of curves on cones to show that under certain numerical assumptions for d, g and r, the scheme $\mathcal{I}_{d,g,r}$ acquires generically smooth components whose general points correspond to curves that are double covers of irrational curves. In particular, in the case $\rho(d, g, r) := g - (r + 1)(g - d + r) \ge 0$ we construct explicitly a *regular component* that is different from the distinguished component of $\mathcal{I}_{d,g,r}$ dominating the moduli space \mathcal{M}_g . Our result implies also that if $g \ge 57$ then $\mathcal{I}_{\frac{4g}{3},g,\frac{g+1}{2}}$ has at least two generically smooth components parametrizing linearly normal curves.

1. INTRODUCTION

Let $\mathcal{I}_{d,g,r}$ be the union of irreducible components of the Hilbert scheme whose general points correspond to smooth irreducible non-degenerate complex curves of degree dand genus g in \mathbb{P}^r . A component of $\mathcal{I}_{d,g,r}$ is called *regular* if it is reduced and of expected dimension $\lambda_{d,g,r} := (r+1)d - (r-3)(g-1)$. Otherwise it is called *superabundant*. For $\rho(d,g,r) := g - (r+1)(g - d + r) \ge 0$, it is known that $\mathcal{I}_{d,g,r}$ has the unique component dominating \mathcal{M}_q , see [18, p. 70]. It is usually referred to as the *distinguished* component.

Historically, Severi claimed in [23] that $\mathcal{I}_{d,g,r}$ is irreducible if $d \ge g + r$. It was proved that $\mathcal{I}_{d,g,r}$ is irreducible if $d \ge g + r$ and r = 3, 4, see [11] and [12]. On the other hand, for $r \ge 5$ (and $\rho(d, g, r) \ge 0$) there have been given several examples in which $\mathcal{I}_{d,g,r}$ possesses additional non-distinguished components ([20], [19], [6], [10], etc), but for none of them it has been proven to be regular. Note that all these examples are given by non-linearly normal curves. We remark that in [6] we showed the existence of a *non-distinguished* component $\mathcal{D}_{d,g,r}$ of $\mathcal{I}_{d,g,r}$ parameterizing curves that are double covers of irrational curves, whereas all other known to us examples of reducible Hilbert schemes of curves have used curves that are *m*-sheeted coverings of \mathbb{P}^1 with $m \ge 3$.

²⁰⁰⁰ Mathematics Subject Classification. Primary 14C05; Secondary 14H10.

Key words and phrases. Hilbert scheme of curves, Brill-Noether theory, double covering.

The first author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(NRF-2016R1D1A3B03933342). The third author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (NRF-2016R1D1A1B03930844).

In [6, Question 4.7, p. 598] we asked about the possibility of $\mathcal{D}_{d,g,r}$ being reduced. In the present paper we reconstruct this component under less constrains, this time using a family of curves on cones, which are double coverings of hyperplane sections of the cones. We construct and characterize the properties of $\mathcal{D}_{d,g,r}$ using tools from the theory of ruled surfaces, while in [6] we only showed its existence using Brill-Noether theory of linear series on curves. Our approach is motivated by the fact that for a given double covering $\varphi : X \to Y$ the curves X and Y can be regarded as curves on the ruled surface S := $\mathbb{P}(\varphi_*\mathcal{O}_X)$, as we explain in section 2. It allows us to construct the additional component in a more geometric way and to obtain its generic smoothness, which gives an affirmative answer to the question raised in [6].

Our main result is as follows.

Theorem A. Assume that g and γ are integers with $g \ge 4\gamma - 2 \ge 38$. Let

$$d := 2g - 4\gamma + 2$$
 and $\max\left\{\gamma, \frac{2(g-1)}{\gamma}\right\} \le r \le R := g - 3\gamma + 2$.

Then the Hilbert scheme $\mathcal{I}_{d,g,r}$ possesses a generically reduced component $\mathcal{D}_{d,g,r}$ for which

$$\dim \mathcal{D}_{d,g,r} = \lambda_{d,g,r} + r\gamma - 2g + 2.$$

Further, let $X_r \subset \mathbb{P}^r$ be a smooth curve corresponding to a general point of $\mathcal{D}_{d,g,r}$.

- (i) If r = R then X_R is the intersection of a general quadric hypersurface with a cone over a smooth curve Y of degree $g 2\gamma + 1$ and genus γ in \mathbb{P}^{R-1} and X_R is embedded in \mathbb{P}^R by the complete linear series $|R_{\varphi}|$ on X_R , where R_{φ} is the ramification divisor of the natural projection morphism $\varphi : X_R \to Y$ of degree 2 given by the ruling of cone;
- (ii) If r < R then X_r is given by a general projection of some X_R as in (i), that is, X_r is embedded in \mathbb{P}^r by a general linear subseries g_d^r of $|R_{\varphi}|$.

In our view, one of the interesting implications of Theorem A is that if $r = \frac{2(g-1)}{\gamma} \ge \gamma \ge 10$ and $d = 2g - 4\gamma + 2$, then the scheme $\mathcal{I}_{d,g,r}$ acquires a second regular component in addition to its distinguished component dominating the moduli space \mathcal{M}_g , see Corollary 9. To our best knowledge, it is the first example in which simultaneous existence of two distinct regular components of $\mathcal{I}_{d,g,r}$ has been observed in the Brill-Noether case $\rho(d,g,r) \ge 0$. We remark also that in the case $g = 6\gamma - 3$ and $r = R = 3\gamma - 1$, the Hilbert scheme $\mathcal{I}_{\frac{4g}{3},g,\frac{g+1}{2}}$ has at least two generically smooth components parametrizing linearly normal curves as it is explained in Remark 11.

The remaining sections of the paper are organized as follows. In section 2, we provide a motivation for the construction of the component described in Theorem A by reviewing the relations between double coverings of curves, ruled surfaces and their embeddings as cones. We also prove there several statements that will be used for the construction of $\mathcal{D}_{d,g,r}$ in section 4. Possibly, some of them might be of independent interest. In section 3 we briefly review several facts about the Gaussian map associated to linear

series on curves and prove a technical result facilitating the computation of the dimension of the tangent space at a general point of $\mathcal{D}_{d,g,r}$. In section 4 we give the proof of Theorem A.

We work over \mathbb{C} . We understand by *curve* a smooth integral projective algebraic curve. We denote by L^{\vee} the dual line bundle for a given line bundle L defined on an algebraic variety X. As usual, ω_X will stand for the canonical line bundle on X. We denote by |L| the complete linear series $\mathbb{P}(H^0(X, L))$. When X is an object of a family, we denote by [X] the corresponding point of the Hilbert scheme representing the family. Throughout the entire paper

$$d := 2g - 4\gamma + 2 \quad \text{and} \quad R := g - 3\gamma + 2.$$

For definitions and properties of the objects not explicitly introduced in the paper refer to [17] and [1].

Acknowledgements

We thank KIAS for the warm hospitality when we were associate members in KIAS and the second author visited there. We would like to thank the referees for the constructive comments and valuable suggestions, which helped to improve the quality of our paper.

2. MOTIVATION AND PRELIMINARY RESULTS

Suppose that $\varphi : X \to Y$ is an m : 1 cover, $m \ge 2$, where X and Y are smooth curves of genus g and γ , correspondingly. As it is well known, the covering induces a short exact sequence of vector bundles on Y

$$0 \to \mathcal{O}_Y \xrightarrow{\varphi^{\sharp}} \varphi_* \mathcal{O}_X \to \mathcal{E}^{\vee} \to 0 \,,$$

where \mathcal{E}^{\vee} is the so called *Tschirnhausen module*, see [21]. It is a rank (m-1)-vector bundle on Y. Since X and Y are curves over \mathbb{C} , the exact sequence splits, i.e. $\varphi_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{E}^{\vee}$. According to [17, Ex. IV.2.6, p. 306], $(\det \varphi_*\mathcal{O}_X)^2 \cong \mathcal{O}_Y(-B)$, where B is the branch divisor of the covering. In particular, $\deg B = 2(g-1) - 2m(\gamma - 1)$.

We focus on the case m = 2. In such a case \mathcal{E} must be a line bundle on Y and we can assume that $\mathcal{E} = \mathcal{O}_Y(E)$ for some divisor E on Y. Since

$$\deg B = -\deg(\det \varphi_* \mathcal{O}_X)^2 = -\deg(\det(\mathcal{O}_Y \oplus \mathcal{O}_Y(-E)))^2 = 2\deg E$$

it follows that deg $E = g - 2\gamma + 1$.

Further we suppose that *E* is a nonspecial and very ample divisor on *Y*. Denote by \mathcal{F} the rank 2 vector bundle $\mathcal{F} := \mathcal{O}_Y \oplus \mathcal{O}_Y(E)$ on *Y* and let *S* be the ruled surface $S := \mathbb{P}(\mathcal{F})$ with natural projection $f : S \to Y$. Since deg E > 0, $\mathcal{F}_0 := \mathcal{O}_Y \oplus \mathcal{O}_Y(-E)$ will be the normalization of the vector bundle \mathcal{F} . As it is decomposable, $f : S \to Y$ has two canonically determined sections. They are Y_0 which corresponds to the short exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y \oplus \mathcal{O}_Y(-E) \to \mathcal{O}_Y(-E) \to 0$$

and Y_1 which corresponds to the short exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y \oplus \mathcal{O}_Y(E) \to \mathcal{O}_Y(E) \to 0$$
.

The section Y_0 is the section with minimal self-intersection on S and $Y_0^2 = \deg(\mathcal{O}_Y(-E)) = -g + 2\gamma - 1$. As it well known, $\operatorname{Pic}(S) \cong \mathbb{Z}[Y_0] \oplus f^*(\operatorname{Pic}(Y))$. For a divisor D on Y we will denote by $D\mathfrak{f}$ the divisor $f^*(D)$ on S. Also, we have for the section Y_1 that $Y_1^2 = \deg(\mathcal{O}_Y(E)) = g - 2\gamma + 1$ and it is not difficult to see that $Y_1 \sim Y_0 + E\mathfrak{f}$. In general, cohomologies like $h^j(S, \mathcal{O}_S(nY_0 + D\mathfrak{f}))$ are calculated using the projection formula, see [17, Ex. III.8.3, p. 253], as

$$h^{j}(S, \mathcal{O}_{S}(nY_{0} + D\mathfrak{f})) = h^{j}(Y, \mathcal{S}ym^{n}(\mathcal{F}_{0}) \otimes \mathcal{O}_{Y}(D))$$

but since S is decomposable, i.e. \mathcal{F}_0 splits, the calculation reduces simply to

(1)
$$h^{j}(S, \mathcal{O}_{S}(nY_{0}+D\mathfrak{f})) = \sum_{k=0}^{n} h^{j}(Y, \mathcal{O}_{Y}(D-kE)),$$

see for example [14]. From here

(2)
$$h^0(S, \mathcal{O}_S(Y_1)) = h^0(S, \mathcal{O}_S(Y_0 + E\mathfrak{f})) = h^0(Y, \mathcal{O}_Y(E)) + h^0(Y, \mathcal{O}_Y) = g - 3\gamma + 3.$$

Using [17, Ex. V.2.11 (a), p. 385], we obtain that the linear series $|\mathcal{O}_S(Y_1)\rangle| \equiv |\mathcal{O}_S(Y_0 + E\mathfrak{f})\rangle|$ is base point free. Therefore it defines a morphism

$$\Psi := \Psi_{|\mathcal{O}_S(Y_1)|} : S \to \mathbb{P}^R,$$

where $R = g - 3\gamma + 2$. Since *E* is very ample, it follows by [14, Proposition 23, p. 38] that Ψ is isomorphism away from Y_0 . Due to $Y_0 \cdot Y_1 = Y_0 \cdot (Y_0 + E\mathfrak{f}) = 0$, the morphism Ψ contracts the curve Y_0 to a point. Therefore $F := \Psi(S) \subset \mathbb{P}^R$ is a cone of degree

$$\deg F = Y_1 \cdot Y_1 = (Y_0 + E\mathfrak{f}) \cdot (Y_0 + E\mathfrak{f}) = \deg E = g - 2\gamma + 1$$

over the image of a smooth integral curve from the linear series $|\mathcal{O}_S(Y_0 + E\mathfrak{f})|$.

By Bertini's theorem, Ψ maps a general element of $|\mathcal{O}_S(Y_1)||$ to a smooth integral curve of genus γ , degree $g - 2\gamma + 1$, which is further linearly normally embedded in some hyperplane \mathbb{P}^{R-1} of \mathbb{P}^R due to (2). A similar fact is true about a general element of $|\mathcal{O}_S(2Y_1)||$. Namely, a general $C \in |\mathcal{O}_S(2Y_1)|| \equiv |\mathcal{O}_S(2Y_0 + 2E\mathfrak{f})||$ is mapped by Ψ to a smooth integral curve $\Psi(C)$ of genus g, degree $2g - 4\gamma + 2 = d$, which is linearly normal in \mathbb{P}^R . Indeed, since $Y_0 \cdot Y_1 = 0$ and Ψ is isomorphism away from Y_0 , it follows by Bertini's theorem that $\Psi(C)$ is smooth and integral. Its degree is deg $\Psi(C) = 2Y_1 \cdot Y_1 = 2g - 4\gamma + 2$, while by the adjunction formula

$$\deg C \cdot (K_S + C) = (2Y_1) \cdot (K_S + 2Y_1) = 2(2\gamma - 2) + 2g - 4\gamma + 2 = 2g - 2$$

we get that its genus is g. Finally, to see that $\Psi(C) \subset \mathbb{P}^R$ is linearly normal, consider the exact sequence

$$0 \to \mathcal{O}_S(-Y_0 - E\mathfrak{f}) \to \mathcal{O}_S(Y_0 + E\mathfrak{f}) \to \mathcal{O}_C(Y_0 + E\mathfrak{f}) \to 0.$$

It is sufficient to see that $h^1(S, \mathcal{O}_S(-Y_0 - E\mathfrak{f})) = 0$, which is not difficult to obtain using the Serre duality.

The arguments above motivate the following statement.

Proposition 1. Assume that Y is a smooth curve of genus γ and E is a very ample non-special divisor on Y of degree e. Let $S := \mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(-E))$, Y_0 be the section of minimal self-intersection of the natural projection $f : S \to Y$ and $Y_1 \in |\mathcal{O}_S(Y_0 + E\mathfrak{f})|$ be a smooth integral curve. Let $\Psi := \Psi_{|\mathcal{O}_S(Y_0 + E\mathfrak{f})|}$ be the morphism induced by the complete linear series $|\mathcal{O}_S(Y_0 + E\mathfrak{f})|$. Then:

- (a) $|\mathcal{O}_S(Y_0 + E\mathfrak{f})|$ is base point free and of dimension $e \gamma + 1$;
- (b) Ψ is an isomorphism away from Y_0 and contracts Y_0 to a point in $\mathbb{P}^{e-\gamma+1}$, in particular, $\Psi(S)$ is a cone over $\Psi(Y_1)$;
- (c) for a general $C \in |\mathcal{O}_S(2Y_0 + 2E\mathfrak{f})|$
 - (c.1) $\Psi(C)$ is a linearly normal smooth irreducible curve of genus $2\gamma + e 1$ and degree 2e in $\mathbb{P}^{e-\gamma+1}$;
 - (c.2) the linear series $|\mathcal{O}_C(R_{\varphi})|$ on C is traced by the linear series $|\mathcal{O}_S(Y_0 + E\mathfrak{f})|$ on S, where R_{φ} is the ramification divisor of the morphism $\varphi : C \to Y$ induced by the ruling of S.

Proof. Statements (a), (b) and (c.1) are obtained by very similar arguments like those in the discussion preceding the proposition. We only need to check (c.2). Recall that $K_S \sim -2Y_0 + (K_Y - E)\mathfrak{f}$. On C we have $K_C - \varphi^* K_Y \sim R_{\varphi}$, i.e. $\mathcal{O}_C(R_{\varphi}) = \omega_C \otimes (\varphi^* \omega_Y)^{\vee}$. The canonical divisor K_C on C is induced by the restriction of $K_S + C \sim K_S + (2Y_0 + 2E\mathfrak{f})$ on C. Similarly, the restriction of $K_S + Y_1 \sim K_S + Y_0 + E\mathfrak{f}$ on Y_1 induces K_{Y_1} . Therefore

$$R_{\varphi} \sim (K_S + (2Y_0 + 2E\mathfrak{f}) - (K_S + Y_0 + E\mathfrak{f}))_{|_C} \sim (Y_0 + E\mathfrak{f})_{|_C}.$$

By (a) and (c.1), $h^0(C, \mathcal{O}_C(Y_0 + E\mathfrak{f}_{|_C})) = h^0(S, \mathcal{O}_S(Y_0 + E\mathfrak{f})) = e - \gamma + 2$. Therefore the linear series $|\mathcal{O}_S(Y_0 + E\mathfrak{f})|$ on S induces the linear series $|\mathcal{O}_C(R_{\varphi})|$ on C.

Remark 2. When $e = g - 2\gamma + 1 \ge 2\gamma - 1$ and the divisor E on Y is very ample, where $\mathcal{O}_Y(-E)$ is the Tschirnhausen module of a double covering $X \to Y$, statement (c.2) implies that $\mathcal{O}_C(R_{\varphi})$ is very ample and $h^0(C, \mathcal{O}_C(R_{\varphi})) = g - 3\gamma + 3$. It improves a similar claim proved in [6, Lemma 4.1] where it was assumed that $g \ge 6\gamma - 1$.

Remark 3. Proposition 1 suggests how to give an alternative construction of the component $\mathcal{D}_{2g-4\gamma+2,g,r}$ constructed in [6, Theorem 4.3, p. 594]. For this take $e = g - 2\gamma + 1 \ge 2\gamma - 1$ and consider the family \mathcal{Z} of surface scrolls $F \subset \mathbb{P}^R$, over a curve Y of genus γ , deg $F = \deg Y = e = g - 2\gamma + 1$ with $h^0(F, \mathcal{O}_F(1)) = g - 3\gamma + 3$ and $h^1(F, \mathcal{O}_F(1)) = \gamma$. According to [3, Lemma 1, p. 7] such a scroll is necessarily a cone, say F, over a projectively normal curve in \mathbb{P}^{R-1} of genus

 γ and degree *e*. Further, let \mathcal{F} be the family of smooth curves in $|\mathcal{O}_F(2)|$ on the cones $F \subset \mathbb{P}^R$ from the family \mathcal{Z} . By a counting of the parameters on which the family \mathcal{Z} depends, similar to the one carried out in [3, Remark 2, p. 15] and [4, Proposition 7.1, p. 150],

$$\dim \mathcal{Z} = + 3\gamma - 3 : number of parameters of curves $Y \in \mathcal{M}_{\gamma} + \gamma : number of parameters of line bundles \mathcal{O}_Y(E) \in \operatorname{Pic}(Y) of degree $g - 2\gamma + 1 \ge 2\gamma - 1$ necessary to fix the geometrically ruled surface $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(-E)) + (R+1)^2 - 1 = \dim(\operatorname{Aut}(\mathbb{P}^R)) - ((g - 2\gamma + 1) - \gamma + 2) = \dim G_F$, where G_F is the subgroup of $\operatorname{Aut}(\mathbb{P}^R)$ fixing the scroll F , see [4, Lemma 6.4, p. 148]$$$

one finds that dim $\mathcal{Z} = 7(\gamma - 1) - g + (R + 1)^2$. On the other hand computing dim $|\mathcal{O}_F(2)|$ using (1) and the Riemann-Roch formula, we get easily dim $|\mathcal{O}_F(2)| = 3g - 8\gamma + 5$. Therefore for the dimension of \mathcal{F} we obtain

$$\dim \mathcal{F} = \dim \mathcal{Z} + \dim |\mathcal{O}_F(2)| = 2g - \gamma - 2 + (g - 3\gamma + 3)^2.$$

It is precisely the dimension of the component $\mathcal{D}_{2g-4\gamma+2,g,r}$ constructed in [6, Theorem 4.3] when $r = R = g - 3\gamma + 2$ and it improves the bound calculated in [6, Lemma 4.1] where it was assumed that $g \ge 6\gamma - 1$.

The above arguments do not imply yet that the family \mathcal{F} gives rise to a component of the Hilbert scheme $\mathcal{I}_{d,g,R}$. To prove this formally, we will compute in section 4 $h^0(C, N_{C/\mathbb{P}^R})$ for a general $C \in \mathcal{F}$. For the purposes of that computation we need several more formal statements about the normal bundles of curves on cones, which we prove below.

Lemma 4. Let X be a smooth non-degenerate curve in \mathbb{P}^r and let H be a hyperplane in \mathbb{P}^r . Assume that $\pi_p : X \to H \subset \mathbb{P}^r$ is a projection from a point $p \notin H \cup X$ such that the image $Y := \pi_p(X)$ is smooth in \mathbb{P}^{r-1} . Then

(3)
$$0 \to O_X(R_{\pi_p}) \otimes \mathcal{O}_X(1) \to N_{X/\mathbb{P}^r} \to \pi_p^* N_{Y/\mathbb{P}^{r-1}} \to 0,$$

where R_{π_p} is the ramification divisor of the covering $\pi_p : X \to Y$.

Proof. Since $\pi_p : X \to \mathbb{P}^{r-1} \subset \mathbb{P}^r$ is a projection from a point $p \notin X$, we have $\pi_p^*(\mathcal{O}_Y(1)) = \mathcal{O}_X(1)$. For the curves X and Y we have the Euler sequences

$$0 \to \mathcal{O}_X \to \oplus^{r+1} \mathcal{O}_X(1) \to T_{\mathbb{P}^r|_X} \to 0$$

and

$$0 \to \mathcal{O}_Y \to \oplus^r \mathcal{O}_Y(1) \to T_{\mathbb{P}^{r-1}|_Y} \to 0$$

Pulling the second sequence to *X* via π_p we obtain

where α is the induced map between the restrictions of $T_{\mathbb{P}^r|_X}$ and $\pi_p^*(T_{\mathbb{P}^{r-1}|_Y})$ and ker (α) is its kernel. By the Snake lemma we obtain

$$0 \to \mathcal{O}_X(1) \to T_{\mathbb{P}^r|_X} \to \pi_p^*\left(T_{\mathbb{P}^{r-1}|_Y}\right) \to 0.$$

Further, using the normal bundle sequence for N_{X/\mathbb{P}^r} and $N_{Y/\mathbb{P}^{r-1}}$, we get the following commutative diagram

where β is the induced map between the normal bundles N_{X/\mathbb{P}^r} and $\pi_p^*(N_{Y/\mathbb{P}^{r-1}})$. Similarly as before, by the Snake lemma we get ker $\beta \cong \mathcal{O}_X(R_{\pi_p}) \otimes \mathcal{O}_X(1)$, and thus we deduce the short exact sequence

$$0 \to \mathcal{O}_X(R_{\pi_p}) \otimes \mathcal{O}_X(1) \to N_{X/\mathbb{P}^r} \to \pi_p^* N_{Y/\mathbb{P}^{r-1}} \to 0$$

Corollary 5. Suppose that $Y \subset \mathbb{P}^{r-1} \subset \mathbb{P}^r$, $r \geq 3$, is a smooth non-degenerate curve of genus γ . Let $p \in \mathbb{P}^r \setminus \mathbb{P}^{r-1}$ be and arbitrary point. Consider the cone $F \subset \mathbb{P}^r$ over Y with vertex p. Suppose that a curve $X \subset F$ is cut by a general hypersurface $Q_m \subset \mathbb{P}^R$ of degree m, i.e. $X \in |\mathcal{O}_F(m)|$ is general. Let $\varphi : X \xrightarrow{m:1} Y$ be the m-sheeted covering map induced by the ruling of the cone. Then there is an exact sequence

(4)
$$0 \to \mathcal{O}_X(m) \to N_{X/\mathbb{P}^r} \to \varphi^* N_{Y/\mathbb{P}^{r-1}} \to 0.$$

Proof. The line bundle $O_X(R_{\varphi})$ associated to the ramification divisor R_{φ} of the covering $\varphi : X \to Y$ has the property $\mathcal{O}_X(R_{\varphi}) \simeq \mathcal{O}_X(m-1)$. To see this, recall that $R_{\varphi} \sim K_X - \varphi^* K_Y$. The canonical divisor K_X on X is cut by the restriction of $K_F + X$ on X and K_Y is cut by the restriction of $K_F + Y$ on Y. Therefore

$$K_X - \varphi^* K_Y = (K_F + X)|_X - (K_F + Y)|_X \sim (X - Y)|_X \sim (m - 1)Y|_X.$$

Hence $\mathcal{O}_X(R_{\varphi}) \simeq \mathcal{O}_X(m-1)$ and Lemma 4 yields the exact sequence (4).

Corollary 6. Let $X, Y \subset F \subset \mathbb{P}^r$ be smooth curves on the cone F with vertex p as in Corollary 5, where $r \geq 6$. Let $W \subset \mathbb{P}^r$ be a general projective subspace of \mathbb{P}^r of dimension r - s - 1, where $5 \leq s \leq r - 1$. Consider the projection $\pi_W : \mathbb{P}^r \setminus W \to \mathbb{P}^s$ with center W to a general projective subspace of \mathbb{P}^r of dimension s. Denote by X_s, Y_s and F_s the images of X, Y and F under π_W . Let $\varphi_s : X_s \to Y_s$ be the covering map induced by the ruling of F_s . Then

(5)
$$0 \to \mathcal{O}_{X_s}(m) \to N_{X_s/\mathbb{P}^s} \to \varphi_s^* N_{Y_s/\mathbb{P}^{s-1}} \to 0.$$

Proof. Since $r \ge s+1 \ge 6$, a general projective subspace of \mathbb{P}^r of dimension r-s-1 does not meet the secant variety of F, which is of dimension at most 5. Therefore X, Y and Fare isomorphic to their images X_s, Y_s and F_s . Also, the m : 1 covering $\varphi : X \to Y$ induced by the ruling on F goes to an m : 1 covering $\varphi_s : X_s \to Y_s$ induced by the ruling on F_s such that $\pi_{W|_Y} \circ \varphi = \varphi_s \circ \pi_{W|_X}$. In particular, $\pi_{W|_X}(R_{\varphi}) = R_{\varphi_s}$. Thus the ramification divisor R_{φ_s} is linearly equivalent to a divisor cut on X_s by a hypersurface of degree m-1in \mathbb{P}^s . Hence $\mathcal{O}_{X_s}(R_{\varphi_s}) \simeq \mathcal{O}_{X_s}(m-1)$ and Lemma 4 gives the exact sequence (5).

3. A SHORT NOTE ON THE GAUSSIAN MAP

Let *Y* be a smooth curve of genus γ and *L* and *M* be line bundles on *Y*. Let $\mu_{L,M}$

(6)
$$\mu_{L,M} : H^0(Y,L) \otimes H^0(Y,M) \to H^0(Y,L \otimes M)$$

be the natural multiplication. The Gaussian map $\Phi_{L,M}$

$$\Phi_{L,M} : \ker \mu_{L,M} \to H^0(Y, L \otimes M \otimes \omega_Y)$$

was introduced by Wahl in [24]. Locally, $\Phi_{L,M} : s \otimes t \mapsto sdt - tds$ for sections $s \in H^0(L)$ and $t \in H^0(M)$. It has been studied by a number of authors. We refer to [24] and [5] for its precise definition and some properties. We recall only several notions that will be used in Proposition 7 needed for the proof of Theorem A.

The notation R(L, M) is often used instead of ker $\mu_{L,M}$ for the map $\mu_{L,M}$ in (6). When $V \subset H^0(Y, L)$ is a vector subspace and $M = \omega_Y$, the map $\mu_{L,M}$ in (6) restricted on $V \otimes H^0(Y, \omega_Y)$ will be denoted by μ_V and the Gaussian map restricted on ker μ_V will be denoted by $\Phi_{\omega_Y,V}$.

The proposition that follows is formulated in the specific form in which it will be used in the proof of Theorem A.

Proposition 7. Let *Y* be a smooth curve of general moduli of genus $\gamma \ge 10$, and let *E* be a general line bundle on *Y* of degree $g - 2\gamma + 1 \ge 2\gamma - 1$. Let $V \subseteq H^0(Y, E)$ be general linear subspace of dimension $r = \dim V \ge \max\left\{\gamma, \frac{2(g-1)}{\gamma}\right\}$. Consider the embedding $Y \subset \mathbb{P}^{r-1} \equiv \mathbb{P}(V^{\vee})$ given by *V*. Then

- the restricted Gaussian mapping $\Phi_{\omega_Y,V}$ is surjective, and
- $h^0(N_{Y/\mathbb{P}^{r-1}}(-1)) = \dim V = r.$

Proof. Denote by μ the cup-product map

$$\mu: H^0(Y, E) \otimes H^0(Y, \omega_Y) \to H^0(Y, \omega_Y \otimes E) .$$

Since deg $E = g - 2\gamma + 1 \ge 2\gamma - 1$, so *E* is very ample, it follows by [16, Theorem (4.e.1) and Theorem (4.e.4)] and [7] that μ is surjective.

The linear series determined by V is very ample since $Y \in \mathcal{M}_{\gamma}$ is general, $\gamma \geq 10$ and $V \subset H^0(Y, E)$ is also general. Consider the restriction μ_V of μ to

$$\mu_V: V \otimes H^0(Y, \omega_Y) \to H^0(Y, \omega_Y \otimes E).$$

Let $R(\omega_Y, E)$ be the kernel of the map μ and consider the Gaussian map $\Phi_{\omega_Y, E}$ defined on $R(\omega_Y, E)$

$$\Phi_{\omega_Y,E}: R(\omega_Y, E) \to H^0(\omega_Y^2 \otimes E)$$

and similarly its restriction $\Phi_{\omega_Y,V}$ defined on the kernel $R(\omega_Y,V)$ of the map μ_V

(7)
$$\Phi_{\omega_Y,V}: R(\omega_Y, V) \to H^0(\omega_Y^2 \otimes E)$$

In the case of complete embedding, i.e. if $V = H^0(Y, E)$, the claim follows by [9, Proposition 1.2], where it is proven that

$$h^{0}(N_{Y/\mathbb{P}^{r-1}}(-1)) = h^{0}(Y, E) + \operatorname{corank}(\Phi_{\omega_{Y}, E}),$$

and by [8, Proposition (2.9)], where it is proven that $\Phi_{\omega_Y,E}$ is surjective for $\gamma \ge 10$ and $\deg E = g - 2\gamma + 1 \ge 2\gamma - 1$. In the case of incomplete embedding, i.e. if $V \subsetneq H^0(Y, E)$, exactly the same argument as in the proof of [9, Proposition 1.2] shows that

(8)
$$h^0(N_{Y/\mathbb{P}^{r-1}}(-1)) = \dim V + \operatorname{corank}(\Phi_{\omega_Y,V}) = r + \operatorname{corank}(\Phi_{\omega_Y,V}) ,$$

provided that μ_V is surjective. This is what we will prove next.

Since μ_V is the restriction of μ to $V \otimes H^0(Y, \omega_Y)$, we have

$$\ker \mu_V = \ker \mu \cap \left(V \otimes H^0(Y, \omega_Y) \right) \,.$$

Due to $\gamma \leq \dim V \leq \dim H^0(Y, E)$, it follows from [2, Proposition 4.3] that

(9) dim $\left(\ker \mu \cap \left(V \otimes H^0(Y, \omega_Y)\right)\right) = \max\{0, \dim \left(\ker \mu\right) - (h^0(Y, E) - \dim V)h^0(Y, \omega_Y)\}.$

Since μ is surjective, dim $(\ker \mu) = (\deg(E) - \gamma + 1)\gamma - (\deg(E) + \gamma - 1) = (g - 3\gamma)(\gamma - 1)$. By assumption $r = \dim V \ge \frac{2(g-1)}{\gamma}$, hence

$$\dim \ker \mu - (h^0(Y, E) - \dim V)\gamma = (\gamma - 1)(g - 3\gamma) - (g - 3\gamma + 2 - r)\gamma$$
$$= \gamma - g + r\gamma > 0.$$

By (9) we obtain

$$\dim \ker \mu_V = \gamma - g + r\gamma.$$

From here we get for the dimension of its image

$$\dim (\operatorname{Im}(\mu_V)) = r\gamma - \dim \ker \mu_V = g - \gamma = h^0(Y, \omega_Y \otimes E) \,.$$

This shows that μ_V is surjective, which proves (8).

It remains to show that $\Phi_{\omega_Y,V}$ is surjective. According to [2, Theorem 4.1], the Gaussian map $\Phi_{\omega_Y,V}$ is of maximal rank. Suppose that it is not surjective. Then it must be injective and its image in $H^0(Y, \omega_Y^2 \otimes E)$ should be proper, hence

$$\gamma - g + r\gamma = \dim \ker \mu_V < h^0(Y, \omega_Y^2 \otimes E) = g + \gamma - 2$$
,

which implies $r < \frac{2(g-1)}{\gamma}$. The last is impossible in view of the assumption that $r = \dim V \ge \max\left\{\gamma, \frac{2(g-1)}{\gamma}\right\}$. Therefore, $\Phi_{\omega_Y,V}$ must be surjective and from (8) we conclude also that $h^0(N_{Y/\mathbb{P}^{r-1}}(-1)) = \dim V = r$.

4. PROOF OF THEOREM A

Before demonstrating the proof of Theorem A we recall a few facts concerning the Hilbert scheme of cones. Proposition 1 and the counting of the number of parameters in Remark 3 gives the idea how to construct explicitly the component $\mathcal{D}_{d,g,R}$. Recall that $d = 2g - 4\gamma + 2$ and $R = g - 3\gamma + 2$.

Let $\gamma \geq 10$ and $g \geq 4\gamma - 2$ be integers. Consider the Hilbert scheme $\mathcal{I}_{d/2,\gamma,R-1}$ of smooth curves of degree d/2 and genus γ in \mathbb{P}^{R-1} . By [18, Theorem on p. 75] and [22, Theorem on p. 26], $\mathcal{I}_{d/2,\gamma,R-1}$ is reduced and irreducible of dimension $\lambda_{d/2,\gamma,R-1} = Rd/2 - (R-4)(\gamma-1)$. Denote by $\mathcal{H}(\mathcal{I}_{d/2,\gamma,R-1})$ the family of cones in \mathbb{P}^R over curves representing points of $\mathcal{I}_{d/2,\gamma,R-1}$. Since $\gamma \geq 10$ it follows by [8, Proposition 2.1] that for a general $[Y] \in \mathcal{I}_{d/2,\gamma,R-1}$ the Gaussian map $\Phi_{\omega_Y,\mathcal{O}_Y(1)}$ is surjective, hence by [8, Proposition 2.18] $\mathcal{H}(\mathcal{I}_{d/2,\gamma,R-1})$ is a generically smooth component of the Hilbert scheme of surfaces of degree d/2 in \mathbb{P}^R and

(10)
$$\dim \mathcal{H}(\mathcal{I}_{d/2,\gamma,R-1}) = h^0(Y, N_{Y/\mathbb{P}^{R-1}}) + R = \lambda_{d/2,\gamma,R-1} + R.$$

First we give the proof of Theorem A in the case r = R.

Proposition 8. Suppose that $\gamma \ge 10$ and $g \ge 4\gamma - 2$. Let $\mathcal{F}_{d,g,R}$ be the family of curves $C \subset \mathbb{P}^R$ obtained as the intersection of a cone F and a general hypersurface of degree 2 in \mathbb{P}^R , where $[F] \in \mathcal{H}(\mathcal{I}_{d/2,\gamma,R-1})$. Let $\mathcal{D}_{d,g,R}$ be the closure of the set of points in $\mathcal{I}_{d,g,R}$ corresponding to curves from the family $\mathcal{F}_{d,g,R}$. Then

- $\mathcal{D}_{d,q,R}$ is a generically smooth irreducible component of $\mathcal{I}_{d,q,R}$, and
- dim $\mathcal{D}_{d,g,R} = 2g \gamma 2 + (R+1)^2 = \lambda_{d,g,R} + R\gamma 2g + 2.$

Proof. First we compute dim $\mathcal{D}_{d,g,R}$. For a general point $[F] \in \mathcal{H}(\mathcal{I}_{d/2,\gamma,R-1})$, the cone F is projectively normal since it is a cone over a general curve Y from $\mathcal{I}_{d/2,\gamma,R-1}$, which is projectively normal by [15, Theorem 1, p. 74]. Therefore the linear series $|\mathcal{O}_F(2)|$ on F is induced by $|\mathcal{O}_{\mathbb{P}^R}(2)|$. By equalities (10) and (1), $h^0(F, \mathcal{O}_F(2)) = 3g - 8\gamma + 6$. Therefore

$$\dim \mathcal{D}_{d,g,R} = \dim \mathcal{H}(\mathcal{I}_{g-2\gamma+1,\gamma,R-1}) + h^0(F,\mathcal{O}_F(2)) - 1$$
$$= \lambda_{d/2,\gamma,R-1} + R + 3g - 8\gamma + 5.$$

Remark that since $\lambda_{d/2,\gamma,R-1} = Rd/2 - (R-4)(\gamma-1)$ and $\lambda_{d,g,R} = (R+1)d - (R-1)(g-1)$, the expression for dim $\mathcal{D}_{d,g,R}$ can also be written as

$$\dim \mathcal{D}_{d,g,R} = \lambda_{d,g,R} + R\gamma - 2g + 2 = (R+1)^2 + 2g - \gamma - 2.$$

To prove that $\mathcal{D}_{d,g,R}$ is a generically smooth component of $\mathcal{I}_{d,g,R}$, it is sufficient to show that for a general $[X] \in \mathcal{D}_{d,g,R}$ we have $h^0(X, N_{X/\mathbb{P}^r}) = (R+1)^2 + 2g - \gamma - 2 =$ $\dim \mathcal{D}_{d,g,R}$. Since $X \subset F$ is cut by a general quadratic hypersurface in \mathbb{P}^R , the ruling of Finduces a double covering $\varphi : X \to Y$, where $Y \subset F$ is cut by a general hyperplane in \mathbb{P}^R and also $[Y] \in \mathcal{I}_{d/2,\gamma,R-1}$ is general. It follows by Proposition 1 and Corollary 5 that

$$0 \to \mathcal{O}_X(2) \to N_{X/\mathbb{P}^R} \to \varphi^* N_{Y/\mathbb{P}^{R-1}} \to 0.$$

Since deg $\mathcal{O}_X(2) = 2d = 4g - 8\gamma + 4 > 2g - 2$, the series $|\mathcal{O}_X(2)|$ is nonspecial, hence $h^1(X, \mathcal{O}_X(2)) = 0$. Therefore, using projection formula, Leray's isomorphism and $\varphi_*O_X = O_Y + O_Y(-E)$, we get

$$h^{0}(X, N_{X/\mathbb{P}^{R}}) = h^{0}(X, \mathcal{O}_{X}(2)) + h^{0}(X, \varphi^{*}N_{Y/\mathbb{P}^{R-1}})$$

= $h^{0}(X, \mathcal{O}_{X}(2)) + h^{0}(Y, N_{Y/\mathbb{P}^{R-1}}) + h^{0}(Y, N_{Y/\mathbb{P}^{R-1}}(-1))$
= $3g - 8\gamma + 5 + \lambda_{d/2,\gamma,R-1} + R$
= $\dim \mathcal{D}_{d,g,R}$.

This implies that for a general $[X] \in \mathcal{D}_{d,g,R}$

$$\dim \mathcal{D}_{d,g,R} = \dim \mathcal{F}_{d,g,R} = h^0(X, N_{X/\mathbb{P}^R}) = \dim T_{[X]}\mathcal{D}_{d,g,R},$$

therefore $\mathcal{D}_{d,g,R}$ is a generically smooth component of $\mathcal{I}_{d,g,R}$.

Now we give the proof of Theorem A for $\max\left\{\gamma, \frac{2(g-1)}{\gamma}\right\} \le r < R$.

Proof of Theorem A. Let $\mathcal{F}_{d,g,r}$ be the family of curves in \mathbb{P}^r obtained from the family $\mathcal{F}_{d,g,R}$ in Proposition 8 by a projection $\pi_W : \mathbb{P}^R \to \mathbb{P}^r$ with center $W \subset \mathbb{P}^R$, where $W \cong \mathbb{P}^{R-r-1}$ is general. Let $\mathcal{D}_{d,g,r}$ be the closure of the set of points in $\mathcal{I}_{d,g,r}$ corresponding to the curves from the family $\mathcal{F}_{d,g,r}$. Since $\operatorname{codim}(W, \mathbb{P}^R) \ge \gamma + 1 \ge 10$, a cone F and curves X and Y as in the proof of Proposition 8 are isomorphic to their images $F_r = \pi_W(F)$, $X_r = \pi_W(X)$ and $Y_r = \pi_W(Y)$, correspondingly. Also, $\varphi_r : X_r \to Y_r$ induced by ruling of F_r is a double covering as in Corollary 6. Note that if p is the vertex of F then $\pi_W(p)$ is the vertex of F_r . Therefore

(11)
$$0 \to \mathcal{O}_{X_r}(2) \to N_{X_r/\mathbb{P}^r} \to \varphi_r^* N_{Y_r/\mathbb{P}^{r-1}} \to 0$$

The embedding of $Y_r \,\subset \mathbb{P}^{r-1}$ is incomplete, but since $\deg Y_r = d/2 = g - 2\gamma + 1 \geq 2\gamma - 1$, it follows by [22] that $\mathcal{I}_{d/2,\gamma,r-1}$ has a unique generically smooth component of the expected dimension $\lambda_{d/2,\gamma,r-1}$. Therefore, for a general $Y_r \in \mathcal{I}_{d/2,\gamma,r-1}$ (as in our case), $h^1(Y_r, N_{Y_r/\mathbb{P}^{r-1}}) = 0$ or equivalently $h^0(Y_r, N_{Y_r/\mathbb{P}^{r-1}}) = \lambda_{d/2,\gamma,r-1}$. Then we can compute $h^0(N_{X_r/\mathbb{P}^r})$ in a similar way as before. Since the projection π_W is general and $r \geq \max\left\{\gamma, \frac{2(g-1)}{\gamma}\right\}$, it follows by Proposition 7 that $h^0(Y_r, N_{Y_r/\mathbb{P}^{r-1}}(-1)) = r$. Since $\deg \mathcal{O}_{X_r}(2) = 2d > 2g - 2$ we have $h^1(X_r, \mathcal{O}_{X_r}(2)) = h^1(X, \mathcal{O}_X(2)) = 0$. Using (11) we find

$$h^{0}(X_{r}, N_{X_{r}/\mathbb{P}^{r}}) = h^{0}(X_{r}, \mathcal{O}_{X_{r}}(2)) + h^{0}(X_{r}, \varphi_{r}^{*}N_{Y_{r}/\mathbb{P}^{r-1}})$$

$$= 2d - g + 1 + h^{0}(Y_{r}, N_{Y_{r}/\mathbb{P}^{r-1}}) + h^{0}(Y_{r}, N_{Y_{r}/\mathbb{P}^{r-1}}(-1))$$

$$= 4g - 8\gamma + 4 - g + 1 + \lambda_{g-2\gamma+1,\gamma,r-1} + r$$

$$= 3g - 8\gamma + 5 + r + (g - 2\gamma + 1)r - (r - 4)(\gamma - 1)$$

$$= (r + 3)g - (3r + 4)\gamma + 3r + 1.$$

Let's compute also the dimension of the family $\mathcal{F}_{d,g,r}$. It is similar to the one carried out in the proof of [6, Theorem 4.3]. Since the curves in $\mathcal{F}_{d,g,r}$ are obtained as generic projections from \mathbb{P}^R to \mathbb{P}^r , we have

$$\dim \mathcal{F}_{d,g,r} = \dim \mathcal{F}_{d,g,R} - \dim \operatorname{Aut} \mathbb{P}^{R} + \dim \operatorname{Aut} \mathbb{P}^{r} + \dim \operatorname{Grass}(r+1, R+1)$$
$$= 2g - \gamma - 2 + (r+1)^{2} + (r+1)(R-r)$$
$$= 2g - \gamma - 2 + (r+1)(g - 3\gamma + 2 + 1)$$
$$= (r+3)g - (3r+4)\gamma + 3r + 1.$$

Notice that this number is exactly equal to the one claimed in Theorem A since

$$\lambda_{d,g,r} + r\gamma - 2g + 2 = (r+1)(2g - 4\gamma + 2) - (r-3)(g-1) + r\gamma - 2g + 2$$
$$= (r+3)g - (3r+4)\gamma + 3r + 1.$$

Hence dim $\mathcal{D}_{d,g,r} = \dim \mathcal{F}_{d,g,r} = h^0(X_r, N_{X_r/\mathbb{P}^r}) = \dim T_{[X_r]}\mathcal{D}_{d,g,r}$, which completes the proof of Theorem A.

Corollary 9. If

(12)
$$\gamma \mid 2(g-1) \quad and \quad r := \frac{2(g-1)}{\gamma} \ge \gamma \ge 10,$$

then $\mathcal{D}_{d,g,r}$ is a regular component of $\mathcal{I}_{d,g,r}$ different from the distinguished one.

Proof. With the particular values of $d = 2g - 4\gamma + 2$ and r we have $r = \frac{2(g-1)}{\gamma} \leq \frac{g-1}{5}$ for $\gamma \geq 10$, hence $5r \leq g - 1$. From here it is easy to see that $d - g - r \geq g + 2 - 5r \geq 3$. Therefore $\rho(d, g, r) = g - (r + 1)(g - d + r) \geq g > 0$, hence the distinguished component of $\mathcal{I}_{d,g,r}$ dominating \mathcal{M}_g exists. Apart from it, Theorem A guarantees the existence of the regular component $\mathcal{D}_{d,g,r}$ which is apparently different from the distinguished one as the former projects properly in \mathcal{M}_g .

Remark 10. It appears that the condition $r \ge \frac{2(g-1)}{\gamma}$ is essential for the family $\mathcal{F}_{d,g,r}$ giving rise to a component of $\mathcal{I}_{d,g,r}$, because in such case $r < \frac{2(g-1)}{\gamma}$ we have dim $\mathcal{F}_{d,g,r} < \lambda_{d,g,r}$. Notice also that in such a case the Gaussian map in Proposition 7 is definitely not surjective.

Remark 11. In their paper [20] Mezzetti and Sacchiero constructed generically smooth irreducible components of $\mathcal{I}_{d,g,r}$, denoted there $W_{d,g,r}^m$, whose general points are m- sheeted coverings of \mathbb{P}^1 , where $m \geq 3$. In the case of $g = 6\gamma - 3$, we have $d = 2g - 4\gamma + 2 = 8\gamma - 4 = \frac{4}{3}g$, $R = g - 3\gamma + 2 = 3\gamma - 1 = \frac{g+1}{2}$, and the Hilbert scheme $\mathcal{I}_{\frac{4g}{3},g,\frac{g+1}{2}}$ has two components parametrizing linearly normal curves. One of them is the component $\mathcal{D}_{\frac{4g}{3},g,\frac{g+1}{2}}$, shown to exist in our Proposition 8, and the other one is the component $W_{d,g,r}^m$ for m = 4, $d = \frac{4}{3}g$, and $r = R = \frac{g+1}{2}$ (it is easy to check that the numerical conditions for the existence of $W_{\frac{4g}{3},g,\frac{g+1}{2}}$ are satisfied when $\gamma \geq 10$). Notice that since $d - g - R = -\gamma < 0$, the existence of these two components do not provide a counterexample to Severi's conjecture claiming that $\mathcal{I}_{d,g,r}$ has a unique irreducible component parametrizing linearly normal curves if $d \geq g + r$.

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