# **COMPONENTS OF THE HILBERT SCHEME OF SMOOTH PROJECTIVE CURVES USING RULED SURFACES**

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ABSTRACT. Let  $\mathcal{I}_{d,g,r}$  be the union of irreducible components of the Hilbert scheme whose general points correspond to smooth irreducible non-degenerate curves of degree d and genus  $g$  in  $\mathbb{P}^r$ . We use families of curves on cones to show that under certain numerical assumptions for  $d$ ,  $g$  and  $r$ , the scheme  $\mathcal{I}_{d,g,r}$  acquires generically smooth components whose general points correspond to curves that are double covers of irrational curves. In particular, in the case  $\rho(d, g, r) := g - (r + 1)(g - d + r) \geq 0$  we construct explicitly a *regular component* that is different from the distinguished component of  $\mathcal{I}_{d,g,r}$  dominating the moduli space  $\mathcal{M}_g$ . Our result implies also that if  $g\geq 57$  then  $\mathcal{I}_{\frac{4g}{3},g,\frac{g+1}{2}}$  has at least two generically smooth components parametrizing linearly normal curves.

#### 1. INTRODUCTION

Let  $\mathcal{I}_{d,q,r}$  be the union of irreducible components of the Hilbert scheme whose general points correspond to smooth irreducible non-degenerate complex curves of degree d and genus  $g$  in  $\mathbb{P}^r$ . A component of  $\mathcal{I}_{d,g,r}$  is called *regular* if it is reduced and of expected dimension  $\lambda_{d,g,r} := (r+1)d - (r-3)(g-1)$ . Otherwise it is called *superabundant*. For  $\rho(d, g, r) := g - (r + 1)(g - d + r) \geq 0$ , it is known that  $\mathcal{I}_{d,q,r}$  has the unique component dominating Mg, see [\[18,](#page-13-0) p. 70]. It is usually referred to as the *distinguished* component.

Historically, Severi claimed in [\[23\]](#page-13-1) that  $\mathcal{I}_{d,q,r}$  is irreducible if  $d \geq g+r$ . It was proved that  $\mathcal{I}_{d,q,r}$  is irreducible if  $d \geq g+r$  and  $r=3,4$ , see [\[11\]](#page-13-2) and [\[12\]](#page-13-3). On the other hand, for  $r \geq 5$  (and  $\rho(d, g, r) \geq 0$ ) there have been given several examples in which  $\mathcal{I}_{d,q,r}$  possesses additional non-distinguished components ([\[20\]](#page-13-4), [\[19\]](#page-13-5), [\[6\]](#page-13-6), [\[10\]](#page-13-7), etc), but for none of them it has been proven to be regular. Note that all these examples are given by non-linearly normal curves. We remark that in [\[6\]](#page-13-6) we showed the existence of a *non-distinguished* component  $\mathcal{D}_{d,q,r}$  of  $\mathcal{I}_{d,q,r}$  parameterizing curves that are double covers of irrational curves, whereas all other known to us examples of reducible Hilbert schemes of curves have used curves that are *m*-sheeted coverings of  $\mathbb{P}^1$  with  $m\geq 3$ .

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In [\[6,](#page-13-6) Question 4.7, p. 598] we asked about the possibility of  $\mathcal{D}_{d,q,r}$  being reduced. In the present paper we reconstruct this component under less constrains, this time using a family of curves on cones, which are double coverings of hyperplane sections of the cones. We construct and characterize the properties of  $\mathcal{D}_{d,g,r}$  using tools from the theory of ruled surfaces, while in [\[6\]](#page-13-6) we only showed its existence using Brill-Noether theory of linear series on curves. Our approach is motivated by the fact that for a given double covering  $\varphi: X \to Y$  the curves X and Y can be regarded as curves on the ruled surface  $S :=$  $\mathbb{P}(\varphi_* \mathcal{O}_X)$ , as we explain in section [2.](#page-2-0) It allows us to construct the additional component in a more geometric way and to obtain its generic smoothness, which gives an affirmative answer to the question raised in [\[6\]](#page-13-6).

Our main result is as follows.

**Theorem A.** *Assume that* g and  $\gamma$  are integers with  $g \geq 4\gamma - 2 \geq 38$ . Let

$$
d := 2g - 4\gamma + 2 \quad \text{and} \quad \max\left\{\gamma, \frac{2(g-1)}{\gamma}\right\} \le r \le R := g - 3\gamma + 2 \, .
$$

*Then the Hilbert scheme*  $\mathcal{I}_{d,g,r}$  *possesses a generically reduced component*  $\mathcal{D}_{d,g,r}$  *for which* 

$$
\dim \mathcal{D}_{d,g,r} = \lambda_{d,g,r} + r\gamma - 2g + 2.
$$

*Further, let*  $X_r \subset \mathbb{P}^r$  be a smooth curve corresponding to a general point of  $\mathcal{D}_{d,g,r}$ .

- (i) If  $r = R$  then  $X_R$  is the intersection of a general quadric hypersurface with a cone over a smooth curve  $Y$  of degree  $g-2\gamma+1$  and genus  $\gamma$  in  $\mathbb{P}^{R-1}$  and  $X_R$  is embedded in  $\mathbb{P}^R$  by *the complete linear series*  $|R_{\varphi}|$  *on*  $X_R$ *, where*  $R_{\varphi}$  *is the ramification divisor of the natural projection morphism*  $\varphi$  :  $X_R \to Y$  *of degree 2 given by the ruling of cone;*
- (ii) If  $r < R$  then  $X_r$  is given by a general projection of some  $X_R$  as in (i), that is,  $X_r$  is embedded in  $\mathbb{P}^r$  by a general linear subseries  $g_d^r$  of  $|R_\varphi|.$

In our view, one of the interesting implications of Theorem A is that if  $r = \frac{2(g-1)}{\gamma} \geq$  $\gamma \ge 10$  and  $d = 2g - 4\gamma + 2$ , then the scheme  $\mathcal{I}_{d,g,r}$  acquires a second regular component in addition to its distinguished component dominating the moduli space  $\mathcal{M}_q$ , see Corollary [9.](#page-12-0) To our best knowledge, it is the first example in which simultaneous existence of two distinct regular components of  $\mathcal{I}_{d,g,r}$  has been observed in the Brill-Noether case  $\rho(d, g, r) \geq 0$ . We remark also that in the case  $g = 6\gamma - 3$  and  $r = R = 3\gamma - 1$ , the Hilbert scheme  ${\cal I}_{\frac{4g}{3},g,\frac{g+1}{2}}$  has at least two generically smooth components parametrizing linearly normal curves as it is explained in Remark [11.](#page-12-1)

The remaining sections of the paper are organized as follows. In section [2,](#page-2-0) we provide a motivation for the construction of the component described in Theorem A by reviewing the relations between double coverings of curves, ruled surfaces and their embeddings as cones. We also prove there several statements that will be used for the construction of  $\mathcal{D}_{d,g,r}$  in section [4.](#page-9-0) Possibly, some of them might be of independent interest. In section [3](#page-7-0) we briefly review several facts about the Gaussian map associated to linear series on curves and prove a technical result facilitating the computation of the dimension of the tangent space at a general point of  $\mathcal{D}_{d,q,r}$ . In section [4](#page-9-0) we give the proof of Theorem A.

We work over C. We understand by *curve* a smooth integral projective algebraic curve. We denote by  $L^{\vee}$  the dual line bundle for a given line bundle  $L$  defined on an algebraic variety X. As usual,  $\omega_X$  will stand for the canonical line bundle on X. We denote by |L| the complete linear series  $\mathbb{P}(H^0(X,L))$ . When X is an object of a family, we denote by  $[X]$  the corresponding point of the Hilbert scheme representing the family. Throughout the entire paper

$$
d := 2g - 4\gamma + 2 \quad \text{and} \quad R := g - 3\gamma + 2 \, .
$$

For definitions and properties of the objects not explicitly introduced in the paper refer to [\[17\]](#page-13-8) and [\[1\]](#page-13-9).

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## 2. MOTIVATION AND PRELIMINARY RESULTS

<span id="page-2-0"></span>Suppose that  $\varphi: X \to Y$  is an  $m: 1$  cover,  $m \geq 2$ , where X and Y are smooth curves of genus g and  $\gamma$ , correspondingly. As it is well known, the covering induces a short exact sequence of vector bundles on Y

$$
0 \to \mathcal{O}_Y \xrightarrow{\varphi^{\sharp}} \varphi_* \mathcal{O}_X \to \mathcal{E}^{\vee} \to 0 \,,
$$

where  $\mathcal{E}^{\vee}$  is the so called *Tschirnhausen module*, see [\[21\]](#page-13-10). It is a rank  $(m-1)$ -vector bundle on Y. Since X and Y are curves over  $\mathbb C$ , the exact sequence splits, i.e.  $\varphi_* \mathcal O_X \cong \mathcal O_Y \oplus \mathcal E^{\vee}$ . According to [\[17,](#page-13-8) Ex. IV.2.6, p. 306],  $(\det \varphi_* \mathcal{O}_X)^2 \cong \mathcal{O}_Y(-B)$ , where B is the branch divisor of the covering. In particular, deg  $B = 2(g - 1) - 2m(\gamma - 1)$ .

We focus on the case  $m = 2$ . In such a case  $\mathcal E$  must be a line bundle on Y and we can assume that  $\mathcal{E} = \mathcal{O}_Y(E)$  for some divisor E on Y. Since

$$
\deg B = -\deg(\det \varphi_* \mathcal{O}_X)^2 = -\deg(\det(\mathcal{O}_Y \oplus \mathcal{O}_Y(-E)))^2 = 2 \deg E
$$

it follows that deg  $E = g - 2\gamma + 1$ .

Further we suppose that  $E$  is a nonspecial and very ample divisor on  $Y$ . Denote by F the rank 2 vector bundle  $\mathcal{F} := \mathcal{O}_Y \oplus \mathcal{O}_Y(E)$  on Y and let S be the ruled surface  $S := \mathbb{P}(\mathcal{F})$  with natural projection  $f : S \to Y$ . Since  $\deg E > 0$ ,  $\mathcal{F}_0 := \mathcal{O}_Y \oplus \mathcal{O}_Y(-E)$ will be the normalization of the vector bundle F. As it is decomposable,  $f : S \rightarrow Y$  has

two canonically determined sections. They are  $Y_0$  which corresponds to the short exact sequence

$$
0 \to \mathcal{O}_Y \to \mathcal{O}_Y \oplus \mathcal{O}_Y(-E) \to \mathcal{O}_Y(-E) \to 0,
$$

and  $Y_1$  which corresponds to the short exact sequence

$$
0 \to \mathcal{O}_Y \to \mathcal{O}_Y \oplus \mathcal{O}_Y(E) \to \mathcal{O}_Y(E) \to 0.
$$

The section  $Y_0$  is the section with minimal self-intersection on S and  $Y_0^2 = \deg(\mathcal{O}_Y(-E)) =$  $-g + 2\gamma - 1$ . As it well known,  $Pic(S) \cong \mathbb{Z}[Y_0] \oplus f^*(Pic(Y))$ . For a divisor D on Y we will denote by Df the divisor  $f^*(D)$  on S. Also, we have for the section  $Y_1$  that  $Y_1^2 =$  $deg(\mathcal{O}_Y(E)) = g - 2\gamma + 1$  and it is not difficult to see that  $Y_1 \sim Y_0 + Ef$ . In general, cohomologies like  $h^j(S, \mathcal{O}_S(nY_0 + Df))$  are calculated using the projection formula, see [\[17,](#page-13-8) Ex. III.8.3, p. 253], as

<span id="page-3-1"></span>
$$
h^{j}(S, \mathcal{O}_{S}(nY_{0}+Df))=h^{j}(Y, \mathcal{S}ym^{n}(\mathcal{F}_{0})\otimes \mathcal{O}_{Y}(D)),
$$

but since S is decomposable, i.e.  $\mathcal{F}_0$  splits, the calculation reduces simply to

(1) 
$$
h^{j}(S, \mathcal{O}_{S}(nY_{0}+Df)) = \sum_{k=0}^{n} h^{j}(Y, \mathcal{O}_{Y}(D-kE)),
$$

see for example [\[14\]](#page-13-11). From here

<span id="page-3-0"></span>(2) 
$$
h^{0}(S, \mathcal{O}_{S}(Y_{1})) = h^{0}(S, \mathcal{O}_{S}(Y_{0} + Ef)) = h^{0}(Y, \mathcal{O}_{Y}(E)) + h^{0}(Y, \mathcal{O}_{Y}) = g - 3\gamma + 3.
$$

Using [\[17,](#page-13-8) Ex. V.2.11 (a), p. 385], we obtain that the linear series  $|\mathcal{O}_S(Y_1)| \equiv |\mathcal{O}_S(Y_0 + Ef)|$ is base point free. Therefore it defines a morphism

$$
\Psi := \Psi_{|\mathcal{O}_S(Y_1))|} : S \to \mathbb{P}^R,
$$

where  $R = g - 3\gamma + 2$ . Since E is very ample, it follows by [\[14,](#page-13-11) Proposition 23, p. 38] that  $\Psi$  is isomorphism away from  $Y_0$ . Due to  $Y_0 \cdot Y_1 = Y_0 \cdot (Y_0 + Ef) = 0$ , the morphism  $\Psi$ contracts the curve  $Y_0$  to a point. Therefore  $F:=\Psi(S)\subset \mathbb{P}^R$  is a cone of degree

$$
\deg F = Y_1 \cdot Y_1 = (Y_0 + E\mathfrak{f}) \cdot (Y_0 + E\mathfrak{f}) = \deg E = g - 2\gamma + 1
$$

over the image of a smooth integral curve from the linear series  $|O_s(Y_0 + Ef)|$ .

By Bertini's theorem,  $\Psi$  maps a general element of  $|\mathcal{O}_S(Y_1)|$  to a smooth integral curve of genus  $\gamma$ , degree  $g - 2\gamma + 1$ , which is further linearly normally embedded in some hyperplane  $\mathbb{P}^{R-1}$  of  $\mathbb{P}^R$  due to [\(2\)](#page-3-0). A similar fact is true about a general element of  $|\mathcal{O}_S(2Y_1)|$ . Namely, a general  $C \in |\mathcal{O}_S(2Y_1)| \equiv |\mathcal{O}_S(2Y_0 + 2Ef)|$  is mapped by  $\Psi$  to a smooth integral curve  $\Psi(C)$  of genus g, degree  $2g - 4\gamma + 2 = d$ , which is linearly normal in  $\mathbb{P}^R.$  Indeed, since  $Y_0\cdot Y_1=0$  and  $\Psi$  is isomorphism away from  $Y_0$ , it follows by Bertini's theorem that  $\Psi(C)$  is smooth and integral. Its degree is deg  $\Psi(C) = 2Y_1 \cdot Y_1 = 2g - 4\gamma + 2$ , while by the adjunction formula

$$
\deg C \cdot (K_S + C) = (2Y_1) \cdot (K_S + 2Y_1) = 2(2\gamma - 2) + 2g - 4\gamma + 2 = 2g - 2
$$

we get that its genus is g. Finally, to see that  $\Psi(C) \subset \mathbb{P}^R$  is linearly normal, consider the exact sequence

$$
0 \to \mathcal{O}_S(-Y_0 - E\mathfrak{f}) \to \mathcal{O}_S(Y_0 + E\mathfrak{f}) \to \mathcal{O}_C(Y_0 + E\mathfrak{f}) \to 0.
$$

It is sufficient to see that  $h^1(S, \mathcal{O}_S(-Y_0 - Ef)) = 0$ , which is not difficult to obtain using the Serre duality.

The arguments above motivate the following statement.

<span id="page-4-0"></span>**Proposition 1.** *Assume that* Y *is a smooth curve of genus* γ *and* E *is a very ample non-special divisor on* Y *of degree e*. Let  $S := \mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(-E))$ ,  $Y_0$  be the section of minimal self-intersection *of the natural projection*  $f : S \to Y$  *and*  $Y_1 \in |O_S(Y_0 + Ef)|$  *be a smooth integral curve. Let*  $\Psi := \Psi_{|O_S(Y_0+E\mathfrak{f})|}$  *be the morphism induced by the complete linear series*  $|O_S(Y_0+E\mathfrak{f})|$ *. Then:* 

- (a)  $|\mathcal{O}_S(Y_0 + E\mathfrak{f})|$  *is base point free and of dimension*  $e \gamma + 1$ *;*
- (b)  $\Psi$  *is an isomorphism away from*  $Y_0$  *and contracts*  $Y_0$  *to a point in*  $\mathbb{P}^{e-\gamma+1}$ *, in particular,*  $\Psi(S)$  *is a cone over*  $\Psi(Y_1)$ *;*
- (c) *for a general*  $C \in |O_S(2Y_0 + 2Ef)|$ 
	- (c.1)  $\Psi(C)$  *is a linearly normal smooth irreducible curve of genus*  $2\gamma + e 1$  *and degree*  $2e$ *in*  $\mathbb{P}^{e-γ+1}$ *;*
	- (c.2) *the linear series*  $|\mathcal{O}_C(R_\varphi)|$  *on* C *is traced by the linear series*  $|\mathcal{O}_S(Y_0 + Ef)|$  *on* S, *where*  $R_{\varphi}$  *is the ramification divisor of the morphism*  $\varphi : C \to Y$  *induced by the ruling of* S*.*

*Proof.* Statements (a), (b) and (c.1) are obtained by very similar arguments like those in the discussion preceding the proposition. We only need to check (c.2). Recall that  $K_S \sim$  $-2Y_0 + (K_Y - E)$ f. On C we have  $K_C - \varphi^* K_Y \sim R_\varphi$ , i.e.  $\mathcal{O}_C(R_\varphi) = \omega_C \otimes (\varphi^* \omega_Y)^\vee$ . The canonical divisor  $K_C$  on C is induced by the restriction of  $K_S + C \sim K_S + (2Y_0 + 2Ef)$  on *C*. Similarly, the restriction of  $K_S + Y_1 \sim K_S + Y_0 + Ef$  on  $Y_1$  induces  $K_{Y_1}$ . Therefore

$$
R_{\varphi} \sim (K_S + (2Y_0 + 2E\mathfrak{f}) - (K_S + Y_0 + E\mathfrak{f}))_{|_C} \sim (Y_0 + E\mathfrak{f})_{|_C}.
$$

By (a) and (c.1),  $h^0(C, \mathcal{O}_C(Y_0 + E\mathfrak{f}_{|_C})) = h^0(S, \mathcal{O}_S(Y_0 + E\mathfrak{f})) = e - \gamma + 2$ . Therefore the linear series  $|\mathcal{O}_S(Y_0 + Ef)|$  on S induces the linear series  $|\mathcal{O}_C(R_\varphi)|$  on C.

**Remark 2.** *When*  $e = g - 2\gamma + 1 \geq 2\gamma - 1$  *and the divisor* E *on* Y *is very ample, where*  $\mathcal{O}_Y(-E)$ *is the Tschirnhausen module of a double covering*  $X \to Y$ , statement (c.2) *implies that*  $\mathcal{O}_C(R_\varphi)$  *is*  $very \ ample \ and \ h^0(C,{\cal O}_{C}(R_{\varphi}))=g-3\gamma+3.$  It improves a similar claim proved in [\[6,](#page-13-6) Lemma 4.1] *where it was assumed that*  $q \geq 6\gamma - 1$ *.* 

<span id="page-4-1"></span>**Remark 3.** Proposition [1](#page-4-0) *suggests how to give an alternative construction of the component*  $\mathcal{D}_{2q-4\gamma+2,q,r}$  *constructed in* [\[6,](#page-13-6) Theorem 4.3, p. 594]*. For this take*  $e = g - 2\gamma + 1 \geq 2\gamma - 1$  *and*  $f$  *consider the family*  ${\cal Z}$  *of surface scrolls*  $F\subset {\mathbb P}^R$ *, over a curve*  $Y$  *of genus*  $\gamma$ *,*  $\deg F=\deg Y=e=0$  $g-2\gamma+1$  with  $h^0(F,{\mathcal O}_F(1))=g-3\gamma+3$  and  $h^1(F,{\mathcal O}_F(1))=\gamma$ . According to [\[3,](#page-13-12) Lemma 1, p. 7] *such a scroll is necessarily a cone, say* F*, over a projectively normal curve in* P R−1 *of genus*

 $\gamma$  and degree  $e.$  Further, let  ${\cal F}$  be the family of smooth curves in  $|{\cal O}_F(2)|$  on the cones  $F\subset {\mathbb P}^R$ *from the family* Z*. By a counting of the parameters on which the family* Z *depends, similar to the one carried out in* [\[3,](#page-13-12) Remark 2, p. 15] *and* [\[4,](#page-13-13) Proposition 7.1, p. 150]*,*

dim *Z* =  
\n+ 3
$$
\gamma
$$
 − 3 : number of parameters of curves  $Y \in M_{\gamma}$   
\n+  $\gamma$  : number of parameters of line bundles  $O_Y(E) \in Pic(Y)$  of degree  $g - 2\gamma + 1 \ge$   
\n2 $\gamma$  − 1 necessary to fix the geometrically ruled surface  $\mathbb{P}(O_Y \oplus O_Y(-E))$   
\n+  $(R + 1)^2 - 1 = \dim(\text{Aut}(\mathbb{P}^R))$   
\n-  $((g - 2\gamma + 1) - \gamma + 2) = \dim G_F$ , where  $G_F$  is the subgroup of  $\text{Aut}(\mathbb{P}^R)$  fixing the  
\nscroll *F*, see [4, Lemma 6.4, p. 148]

*one finds that*  $\dim \mathcal{Z} = 7(\gamma-1)-g + (R+1)^2.$  *On the other hand computing*  $\dim |\mathcal{O}_F(2)|$  *using [\(1\)](#page-3-1)* and the Riemann-Roch formula, we get easily  $\dim |\mathcal{O}_F(2)| = 3g - 8\gamma + 5$ . Therefore for the *dimension of* F *we obtain*

$$
\dim \mathcal{F} = \dim \mathcal{Z} + \dim |\mathcal{O}_F(2)| = 2g - \gamma - 2 + (g - 3\gamma + 3)^2.
$$

*It is precisely the dimension of the component*  $D_{2q-4γ+2,q,r}$  *constructed in* [\[6,](#page-13-6) Theorem 4.3] *when* r = R = g−3γ+2 *and it improves the bound calculated in* [\[6,](#page-13-6) Lemma 4.1] *where it was assumed that*  $q > 6\gamma - 1$ *.* 

The above arguments do not imply yet that the family  $\mathcal F$  gives rise to a component of the Hilbert scheme  $\mathcal{I}_{d,q,R}$ . To prove this formally, we will compute in section [4](#page-9-0)  $h^0(C, N_{C/\mathbb{P}^R})$  for a general  $C \in \mathcal{F}$ . For the purposes of that computation we need several more formal statements about the normal bundles of curves on cones, which we prove below.

<span id="page-5-0"></span>**Lemma 4.** Let X be a smooth non-degenerate curve in  $\mathbb{P}^r$  and let H be a hyperplane in  $\mathbb{P}^r$ . Assume that  $\pi_p: X \to H \subset \mathbb P^r$  is a projection from a point  $p \notin H \cup X$  such that the image  $Y:=\pi_p(X)$  is smooth in  $\mathbb P^{r-1}.$  Then

(3) 
$$
0 \to O_X(R_{\pi_p}) \otimes \mathcal{O}_X(1) \to N_{X/\mathbb{P}^r} \to \pi_p^* N_{Y/\mathbb{P}^{r-1}} \to 0,
$$

where  $R_{\pi_p}$  is the ramification divisor of the covering  $\pi_p:X\to Y.$ 

*Proof.* Since  $\pi_p: X \to \mathbb{P}^{r-1} \subset \mathbb{P}^r$  is a projection from a point  $p \notin X$ , we have  $\pi_p^*(\mathcal{O}_Y(1)) =$  $\mathcal{O}_X(1)$ . For the curves X and Y we have the Euler sequences

$$
0 \to \mathcal{O}_X \to \bigoplus^{r+1} \mathcal{O}_X(1) \to T_{\mathbb{P}^r|_X} \to 0
$$

and

$$
0 \to \mathcal{O}_Y \to \bigoplus^r \mathcal{O}_Y(1) \to T_{\mathbb{P}^{r-1}|_Y} \to 0
$$

Pulling the second sequence to X via  $\pi_p$  we obtain

0 0 ↓ ↓ O<sup>X</sup> (1) ≃ ker(α) ↓ ↓ 0 → O<sup>X</sup> → ⊕r+1OX(1) → TPr|<sup>X</sup> → 0 ↓ ↓ ↓ α 0 → O<sup>X</sup> → π ∗ p (⊕<sup>r</sup> <sup>1</sup>O<sup>Y</sup> (1)) → π ∗ p TPr−1|<sup>Y</sup> → 0 ↓ 0

where  $\alpha$  is the induced map between the restrictions of  $T_{\mathbb P^r|_X}$  and  $\pi_p^*\left(T_{\mathbb P^{r-1}|_Y}\right)$  and  $\ker\left(\alpha\right)$ is its kernel. By the Snake lemma we obtain

$$
0 \to \mathcal{O}_X(1) \to T_{\mathbb{P}^r|_X} \to \pi_p^* \left( T_{\mathbb{P}^{r-1}|_Y} \right) \to 0.
$$

Further, using the normal bundle sequence for  $N_{X/\mathbb{P}^r}$  and  $N_{Y/\mathbb{P}^{r-1}}$ , we get the following commutative diagram

$$
\begin{array}{ccccccc}\n & & & & & 0 & & 0 & \\
 & & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_X(1) & & & \ker(\beta) & & \\
 & & & & \downarrow & & & \downarrow & & \\
0 & \rightarrow & T_X & \rightarrow & T_{\mathbb{P}^r|_X} & \rightarrow & N_{X/\mathbb{P}^r} & \rightarrow & 0 & \\
 & & & & \downarrow & & & \downarrow & \beta & & \\
0 & \rightarrow & \pi_p^*(T_Y) & \rightarrow & \pi_p^*\left(T_{\mathbb{P}^{r-1}|_Y}\right) & \rightarrow & \pi_p^*\left(N_{Y/\mathbb{P}^{r-1}}\right) & \rightarrow & 0 & \\
 & & & & \downarrow & & & \downarrow & & \\
 & & & & \mathcal{O}_{R_{\pi_p}} & & 0 & & \\
 & & & & \downarrow & & & & 0 & & \\
 & & & & & 0 & & & & \\
\end{array}
$$

where  $\beta$  is the induced map between the normal bundles  $N_{X/\mathbb{P}^r}$  and  $\pi_p^*\left(N_{Y/\mathbb{P}^{r-1}}\right)$ . Similarly as before, by the Snake lemma we get  $\ker \beta \cong \mathcal{O}_X(R_{\pi_p})\otimes \mathcal{O}_X(1)$ , and thus we deduce the short exact sequence

$$
0 \to \mathcal{O}_X(R_{\pi_p}) \otimes \mathcal{O}_X(1) \to N_{X/\mathbb{P}^r} \to \pi_p^* N_{Y/\mathbb{P}^{r-1}} \to 0
$$



<span id="page-7-2"></span>**Corollary 5.** *Suppose that*  $Y \subset \mathbb{P}^{r-1} \subset \mathbb{P}^r$ ,  $r \geq 3$ , is a smooth non-degenerate curve of genus  $\gamma$ . Let  $p\in\mathbb P^r\setminus \mathbb P^{r-1}$  be and arbitrary point. Consider the cone  $F\subset \mathbb P^r$  over  $Y$  with vertex  $p.$  Suppose that a curve  $X\subset F$  is cut by a general hypersurface  $Q_m\subset \mathbb P^R$  of degree  $m$ , i.e.  $X\in |{\mathcal O}_F(m)|$  is general. Let  $\varphi\,:\,X \xrightarrow{m:1} Y$  be the m-sheeted covering map induced by the ruling of the cone. Then *there is an exact sequence*

<span id="page-7-1"></span>(4) 
$$
0 \to \mathcal{O}_X(m) \to N_{X/\mathbb{P}^r} \to \varphi^* N_{Y/\mathbb{P}^{r-1}} \to 0.
$$

*Proof.* The line bundle  $O_X(R_{\varphi})$  associated to the ramification divisor  $R_{\varphi}$  of the covering  $\varphi: X \to Y$  has the property  $\mathcal{O}_X(R_\varphi) \simeq \mathcal{O}_X(m-1)$ . To see this, recall that  $R_\varphi \sim K_X - \varphi^* K_Y$ . The canonical divisor  $K_X$  on X is cut by the restriction of  $K_F + X$  on X and  $K_Y$  is cut by the restriction of  $K_F + Y$  on Y. Therefore

$$
K_X - \varphi^* K_Y = (K_F + X)|_X - (K_F + Y)|_X \sim (X - Y)|_X \sim (m - 1)Y|_X.
$$

Hence  $\mathcal{O}_X(R_\varphi) \simeq \mathcal{O}_X(m-1)$  and Lemma [4](#page-5-0) yields the exact sequence [\(4\)](#page-7-1).

<span id="page-7-5"></span>**Corollary 6.** Let  $X, Y \subset F \subset \mathbb{P}^r$  be smooth curves on the cone F with vertex p as in Corollary  $5$ , where  $r\geq 6.$  Let  $W\subset \mathbb P^r$  be a general projective subspace of  $\mathbb P^r$  of dimension  $r-s-1$ , where  $5 \leq s \leq r-1$ . Consider the projection  $\pi_W : \mathbb P^r \setminus W \to \mathbb P^s$  with center  $W$  to a general projective  $s$ ubspace of  $\mathbb{P}^r$  of dimension  $s$ . Denote by  $X_s$ ,  $Y_s$  and  $F_s$  the images of  $X$ ,  $Y$  and  $F$  under  $\pi_W$ . Let  $\varphi_s: X_s \to Y_s$  be the covering map induced by the ruling of  $F_s$ . Then

<span id="page-7-3"></span>(5) 
$$
0 \to \mathcal{O}_{X_s}(m) \to N_{X_s/\mathbb{P}^s} \to \varphi_s^* N_{Y_s/\mathbb{P}^{s-1}} \to 0.
$$

*Proof.* Since  $r \geq s + 1 \geq 6$ , a general projective subspace of  $\mathbb{P}^r$  of dimension  $r - s - 1$  does not meet the secant variety of  $F$ , which is of dimension at most 5. Therefore  $X$ ,  $Y$  and  $F$ are isomorphic to their images  $X_s$ ,  $Y_s$  and  $F_s$ . Also, the  $m:1$  covering  $\varphi: X \to Y$  induced by the ruling on F goes to an  $m:1$  covering  $\varphi_s: X_s \to Y_s$  induced by the ruling on  $F_s$ such that  $\pi_{W|_Y}\circ\varphi\,=\,\varphi_s\circ\pi_{W|_X}.$  In particular,  $\pi_{W|_X}(R_\varphi)\,=\,R_{\varphi_s}.$  Thus the ramification divisor  $R_{\varphi_s}$  is linearly equivalent to a divisor cut on  $X_s$  by a hypersurface of degree  $m-1$ in  $\mathbb{P}^s$ . Hence  $\mathcal{O}_{X_s}(R_{\varphi_s}) \simeq \mathcal{O}_{X_s}(m-1)$  and Lemma [4](#page-5-0) gives the exact sequence [\(5\)](#page-7-3).

### <span id="page-7-4"></span>3. A SHORT NOTE ON THE GAUSSIAN MAP

<span id="page-7-0"></span>Let *Y* be a smooth curve of genus  $\gamma$  and *L* and *M* be line bundles on *Y*. Let  $\mu_{L,M}$ 

(6) 
$$
\mu_{L,M}: H^0(Y,L)\otimes H^0(Y,M)\to H^0(Y,L\otimes M)
$$

be the natural multiplication. The Gaussian map  $\Phi_{L,M}$ 

$$
\Phi_{L,M} : \ker \mu_{L,M} \to H^0(Y, L \otimes M \otimes \omega_Y)
$$

was introduced by Wahl in [\[24\]](#page-13-14). Locally,  $\Phi_{L,M}$  :  $s \otimes t \mapsto sdt - tds$  for sections  $s \in H^0(L)$ and  $t \in H^0(M)$ . It has been studied by a number of authors. We refer to [\[24\]](#page-13-14) and [\[5\]](#page-13-15) for its

precise definition and some properties. We recall only several notions that will be used in Proposition [7](#page-8-0) needed for the proof of Theorem A.

The notation  $R(L, M)$  is often used instead of ker  $\mu_{L,M}$  for the map  $\mu_{L,M}$  in [\(6\)](#page-7-4). When  $V \subset H^0(Y, L)$  is a vector subspace and  $M = \omega_Y$ , the map  $\mu_{L,M}$  in [\(6\)](#page-7-4) restricted on  $V\otimes H^0(Y,\omega_Y)$  will be denoted by  $\mu_V$  and the Gaussian map restricted on ker  $\mu_V$  will be denoted by  $\Phi_{\omega_Y, V}$ .

The proposition that follows is formulated in the specific form in which it will be used in the proof of Theorem A.

<span id="page-8-0"></span>**Proposition 7.** Let Y be a smooth curve of general moduli of genus  $\gamma \geq 10$ , and let E be a general *line bundle on Y of degree*  $g - 2\gamma + 1 \geq 2\gamma - 1$ *. Let*  $V \subseteq H^0(Y, E)$  *be general linear subspace of dimension*  $r = \dim V \ge \max\left\{\gamma, \frac{2(g-1)}{\gamma}\right\}$  $\left\{ \frac{p-1}{\gamma} \right\}$ . Consider the embedding  $Y \subset \mathbb P^{r-1} \equiv \mathbb P(V^\vee)$  given by V *. Then*

- *the restricted Gaussian mapping*  $\Phi_{\omega_Y,V}$  *is surjective, and*
- $h^0(N_{Y/\mathbb{P}^{r-1}}(-1)) = \dim V = r.$

*Proof.* Denote by  $\mu$  the cup-product map

$$
\mu: H^0(Y, E) \otimes H^0(Y, \omega_Y) \to H^0(Y, \omega_Y \otimes E).
$$

Since deg  $E = g - 2\gamma + 1 \geq 2\gamma - 1$ , so E is very ample, it follows by [\[16,](#page-13-16) Theorem (4.e.1) and Theorem (4.e.4)] and [\[7\]](#page-13-17) that  $\mu$  is surjective.

The linear series determined by V is very ample since  $Y \in \mathcal{M}_{\gamma}$  is general,  $\gamma \geq 10$ and  $V \subset H^0(Y, E)$  is also general. Consider the restriction  $\mu_V$  of  $\mu$  to

$$
\mu_V: V \otimes H^0(Y, \omega_Y) \to H^0(Y, \omega_Y \otimes E).
$$

Let  $R(\omega_Y, E)$  be the kernel of the map  $\mu$  and consider the Gaussian map  $\Phi_{\omega_Y, E}$  defined on  $R(\omega_Y, E)$ 

$$
\Phi_{\omega_Y,E}: R(\omega_Y, E) \to H^0(\omega_Y^2 \otimes E),
$$

and similarly its restriction  $\Phi_{\omega_Y,V}$  defined on the kernel  $R(\omega_Y, V)$  of the map  $\mu_V$ 

(7) 
$$
\Phi_{\omega_Y,V}: R(\omega_Y, V) \to H^0(\omega_Y^2 \otimes E).
$$

In the case of complete embedding, i.e. if  $V = H^0(Y, E)$ , the claim follows by [\[9,](#page-13-18) Proposition 1.2], where it is proven that

$$
h^{0}(N_{Y/\mathbb{P}^{r-1}}(-1)) = h^{0}(Y, E) + \text{corank}(\Phi_{\omega_{Y}, E}),
$$

and by [\[8,](#page-13-19) Proposition (2.9)], where it is proven that  $\Phi_{\omega_Y,E}$  is surjective for  $\gamma \geq 10$  and  $\deg E = g - 2\gamma + 1 \geq 2\gamma - 1$ . In the case of incomplete embedding, i.e. if  $V \subsetneq H^0(Y, E)$ , exactly the same argument as in the proof of [\[9,](#page-13-18) Proposition 1.2] shows that

<span id="page-8-1"></span>(8) 
$$
h^{0}(N_{Y/\mathbb{P}^{r-1}}(-1)) = \dim V + \text{corank}(\Phi_{\omega_{Y},V}) = r + \text{corank}(\Phi_{\omega_{Y},V}),
$$

provided that  $\mu_V$  is surjective. This is what we will prove next.

Since  $\mu_V$  is the restriction of  $\mu$  to  $V\otimes H^0(Y,\omega_Y)$ , we have

$$
\ker \mu_V = \ker \mu \cap (V \otimes H^0(Y, \omega_Y)) \ .
$$

Due to  $\gamma \le \dim V \le \dim H^0(Y,E)$ , it follows from [\[2,](#page-13-20) Proposition 4.3] that

<span id="page-9-1"></span>(9) dim  $(\ker \mu \cap (V \otimes H^0(Y, \omega_Y))) = \max\{0, \dim (\ker \mu) - (h^0(Y, E) - \dim V)h^0(Y, \omega_Y)\}.$ 

Since  $\mu$  is surjective, dim (ker  $\mu$ ) = (deg(E) –  $\gamma + 1)\gamma$  – (deg(E) +  $\gamma$  – 1) = (g – 3 $\gamma$ )( $\gamma$  – 1). By assumption  $r = \dim V \geq \frac{2(g-1)}{2}$  $\frac{\gamma^{(-1)}}{\gamma}$ , hence

dim ker 
$$
\mu - (h^0(Y, E) - \dim V)\gamma = (\gamma - 1)(g - 3\gamma) - (g - 3\gamma + 2 - r)\gamma
$$
  
=  $\gamma - g + r\gamma > 0$ .

By [\(9\)](#page-9-1) we obtain

dim ker 
$$
\mu_V = \gamma - g + r\gamma
$$
.

From here we get for the dimension of its image

$$
\dim\left(\text{Im}(\mu_V)\right) = r\gamma - \dim \ker \mu_V = g - \gamma = h^0(Y, \omega_Y \otimes E).
$$

This shows that  $\mu_V$  is surjective, which proves [\(8\)](#page-8-1).

It remains to show that  $\Phi_{\omega_Y,V}$  is surjective. According to [\[2,](#page-13-20) Theorem 4.1], the Gaussian map  $\Phi_{\omega_Y,V}$  is of maximal rank. Suppose that it is not surjective. Then it must be injective and its image in  $H^0(Y,\omega_Y^2\otimes E)$  should be proper, hence

$$
\gamma - g + r\gamma = \dim \ker \mu_V < h^0(Y, \omega_Y^2 \otimes E) = g + \gamma - 2,
$$

which implies  $r < \frac{2(g-1)}{\gamma}$ . The last is impossible in view of the assumption that  $r =$  $\dim V \geq \max\left\{\gamma, \frac{2(g-1)}{\gamma}\right\}$  $\left\{\frac{y-1}{\gamma}\right\}$ . Therefore,  $\Phi_{\omega_Y,V}$  must be surjective and from [\(8\)](#page-8-1) we conclude also that  $h^0(N_{Y/\mathbb{P}^{r-1}}(-1)) = \dim V = r.$ 

#### 4. PROOF OF THEOREM A

<span id="page-9-0"></span>Before demonstrating the proof of Theorem A we recall a few facts concerning the Hilbert scheme of cones. Proposition [1](#page-4-0) and the counting of the number of parameters in Remark [3](#page-4-1) gives the idea how to construct explicitly the component  $\mathcal{D}_{d,g,R}$ . Recall that  $d = 2g - 4\gamma + 2$  and  $R = g - 3\gamma + 2$ .

Let  $\gamma \geq 10$  and  $g \geq 4\gamma - 2$  be integers. Consider the Hilbert scheme  $\mathcal{I}_{d/2,\gamma,R-1}$ of smooth curves of degree  $d/2$  and genus  $\gamma$  in  $\mathbb{P}^{R-1}$ . By [\[18,](#page-13-0) Theorem on p. 75] and [\[22,](#page-13-21) Theorem on p. 26],  $\mathcal{I}_{d/2,\gamma,R-1}$  is reduced and irreducible of dimension  $\lambda_{d/2,\gamma,R-1}$  =  $Rd/2 - (R-4)(\gamma - 1)$ . Denote by  $\mathcal{H}(\mathcal{I}_{d/2,\gamma,R-1})$  the family of cones in  $\mathbb{P}^R$  over curves representing points of  $\mathcal{I}_{d/2,\gamma,R-1}$ . Since  $\gamma \geq 10$  it follows by [\[8,](#page-13-19) Proposition 2.1] that for a general  $[Y]\in\mathcal{I}_{d/2,\gamma,R-1}$  the Gaussian map  $\Phi_{\omega_Y,{\cal O}_Y(1)}$  is surjective, hence by [\[8,](#page-13-19) Proposition

2.18]  $\mathcal{H}(\mathcal{I}_{d/2,\gamma,R-1})$  is a generically smooth component of the Hilbert scheme of surfaces of degree  $d/2$  in  $\mathbb{P}^R$  and

(10) 
$$
\dim \mathcal{H}(\mathcal{I}_{d/2,\gamma,R-1}) = h^0(Y,N_{Y/\mathbb{P}^{R-1}}) + R = \lambda_{d/2,\gamma,R-1} + R.
$$

<span id="page-10-0"></span>First we give the proof of Theorem A in the case  $r = R$ .

<span id="page-10-1"></span>**Proposition 8.** *Suppose that*  $\gamma \ge 10$  *and*  $g \ge 4\gamma - 2$ . Let  $\mathcal{F}_{d,g,R}$  be the family of curves  $C \subset \mathbb{P}^R$ *obtained as the intersection of a cone* F *and a general hypersurface of degree 2 in* P <sup>R</sup>*, where*  $[F] \in \mathcal{H}(\mathcal{I}_{d/2,\gamma,R-1})$ . Let  $\mathcal{D}_{d,g,R}$  be the closure of the set of points in  $\mathcal{I}_{d,g,R}$  corresponding to curves *from the family*  $\mathcal{F}_{d,q,R}$ *. Then* 

- $\mathcal{D}_{d,q,R}$  *is a generically smooth irreducible component of*  $\mathcal{I}_{d,q,R}$ *, and*
- dim  $\mathcal{D}_{d,g,R} = 2g \gamma 2 + (R+1)^2 = \lambda_{d,g,R} + R\gamma 2g + 2.$

*Proof.* First we compute dim  $\mathcal{D}_{d,q,R}$ . For a general point  $[F] \in \mathcal{H}(\mathcal{I}_{d/2,\gamma,R-1})$ , the cone F is projectively normal since it is a cone over a general curve Y from  $\mathcal{I}_{d/2,\gamma,R-1}$ , which is projectively normal by [\[15,](#page-13-22) Theorem 1, p. 74]. Therefore the linear series  $|\mathcal{O}_F(2)|$  on F is induced by  $|\mathcal{O}_{\mathbb{P}^R}(2)|$ . By equalities [\(10\)](#page-10-0) and [\(1\)](#page-3-1),  $h^0(F, \mathcal{O}_F(2)) = 3g-8\gamma+6$ . Therefore

> $\dim \mathcal{D}_{d,g,R} = \dim \mathcal{H}(\mathcal{I}_{g-2\gamma+1,\gamma,R-1}) + h^0(F,\mathcal{O}_F(2)) - 1$  $=\lambda_{d/2,\gamma,R-1} + R + 3q - 8\gamma + 5$ .

Remark that since  $\lambda_{d/2,\gamma,R-1} = Rd/2 - (R-4)(\gamma - 1)$  and  $\lambda_{d,q,R} = (R+1)d - (R-1)(g-1)$ , the expression for dim  $\mathcal{D}_{d,q,R}$  can also be written as

$$
\dim \mathcal{D}_{d,g,R} = \lambda_{d,g,R} + R\gamma - 2g + 2 = (R+1)^2 + 2g - \gamma - 2.
$$

To prove that  $\mathcal{D}_{d,q,R}$  is a generically smooth component of  $\mathcal{I}_{d,q,R}$ , it is sufficient to show that for a general  $[X] \in \mathcal{D}_{d,g,R}$  we have  $h^0(X,N_{X/\mathbb{P}^r}) = (R+1)^2 + 2g - \gamma - 2 = 0$  $\dim \mathcal{D}_{d,g,R}.$  Since  $X \subset F$  is cut by a general quadratic hypersurface in  $\mathbb{P}^R$ , the ruling of  $F$ induces a double covering  $\varphi: X \to Y$ , where  $Y \subset F$  is cut by a general hyperplane in  $\mathbb{P}^R$ and also  $[Y] \in \mathcal{I}_{d/2,\gamma,R-1}$  is general. It follows by Proposition [1](#page-4-0) and Corollary [5](#page-7-2) that

$$
0 \to \mathcal{O}_X(2) \to N_{X/\mathbb{P}^R} \to \varphi^* N_{Y/\mathbb{P}^{R-1}} \to 0.
$$

Since deg  $\mathcal{O}_X(2) = 2d = 4g - 8\gamma + 4 > 2g - 2$ , the series  $|\mathcal{O}_X(2)|$  is nonspecial, hence  $h^1(X, \mathcal{O}_X(2)) = 0$ . Therefore, using projection formula, Leray's isomorphism and  $\varphi_* O_X =$  $O_Y + O_Y(-E)$ , we get

$$
h^{0}(X, N_{X/\mathbb{P}^{R}}) = h^{0}(X, \mathcal{O}_{X}(2)) + h^{0}(X, \varphi^{*}N_{Y/\mathbb{P}^{R-1}})
$$
  
=  $h^{0}(X, \mathcal{O}_{X}(2)) + h^{0}(Y, N_{Y/\mathbb{P}^{R-1}}) + h^{0}(Y, N_{Y/\mathbb{P}^{R-1}}(-1))$   
=  $3g - 8\gamma + 5 + \lambda_{d/2, \gamma, R-1} + R$   
=  $\dim \mathcal{D}_{d,g,R}$ .

This implies that for a general  $[X] \in \mathcal{D}_{d,q,R}$ 

$$
\dim \mathcal{D}_{d,g,R} = \dim \mathcal{F}_{d,g,R} = h^0(X,N_{X/\mathbb{P}^R}) = \dim T_{[X]}\mathcal{D}_{d,g,R},
$$

therefore  $\mathcal{D}_{d,g,R}$  is a generically smooth component of  $\mathcal{I}_{d,g,R}$ .

Now we give the proof of Theorem A for  $\max\left\{\gamma,\frac{2(g-1)}{\gamma}\right\}$  $\left\{\frac{n-1}{\gamma}\right\} \leq r < R.$ 

**Proof of Theorem A.** Let  $\mathcal{F}_{d,g,r}$  be the family of curves in  $\mathbb{P}^r$  obtained from the family  $\mathcal{F}_{d,g,R}$  in Proposition [8](#page-10-1) by a projection  $\pi_W : \mathbb{P}^R \to \mathbb{P}^r$  with center  $W \subset \mathbb{P}^R$ , where  $W \cong \mathbb{P}^r$  $\mathbb{P}^{R-r-1}$  is general. Let  $\mathcal{D}_{d,g,r}$  be the closure of the set of points in  $\mathcal{I}_{d,g,r}$  corresponding to the curves from the family  $\mathcal{F}_{d,g,r}.$  Since  $\text{codim}(W,\mathbb{P}^R) \geq \gamma + 1 \geq 10$ , a cone  $F$  and curves X and Y as in the proof of Proposition [8](#page-10-1) are isomorphic to their images  $F_r = \pi_W(F)$ ,  $X_r = \pi_W(X)$  and  $Y_r = \pi_W(Y)$ , correspondingly. Also,  $\varphi_r : X_r \to Y_r$  induced by ruling of  $F_r$  is a double covering as in Corollary [6.](#page-7-5) Note that if p is the vertex of F then  $\pi_W(p)$  is the vertex of  $F_r$ . Therefore

<span id="page-11-0"></span>(11) 
$$
0 \to \mathcal{O}_{X_r}(2) \to N_{X_r/\mathbb{P}^r} \to \varphi_r^* N_{Y_r/\mathbb{P}^{r-1}} \to 0.
$$

The embedding of  $Y_r \subset \mathbb{P}^{r-1}$  is incomplete, but since  $\deg Y_r = d/2 = g - 2\gamma + 1 \geq$  $2\gamma - 1$ , it follows by [\[22\]](#page-13-21) that  $\mathcal{I}_{d/2,\gamma,r-1}$  has a unique generically smooth component of the expected dimension  $\lambda_{d/2,\gamma,r-1}$ . Therefore, for a general  $Y_r \in \mathcal{I}_{d/2,\gamma,r-1}$  (as in our case),  $h^1(Y_r, N_{Y_r/\mathbb{P}^{r-1}}) = 0$  or equivalently  $h^0(Y_r, N_{Y_r/\mathbb{P}^{r-1}}) = \lambda_{d/2,\gamma,r-1}$ . Then we can compute  $h^0(N_{X_r/\mathbb{P}^r})$  in a similar way as before. Since the projection  $\pi_W$  is general and  $r \geq \max\left\{\gamma, \frac{2(g-1)}{\gamma}\right\}$  $\left\{ \frac{p-1}{\gamma} \right\}$ , it follows by Proposition [7](#page-8-0) that  $h^0(Y_r, N_{Y_r/\mathbb{P}^{r-1}}(-1)) = r$ . Since  $deg \mathcal{O}_{X_r}(2) = 2d > 2g - 2$  we have  $h^1(X_r, \mathcal{O}_{X_r}(2)) = h^1(X, \mathcal{O}_X(2)) = 0$ . Using [\(11\)](#page-11-0) we find

$$
h^{0}(X_{r}, N_{X_{r}/\mathbb{P}^{r}}) = h^{0}(X_{r}, \mathcal{O}_{X_{r}}(2)) + h^{0}(X_{r}, \varphi_{r}^{*}N_{Y_{r}/\mathbb{P}^{r-1}})
$$
  
= 2d - g + 1 + h^{0}(Y\_{r}, N\_{Y\_{r}/\mathbb{P}^{r-1}}) + h^{0}(Y\_{r}, N\_{Y\_{r}/\mathbb{P}^{r-1}}(-1))  
= 4g - 8\gamma + 4 - g + 1 + \lambda\_{g-2\gamma+1,\gamma,r-1} + r  
= 3g - 8\gamma + 5 + r + (g - 2\gamma + 1)r - (r - 4)(\gamma - 1)  
= (r + 3)g - (3r + 4)\gamma + 3r + 1.

Let's compute also the dimension of the family  $\mathcal{F}_{d,q,r}$ . It is similar to the one carried out in the proof of [\[6,](#page-13-6) Theorem 4.3]. Since the curves in  $\mathcal{F}_{d,q,r}$  are obtained as generic projections from  $\mathbb{P}^R$  to  $\mathbb{P}^r$ , we have

$$
\dim \mathcal{F}_{d,g,r} = \dim \mathcal{F}_{d,g,R} - \dim \operatorname{Aut} \mathbb{P}^R + \dim \operatorname{Aut} \mathbb{P}^r + \dim Grass(r+1, R+1)
$$
  
= 2g - \gamma - 2 + (r + 1)^2 + (r + 1)(R - r)  
= 2g - \gamma - 2 + (r + 1)(g - 3\gamma + 2 + 1)  
= (r + 3)g - (3r + 4)\gamma + 3r + 1.

Notice that this number is exactly equal to the one claimed in Theorem A since

$$
\lambda_{d,g,r} + r\gamma - 2g + 2 = (r+1)(2g - 4\gamma + 2) - (r-3)(g-1) + r\gamma - 2g + 2
$$
  
=  $(r+3)g - (3r+4)\gamma + 3r + 1$ .

Hence  $\dim \mathcal{D}_{d,g,r} = \dim \mathcal{F}_{d,g,r} = h^0(X_r,N_{X_r/\mathbb{P}^r}) = \dim T_{[X_r]} \mathcal{D}_{d,g,r}$ , which completes the proof of Theorem A.

## <span id="page-12-0"></span>**Corollary 9.** *If*

(12) 
$$
\gamma \mid 2(g-1) \quad \text{and} \quad r := \frac{2(g-1)}{\gamma} \ge \gamma \ge 10,
$$

*then*  $\mathcal{D}_{d,q,r}$  *is a regular component of*  $\mathcal{I}_{d,q,r}$  *different from the distinguished one.* 

*Proof.* With the particular values of  $d = 2g - 4\gamma + 2$  and  $r$  we have  $r = \frac{2(g-1)}{\gamma} \leq \frac{g-1}{5}$  $\frac{-1}{5}$  for  $\gamma \ge 10$ , hence  $5r \le g - 1$ . From here it is easy to see that  $d - g - r \ge g + 2 - 5r \ge 3$ . Therefore  $\rho(d, g, r) = g - (r + 1)(g - d + r) \ge g > 0$ , hence the distinguished component of  $\mathcal{I}_{d,q,r}$  dominating  $\mathcal{M}_q$  exists. Apart from it, Theorem A guarantees the existence of the regular component  $\mathcal{D}_{d,g,r}$  which is apparently different from the distinguished one as the former projects properly in  $\mathcal{M}_q$ .

**Remark 10.** It appears that the condition  $r \geq \frac{2(g-1)}{2}$  $\frac{q-1}{\gamma}$  is essential for the family  $\mathcal{F}_{d, g, r}$  giving rise *to a component of*  $\mathcal{I}_{d,g,r}$ *, because in such case*  $r<\frac{2(g-1)}{\gamma}$  *we have*  $\dim\mathcal{F}_{d,g,r}<\lambda_{d,g,r}.$  *Notice also that in such a case the Gaussian map in* Proposition [7](#page-8-0) *is definitely not surjective.*

<span id="page-12-1"></span>**Remark 11.** *In their paper*[\[20\]](#page-13-4) *Mezzetti and Sacchiero constructed generically smooth irreducible*  $\alpha$  *components of*  $\mathcal{I}_{d,g,r}$ *, denoted there*  $W^m_{d,g,r}$ *, whose general points are*  $m-$  *sheeted coverings of*  $\mathbb{P}^1$ *,* where  $m \geq 3$ . In the case of  $g = 6\gamma - 3$ , we have  $d = 2g - 4\gamma + 2 = 8\gamma - 4 = \frac{4}{3}g$ ,  $R =$  $g-3\gamma+2=3\gamma-1=\frac{g+1}{2}$ , and the Hilbert scheme  $\mathcal{I}_{\frac{4g}{3},g,\frac{g+1}{2}}$  has two components parametrizing linearly normal curves. One of them is the component  $\mathcal{D}_{\frac{4g}{3},g,\frac{g+1}{2}}$ , shown to exist in our Proposition *[8,](#page-10-1) and the other one is the component*  $W_{d,g,r}^m$  *for*  $m=4$ *,*  $d=\frac{4}{3}$  $\frac{4}{3}g$ , and  $r = R = \frac{g+1}{2}$  $\frac{+1}{2}$  (it is easy *to check that the numerical conditions for the existence of*  $W_{\frac{4g}{3},g,\frac{g+1}{2}}^4$  *are satisfied when*  $\gamma\geq 10$ *). Notice that since*  $d - g - R = -\gamma < 0$ , the existence of these two components do not provide *a counterexample to* Severi's conjecture *claiming that* Id,g,r *has a unique irreducible component parametrizing linearly normal curves if*  $d \geq g + r$ .

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