EQUIVARIANT HOLOMORPHIC ANOMALY EQUATION

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ABSTRACT. In [16] the fundamental relationship between stable quotient invariants and the B-model for local \mathbb{P}^2 in all genera was studied under some specialization of equivariant variables. We generalize the argument of [16] to full equivariant settings without the specialization. Our main results are the proof of holomorphic anomaly equations for the equivariant Gromov-Witten theories of local \mathbb{P}^2 and local \mathbb{P}^3 . We also state the generalization to full equivariant formal quintic theory of the result in [17].

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0. INTRODUCTION

0.1. Equivariant local \mathbb{P}^n theories. Equivariant local \mathbb{P}^n theories can be constructed as follows. Let the algebraic torus

$$\mathsf{T}_{n+1} = (\mathbb{C}^*)^{n+1}$$

act with the standard linearization on \mathbb{P}^n with weights $\lambda_0, \ldots, \lambda_n$ on the vector space $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Let $\overline{M}_g(\mathbb{P}^n, d)$ be the moduli space of stable maps to \mathbb{P}^n equipped with the canonical T_{n+1} -action, and let

 $\mathsf{C} \to \overline{M}_g(\mathbb{P}^n, d) \,, \ f: \mathcal{C} \to \mathbb{P}^n \,, \ \mathsf{S} = f^* \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathsf{C}$

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be the standard universal structures.

The equivariant Gromov-Witten invariants of the local \mathbb{P}^n are defined via the equivariant integrals

(1)
$$N_{g,d}^{\mathsf{GW}} = \int_{[\overline{M}_g(\mathbb{P}^n,d)]^{\mathrm{vir}}} e\Big(-R\pi_*f^*\mathcal{O}_{\mathbb{P}^n}(-n-1)\Big).$$

The integral (1) defines a rational function in λ_i

$$N_{g,d}^{\mathsf{GW}} \in \mathbb{Q}(\lambda_0, \dots, \lambda_n)$$
.

Over the moduli space of stable quotients, there is a universal curve

(2)
$$\pi: \mathcal{C} \to \overline{Q}_q(\mathbb{P}^n, d)$$

with a universal quotient

$$0 \longrightarrow \mathsf{S} \longrightarrow \mathbb{C}^N \otimes \mathcal{O}_{\mathcal{C}} \xrightarrow{q_U} \mathsf{Q} \longrightarrow 0.$$

The equivariant stable quotient invariants of the local \mathbb{P}^n are defined via the equivariant integrals

(3)
$$N_{g,d}^{\mathsf{SQ}} = \int_{[\overline{Q}_g(\mathbb{P}^n,d)]^{\mathrm{vir}}} e\Big(-R\pi_*\mathsf{S}\Big) \,.$$

The integral (3) also defines a rational function in λ_i

$$N_{g,d}^{\mathsf{SQ}} \in \mathbb{Q}(\lambda_0, \dots, \lambda_n)$$
.

We refer the reader to [16, Section 1] for a more leisurely treatment of stable quotients.

In [16] it was observed that the analysis of *I*-function in [21] plays important role in the study of local \mathbb{P}^n theories. But the result in [21] holds only after the specialization to (n + 1)-th root of unity ζ_{n+1} ,

$$\lambda_i = \zeta_{n+1}^i$$

In order to generalize the results in [16] to full equivariant theories, one needs the analogous generalization of the results in [21] to full equivariant settings. This will be studied in Appendix.

0.2. Holomorphic anomaly for $K\mathbb{P}^2$. We state the precise form of the holomorphic anomaly equations for local \mathbb{P}^2 . Denote by $K\mathbb{P}^2$ the total space of the canonical bundle over \mathbb{P}^2 . Let $H \in H^2(K\mathbb{P}^2, \mathbb{Q})$ be the hyperplane class obtained from \mathbb{P}^2 , and let

$$\mathcal{F}_{g,m}^{\mathsf{GW}}(Q) = \langle H, \dots, H \rangle_{g,m}^{\mathsf{GW}} = \sum_{d=0}^{\infty} Q^d \int_{[\overline{M}_{g,m}(K\mathbb{P}^2,d)]^{\mathrm{vir}}} \prod_{i=1}^m \operatorname{ev}_i^*(H) ,$$
$$\mathcal{F}_{g,m}^{\mathsf{SQ}}(q) = \langle H, \dots, H \rangle_{g,m}^{\mathsf{SQ}} = \sum_{d=0}^{\infty} Q^d \int_{[\overline{Q}_{g,m}(K\mathbb{P}^2,d)]^{\mathrm{vir}}} \prod_{i=1}^m \operatorname{ev}_i^*(H) .$$

be the Gromov-Witten and stable quotient series respectively (involving the evaluation morphisms at the markings). The relationship between the Gromov-Witten and stable quotient invariants of $K\mathbb{P}^2$ is proven in [7] in case 2g - 2 + n > 0:

(4)
$$\mathcal{F}_{g,m}^{\mathsf{GW}}(Q(q)) = \mathcal{F}_{g,m}^{\mathsf{SQ}}(q) \,,$$

where Q(q) is the mirror map,

$$I_1^{K\mathbb{P}^2}(q) = \log(q) + 3\sum_{d=1}^{\infty} (-q)^d \frac{(3d-1)!}{(d!)^3},$$
$$Q(q) = \exp\left(I_1^{K\mathbb{P}^2}(q)\right) = q \cdot \exp\left(3\sum_{d=1}^{\infty} (-q)^d \frac{(3d-1)!}{(d!)^3}\right)$$

To state the holomorphic anomaly equations, we need the following additional series in q.

$$L(q) = (1 + 27q)^{-\frac{1}{3}} = 1 - 9q + 162q^{2} + \dots ,$$

$$C_{1}(q) = q \frac{d}{dq} I_{1}^{K\mathbb{P}^{2}} ,$$

$$A_{2}(q) = \frac{1}{L^{3}} \left(3 \frac{q \frac{d}{dq} C_{1}}{C_{1}} + 1 - \frac{L^{3}}{2} \right) .$$

We also need new series $L_i(q)$ defined by roots of following degree 3 polynomial in \mathcal{L} for i = 0, 1, 2:

$$(1+27q)\mathcal{L}^3 - (\lambda_0 + \lambda_1 + \lambda_2)\mathcal{L}^2 + (\lambda_0\lambda_1 + \lambda_1\lambda_2 + \lambda_2\lambda_0)\mathcal{L} - \lambda_0\lambda_2\lambda_3,$$

with initial conditions,

$$L_i(0) = \lambda_i$$
.

Let f_2 be the polynomial of degree 2 in variable x over $\mathbb{C}(\lambda_0, \lambda_1, \lambda_2)$ defined by

$$f_2(x) := (\lambda_0 + \lambda_1 + \lambda_2)x^2 - 2(\lambda_0\lambda_1 + \lambda_1\lambda_2 + \lambda_2\lambda_0)x + 3\lambda_0\lambda_1\lambda_2.$$

The ring

$$\mathbf{G}_2 := \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[L_0^{\pm 1}, L_1^{\pm 1}, L_2^{\pm 1}, f(L_0)^{-\frac{1}{2}}, f(L_1)^{-\frac{1}{2}}, f(L_2)^{-\frac{1}{2}}]$$

will play a basic role in our paper. Consider the free polynomial rings in the variables A_2 and C_1^{-1} over \mathbb{G}_2 ,

(5)
$$G_2[A_2], G_2[A_2, C_1^{-1}].$$

We have canonical maps

(6)
$$\mathbb{G}_2[A_2] \to \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]], \quad \mathbb{G}_2[A_2, C_1^{-1}] \to \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]]$$

given by assigning the above defined series $A_2(q)$ and C_1^{-1} to the variables A_2 and C_1^{-1} respectively. Therefore we can consider elements of the rings (5) either as free polynomials in the variables A_2 and C_1^{-1} or as series in q.

Let $F(q) \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]]$ be a series in q. When we write

$$F(q) \in \mathbb{G}_2[A_2]$$

we mean there is a cononical lift $F \in \mathbb{G}_2[A_2]$ for which

$$F \to F(q) \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]]$$

under the map (6). The symbol F without the argument q is the canonical lift. The notation

$$F(q) \in \mathbb{G}_2[A_2, C_1^{-1}]$$

is parallel.

Let T be the standard coordinate mirror to $t = \log(q)$,

 $T = I_1^{K\mathbb{P}^2}(q) \,.$

Then $Q(q) = \exp(T)$ is the mirror map.

Conjecture 1. For the stable quotient invariants of $K\mathbb{P}^2$,

- (i) $\mathcal{F}_{g}^{SQ}(q) \in \mathbb{G}_{2}[A_{2}] \text{ for } g \geq 2,$ (ii) \mathcal{F}_{g}^{SQ} is of degree at most 3g 3 with respect to A_{2} , (iii) $\frac{\partial^{k}\mathcal{F}^{SQ}}{\partial T^{k}}(q) \in \mathbb{G}_{2}[A_{2}, C_{1}^{-1}] \text{ for } g \geq 1 \text{ and } k \geq 1,$ (iv) $\frac{\partial^{k}\mathcal{F}^{SQ}_{g}}{\partial T^{k}}$ is homogeneous of degree k with respect to C_{1}^{-1} .

Here, $\mathcal{F}_{a}^{SQ} = \mathcal{F}_{a,0}^{SQ}$.

Conjecture 2. The holomorphic anomaly equations for the stable quotient invariants of $K\mathbb{P}^2$ hold for $g \geq 2$:

$$\frac{1}{C_1^2} \frac{\partial \mathcal{F}_g^{\mathsf{SQ}}}{\partial A_2} = \frac{1}{2} \sum_{i=1}^{g-1} \frac{\partial \mathcal{F}_{g-i}^{\mathsf{SQ}}}{\partial T} \frac{\partial \mathcal{F}_i^{\mathsf{SQ}}}{\partial T} + \frac{1}{2} \frac{\partial^2 \mathcal{F}_{g-1}^{\mathsf{SQ}}}{\partial T^2} \,.$$

The derivative of \mathcal{F}_{g}^{SQ} with respect to A_{2} in the above equation is well-defined since

$$\mathcal{F}_g^{\mathsf{SQ}} \in \mathbb{G}_2[A_2]$$

by part (i) of Conjecture 2. By parts (ii) and (iii),

$$\frac{\partial \mathcal{F}_{g-i}^{\mathsf{SQ}}}{\partial T} \frac{\partial \mathcal{F}_{i}^{\mathsf{SQ}}}{\partial T} \ , \ \frac{\partial^2 \mathcal{F}_{g-1}^{\mathsf{SQ}}}{\partial T^2} \in \mathbb{G}_2[A_2, C_1^{-1}]$$

are both of degree 2 in C_1^{-1} . Hence, the holomorphic anomaly equation of Conjecture 2 may be viewed as holding in $\mathbb{G}[A_2]$ since the factors of C_1^{-1} on both sides cancel. If we use the specializations by primitive third root of unity ζ

$$\lambda_i = \zeta^i \,,$$

the holomorphic anomaly equations here for $K\mathbb{P}^2$ recover the precise form presented in [1, (4.27)] via B-model physics.

Conjecture 2 determine $\mathcal{F}_g^{SQ} \in \mathbb{G}_2[A_2]$ uniquely as a polynomial in A_2 up to a constant term in \mathbb{G}_2 . The degree of the constant term can be bounded. Therefore Conjecture 2 determine \mathcal{F}_g^{SQ} from the lower genus theory together with a finite amount of date.

We will prove the following special cases of the conjectures in Section 6.

Theorem 3. Conjecture 1 holds for the choices of $\lambda_0, \lambda_1, \lambda_2$ such that

$$\lambda_i \neq \lambda_j \text{ for } i \neq j,$$
$$(\lambda_0 \lambda_1 + \lambda_1 \lambda_2 + \lambda_2 \lambda_0)^2 - 3\lambda_0 \lambda_1 \lambda_2 (\lambda_0 + \lambda_1 + \lambda_2) = 0.$$

Theorem 4. Conjecture 2 holds for the choices of $\lambda_0, \lambda_1, \lambda_2$ such that

$$\lambda_i \neq \lambda_j \text{ for } i \neq j$$
,
 $(\lambda_0 \lambda_1 + \lambda_1 \lambda_2 + \lambda_2 \lambda_0)^2 - 3\lambda_0 \lambda_1 \lambda_2 (\lambda_0 + \lambda_1 + \lambda_2) = 0$.

0.3. Holomorphic anomaly equations for $K\mathbb{P}^3$. We state the precise form of the holomorphic anomaly equations for local \mathbb{P}^3 . Since the study of local \mathbb{P}^3 will be parallel to the study of local \mathbb{P}^2 , we will sometime use the same notations for local \mathbb{P}^2 and local \mathbb{P}^3 . Since the study of two theories are logically independent in our paper, the indication of each notation will be clear from the context. Denote by $K\mathbb{P}^3$ the total space of the canonical bundle over \mathbb{P}^3 . Let $H \in H^2(K\mathbb{P}^3, \mathbb{Q})$ be the hyperplane class obtained from \mathbb{P}^3 , and let

$$\mathcal{F}_{g,m}^{\mathsf{GW}}[a,b](Q) = \langle \tau_0(H)^a \tau_0(H^2)^b \rangle_{g,m}^{\mathsf{GW}} = \sum_{d=0}^{\infty} Q^d \int_{[\overline{M}_{g,m}(K\mathbb{P}^3,d)]^{\mathrm{vir}}} \prod_{i=1}^a \operatorname{ev}_i^*(H) \prod_{i=a+1}^{a+b} \operatorname{ev}_i^*(H^2),$$
$$\mathcal{F}_{g,m}^{\mathsf{SQ}}[a,b](q) = \langle \tau_0(H)^a \tau_0(H^2)^b \rangle_{g,m}^{\mathsf{SQ}} = \sum_{d=0}^{\infty} q^d \int_{[\overline{Q}_{g,m}(K\mathbb{P}^3,d)]^{\mathrm{vir}}} \prod_{i=1}^a \operatorname{ev}_i^*(H) \prod_{i=a+1}^{a+b} \operatorname{ev}_i^*(H^2).$$

be the Gromov-Witten and stable quotient series respectively with a + b = m (involving the evaluation morphisms at the markings). The relationship between the Gromov-Witten and stable quotient invariants of $K\mathbb{P}^3$ is proven in [7] in case 2g - 2 + n > 0:

(7)
$$\mathcal{F}_{g,m}^{\mathsf{GW}}[a,b](Q(q)) = \mathcal{F}_{g,m}^{\mathsf{SQ}}[a,b](q) \,,$$

where Q(q) is the mirror map,

$$I_1^{K\mathbb{P}^3}(q) = \log(q) + 4\sum_{d=1}^{\infty} q^d \frac{(4d-1)!}{(d!)^4},$$
$$Q(q) = \exp\left(I_1^{K\mathbb{P}^3}(q)\right) = q \cdot \exp\left(4\sum_{d=1}^{\infty} q^d \frac{(4d-1)!}{(d!)^4}\right)$$

To state the holomorphic anomaly equations, we need the following additional series in q.

$$L(q) = (1 - 4^4 q)^{-\frac{1}{4}} = 1 + 64q + 10240q^2 + \dots ,$$

$$C_1(q) = q \frac{d}{dq} I_1^{K\mathbb{P}^3} ,$$

$$A_2(q) = \frac{q \frac{d}{dq} C_1}{C_1} .$$

We will define extra series $B_2(q)$, $B_4(q) \in \mathbb{C}[[q]]$ in (56). We also need new series $L_i(q)$ defined by roots of following degree 4 polynomial in \mathcal{L} for i = 0, 1, 2, 3:

$$(1-4^4q)\mathcal{L}^4 - s_1\mathcal{L}^3 + s_2\mathcal{L}^2 - s_3\mathcal{L} + s_4,$$

with initial conditions,

$$L_i(0) = \lambda_i$$
.

Here, s_k is k-th elementary symmetric function in $\lambda_0, \ldots, \lambda_3$. Let f_3 be the polynomial of degree 3 in variable x over $\mathbb{C}(\lambda_0, \ldots, \lambda_3)$ defined by

$$f_3(x) := s_1 x^3 - 2s_2 x^2 + 3s_3 x - 4s_4.$$

The ring

$$\mathbb{G}_3 := \mathbb{C}(\lambda_0, \dots, \lambda_3)[L_0^{\pm 1}, \dots, L_3^{\pm 1}, f(L_0)^{-\frac{1}{2}}, \dots, f(L_3)^{-\frac{1}{2}}]$$

will play a basic role in our paper. Consider the free polynomial rings in the variables A_2 , B_2 , B_4 and C_1^{-1} over \mathbb{G}_3 ,

(8)
$$\mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}]$$

We have canonical map

(9)
$$\mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}] \to \mathbb{C}(\lambda_0, \dots, \lambda_3)[[q]]$$

given by assigning the above defined series $A_2(q)$, $B_2(q)$, $B_4(q)$ and $C_1(q)$ to the variables A_2 , B_2 , B_4 and C_1 respectively. Therefore we can consider elements of the rings (8) either as free polynomials in the variables A_2 , B_2 , B_4 and C_1 or as series in q.

Let $F(q) \in \mathbb{C}(\lambda_0, \ldots, \lambda_3)[[q]]$ be a series in q. When we write

$$F(q) \in \mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}]$$

we mean there is a cononical lift $F \in \mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}]$ for which

 $F \to F(q) \in \mathbb{C}(\lambda_0, \dots, \lambda_2)[[q]]$

under the map (9). The symbol F without the argument q is the canonical lift. The notation

$$F(q) \in \mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}]$$

is parallel.

Let T be the standard coordinate mirror to $t = \log(q)$,

$$T = I_1^{K\mathbb{P}^3}(q) \,.$$

Then $Q(q) = \exp(T)$ is the mirror map.

Conjecture 5. For the stable quotient invariants of $K\mathbb{P}^3$,

- (i) $\mathcal{F}_{g,a+b}^{SQ}[a,b](q) \in \mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}] \text{ for } g \geq 2,$ (ii) \mathcal{F}_g^{SQ} is of degree at most 2(3g-3) with respect to A_2 , (iii) $\frac{\partial^k \mathcal{F}^{SQ}}{\partial T^k}(q), \in \mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}] \text{ for } g \geq 1 \text{ and } k \geq 1.$

Here, $\mathcal{F}_{q}^{\mathsf{SQ}} = \mathcal{F}_{q,0}^{\mathsf{SQ}}[0,0].$

Conjecture 6. The holomorphic anomaly equations for the stable quotient invariants of $K\mathbb{P}^3$ hold for q > 2:

$$\frac{L^2}{4C_1} \frac{\partial \mathcal{F}_g^{\mathsf{SQ}}}{\partial A_2} + \frac{-2s_1 L^4 - C_1 (3B_2 L^2 - s_1 L^6)}{4C_1^2} \frac{\partial \mathcal{F}_g^{\mathsf{SQ}}}{\partial B_4} = \sum_{i=1}^{g-1} \mathcal{F}_{g-i,1}^{\mathsf{SQ}}[0,1] \mathcal{F}_{i,1}^{\mathsf{SQ}}[1,0] + \mathcal{F}_{g-1,1}^{\mathsf{SQ}}[1,1] ,$$

$$\frac{2L^4}{C_1^2(L^4 - 1 - 2A_2)} \frac{\partial \mathcal{F}_g^{\mathsf{SQ}}}{\partial B_2} = \sum_{i=1}^{g-1} \frac{\partial \mathcal{F}_{g-i}^{\mathsf{SQ}}}{\partial T} \frac{\partial \mathcal{F}_i^{\mathsf{SQ}}}{\partial T} + \frac{\partial^2 \mathcal{F}_{g-1}^{\mathsf{SQ}}}{\partial T^2}$$

The derivative of \mathcal{F}_g^{SQ} with respect to A_2 , B_2 and B_4 in the above equations is well-defined since

$$\mathcal{F}_g^{\mathsf{SQ}} \in \mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}]$$

by part (i) of Conjecture 6.

We will prove the following special cases of the conjectures in Section 6.

Theorem 7. Conjecture 5 holds for the choices of $\lambda_0, \ldots, \lambda_3$ such that

$$\lambda_i \neq \lambda_j \text{ for } i \neq j$$
,
 $4s_2^2 - s_1s_3 = 0$,
 $2s_2^3 - 27s_1^2s_4 = 0$.

Theorem 8. Conjecture 6 holds for the choices of $\lambda_0, \ldots, \lambda_3$ such that

$$\lambda_i \neq \lambda_j \text{ for } i \neq j ,$$

 $4s_2^2 - s_1 s_3 = 0 ,$
 $2s_2^3 - 27s_1^2 s_4 = 0 .$

For Calabi-Yau 3-folds, holomorphic anomaly equations were first discovered in physics ([2]). Also there were many studies to understand holomorphic anomaly equations mathematically ([12, 16, 17, 19]). But less is known for higher dimensional Calabi-Yau manifolds in physics. It might be interesting question to find physical arguments for the holomorphic anomaly equation for $K\mathbb{P}^3$ proposed in our paper.

0.4. Holomorphic anomaly for equivariant formal quintic invariants. A particular twisted theory on \mathbb{P}^4 related to the quintic 3fold was introduced in [17]. Let the algebraic torus

$$\mathsf{T} = (\mathbb{C}^*)^5$$

act with the standard linearization on \mathbb{P}^4 with weights $\lambda_0, \ldots, \lambda_4$ on the vector space $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1))$.

Let

(10)
$$\mathsf{C} \to \overline{M}_g(\mathbb{P}^4, d), \quad f: \mathsf{C} \to \mathbb{P}^4, \quad \mathsf{S} = f^* \mathcal{O}_{\mathbb{P}^4}(-1) \to \mathsf{C}$$

be the universal curve, the universal map, and the universal bundle over the moduli space of stable maps — all equipped with canonical T-actions. We define the formal quintic invariants by^1

(11)
$$\widetilde{N}_{g,d}^{\mathsf{GW}} = \int_{[\overline{M}_g(\mathbb{P}^4,d)]^{vir}} e(R\pi_*(\mathsf{S}^{-5})),$$

where $e(R\pi_*(S^{-5}))$ is the equivariant Euler class defined *after* localization. More precisely, on each T-fixed locus of $\overline{M}_g(\mathbb{P}^4, d)$, both

 $R^0 \pi_*(\mathsf{S}^{-5})$ and $R^1 \pi_*(\mathsf{S}^{-5})$

are vector bundles with moving weights, so

$$e(R\pi_*(\mathsf{S}^{-5})) = \frac{c_{\rm top}(R^0\pi_*(\mathsf{S}^{-5}))}{c_{\rm top}(R^1\pi_*(\mathsf{S}^{-5}))}$$

is well-defined. The integral (11) is homogeneous of degree 0 in localized equivariant cohomology and defines a rational function in λ_i ,

$$\widetilde{N}_{g,d}^{\mathsf{GW}} \in \mathbb{C}(\lambda_0, \dots, \lambda_4)$$

Let $g \geq 2$. The associated Gromov-Witten generating series is

$$\widetilde{\mathcal{F}}_{g}^{\mathsf{GW}}(Q) \ = \ \sum_{d=0}^{\infty} \widetilde{N}_{g,d}^{\mathsf{GW}} Q^{d} \ \in \ \mathbb{C}[[Q]]$$

Let

$$I_0^{\mathbf{Q}}(q) = \sum_{d=0}^{\infty} q^d \frac{(5d)!}{(d!)^5}, \quad I_1^{\mathbf{Q}}(q) = \log(q) I_0^{\mathbf{Q}}(q) + 5 \sum_{d=1}^{\infty} q^d \frac{(5d)!}{(d!)^5} \left(\sum_{r=d+1}^{5d} \frac{1}{r} \right).$$

We *define* the generating series of stable quotient invariants for formal quintic theory by the wall-crossing formula for the true quintic theory which has been recently proven by Ciocan-Fontanine and Kim in [6],

(12)
$$\widetilde{\mathcal{F}}_{g}^{\mathsf{GW}}(Q(q)) = I_{0}^{\mathsf{Q}}(q)^{2g-2} \cdot \widetilde{\mathcal{F}}_{g}^{\mathsf{SQ}}(q)$$

with respect to the true quintic mirror map

$$Q(q) = \exp\left(\frac{I_1^{\mathsf{Q}}(q)}{I_0^{\mathsf{Q}}(q)}\right) = q \cdot \exp\left(\frac{5\sum_{d=1}^{\infty} q^d \frac{(5d)!}{(d!)^5} \left(\sum_{r=d+1}^{5d} \frac{1}{r}\right)}{\sum_{d=0}^{\infty} \frac{(5d)!}{(d!)^5}}\right)$$

Denote the B-model side of (12) by

$$\widetilde{F}^{\mathsf{B}}_{g}(q) = I^{\mathsf{Q}}_{0}(q)^{2g-2} \widetilde{\mathcal{F}}^{\mathsf{SQ}}_{g}(q) \,.$$

In order to state the holomorphic anomaly equations, we require several series in q. First, let

$$L(q) = (1 - 5^5 q)^{-\frac{1}{5}} = 1 + 625q + 117185q^2 + \dots$$

¹The negative exponent denotes the dual: S is a line bundle and $S^{-5} = (S^{\star})^{\otimes 5}$.

Let $\mathsf{D} = q \frac{d}{dq}$, and let

$$C_0(q) = I_0^{\mathsf{Q}}, \quad C_1(q) = \mathsf{D}\left(\frac{I_1^{\mathsf{Q}}}{I_0^{\mathsf{Q}}}\right).$$

We define

$$\begin{split} K_2(q) &= -\frac{1}{L^5} \frac{\mathsf{D}C_0}{C_0}, \\ A_2(q) &= \frac{1}{L^5} \left(-\frac{1}{5} \frac{\mathsf{D}C_1}{C_1} - \frac{2}{5} \frac{\mathsf{D}C_0}{C_0} - \frac{3}{25} \right), \\ A_4(q) &= \frac{1}{L^{10}} \left(-\frac{1}{25} \left(\frac{\mathsf{D}C_0}{C_0} \right)^2 - \frac{1}{25} \left(\frac{\mathsf{D}C_0}{C_0} \right) \left(\frac{\mathsf{D}C_1}{C_1} \right) \right. \\ &\qquad + \frac{1}{25} \mathsf{D} \left(\frac{\mathsf{D}C_0}{C_0} \right) + \frac{2}{25^2} \right), \\ A_6(q) &= \frac{1}{31250L^{15}} \left(4 + 125\mathsf{D} \left(\frac{\mathsf{D}C_0}{C_0} \right) + 50 \left(\frac{\mathsf{D}C_0}{C_0} \right) \left(1 + 10\mathsf{D} \left(\frac{\mathsf{D}C_0}{C_0} \right) \right) \right. \\ &\qquad - 5L^5 \left(1 + 10 \left(\frac{\mathsf{D}C_0}{C_0} \right) + 25 \left(\frac{\mathsf{D}C_0}{C_0} \right)^2 + 25\mathsf{D} \left(\frac{q\frac{d}{dq}C_0}{C_0} \right) \right) \\ &\qquad + 125\mathsf{D}^2 \left(\frac{\mathsf{D}C_0}{C_0} \right) - 125 \left(\frac{\mathsf{D}C_0}{C_0} \right)^2 \left(\left(\frac{\mathsf{D}C_1}{C_1} \right) - 1 \right) \right). \end{split}$$

For the full equivariant formal quintic theory, we need extra series $B_1, B_2, B_3, B_4 \in \mathbb{C}[[q]]$ obtained from *I*-function of quintic. We will give the exact definitions of these extra series in forthcoming paper [15]. These series are closely related to the extra generators in [12], where the formal quintic theory was studied in genus 2 case with connections to real quintic theory.

We also need new series $L_i(q)$ defined by roots of following degree 5 polynomial in \mathcal{L} for $i = 0, \ldots, 4$:

$$(1-5^5q)\mathcal{L}^5 - s_1\mathcal{L}^4 + s_2\mathcal{L}^3 - s_3\mathcal{L}^2 + s_4\mathcal{L} - s_5,$$

with initial conditions,

$$L_i(0) = \lambda_i \, .$$

Here s_k is k-th elementary symmetric function in $\lambda_0, \ldots, \lambda_4$. Let f_4 be the polynomial of degree 4 in variable x over $\mathbb{C}(\lambda_0, \ldots, \lambda_4)$ defined by

$$f_4(x) := s_1 x^4 - 2s_2 x^3 + 3s_3 x^2 - 4s_3 x + 5s_4 \,.$$

The ring

$$\mathbb{G}_{\mathsf{Q}} := \mathbb{C}(\lambda_0, \dots, \lambda_4)[L_0^{\pm 1}, \dots, L_4^{\pm 1}, f_4(L_0)^{-\frac{1}{2}}, \dots, f_4(L_4)^{-\frac{1}{2}}]$$

will play a basic role in formal quintic theory.

Let T be the standard coordinate mirror to $t = \log(q)$,

$$T = \frac{I_1^{\mathsf{Q}}(q)}{I_0^{\mathsf{Q}}(q)} \,.$$

Then $Q(q) = \exp(T)$ is the mirror map. Let

$$\mathbb{G}_{\mathsf{Q}}[A_2, A_4, A_6, B_1, B_2, B_3, B_4, C_0^{\pm 1}, C_1^{-1}, K_2]$$

be the free polynomial ring over \mathbb{G}_{Q} .

Conjecture 9. For the series $\widetilde{\mathcal{F}}_{q}^{\mathsf{B}}$ associated to the formal quintic,

- (i) $\widetilde{\mathcal{F}}_{g}^{\mathsf{B}}(q) \in \mathbb{G}_{\mathsf{Q}}[A_{2}, A_{4}, A_{6}, B_{1}, \dots, B_{4}, C_{0}^{\pm 1}, C_{1}^{-1}, K_{2}] \text{ for } g \geq 2,$ (ii) $\frac{\partial^{k} \widetilde{\mathcal{F}}_{g}^{\mathsf{B}}}{\partial T^{k}}(q) \in \mathbb{G}_{\mathsf{Q}}[A_{2}, A_{4}, A_{6}, B_{1}, \dots, B_{4}, C_{0}^{\pm 1}, C_{1}^{-1}, K_{2}] \text{ for } g \geq 1,$ $k \geq 1,$
- (iii) $\frac{\partial^k \tilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial T^k}$ is homogeneous with respect to C_1^{-1} of degree k.

Conjecture 10. The series $\widetilde{\mathcal{F}}_{g}^{\mathsf{B}}$ associated to the formal quintic satisfy some holomorphic anomaly equations which specialize to

$$\begin{split} \frac{1}{C_0^2 C_1^2} \frac{\partial \widetilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial A_2} &- \frac{1}{5C_0^2 C_1^2} \frac{\partial \widetilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial A_4} K_2 + \frac{1}{50C_0^2 C_1^2} \frac{\partial \widetilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial A_6} K_2^2 = \\ & \frac{1}{2} \sum_{i=1}^{g-1} \frac{\partial \widetilde{\mathcal{F}}_{g-i}^{\mathsf{B}}}{\partial T} \frac{\partial \widetilde{\mathcal{F}}_i^{\mathsf{B}}}{\partial T} + \frac{1}{2} \frac{\partial^2 \widetilde{\mathcal{F}}_{g-1}^{\mathsf{B}}}{\partial T^2} ,\\ & \frac{\partial \widetilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial K_2} = 0 , \end{split}$$

with the choice of $\lambda_i = \zeta_5^i$.

We expect holomorphic anomaly equations in Conjecture 10 to hold in the ring

(13) $\mathbb{G}_{\mathsf{Q}}[A_2, A_4, A_6, B_1, B_2, B_3, B_4, C_0^{\pm 1}, C_1^{-1}, K_2].$

Remark 11. If we specialize λ_i to (the power of) primitive fifth root of unity ζ_5 ,

(14)
$$\lambda_i = \zeta_5^i \,,$$

the expected equations in Conjecture 10 exactly matches² the conjectural holomorphic anomaly equation [1, (2.52)] for the true quintic theory and this was the main result in [17]. Also the ring (13) can be reduced to the Yamaguchi-Yau ring introduced in [20] for the true quintic theory only with the choice of specialization (14). This explains why the specialization (14) used in [17] is the natural choice.

Theorem 12. Conjecture 9 holds for the choices of $\lambda_0, \ldots, \lambda_4$ such that

$$\lambda_i \neq \lambda_j \text{ for } i \neq j,$$

$$2s_1s_3 = s_2^2,$$

$$8s_1^2s_4 = s_2^3,$$

$$80s_1^3s_5 = s_2^4.$$

where s_k is k-th elementary symmetric function in $\lambda_0, \ldots, \lambda_4$.

Theorems 12 will be proven and the precise form of holomorphic anomaly equations in Conjecture 10 will appear in [15].

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1. LOCALIZATION GRAPH

1.1. Torus action. Let $\mathsf{T} = (\mathbb{C}^*)^{n+1}$ act diagonally on the vector space \mathbb{C}^{n+1} with weights

$$-\lambda_0,\ldots,-\lambda_n$$

Denote the T-fixed points of the induced T-action on \mathbb{P}^n by

$$p_0,\ldots,p_n$$

The weights of T on the tangent space $T_{p_j}(\mathbb{P}^n)$ are

$$\lambda_j - \lambda_0, \ldots, \widehat{\lambda_j - \lambda_j}, \ldots, \lambda_j - \lambda_n.$$

²Our functions K_2 and A_{2k} are normalized differently with respect to C_0 and C_1 . The dictionary to exactly match the notation of [1, (2.52)] is to multiply our K_2 by $(C_0C_1)^2$ and our A_{2k} by $(C_0C_1)^{2k}$.

There is an induced T-action on the moduli space $\overline{Q}_{g,k}(\mathbb{P}^n, d)$. The localization formula of [11] applied to the virtual fundamental class $[\overline{Q}_{g,k}(\mathbb{P}^n, d)]^{vir}$ will play a fundamental role our paper. The T-fixed loci are represented in terms of dual graphs, and the contributions of the T-fixed loci are given by tautological classes. The formulas here are standard, see [13, 18].

1.2. Graphs. Let the genus g and the number of markings k for the moduli space be in the stable range

(15)
$$2g - 2 + k > 0$$
.

We can organize the T-fixed loci of $\overline{Q}_{g,k}(\mathbb{P}^n, d)$ according to decorated graphs. A *decorated graph* $\Gamma \in \mathsf{G}_{g,k}(\mathbb{P}^n)$ consists of the data $(\mathsf{V}, \mathsf{E}, \mathsf{N}, \mathsf{g}, \mathsf{p})$ where

- (i) V is the vertex set,
- (ii) E is the edge set (including possible self-edges),
- (iii) $\mathsf{N}: \{1, 2, ..., k\} \to \mathsf{V}$ is the marking assignment,
- (iv) $g:V\to\mathbb{Z}_{\geq 0}$ is a genus assignment satisfying

$$g = \sum_{v \in V} \mathsf{g}(v) + h^1(\Gamma)$$

and for which (V, E, N, g) is stable graph³,

(v) $\mathbf{p}: \mathbf{V} \to (\mathbb{P}^n)^\mathsf{T}$ is an assignment of a T-fixed point $\mathbf{p}(v)$ to each vertex $v \in \mathbf{V}$.

The markings $L = \{1, \ldots, k\}$ are often called *legs*.

To each decorated graph $\Gamma \in \mathsf{G}_{g,k}(\mathbb{P}^n)$, we associate the set of fixed loci of

$$\sum_{d\geq 0} \left[\overline{Q}_{g,k}(\mathbb{P}^n, d) \right]^{\mathrm{vir}} q^d$$

with elements described as follows:

- (a) If $\{v_{i_1}, \ldots, v_{i_j}\} = \{v \mid \mathbf{p}(v) = p_i\}$, then $f^{-1}(p_i)$ is a disjoint union of connected stable curves of genera $\mathbf{g}(v_{i_1}), \ldots, \mathbf{g}(v_{i_j})$ and finitely many points.
- (b) There is a bijective correspondence between the connected components of C \ D and the set of edges and legs of Γ respecting vertex incidence where C is domain curve and D is union of all subcurves of C which appear in (a).

We write the localization formula as

$$\sum_{d\geq 0} \left[\overline{Q}_{g,k}(\mathbb{P}^n,d)\right]^{\operatorname{vir}} q^d = \sum_{\Gamma\in\mathsf{G}_{g,k}(\mathbb{P}^n)} \operatorname{Cont}_{\Gamma}.$$

³Corresponding to a stratum of the moduli space of stable curves $\overline{M}_{g,n}$.

While $\mathsf{G}_{g,k}(\mathbb{P}^n)$ is a finite set, each contribution $\operatorname{Cont}_{\Gamma}$ is a series in q obtained from an infinite sum over all edge possibilities (b).

1.3. Unstable graphs. The moduli spaces of stable quotients

 $\overline{Q}_{0,2}(\mathbb{P}^n,d)$ and $\overline{Q}_{1,0}(\mathbb{P}^n,d)$

for d > 0 are the only⁴ cases where the pair (g, k) does *not* satisfy the Deligne-Mumford stability condition (15).

An appropriate set of decorated graphs $G_{0,2}(\mathbb{P}^n)$ is easily defined: The graphs $\Gamma \in G_{0,2}(\mathbb{P}^n)$ all have 2 vertices connected by a single edge. Each vertex carries a marking. All of the conditions (i)-(v) of Section 1.2 are satisfied except for the stability of $(V, \mathsf{E}, \mathsf{N}, \gamma)$. The localization formula holds,

(16)
$$\sum_{d\geq 1} \left[\overline{Q}_{0,2}(\mathbb{P}^n, d) \right]^{\operatorname{vir}} q^d = \sum_{\Gamma\in\mathsf{G}_{0,2}(\mathbb{P}^n)} \operatorname{Cont}_{\Gamma},$$

For $\overline{Q}_{1,0}(\mathbb{P}^n, d)$, the matter is more problematic — usually a marking is introduced to break the symmetry.

2. Basic correlators

2.1. **Overview.** We review here basic generating series in q which arise in the genus 0 theory of quasimap invariants. The series will play a fundamental role in the calculations of Sections 3 - 6 related to the holomorphic anomaly equation for $K\mathbb{P}^2$.

We fix a torus action $\mathsf{T} = (\mathbb{C}^*)^3$ on \mathbb{P}^2 with weights⁵

$$-\lambda_0, -\lambda_1, -\lambda_2$$

on the vector space \mathbb{C}^3 . The T-weight on the fiber over p_i of the canonical bundle

(17)
$$\mathcal{O}_{\mathbb{P}^2}(-3) \to \mathbb{P}^2$$

is $-3\lambda_i$. The toric Calabi-Yau $K\mathbb{P}^2$ is the total space of (17).

⁴The moduli spaces $\overline{Q}_{0,0}(\mathbb{P}^n, d)$ and $\overline{Q}_{0,1}(\mathbb{P}^n, d)$ are empty by the definition of a stable quotient.

⁵The associated weights on $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ are $\lambda_0, \lambda_1, \lambda_2$ and so match the conventions of Section 0.1.

2.2. First correlators. We require several correlators defined via the Euler class of the obstruction bundle,

$$e(\mathrm{Obs}) = e(R^1 \pi_* \mathsf{S}^3),$$

associated to the $K\mathbb{P}^2$ geometry on the moduli space $\overline{Q}_{g,n}(\mathbb{P}^2, d)$. The first two are obtained from standard stable quotient invariants. For $\gamma_i \in H^*_{\mathsf{T}}(\mathbb{P}^2)$, let

$$\left\langle \gamma_{1}\psi^{a_{1}},...,\gamma_{n}\psi^{a_{n}}\right\rangle_{g,n,d}^{\mathsf{SQ}} = \int_{[\overline{Q}_{g,n}(\mathbb{P}^{2},d)]^{\mathrm{vir}}} e(\mathrm{Obs})\cdot\prod_{i=1}^{n}\mathrm{ev}_{i}^{*}(\gamma_{i})\psi_{i}^{a_{i}},$$
$$\left\langle \left\langle \gamma_{1}\psi^{a_{1}},...,\gamma_{n}\psi^{a_{n}}\right\rangle \right\rangle_{0,n}^{\mathsf{SQ}} = \sum_{d\geq0}\sum_{k\geq0}\frac{q^{d}}{k!}\left\langle \gamma_{1}\psi^{a_{1}},...,\gamma_{n}\psi^{a_{n}},t,...,t\right\rangle_{0,n+k,d}^{\mathsf{SQ}},$$

where, in the second series, $t \in H^*_{\mathsf{T}}(\mathbb{P}^2)$. We will systematically use the quasimap notation 0+ for stable quotients,

$$\left\langle \gamma_{1}\psi^{a_{1}},...,\gamma_{n}\psi^{a_{n}}\right\rangle_{g,n,d}^{0+} = \left\langle \gamma_{1}\psi^{a_{1}},...,\gamma_{n}\psi^{a_{n}}\right\rangle_{g,n,d}^{\mathsf{SQ}}$$

$$\left\langle \left\langle \gamma_{1}\psi^{a_{1}},...,\gamma_{n}\psi^{a_{n}}\right\rangle \right\rangle_{0,n}^{0+} = \left\langle \left\langle \gamma_{1}\psi^{a_{1}},...,\gamma_{n}\psi^{a_{n}}\right\rangle \right\rangle_{0,n}^{\mathsf{SQ}}$$

2.3. Light markings. Moduli of quasimaps can be considered with n ordinary (weight 1) markings and k light (weight ϵ) markings⁶,

$$\overline{Q}_{g,n|k}^{0+,0+}(\mathbb{P}^2,d)$$
.

Let $\gamma_i \in H^*_{\mathsf{T}}(\mathbb{P}^2)$ be equivariant cohomology classes, and let

$$\delta_j \in H^*_{\mathsf{T}}([\mathbb{C}^3/\mathbb{C}^*])$$

be classes on the stack quotient. Following the notation of [13], we define series for the $K\mathbb{P}^2$ geometry,

$$\begin{split} \left\langle \gamma_{1}\psi^{a_{1}},\ldots,\gamma_{n}\psi^{a_{n}};\delta_{1},\ldots,\delta_{k}\right\rangle_{g,n|k,d}^{0+,0+} = \\ \int_{[\overline{Q}_{g,n|k}^{0+,0+}(\mathbb{P}^{2},d)]^{\mathrm{vir}}} e(\mathrm{Obs})\cdot\prod_{i=1}^{n}\mathrm{ev}_{i}^{*}(\gamma_{i})\psi_{i}^{a_{i}}\cdot\prod_{j=1}^{k}\widehat{\mathrm{ev}}_{j}^{*}(\delta_{j}), \\ \left\langle \left\langle \gamma_{1}\psi^{a_{1}},\ldots,\gamma_{n}\psi^{a_{n}}\right\rangle \right\rangle_{0,n}^{0+,0+} = \\ \sum_{d\geq0}\sum_{k\geq0}\frac{q^{d}}{k!}\left\langle \gamma_{1}\psi^{a_{1}},\ldots,\gamma_{n}\psi^{a_{n}};t,\ldots,t\right\rangle_{0,n|k,d}^{0+,0+}, \end{split}$$

⁶See Sections 2 and 5 of [5].

where, in the second series, $t \in H^*_{\mathsf{T}}([\mathbb{C}^3/\mathbb{C}^*])$. For each T-fixed point $p_i \in \mathbb{P}^2$, let

$$e_i = e(T_{p_i}(\mathbb{P}^2)) \cdot (-3\lambda_i)$$

be the equivariant Euler class of the tangent space of $K\mathbb{P}^2$ at p_i . Let

$$\phi_i = \frac{-3\lambda_i \prod_{j \neq i} (H - \lambda_j)}{e_i}, \quad \phi^i = e_i \phi_i \quad \in H^*_{\mathsf{T}}(\mathbb{P}^2)$$

be cycle classes. Crucial for us are the series

$$\begin{split} \mathbb{S}_{i}(\gamma) &= e_{i} \left\langle \left\langle \frac{\phi_{i}}{z - \psi}, \gamma \right\rangle \right\rangle_{0,2}^{0+,0+} ,\\ \mathbb{V}_{ij} &= \left\langle \left\langle \frac{\phi_{i}}{x - \psi}, \frac{\phi_{j}}{y - \psi} \right\rangle \right\rangle_{0,2}^{0+,0+} . \end{split}$$

Unstable degree 0 terms are included by hand in the above formulas. For $S_i(\gamma)$, the unstable degree 0 term is $\gamma|_{p_i}$. For V_{ij} , the unstable degree 0 term is $\frac{\delta_{ij}}{e_i(x+y)}$.

We also write

$$\mathbb{S}(\gamma) = \sum_{i=0}^{2} \phi_i \mathbb{S}_i(\gamma) \,.$$

The series S_i and V_{ij} satisfy the basic relation

(18)
$$e_i \mathbb{V}_{ij}(x, y) e_j = \frac{\sum_{k=0}^2 \mathbb{S}_i(\phi_k)|_{z=x} \mathbb{S}_j(\phi^k)|_{z=y}}{x+y}$$

proven⁷ in [7].

Associated to each T-fixed point $p_i \in \mathbb{P}^2$, there is a special T-fixed point locus,

(19)
$$\overline{Q}_{0,k|m}^{0+,0+}(\mathbb{P}^2,d)^{\mathsf{T},p_i} \subset \overline{Q}_{0,k|m}^{0+,0+}(\mathbb{P}^2,d)$$

where all markings lie on a single connected genus 0 domain component contracted to p_i . Let Nor denote the equivariant normal bundle of $Q_{0,n|k}^{0+,0+}(\mathbb{P}^2,d)^{\mathsf{T},p_i}$ with respect to the embedding (19). Define

$$\left\langle \gamma_1 \psi^{a_1}, \dots, \gamma_n \psi^{a_n}; \delta_1, \dots, \delta_k \right\rangle_{0, n \mid k, d}^{0+, 0+, p_i} = \int_{\left[\overline{Q}_{0, n \mid k}^{0+, 0+}(\mathbb{P}^2, d)^{\mathsf{T}, p_i}\right]} \frac{e(\mathrm{Obs})}{e(\mathrm{Nor})} \cdot \prod_{i=1}^n \mathrm{ev}_i^*(\gamma_i) \psi_i^{a_i} \cdot \prod_{j=1}^k \widehat{\mathrm{ev}}_j^*(\delta_j) \,,$$

⁷In Gromov-Witten theory, a parallel relation is obtained immediately from the WDDV equation and the string equation. Since the map forgetting a point is not always well-defined for quasimaps, a different argument is needed here [7]

$$\left\langle \left\langle \gamma_1 \psi^{a_1}, \dots, \gamma_n \psi^{a_n} \right\rangle \right\rangle_{0,n}^{0+,0+,p_i} = \sum_{d \ge 0} \sum_{k \ge 0} \frac{q^d}{k!} \left\langle \gamma_1 \psi^{a_1}, \dots, \gamma_n \psi^{a_n}; t, \dots, t \right\rangle_{0,n|k,\beta}^{0+,0+,p_i}$$

2.4. Graph spaces and I-functions.

2.4.1. *Graph spaces.* The big I-function is defined in [5] via the geometry of weighted quasimap graph spaces. We briefly summarize the constructions of [5] in the special case of (0+, 0+)-stability. The more general weightings discussed in [5] will not be needed here.

As in Section 2.3, we consider the quotient

 $\mathbb{C}^3/\mathbb{C}^*$

associated to \mathbb{P}^2 . Following [5], there is a (0+, 0+)-stable quasimap graph space

(20)
$$\mathsf{QG}_{a,n|k,d}^{0+,0+}([\mathbb{C}^3/\mathbb{C}^*]).$$

A \mathbb{C} -point of the graph space is described by data

$$((C, \mathbf{x}, \mathbf{y}), (f, \varphi) : C \longrightarrow [\mathbb{C}^3/\mathbb{C}^*] \times [\mathbb{C}^2/\mathbb{C}^*]).$$

By the definition of stability, φ is a regular map to

$$\mathbb{P}^1 = \mathbb{C}^2 /\!\!/ \mathbb{C}^*$$

of class 1. Hence, the domain curve C has a distinguished irreducible component C_0 canonically isomorphic to \mathbb{P}^1 via φ . The standard \mathbb{C}^* action,

(21)
$$t \cdot [\xi_0, \xi_1] = [t\xi_0, \xi_1], \text{ for } t \in \mathbb{C}^*, [\xi_0, \xi_1] \in \mathbb{P}^1.$$

induces a \mathbb{C}^* -action on the graph space.

The \mathbb{C}^* -equivariant cohomology of a point is a free algebra with generator z,

$$H^*_{\mathbb{C}^*}(\operatorname{Spec}(\mathbb{C})) = \mathbb{Q}[z].$$

Our convention is to define z as the \mathbb{C}^* -equivariant first Chern class of the tangent line $T_0\mathbb{P}^1$ at $0 \in \mathbb{P}^1$ with respect to the action (21),

$$z = c_1(T_0\mathbb{P}^1)$$
.

The T-action on \mathbb{C}^3 lifts to a T-action on the graph space (20) which commutes with the \mathbb{C}^* -action obtained from the distinguished domain

component. As a result, we have a $T \times \mathbb{C}^*$ -action on the graph space and $T \times \mathbb{C}^*$ -equivariant evaluation morphisms

$$\operatorname{ev}_{i} : \mathsf{QG}_{g,n|k,\beta}^{0+,0+}([\mathbb{C}^{3}/\mathbb{C}^{*}]) \to \mathbb{P}^{2}, \qquad i = 1, \dots, n,$$

$$\widehat{\operatorname{ev}}_{j} : \mathsf{QG}_{g,n|k,\beta}^{0+,0+}([\mathbb{C}^{3}/\mathbb{C}^{*}]) \to [\mathbb{C}^{3}/\mathbb{C}^{*}], \qquad j = 1, \dots, k.$$

Since a morphism

$$f: C \to [\mathbb{C}^3/\mathbb{C}^*]$$

is equivalent to the data of a principal **G**-bundle P on C and a section u of $P \times_{\mathbb{C}^*} \mathbb{C}^3$, there is a natural morphism

$$C \to E\mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^3$$

and hence a pull-back map

$$f^*: H^*_{\mathbb{C}^*}([\mathbb{C}^3/\mathbb{C}^*]) \to H^*(C).$$

The above construction applied to the universal curve over the moduli space and the universal morphism to $[\mathbb{C}^3/\mathbb{C}^*]$ is T-equivariant. Hence, we obtain a pull-back map

$$\widehat{\operatorname{ev}}_{j}^{*}: H^{*}_{\mathsf{T}}(\mathbb{C}^{3}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[z] \to H^{*}_{\mathsf{T} \times \mathbb{C}^{*}}(\mathsf{QG}_{g, n|k, \beta}^{0+, 0+}([\mathbb{C}^{3}/\mathbb{C}^{*}]), \mathbb{Q})$$

associated to the evaluation map \widehat{ev}_i .

2.4.2. I-functions. The description of the fixed loci for the \mathbb{C}^* -action on

$$\mathsf{QG}_{g,0|k,d}^{0+,0+}([\mathbb{C}^3/\mathbb{C}^*])$$

is parallel to the description in [4, §4.1] for the unweighted case. In particular, there is a distinguished subset $\mathsf{M}_{k,d}$ of the \mathbb{C}^* -fixed locus for which all the markings and the entire curve class d lie over $0 \in \mathbb{P}^1$. The locus $\mathsf{M}_{k,d}$ comes with a natural *proper* evaluation map ev_{\bullet} obtained from the generic point of \mathbb{P}^1 :

$$\operatorname{ev}_{\bullet}: \mathsf{M}_{k,d} \to \mathbb{C}^3 /\!\!/ \mathbb{C}^* = \mathbb{P}^2.$$

We can explicitly write

$$\mathsf{M}_{k,d} \cong \mathsf{M}_d \times 0^k \subset \mathsf{M}_d \times (\mathbb{P}^1)^k$$

where M_d is the \mathbb{C}^* -fixed locus in $\mathsf{QG}_{0,0,d}^{0+}([\mathbb{C}^3/\mathbb{C}^*])$ for which the class d is concentrated over $0 \in \mathbb{P}^1$. The locus M_d parameterizes quasimaps of class d,

$$f:\mathbb{P}^1\longrightarrow \left[\mathbb{C}^3/\mathbb{C}^*\right],$$

with a base-point of length d at $0 \in \mathbb{P}^1$. The restriction of f to $\mathbb{P}^1 \setminus \{0\}$ is a constant map to \mathbb{P}^2 defining the evaluation map ev_{\bullet} .

As in [3, 4, 8], we define the big I-function as the generating function for the push-forward via ev_{\bullet} of localization residue contributions of $\mathsf{M}_{k,d}$. For $\mathbf{t} \in H^*_{\mathsf{T}}([\mathbb{C}^3/\mathbb{C}^*], \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[z]$, let

$$\operatorname{Res}_{\mathsf{M}_{k,d}}(\mathbf{t}^k) = \prod_{j=1}^k \widehat{\operatorname{ev}}_j^*(\mathbf{t}) \cap \operatorname{Res}_{\mathsf{M}_{k,d}}[\mathsf{QG}_{g,0|k,d}^{0+,0+}([\mathbb{C}^3/\mathbb{C}^*])]^{\operatorname{vir}}$$
$$= \frac{\prod_{j=1}^k \widehat{\operatorname{ev}}_j^*(\mathbf{t}) \cap [\mathsf{M}_{k,d}]^{\operatorname{vir}}}{\operatorname{e}(\operatorname{Nor}_{\mathsf{M}_{k,d}}^{\operatorname{vir}})},$$

where $\operatorname{Nor}_{M_{k,d}}^{\operatorname{vir}}$ is the virtual normal bundle.

Definition 13. The big \mathbb{I} -function for the (0+, 0+)-stability condition, as a formal function in \mathbf{t} , is

$$\mathbb{I}(q, \mathbf{t}, z) = \sum_{d \ge 0} \sum_{k \ge 0} \frac{q^d}{k!} \operatorname{ev}_{\bullet *} \left(\operatorname{Res}_{\mathsf{M}_{k, d}}(\mathbf{t}^k) \right).$$

2.4.3. Evaluations. Let $\widetilde{H} \in H^*_{\mathsf{T}}([\mathbb{C}^3/\mathbb{C}^*])$ and $H \in H^*_{\mathsf{T}}(\mathbb{P}^2)$ denote the respective hyperplane classes. The I-function of Definition 13 is evaluated in [5].

Proposition 14. For $\mathbf{t} = t\widetilde{H} \in H^*_{\mathsf{T}}([\mathbb{C}^3/\mathbb{C}^*], \mathbb{Q})$,

(22)
$$\mathbb{I}(t) = \sum_{d=0}^{\infty} q^d e^{t(H+dz)/z} \frac{\prod_{k=0}^{3d-1} (-3H-kz)}{\prod_{i=0}^2 \prod_{k=1}^d (H-\lambda_i+kz)}.$$

Observe that the I-function has following expandsion after restriction t = 0,

$$\mathbb{I}|_{t=0} = 1 + \frac{I_1 H}{z} + \frac{I_{2,0} H^2 + I_{2,1} (\lambda_0 + \lambda_1 + \lambda_2) H}{z^2} + \mathcal{O}(\frac{1}{z^3}),$$

where

$$I_{1}(q) = \sum_{d=1}^{\infty} 3 \frac{(3d-1)!}{(d!)^{3}} (-q)^{d},$$

$$I_{2,0}(q) = \sum_{d=1}^{\infty} 3 \frac{(3d-1)!}{(d!)^{3}} \left(3 \operatorname{Har}[3d-1] - 3 \operatorname{Har}[d] \right) (-q)^{d},$$

$$I_{2,1}(q) = \sum_{d=1}^{\infty} 3 \frac{(3d-1)!}{(d!)^{3}} \operatorname{Har}[d] (-q)^{d}.$$

Here $\operatorname{Har}[d] := \sum_{k=1}^{d} \frac{1}{k}$.

We return now to the functions $S_i(\gamma)$ defined in Section 2.3. Using Birkhoff factorization, an evaluation of the series $S(H^j)$ can be obtained from the I-function, see [13]:

 $\mathbb{S}(1) = \mathbb{I},$

(23)
$$S(H) = \frac{z \frac{d}{dt} S(1)}{z \frac{d}{dt} S(1)|_{t=0,H=1,z=\infty}},$$

$$S(H^2) = \frac{z \frac{d}{dt} S(H) - (\lambda_0 + \lambda_1 + \lambda_2) N_2 S(H)}{\left(z \frac{d}{dt} S(H) - (\lambda_0 + \lambda_1 + \lambda_2) N_2 S(H)\right)|_{t=0,H=1,z=\infty}}$$

For a series $F \in \mathbb{C}[[\frac{1}{z}]]$, the specialization $F|_{z=\infty}$ denotes constant term of F with respect to $\frac{1}{z}$. Here, N_2 is series in q defined by

$$N_2(q) := d\frac{q}{dq} \left(\frac{q\frac{d}{dq}I_{2,1}}{1 + q\frac{d}{dq}I_{1,0}}\right).$$

2.4.4. Further calculations. Define small I-function

 $\overline{\mathbb{I}}(q) \in H^*_{\mathsf{T}}(\mathbb{P}^2, \mathbb{Q})[[q]]$

by the restriction

$$\overline{\mathbb{I}}(q) = \mathbb{I}(q,t)|_{t=0} \,.$$

Define differential operators

$$\mathsf{D} = q \frac{d}{dq}, \quad M = H + z \mathsf{D}.$$

Applying $z\frac{d}{dt}$ to \mathbb{I} and then restricting to t = 0 has same effect as applying M to $\overline{\mathbb{I}}$

$$\left[\left(z \frac{d}{dt} \right)^k \mathbb{I} \right] \Big|_{t=0} = M^k \,\overline{\mathbb{I}}$$

The function $\overline{\mathbb{I}}$ satisfies following Picard-Fuchs equation

(24)
$$\left((M - \lambda_0)(M - \lambda_1)(M - \lambda_2) + 3qM(3M + z)(3M + 2z) \right) \overline{\mathbb{I}} = 0$$

implied by the Picard-Fuchs equation for \mathbb{I} ,

$$\left(\prod_{j=0}^{2} \left(z\frac{d}{dt} - \lambda_{j}\right) + 3q\left(z\frac{d}{dt}\right)\left(3\left(z\frac{d}{dt}\right) + z\right)\left(3\left(z\frac{d}{dt}\right) + 2z\right)\right)\mathbb{I} = 0.$$

The restriction $\overline{\mathbb{I}}|_{H=\lambda_i}$ admits following asymptotic form

(25)
$$\overline{\mathbb{I}}|_{H=\lambda_i} = e^{\mu_i/z} \left(R_{0,i} + R_{1,i}z + R_{2,i}z^2 + \ldots \right)$$

with series $\mu_i, R_{k,i} \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]].$

A derivation of (25) is obtained in [21] via the Picard-Fuchs equation (24) for $\overline{\mathbb{I}}|_{H=\lambda_i}$. The series μ_i and $R_{k,i}$ are found by solving differential equations obtained from the coefficient of z^k . For example,

$$\begin{aligned} \lambda_i + \mathsf{D}\mu_i &= L_i, \\ R_{0,i} &= \left(\frac{\lambda_i \prod_{j \neq i} (\lambda_i - \lambda_j)}{f(L_i)}\right)^{\frac{1}{2}}. \end{aligned}$$

Define the series C_1 and C_2 by the equations

(26)
$$C_{1} = z \frac{d}{dt} \mathbb{S}(1)|_{z=\infty,t=0,H=1},$$
$$C_{2} = \left(z \frac{d}{dt} \mathbb{S}(H) - (\lambda_{0} + \lambda_{1} + \lambda_{2}) N_{2} \mathbb{S}(H) \right)|_{z=\infty,t=0,H=1}.$$

The following relation was proven in [21],

(27)
$$C_1^2 C_2 = (1+27q)^{-1}$$

From the equations (23) and (25), we can show the series

$$\overline{\mathbb{S}}_i(1) = \overline{\mathbb{S}}(1)|_{H=\lambda_i}, \quad \overline{\mathbb{S}}_i(H) = \overline{\mathbb{S}}(H)|_{H=\lambda_i}, \quad \overline{\mathbb{S}}_i(H^2) = \overline{\mathbb{S}}(H^2)|_{H=\lambda_i}$$

have the following asymptotic expansions:

$$\overline{\mathbb{S}}_{i}(1) = e^{\frac{\mu_{i}}{z}} \left(R_{00,i} + R_{01,i}z + R_{02,i}z^{2} + \dots \right),$$
(28)
$$\overline{\mathbb{S}}_{i}(H) = e^{\frac{\mu_{i}}{z}} \frac{L_{i}}{C_{1}} \left(R_{10,i} + R_{11,i}z + R_{12}z^{2} + \dots \right),$$

$$\overline{\mathbb{S}}_{i}(H^{2}) = e^{\frac{\mu_{i}}{z}} \frac{L_{i}^{2}}{C_{1}C_{2}} \left(R_{20,i} + R_{21,i}z + R_{22,i}z^{2} + \dots \right).$$

We follow here the normalization of [21]. Note

$$R_{0k,i} = R_{k,i}.$$

As in [21, Theorem 4], we expect the following constraints.

Conjecture 15. For all $k \ge 0$, we have

$$R_{k,i} \in \mathbb{G}_2$$

Conjecture 15 is the main obstruction for the proof of Conjecture 1 and 2. By the same argument of Section 6, we obtain the following result.

Theorem 16. Conjecture 15 implies Conjecture 1 and 2.

By applying asymptotic expansions (28) to (23), we obtain the following results. Lemma 17. We have

$$\begin{aligned} R_{1\,p+1,i} &= R_{0\,p+1,i} + \frac{\mathsf{D}R_{0\,p,i}}{L_i} \,, \\ R_{2\,p+1,i} &= R_{1\,p+1,i} + \frac{\mathsf{D}R_{1\,p,i}}{L_i} + \left(\frac{\mathsf{D}L_i}{L_i^2} - \frac{X}{L_i}\right) - (\lambda_0 + \lambda_1 + \lambda_2)N_2 \frac{R_{1\,k,i}}{L_i} \,, \\ with \ X &= \frac{\mathsf{D}C_1}{C_1} \,. \end{aligned}$$

From Lemma 17, we obtain results for $\overline{\mathbb{S}}(H)|_{H=\lambda_i}$ and $\overline{\mathbb{S}}(H^2)|_{H=\lambda_i}$.

Lemma 18. Suppose Conjecture 15 is true. Then for all $k \ge 0$, we have for all $k \ge 0$,

$$R_{1\,k,i} \in \mathbb{G}_2,$$

$$R_{2\,k,i} = Q_{2\,k,i} - \frac{R_{1\,k-1,i}}{L} X - (\lambda_0 + \lambda_1 + \lambda_2) N_2 \frac{R_{1\,k,i}}{L_i},$$

with $Q_{2k,i} \in \mathbb{G}_2$.

2.5. Determining DX and N_2 . The following relation was proven in [16].

(29)
$$X^{2} - (L^{3} - 1)X + \mathsf{D}X - \frac{2}{9}(L^{3} - 1) = 0.$$

By the above result, the differential ring

(30)
$$\mathbb{G}_2[X, \mathsf{D}X, \mathsf{D}\mathsf{D}X, \ldots]$$

is just the polynomial ring $\mathbb{G}_2[X]$. Denote by $\operatorname{Coeff}(x^i y^j)$ the coefficient of $x^i y^j$ in

$$\sum_{k=0}^{2} e^{-\frac{\mu_{i}}{x} - \frac{\mu_{i}}{y}} \mathbb{S}_{i}(\phi_{k})|_{z=x} \mathbb{S}_{i}(\phi^{k})|_{z=y}.$$

From (18) and (28), we obtain the following equation.

$$\operatorname{Coeff}(x^2) + \operatorname{Coeff}(y^2) - \operatorname{Coeff}(xy) = 0$$

Above equation immediately yields the following relation.

(31)
$$N_2 = -\frac{1}{2}C_2 + \frac{1}{2}L^3.$$

3. Higher genus series on $\overline{M}_{g,n}$

3.1. Intersection theory on $\overline{M}_{g,n}$. We review here the now standard method used by Givental [9, 10, 14] to express genus g descendent correlators in terms of genus 0 data.

Let t_0, t_1, t_2, \ldots be formal variables. The series

$$T(c) = t_0 + t_1 c + t_2 c^2 + \dots$$

in the additional variable c plays a basic role. The variable c will later be replaced by the first Chern class ψ_i of a cotangent line over $\overline{M}_{q,n}$,

$$T(\psi_i) = t_0 + t_1\psi_i + t_2\psi_i^2 + \dots$$

with the index i depending on the position of the series T in the correlator.

Let 2g - 2 + n > 0. For $a_i \in \mathbb{Z}_{\geq 0}$ and $\gamma \in H^*(\overline{M}_{g,n})$, define the correlator

$$\langle \langle \psi^{a_1}, \dots, \psi^{a_n} | \gamma \rangle \rangle_{g,n} = \sum_{k \ge 0} \frac{1}{k!} \int_{\overline{M}_{g,n+k}} \gamma \psi_1^{a_1} \cdots \psi_n^{a_n} \prod_{i=1}^k T(\psi_{n+i}).$$

In the above summation, the k = 0 term is

$$\int_{\overline{M}_{g,n}} \gamma \, \psi_1^{a_1} \cdots \psi_n^{a_n} \, .$$

We also need the following correlator defined for the unstable case,

$$\langle \langle 1,1 \rangle \rangle_{0,2} = \sum_{k>0} \frac{1}{k!} \int_{\overline{M}_{0,2+k}} \prod_{i=1}^{k} T(\psi_{2+i}).$$

For formal variables x_1, \ldots, x_n , we also define the correlator

(32)
$$\left\langle \left\langle \frac{1}{x_1 - \psi}, \dots, \frac{1}{x_n - \psi} \middle| \gamma \right\rangle \right\rangle_{g,n}$$

in the standard way by expanding $\frac{1}{x_i-\psi}$ as a geometric series. Denote by \mathbb{L} the differential operator

$$\mathbb{L} = \frac{\partial}{\partial t_0} - \sum_{i=1}^{\infty} t_i \frac{\partial}{\partial t_{i-1}} = \frac{\partial}{\partial t_0} - t_1 \frac{\partial}{\partial t_0} - t_2 \frac{\partial}{\partial t_1} - \dots$$

The string equation yields the following result.

Lemma 19. For 2g - 2 + n > 0, we have $\mathbb{L}\langle \langle 1, \ldots, 1 | \gamma \rangle \rangle_{g,n} = 0$ and

$$\mathbb{L}\left\langle \left\langle \frac{1}{x_1 - \psi}, \dots, \frac{1}{x_n - \psi} \middle| \gamma \right\rangle \right\rangle_{g,n} = \left(\frac{1}{x_1} + \dots + \frac{1}{x_n} \right) \left\langle \left\langle \frac{1}{x_1 - \psi}, \dots, \frac{1}{x_n - \psi} \middle| \gamma \right\rangle \right\rangle_{g,n}.$$

After the restriction $t_0 = 0$ and application of the dilaton equation, the correlators are expressed in terms of finitely many integrals (by the dimension constraint). For example,

$$\langle \langle 1, 1, 1 \rangle \rangle_{0,3} |_{t_0=0} = \frac{1}{1-t_1}, \langle \langle 1, 1, 1, 1 \rangle \rangle_{0,4} |_{t_0=0} = \frac{t_2}{(1-t_1)^3}, \langle \langle 1, 1, 1, 1, 1 \rangle \rangle_{0,5} |_{t_0=0} = \frac{t_3}{(1-t_1)^4} + \frac{3t_2^2}{(1-t_1)^5}, \langle \langle 1, 1, 1, 1, 1, 1 \rangle \rangle_{0,6} |_{t_0=0} = \frac{t_4}{(1-t_1)^5} + \frac{10t_2t_3}{(1-t_1)^6} + \frac{15t_2^3}{(1-t_1)^7}.$$

We consider $\mathbb{C}(t_1)[t_2, t_3, ...]$ as \mathbb{Z} -graded ring over $\mathbb{C}(t_1)$ with $\deg(t_i) = i - 1$ for $i \ge 2$.

Define a subspace of homogeneous elements by

$$\mathbb{C}\left[\frac{1}{1-t_1}\right][t_2,t_3,\ldots]_{\text{Hom}} \subset \mathbb{C}(t_1)[t_2,t_3,\ldots].$$

We easily see

$$\langle \langle \psi^{a_1}, \dots, \psi^{a_n} | \gamma \rangle \rangle_{g,n} |_{t_0=0} \in \mathbb{C}\left[\frac{1}{1-t_1}\right] [t_2, t_3, \dots]_{\text{Hom}}.$$

Using the leading terms (of lowest degree in $\frac{1}{(1-t_1)}$), we obtain the following result.

Lemma 20. The set of genus 0 correlators

$$\left\{\left.\left\langle\left\langle 1,\ldots,1\right\rangle\right\rangle_{0,n}\right|_{t_{0}=0}\right\}_{n\geq4}$$

freely generate the ring $\mathbb{C}(t_1)[t_2, t_3, ...]$ over $\mathbb{C}(t_1)$.

By Lemma 20, we can find a unique representation of $\langle \langle \psi^{a_1}, \ldots, \psi^{a_n} \rangle \rangle_{g,n}|_{t_0=0}$ in the variables

(33)
$$\left\{ \langle \langle 1, \dots, 1 \rangle \rangle_{0,n} |_{t_0=0} \right\}_{n \ge 3}.$$

The n = 3 correlator is included in the set (33) to capture the variable t_1 . For example, in g = 1,

$$\begin{split} \langle \langle 1,1 \rangle \rangle_{1,2}|_{t_0=0} &= \frac{1}{24} \left(\frac{\langle \langle 1,1,1,1,1 \rangle \rangle_{0,5}|_{t_0=0}}{\langle 1,1,1 \rangle \rangle_{0,3}|_{t_0=0}} - \frac{\langle \langle 1,1,1,1 \rangle \rangle_{0,4}|_{t_0=0}}{\langle \langle 1,1,1 \rangle \rangle_{0,3}|_{t_0=0}} \right) ,\\ \langle \langle 1 \rangle \rangle_{1,1}|_{t_0=0} &= \frac{1}{24} \frac{\langle \langle 1,1,1,1 \rangle \rangle_{0,4}|_{t_0=0}}{\langle \langle 1,1,1 \rangle \rangle_{0,3}|_{t_0=0}} \end{split}$$

A more complicated example in g = 2 is

$$\langle \langle \rangle \rangle_{2,0}|_{t_0=0} = \frac{1}{1152} \frac{\langle \langle 1, 1, 1, 1, 1, 1 \rangle \rangle_{0,6}|_{t_0=0}}{\langle \langle 1, 1, 1 \rangle \rangle_{0,3}|_{t_0=0}^2} \\ - \frac{7}{1920} \frac{\langle \langle 1, 1, 1, 1, 1 \rangle \rangle_{0,5}|_{t_0=0} \langle \langle 1, 1, 1, 1 \rangle \rangle_{0,4}|_{t_0=0}}{\langle \langle 1, 1, 1 \rangle \rangle_{0,3}|_{t_0=0}^3} \\ + \frac{1}{360} \frac{\langle \langle 1, 1, 1, 1 \rangle \rangle_{0,4}|_{t_0=0}^3}{\langle \langle 1, 1, 1 \rangle \rangle_{0,3}|_{t_0=0}^4} .$$

Definition 21. For $\gamma \in H^*(\overline{M}_{g,k})$, let

$$\mathsf{P}_{g,n}^{a_1,...,a_n,\gamma}(s_0,s_1,s_2,...) \in \mathbb{Q}(s_0,s_1,..)$$

be the unique rational function satisfying the condition

$$\langle \langle \psi^{a_1}, \dots, \psi^{a_n} | \gamma \rangle \rangle_{g,n} |_{t_0=0} = \mathsf{P}_{g,n}^{a_1, a_2, \dots, a_n, \gamma} |_{s_i = \langle \langle 1, \dots, 1 \rangle \rangle_{0,i+3} |_{t_0=0}}$$

Proposition 22. For 2g - 2 + n > 0, we have

$$\langle \langle 1, \ldots, 1 | \gamma \rangle \rangle_{g,n} = \mathsf{P}_{g,n}^{0,\ldots,0,\gamma}|_{s_i = \langle \langle 1,\ldots,1 \rangle \rangle_{0,i+3}}.$$

Proof. Both sides of the equation satisfy the differential equation

$$\mathbb{L}=0$$

By definition, both sides have the same initial conditions at $t_0 = 0$. \Box **Proposition 23.** For 2g - 2 + n > 0,

$$\left\langle \left\langle \frac{1}{x_1 - \psi_1}, \dots, \frac{1}{x_n - \psi_n} \middle| \gamma \right\rangle \right\rangle_{g,n} = e^{\langle \langle 1, 1 \rangle \rangle_{0,2} (\sum_i \frac{1}{x_i})} \sum_{a_1, \dots, a_n} \frac{\mathsf{P}_{g,n}^{a_1, \dots, a_n, \gamma} \middle|_{s_i = \langle \langle 1, \dots, 1 \rangle \rangle_{0,i+3}}}{x_1^{a_1 + 1} \cdots x_n^{a_n + 1}}$$

Proof. Both sides of the equation satisfy differential equation

$$\mathbb{L} - \sum_{i} \frac{1}{x_i} = 0$$

Both sides have the same initial conditions at $t_0 = 0$. We use here

$$\mathbb{L}\langle\langle 1,1\rangle\rangle_{0,2}=1\,,\quad \langle\langle 1,1\rangle\rangle_{0,2}|_{t_0=0}=0\,.$$

There is no conflict here with Lemma 19 since (g, n) = (0, 2) is not in the stable range.

3.2. The unstable case (0,2). The definition given in (32) of the correlator is valid in the stable range

$$2g - 2 + n > 0.$$

The unstable case (g, n) = (0, 2) plays a special role. We define

$$\left\langle \left\langle \frac{1}{x_1 - \psi_1}, \frac{1}{x_2 - \psi_2} \right\rangle \right\rangle_{0,2}$$

by adding the degenerate term

$$\frac{1}{x_1 + x_2}$$

to the terms obtained by the expansion of $\frac{1}{x_i - \psi_i}$ as a geometric series. The degenerate term is associated to the (unstable) moduli space of genus 0 with 2 markings.

Proposition 24. We have

$$\left\langle \left\langle \frac{1}{x_1 - \psi_1}, \frac{1}{x_2 - \psi_2} \right\rangle \right\rangle_{0,2} = e^{\left\langle \langle 1, 1 \rangle \right\rangle_{0,2} \left(\frac{1}{x_1} + \frac{1}{x_2}\right)} \left(\frac{1}{x_1 + x_2}\right) \,.$$

Proof. Both sides of the equation satisfy differential equation

$$\mathbb{L} - \sum_{i=1}^2 \frac{1}{x_i} = 0$$

Both sides have the same initial conditions at $t_0 = 0$.

3.3. Local invariants and wall crossing. The torus T acts on the moduli spaces $\overline{M}_{g,n}(\mathbb{P}^2, d)$ and $\overline{Q}_{g,n}(\mathbb{P}^2, d)$. We consider here special localization contributions associated to the fixed points $p_i \in \mathbb{P}^2$.

Consider first the moduli of stable maps. Let

$$\overline{M}_{g,n}(\mathbb{P}^2,d)^{\mathsf{T},p_i} \subset \overline{M}_{g,n}(\mathbb{P}^2,d)$$

be the union of T-fixed loci which parameterize stable maps obtained by attaching T-fixed rational tails to a genus g, *n*-pointed Deligne-Mumford stable curve contracted to the point $p_i \in \mathbb{P}^2$. Similarly, let

$$\overline{Q}_{g,n}(\mathbb{P}^2,d)^{\mathsf{T},p_i} \subset \overline{Q}_{g,n}(\mathbb{P}^2,d)$$

be the parallel T-fixed locus parameterizing stable quotients obtained by attaching base points to a genus g, n-pointed Deligne-Mumford stable curve contracted to the point $p_i \in \mathbb{P}^2$.

Let Λ_i denote the localization of the ring

$$\mathbb{C}[\lambda_0^{\pm 1}, \lambda_1^{\pm 1}, \lambda_2^{\pm 1}]$$

at the three tangent weights at $p_i \in \mathbb{P}^2$. Using the virtual localization formula [11], there exist unique series

$$S_{p_i} \in \Lambda_i[\psi][[Q]]$$

for which the localization contribution of the T-fixed locus $\overline{M}_{g,n}(\mathbb{P}^2, d)^{\mathsf{T}, p_i}$ to the equivariant Gromov-Witten invariants of $K\mathbb{P}^2$ can be written as

$$\sum_{d=0}^{\infty} Q^d \int_{[\overline{M}_{g,n}(K\mathbb{P}^2,d)^{\mathsf{T},p_i}]^{\mathrm{vir}}} \psi_1^{a_1} \cdots \psi_n^{a_n} = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\overline{M}_{g,n+k}} \mathsf{H}_g^{p_i} \psi_1^{a_1} \cdots \psi_n^{a_n} \prod_{j=1}^k S_{p_i}(\psi_{n+j}).$$

Here, $\mathsf{H}_{g}^{p_{i}}$ is the standard vertex class,

(34)
$$\frac{e(\mathbb{E}_g^* \otimes T_{p_i}(\mathbb{P}^2))}{e(T_{p_i}(\mathbb{P}^2))} \cdot \frac{e(\mathbb{E}_g^* \otimes (-3\lambda_i))}{(-3\lambda_i)},$$

obtained the Hodge bundle $\mathbb{E}_g \to \overline{M}_{g,n+k}$.

Similarly, the application of the virtual localization formula to the moduli of stable quotients yields classes

$$F_{p_i,k} \in H^*(\overline{M}_{g,n|k}) \otimes_{\mathbb{C}} \Lambda_i$$

for which the contribution of $\overline{Q}_{g,n}(\mathbb{P}^2, d)^{T,p_i}$ is given by

$$\sum_{d=0}^{\infty} q^d \int_{[\overline{Q}_{g,n}(K\mathbb{P}^2,d)^{\mathsf{T},p_i}]^{\mathrm{vir}}} \psi_1^{a_1} \cdots \psi_n^{a_n} = \sum_{k=0}^{\infty} \frac{q^k}{k!} \int_{\overline{M}_{g,n|k}} \mathsf{H}_g^{p_i} \, \psi_1^{a_1} \cdots \psi_n^{a_n} \, F_{p_i,k}.$$

Here $\overline{M}_{g,n|k}$ is the moduli space of genus g curves with markings

$$\{p_1, \cdots, p_n\} \cup \{\hat{p}_1 \cdots \hat{p}_k\} \in C^{\mathrm{ns}} \subset C$$

satisfying the conditions

(i) the points p_i are distinct,

(ii) the points \hat{p}_j are distinct from the points p_i , with stability given by the ampleness of

$$\omega_C(\sum_{i=1}^m p_i + \epsilon \sum_{j=1}^k \hat{p}_j)$$

for every strictly positive $\epsilon \in \mathbb{Q}$.

The Hodge class $\mathsf{H}_g^{p_i}$ is given again by formula (34) using the Hodge bundle

$$\mathbb{E}_g \to \overline{M}_{g,n|k} \,.$$

Definition 25. For $\gamma \in H^*(\overline{M}_{g,n})$, let

$$\langle \langle \psi_1^{a_1}, \dots, \psi_n^{a_n} | \gamma \rangle \rangle_{g,n}^{p_i,\infty} = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\overline{M}_{g,n+k}} \gamma \, \psi_1^{a_1} \cdots \psi_n^{a_n} \prod_{j=1}^k S_{p_i}(\psi_{n+j}) \,,$$

$$\langle \langle \psi_1^{a_1}, \dots, \psi_n^{a_n} | \gamma \rangle \rangle_{g,n}^{p_i,0+} = \sum_{k=0}^{\infty} \frac{q^k}{k!} \int_{\overline{M}_{g,n|k}} \gamma \, \psi_1^{a_1} \cdots \psi_n^{a_n} F_{p_i,k} \,.$$

Proposition 26 (Ciocan-Fontanine, Kim [7]). For 2g - 2 + n > 0, we have the wall crossing relation

$$\langle \langle \psi_1^{a_1}, \dots, \psi_n^{a_n} \mid \gamma \rangle \rangle_{g,n}^{p_i,\infty}(Q(q)) = \langle \langle \psi_1^{a_1}, \dots, \psi_n^{a_n} \mid \gamma \rangle \rangle_{g,n}^{p_i,0+}(q)$$

where Q(q) is the mirror map

$$Q(q) = \exp(I_1^{K\mathbb{P}^2}(q)).$$

Proposition 26 is a consequence of [7, Lemma 5.5.1]. The mirror map here is the mirror map for $K\mathbb{P}^2$ discussed in Section 0.2. Propositions 22 and 26 together yield

$$\begin{array}{lll} \langle \langle 1, \dots, 1 \mid \gamma \rangle \rangle_{g,n}^{p_i, \infty} &= \mathsf{P}_{g,n}^{0, \dots, 0, \gamma} \big(\langle \langle 1, 1, 1 \rangle \rangle_{0,3}^{p_i, \infty}, \langle \langle 1, 1, 1, 1 \rangle \rangle_{0,4}^{p_i, \infty}, \dots \big) \,, \\ \langle \langle 1, \dots, 1 \mid \gamma \rangle \rangle_{g,n}^{p_i, 0+} &= \mathsf{P}_{g,n}^{0, \dots, 0, \gamma} \big(\langle \langle 1, 1, 1 \rangle \rangle_{0,3}^{p_i, 0+}, \langle \langle 1, 1, 1, 1 \rangle \rangle_{0,4}^{p_i, 0+}, \dots \big) \,. \end{array}$$

Similarly, using Propositions 23 and 26, we obtain

$$\left\langle \left\langle \frac{1}{x_1 - \psi}, \dots, \frac{1}{x_n - \psi} \middle| \gamma \right\rangle \right\rangle_{g,n}^{p_i, \infty} = e^{\langle \langle 1, 1 \rangle \rangle_{0,2}^{p_i, \infty} \left(\sum_i \frac{1}{x_i} \right)} \sum_{a_1, \dots, a_n} \frac{\mathsf{P}_{g,n}^{a_1, \dots, a_n, \gamma} \left(\langle \langle 1, 1, 1 \rangle \rangle_{0,3}^{p_i, \infty}, \langle \langle 1, 1, 1, 1 \rangle \rangle_{0,4}^{p_i, \infty}, \dots \right)}{x_1^{a_1 + 1} \cdots x_n^{a_n + 1}},$$

$$(35) \quad \left\langle \left\langle \frac{1}{x_1 - \psi}, \dots, \frac{1}{x_n - \psi} \middle| \gamma \right\rangle \right\rangle_{g,n}^{p_i, 0+} = e^{\langle \langle 1, 1 \rangle \rangle_{0,2}^{p_i, 0+} \left(\sum_{i \ \overline{x_i}} \right)} \sum_{a_1, \dots, a_n} \frac{\mathsf{P}_{g,n}^{a_1, \dots, a_n, \gamma} \left(\langle \langle 1, 1, 1 \rangle \rangle_{0,3}^{p_i, 0+}, \langle \langle 1, 1, 1, 1 \rangle \rangle_{0,4}^{p_i, 0+}, \dots \right)}{x_1^{a_1 + 1} \cdots x_n^{a_n + 1}}.$$

4. Higher genus series on $K\mathbb{P}^2$

4.1. **Overview.** We apply the localization strategy introduced first by Givental [9, 10, 14] for Gromov-Witten theory to the stable quotient invariants of local \mathbb{P}^2 . The contribution $\operatorname{Cont}_{\Gamma}(q)$ discussed in Section 1 of a graph $\Gamma \in \mathsf{G}_g(\mathbb{P}^2)$ can be separated into vertex and edge contributions. We express the vertex and edge contributions in terms of the series S_i and V_{ij} of Section 2.3.

4.2. Edge terms. Recall the definition⁸ of \mathbb{V}_{ij} given in Section 2.3,

(36)
$$\mathbb{V}_{ij} = \left\langle \left\langle \frac{\phi_i}{x - \psi}, \frac{\phi_j}{y - \psi} \right\rangle \right\rangle_{0,2}^{0+,0+}$$

Let $\overline{\mathbb{V}}_{ij}$ denote the restriction of \mathbb{V}_{ij} to t = 0. Via formula (16), $\overline{\mathbb{V}}_{ij}$ is a summation of contributions of fixed loci indexed by a graph Γ consisting of two vertices connected by a unique edge. Let w_1 and w_2 be T-weights. Denote by

$$\overline{\mathbb{V}}_{ij}^{w_1,w_2}$$

the summation of contributions of T-fixed loci with tangent weights precisely w_1 and w_2 on the first rational components which exit the vertex components over p_i and p_j .

The series $\overline{\mathbb{V}}_{ij}^{w_1,w_2}$ includes *both* vertex and edge contributions. By definition (36) and the virtual localization formula, we find the following relationship between $\overline{\mathbb{V}}_{ij}^{w_1,w_2}$ and the corresponding pure edge contribution $\mathsf{E}_{ij}^{w_1,w_2}$,

$$e_{i}\overline{V}_{ij}^{w_{1},w_{2}}e_{j} = \left\langle \left\langle \frac{1}{w_{1}-\psi}, \frac{1}{x_{1}-\psi} \right\rangle \right\rangle_{0,2}^{p_{i},0+} \mathsf{E}_{ij}^{w_{1},w_{2}} \left\langle \left\langle \frac{1}{w_{2}-\psi}, \frac{1}{x_{2}-\psi} \right\rangle \right\rangle_{0,2}^{p_{j},0+} \\ = \frac{e^{\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_{i},0+}}{w_{1}} + \frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_{j},0+}}{x_{1}}}{w_{1}+x_{1}} \mathsf{E}_{ij}^{w_{1},w_{2}} \frac{e^{\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_{i},0+}}{w_{2}} + \frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_{j},0+}}{x_{2}}}{w_{2}+x_{2}}}{w_{2}+x_{2}}$$

$$=\sum_{a_1,a_2}e^{\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_j,0+}}{x_1}+\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_j,0+}}{w_1}}e^{\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_j,0+}}{x_2}+\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_j,0+}}{w_2}}(-1)^{a_1+a_2}\frac{\mathsf{E}_{ij}^{w_1,w_2}}{w_1^{a_1}w_2^{a_2}}x_1^{a_1-1}x_2^{a_2-1}.$$

After summing over all possible weights, we obtain

$$e_i\left(\overline{\mathbb{V}}_{ij} - \frac{\delta_{ij}}{e_i(x+y)}\right)e_j = \sum_{w_1,w_2} e_i\overline{\mathbb{V}}_{ij}^{w_1,w_2}e_j.$$

The above calculations immediately yield the following result.

⁸We use the variables x_1 and x_2 here instead of x and y.

Lemma 27. We have

$$\begin{bmatrix} e^{-\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_{i},0+}}{x_{1}}} e^{-\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_{j},0+}}{x_{2}}} e_{i} \left(\overline{\mathbb{V}}_{ij} - \frac{\delta_{ij}}{e_{i}(x+y)}\right) e_{j} \end{bmatrix}_{x_{1}^{a_{1}-1}x_{2}^{a_{2}-1}} = \\ \sum_{w_{1},w_{2}} e^{\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_{i},0+}}{w_{1}}} e^{\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_{j},0+}}{w_{2}}} (-1)^{a_{1}+a_{2}} \frac{\mathsf{E}_{ij}^{w_{1},w_{2}}}{w_{1}^{a_{1}}w_{2}^{a_{2}}} \end{bmatrix}$$

The notation $[\ldots]_{x_1^{a_1-1}x_2^{a_2-1}}$ in Lemma 27 denotes the coefficient of $x_1^{a_1-1}x_2^{a_2-1}$ in the series expansion of the argument.

4.3. A simple graph. Before treating the general case, we present the localization formula for a simple graph⁹. Let $\Gamma \in \mathsf{G}_g(\mathbb{P}^2)$ consist of two vertices and one edge,

$$v_1, v_2 \in \Gamma(V), \quad e \in \Gamma(E)$$

with genus and T-fixed point assignments

$$g(v_i) = g_i, \quad p(v_i) = p_i.$$

Let w_1 and w_2 be tangent weights at the vertices p_1 and p_2 respectively. Denote by $\text{Cont}_{\Gamma,w_1,w_2}$ the summation of contributions to

(37)
$$\sum_{d>0} q^d \left[\overline{Q}_g(K\mathbb{P}^2, d) \right]^{\text{vir}}$$

of T-fixed loci with tangent weights precisely w_1 and w_2 on the first rational components which exit the vertex components over p_1 and p_2 . We can express the localization formula for (37) as

$$\left\langle \left\langle \left\langle \frac{1}{w_1 - \psi} \left| \mathsf{H}_{g_1}^{p_1} \right\rangle \right\rangle_{g_1, 1}^{p_1, 0+} \mathsf{E}_{12}^{w_1, w_2} \left\langle \left\langle \left\langle \frac{1}{w_2 - \psi} \left| \mathsf{H}_{g_2}^{p_2} \right\rangle \right\rangle_{g_2, 1}^{p_2, 0+} \right\rangle_{g_2, 1}^{p_2, 0+} \right\rangle \right\rangle_{g_2, 1}^{p_2, 0+} \left\langle \left\langle \frac{1}{w_2 - \psi} \right| \mathsf{H}_{g_2}^{p_2} \right\rangle \right\rangle_{g_2, 1}^{p_2, 0+} \left\langle \left\langle \frac{1}{w_2 - \psi} \right| \mathsf{H}_{g_2}^{p_2} \right\rangle \right\rangle_{g_2, 1}^{p_2, 0+} \left\langle \left\langle \frac{1}{w_2 - \psi} \right| \mathsf{H}_{g_2}^{p_2} \right\rangle \right\rangle_{g_2, 1}^{p_2, 0+} \left\langle \left\langle \frac{1}{w_2 - \psi} \right| \mathsf{H}_{g_2}^{p_2} \right\rangle \right\rangle_{g_2, 1}^{p_2, 0+} \left\langle \left\langle \frac{1}{w_2 - \psi} \right| \mathsf{H}_{g_2}^{p_2} \right\rangle \right\rangle_{g_2, 1}^{p_2, 0+} \left\langle \frac{1}{w_2 - \psi} \right\rangle_{g_2, 1}^{p_2$$

which equals

$$\sum_{a_{1},a_{2}} e^{\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_{1},0+}}{w_{1}}} \frac{\mathsf{P}\left[\psi^{a_{1}-1}\left|\mathsf{H}_{g_{1}}^{p_{1}}\right]_{g_{1},1}^{p_{1},0+}}{w_{1}^{a_{1}}} \mathsf{E}_{12}^{w_{1},w_{2}} e^{\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_{2},0+}}{w_{2}}} \frac{\mathsf{P}\left[\psi^{a_{2}-1}\left|\mathsf{H}_{g_{2}}^{p_{2}}\right]_{g_{2},1}^{p_{2},0+}}{w_{2}^{a_{2}}}$$

where $\mathsf{H}_{g_i}^{p_i}$ is the Hodge class (34). We have used here the notation

$$\mathsf{P} \left[\psi_{1}^{k_{1}}, \dots, \psi_{n}^{k_{n}} \middle| \mathsf{H}_{h}^{p_{i}} \right]_{h,n}^{p_{i},0+} = \\ \mathsf{P}_{h,1}^{k_{1},\dots,k_{n},\mathsf{H}_{h}^{p_{i}}} \left(\langle \langle 1,1,1 \rangle \rangle_{0,3}^{p_{i},0+}, \langle \langle 1,1,1,1 \rangle \rangle_{0,4}^{p_{i},0+}, \dots \right)$$

and applied (35).

 $^{^{9}}$ We follow here the notation of Section 1.

After summing over all possible weights w_1, w_2 and applying Lemma 27, we obtain the following result for the full contribution

$$\operatorname{Cont}_{\Gamma} = \sum_{w_1, w_2} \operatorname{Cont}_{\Gamma, w_1, w_2}$$

of Γ to $\sum_{d\geq 0} q^d \left[\overline{Q}_g(K\mathbb{P}^2, d)\right]^{\text{vir}}$.

Proposition 28. We have

$$\begin{split} \operatorname{Cont}_{\Gamma} &= \sum_{a_1, a_2 > 0} \mathsf{P} \left[\psi^{a_1 - 1} \left| \mathsf{H}_{g_1}^{p_i} \right]_{g_1, 1}^{p_i, 0+} \mathsf{P} \left[\psi^{a_2 - 1} \left| \mathsf{H}_{g_2}^{p_j} \right]_{g_2, 1}^{p_j, 0+} \right. \\ & \left. \cdot (-1)^{a_1 + a_2} \left[e^{-\frac{\langle \langle 1, 1 \rangle \rangle_{0, 2}^{p_i, 0+}}{x_1}} e^{-\frac{\langle \langle 1, 1 \rangle \rangle_{0, 2}^{p_j, 0+}}{x_2}} e_i \left(\overline{\mathbb{V}}_{ij} - \frac{\delta_{ij}}{e_i(x_1 + x_2)} \right) e_j \right]_{x_1^{a_1 - 1} x_2^{a_2 - 1}} \,. \end{split}$$

4.4. A general graph. We apply the argument of Section 4.3 to obtain a contribution formula for a general graph Γ .

Let $\Gamma \in \mathsf{G}_{g,0}(\mathbb{P}^2)$ be a decorated graph as defined in Section 1. The *flags* of Γ are the half-edges¹⁰. Let F be the set of flags. Let

$$\mathsf{w}:\mathsf{F}\to\operatorname{Hom}(\mathsf{T},\mathbb{C}^*)\otimes_{\mathbb{Z}}\mathbb{Q}$$

be a fixed assignment of T-weights to each flag.

We first consider the contribution $\operatorname{Cont}_{\Gamma,w}$ to

$$\sum_{d\geq 0}q^d\left[\overline{Q}_g(K\mathbb{P}^2,d)\right]^{\rm vir}$$

of the T-fixed loci associated Γ satisfying the following property: the tangent weight on the first rational component corresponding to each $f \in \mathsf{F}$ is exactly given by $\mathsf{w}(f)$. We have

(38)
$$\operatorname{Cont}_{\Gamma,\mathsf{w}} = \frac{1}{|\operatorname{Aut}(\Gamma)|} \sum_{\mathsf{A} \in \mathbb{Z}_{>0}^{\mathsf{F}}} \prod_{v \in \mathsf{V}} \operatorname{Cont}_{\Gamma,\mathsf{w}}^{\mathsf{A}}(v) \prod_{e \in \mathsf{E}} \operatorname{Cont}_{\Gamma,\mathsf{w}}(e) \,.$$

The terms on the right side of (38) require definition:

• The sum on the right is over the set $\mathbb{Z}_{>0}^{\mathsf{F}}$ of all maps

$$\mathsf{A}:\mathsf{F}\to\mathbb{Z}_{>0}$$

corresponding to the sum over a_1, a_2 in Proposition 28.

¹⁰Flags are either half-edges or markings.

• For $v \in V$ with *n* incident flags with w-values (w_1, \ldots, w_n) and A-values (a_1, a_2, \ldots, a_n) ,

$$\operatorname{Cont}_{\Gamma,\mathsf{w}}^{\mathsf{A}}(v) = \frac{\mathsf{P}\left[\psi_{1}^{a_{1}-1},\ldots,\psi_{n}^{a_{n}-1} \left| \mathsf{H}_{\mathsf{g}(v)}^{\mathsf{p}(v)} \right]_{\mathsf{g}(v),n}^{\mathsf{p}(v),0+}\right]}{w_{1}^{a_{1}}\cdots w_{n}^{a_{n}}}.$$

• For $e \in \mathsf{E}$ with assignments $(\mathsf{p}(v_1), \mathsf{p}(v_2))$ for the two associated vertices¹¹ and w-values (w_1, w_2) for the two associated flags,

$$\operatorname{Cont}_{\Gamma,\mathsf{w}}(e) = e^{\frac{\langle \langle 1,1 \rangle \rangle_{0,2}^{\mathsf{p}(v_1),0+}}{w_1}} e^{\frac{\langle \langle 1,1 \rangle \rangle_{0,2}^{\mathsf{p}(v_2),0+}}{w_2}} \mathsf{E}_{\mathsf{p}(v_1),\mathsf{p}(v_2)}^{w_1,w_2}.$$

The localization formula then yields (38) just as in the simple case of Section 4.3.

By summing the contribution (38) of Γ over all the weight functions w and applying Lemma 27, we obtain the following result which generalizes Proposition 28.

Proposition 29. We have

$$\operatorname{Cont}_{\Gamma} = \frac{1}{|\operatorname{Aut}(\Gamma)|} \sum_{\mathsf{A} \in \mathbb{Z}_{>0}^{\mathsf{F}}} \prod_{v \in \mathsf{V}} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(v) \prod_{e \in \mathsf{E}} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(e)$$

where the vertex and edge contributions with incident flag A-values (a_1, \ldots, a_n) and (b_1, b_2) respectively are

$$\operatorname{Cont}_{\Gamma}^{\mathsf{A}}(v) = \mathsf{P}\left[\psi_{1}^{a_{1}-1}, \dots, \psi_{n}^{a_{n}-1} \middle| \mathsf{H}_{\mathsf{g}(v)}^{\mathsf{p}(v)}\right]_{\mathsf{g}(v),n}^{\mathsf{p}(v),0+},$$

$$\operatorname{Cont}_{\Gamma}^{\mathsf{A}}(e) = (-1)^{b_{1}+b_{2}} \left[e^{-\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{\mathsf{p}(v_{1}),0+}}{x_{1}}} e^{-\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{\mathsf{p}(v_{2}),0+}}{x_{2}}} e_{i}\left(\overline{\mathbb{V}}_{ij} - \frac{1}{e_{i}(x+y)}\right) e_{j} \right]_{x_{1}^{b_{1}-1}x_{2}^{b_{2}-1}}$$

where $\mathbf{p}(v_1) = p_i$ and $\mathbf{p}(v_2) = p_j$ in the second equation.

4.5. Legs. Let $\Gamma \in \mathsf{G}_{g,n}(\mathbb{P}^2)$ be a decorated graph with markings. While no markings are needed to define the stable quotient invariants of $K\mathbb{P}^2$, the contributions of decorated graphs with markings will appear in the proof of the holomorphic anomaly equation. The formula for the contribution $\operatorname{Cont}_{\Gamma}(H, \ldots, H)$ of Γ to

$$\sum_{d\geq 0} q^d \prod_{j=0}^n \operatorname{ev}^*(H) \cap \left[\overline{Q}_{g,n}(K\mathbb{P}^2, d)\right]^{\operatorname{vir}}$$

is given by the following result.

¹¹In case e is self-edge, $v_1 = v_2$.

Proposition 30. We have

$$\operatorname{Cont}_{\Gamma}(H,\ldots,H) = \frac{1}{|\operatorname{Aut}(\Gamma)|} \sum_{\mathsf{A} \in \mathbb{Z}_{>0}^{\mathsf{F}}} \prod_{v \in \mathsf{V}} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(v) \prod_{e \in \mathsf{E}} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(e) \prod_{l \in \mathsf{L}} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(l),$$

where the leg contribution is

$$\operatorname{Cont}_{\Gamma}^{\mathsf{A}}(l) = (-1)^{\mathsf{A}(l)-1} \left[e^{-\frac{\langle \langle 1,1 \rangle \rangle_{0,2}^{\mathsf{p}(l),0+}}{z}} \overline{\mathbb{S}}_{\mathsf{p}(l)}(H) \right]_{z^{\mathsf{A}(l)-1}}$$

The vertex and edge contributions are same as before.

The proof of Proposition 30 follows the vertex and edge analysis. We leave the details as an exercise for the reader. The parallel statement for Gromov-Witten theory can be found in [9, 10, 14].

5. Vertices, edges, and legs

5.1. **Overview.** Using the results of Givental [9, 10, 14] combined with wall-crossing [7], we calculate here the vertex and edge contributions in terms of the function R_k of Section 2.4.4.

5.2. Calculations in genus 0. We follow the notation introduced in Section 3.1. Recall the series

$$T(c) = t_0 + t_1 c + t_2 c^2 + \dots$$

Proposition 31. (Givental [9, 10, 14]) For $n \ge 3$, we have

$$\langle \langle 1, \dots, 1 \rangle \rangle_{0,n}^{p_i, \infty} = (\sqrt{\Delta_i})^{2g-2+n} \left(\sum_{k \ge 0} \frac{1}{k!} \int_{\overline{M}_{0,n+k}} T(\psi_{n+1}) \cdots T(\psi_{n+k}) \right) \Big|_{t_0 = 0, t_1 = 0, t_j \ge 2 = (-1)^j Q_{j-1,i}}$$

where the functions $\sqrt{\Delta_i}$, $Q_{l,i}$ are defined by

$$\overline{\mathbb{S}}_{i}^{\infty}(1) = e_{i} \left\langle \left\langle \frac{\phi_{i}}{z - \psi}, 1 \right\rangle \right\rangle_{0,2}^{p_{i},\infty} = \frac{e^{\frac{\langle \langle 1,1 \rangle \rangle_{0,2}^{p_{i},\infty}}{z}}}{\sqrt{\Delta_{i}}} \left(1 + \sum_{l=1}^{\infty} Q_{l,i} z^{l} \right) \,.$$

The existence of the above asymptotic expansion of $\overline{\mathbb{S}}_i^{\infty}(1)$ can also be proven by the argument of [4, Theorem 5.4.1]. Similarly, we have an asymptotic expansion of $\overline{\mathbb{S}}_i(1)$,

$$\overline{\mathbb{S}}_{i}(1) = e^{\frac{\langle \langle 1,1 \rangle \rangle_{0,2}^{p_{i},0+}}{z}} \left(\sum_{l=0}^{\infty} R_{l,i} z^{l} \right).$$

By (28), we have

$$\langle \langle 1,1 \rangle \rangle_{0,2}^{p_i,0+} = \mu_i$$

After applying the wall-crossing result of Proposition 26, we obtain

$$\langle \langle 1, \dots, 1 \rangle \rangle_{0,n}^{p_i, \infty}(Q(q)) = \langle \langle 1, \dots, 1 \rangle \rangle_{0,n}^{p_i, 0+}(q), \\ \overline{\mathbb{S}}_i^{\infty}(1)(Q(q)) = \overline{\mathbb{S}}_i(1)(q),$$

where Q(q) is mirror map for $K\mathbb{P}^2$ as before. By comparing asymptotic expansions of $\overline{\mathbb{S}}_i^{\infty}(1)$ and $\overline{\mathbb{S}}_i(1)$, we get a wall-crossing relation between $Q_{l,i}$ and $R_{l,i}$,

$$\sqrt{\Delta_i}(Q(q)) = \frac{1}{R_{0,i}(q)},$$
$$Q_{l,i}(Q(q)) = \frac{R_{l,i}(q)}{R_{0,i}(q)} \text{ for } l \ge 1.$$

We have proven the following result.

Proposition 32. For $n \geq 3$, we have

$$\langle \langle 1, \dots, 1 \rangle \rangle_{0,n}^{p_i,0+} = R_{0,i}^{2g-2+n} \left(\sum_{k \ge 0} \frac{1}{k!} \int_{\overline{M}_{0,n+k}} T(\psi_{n+1}) \cdots T(\psi_{n+k}) \right) \Big|_{t_0 = 0, t_1 = 0, t_j \ge 2 = (-1)^j \frac{R_{j-1,i}}{R_{0,i}}} .$$

Proposition 32 immediately implies the evaluation

(39)
$$\langle \langle 1, 1, 1 \rangle \rangle_{0,3}^{p_i,0+} = \frac{1}{R_{0,i}}$$

Another simple consequence of Proposition 32 is the following basic property.

Corollary 33. For $n \geq 3$, we have

$$\langle \langle 1, \dots, 1 \rangle \rangle_{0,n}^{p_i, 0+} \in \mathbb{C}[R_{0,i}^{\pm 1}, R_{1,i}, R_{2,i}, \dots].$$

5.3. Vertex and edge analysis. By Proposition 29, we have decomposition of the contribution to $\Gamma \in \mathsf{G}_g(\mathbb{P}^2)$ to the stable quotient theory of $K\mathbb{P}^2$ into vertex terms and edge terms

$$\operatorname{Cont}_{\Gamma} = \frac{1}{|\operatorname{Aut}(\Gamma)|} \sum_{\mathsf{A} \in \mathbb{Z}_{>0}^{\mathsf{F}}} \prod_{v \in \mathsf{V}} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(v) \prod_{e \in \mathsf{E}} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(e) \,.$$

Lemma 34. Suppose Conjecture 15 is true. Then we have

$$\operatorname{Cont}_{\Gamma}^{\mathsf{A}}(v) \in \mathbb{G}_2$$
.

Proof. By Proposition 29,

$$\operatorname{Cont}_{\Gamma}^{\mathsf{A}}(v) = \mathsf{P}\left[\psi_{1}^{a_{1}-1}, \dots, \psi_{n}^{a_{n}-1} \middle| \mathsf{H}_{\mathsf{g}(v)}^{\mathsf{p}(v)}\right]_{\mathsf{g}(v), n}^{\mathsf{p}(v), 0+}$$

The right side of the above formula is a polynomial in the variables

$$\frac{1}{\langle\langle 1,1,1\rangle\rangle_{0,3}^{\mathsf{p}(v),0+}} \quad \text{and} \quad \left\{ \left\langle\langle 1,\ldots,1\rangle\rangle_{0,n}^{\mathsf{p}(v),0+} \middle|_{t_0=0} \right. \right\}_{n\geq 4}$$

with coefficients in $\mathbb{C}(\lambda_0, \lambda_1, \lambda_2)$. The Lemma then follows from the evaluation (39), Corollary 33, and Conjecture 15.

Let $e \in \mathsf{E}$ be an edge connecting the T-fixed points $p_i, p_j \in \mathbb{P}^2$. Let the A-values of the respective half-edges be (k, l).

Lemma 35. Suppose Conjecture 15 is true. Then we have $\operatorname{Cont}_{\Gamma}^{\mathsf{A}}(e) \in$ $\mathbb{G}[X]$ and

- the degree of Cont^A_Γ(e) with respect to X is 1,
 the coefficient of X in Cont^A_Γ(e) is

$$(-1)^{k+l} \frac{3L_i L_j R_{1\,k-1,i} R_{1\,l-1,j}}{L^3}$$

Proof. By Proposition 29,

$$\operatorname{Cont}_{\Gamma}^{\mathsf{A}}(e) = (-1)^{k+l} \left[e^{-\frac{\mu\lambda_i}{x} - \frac{\mu\lambda_j}{y}} e_i \left(\overline{\mathbb{V}}_{ij} - \frac{\delta_{ij}}{e_i(x+y)} \right) e_j \right]_{x^{k-1}y^{l-1}}$$

Using also the equation

$$e_i \overline{\mathbb{V}}_{ij}(x, y) e_j = \frac{\sum_{r=0}^2 \overline{\mathbb{S}}_i(\phi_r)|_{z=x} \overline{\mathbb{S}}_j(\phi^r)|_{z=y}}{x+y},$$

we write $\operatorname{Cont}_{\Gamma}^{\mathsf{A}}(e)$ as

$$\left[(-1)^{k+l} e^{-\frac{\mu\lambda_i}{x} - \frac{\mu\lambda_j}{y}} \sum_{r=0}^2 \overline{\mathbb{S}}_i(\phi_r)|_{z=x} \overline{\mathbb{S}}_j(\phi^r)|_{z=y} \right]_{x^k y^{l-1} - x^{k+1} y^{l-2} + \dots + (-1)^{k-1} x^{k+l-1}}$$

where the subscript signifies a (signed) sum of the respective coefficients. If we substitute the asymptotic expansions (28) for

$$\overline{\mathbb{S}}_i(1)$$
, $\overline{\mathbb{S}}_i(H)$, $\overline{\mathbb{S}}_i(H^2)$

in the above expression, the Lemma follows from Conjecture 15, Lemma 18 and (31). 5.4. Legs. Using the contribution formula of Proposition 30,

$$\operatorname{Cont}_{\Gamma}^{\mathsf{A}}(l) = (-1)^{\mathsf{A}(l)-1} \left[e^{-\frac{\langle \langle 1,1 \rangle \rangle_{0,2}^{\mathsf{p}(l),0+}}{z}} \overline{\mathbb{S}}_{\mathsf{p}(l)}(H) \right]_{z^{\mathsf{A}(l)-1}},$$

we easily conclude under the assumption of Conjecture 15

$$C_1 \cdot \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(l) \in \mathbb{G}_2$$
.

6. Holomorphic anomaly for $K\mathbb{P}^2$

6.1. Proof of Theorem 3. By definition, we have

(40)
$$A_2(q) = \frac{1}{L^3} \left(3X + 1 - \frac{L^3}{2} \right) \,.$$

Conjecture 15 was proven in Appendix for the choices of $\lambda_0, \lambda_1, \lambda_2$ such that

$$(\lambda_0\lambda_1 + \lambda_1\lambda_2 + \lambda_2\lambda_0)^2 - 3\lambda_0\lambda_1\lambda_2(\lambda_0 + \lambda_1 + \lambda_2) = 0.$$

Hence, statement (i),

$$\mathcal{F}_g^{\mathsf{SQ}}(q) \in \mathbb{G}_2[A_2]\,,$$

follows from Proposition 29 and Lemmas 34 - 35. Statement (ii), \mathcal{F}_g^{SQ} has at most degree 3g-3 with respect to A_2 , holds since a stable graph of genus g has at most 3g-3 edges. Since

$$\frac{\partial}{\partial T} = \frac{q}{C_1} \frac{\partial}{\partial q} \,,$$

statement (iii),

(41)
$$\frac{\partial^k \mathcal{F}_g^{\mathsf{SQ}}}{\partial T^k}(q) \in \mathbb{G}_2[A_2][C_1^{-1}],$$

follows since the ring

$$\mathbb{G}_2[A_2] = \mathbb{G}_2[X]$$

is closed under the action of the differential operator

$$\mathsf{D} = q \frac{\partial}{\partial q}$$

by (29). The degree of C_1^{-1} in (41) is 1 which yields statement (iv).

6.2. **Proof of Theorem 2.** Let $\Gamma \in \mathsf{G}_g(\mathbb{P}^2)$ be a decorated graph. Let us fix an edge $f \in \mathsf{E}(\Gamma)$:

• if Γ is connected after deleting f, denote the resulting graph by

$$\Gamma_f^0 \in \mathsf{G}_{g-1,2}(\mathbb{P}^2)\,,$$

•• if Γ is disconnected after deleting f, denote the resulting two graphs by

$$\Gamma_f^1 \in \mathsf{G}_{g_1,1}(\mathbb{P}^2)$$
 and $\Gamma_f^2 \in \mathsf{G}_{g_2,1}(\mathbb{P}^2)$

where $g = g_1 + g_2$.

There is no canonical order for the 2 new markings. We will always sum over the 2 labellings. So more precisely, the graph Γ_f^0 in case • should be viewed as sum of 2 graphs

$$\Gamma^0_{f,(1,2)} + \Gamma^0_{f,(2,1)}$$

Similarly, in case ••, we will sum over the ordering of g_1 and g_2 . As usual, the summation will be later compensated by a factor of $\frac{1}{2}$ in the formulas.

By Proposition 29, we have the following formula for the contribution of the graph Γ to the stable quotient theory of $K\mathbb{P}^2$,

$$\operatorname{Cont}_{\Gamma} = \frac{1}{|\operatorname{Aut}(\Gamma)|} \sum_{\mathsf{A} \in \mathbb{Z}_{\geq 0}^{\mathsf{F}}} \prod_{v \in \mathsf{V}} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(v) \prod_{e \in \mathsf{E}} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(e) \,.$$

Let f connect the T-fixed points $p_i, p_j \in \mathbb{P}^2$. Let the A-values of the respective half-edges be (k, l). By Lemma 35, we have

(42)
$$\frac{\partial \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(f)}{\partial X} = (-1)^{k+l} \frac{3L_i L_j R_{1\,k-1,i} R_{1\,l-1,j}}{L^3}$$

• If Γ is connected after deleting f, we have

$$\begin{split} \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\mathbf{A} \in \mathbb{Z}_{\geq 0}^{\mathbf{F}}} \left(\frac{L^3}{3C_1^2}\right) \frac{\partial \mathrm{Cont}_{\Gamma}^{\mathbf{A}}(f)}{\partial X} \prod_{v \in \mathbf{V}} \mathrm{Cont}_{\Gamma}^{\mathbf{A}}(v) \prod_{e \in \mathbf{E}, \, e \neq f} \mathrm{Cont}_{\Gamma}^{\mathbf{A}}(e) \\ &= \frac{1}{2} \operatorname{Cont}_{\Gamma_f^0}(H, H) \,. \end{split}$$

The derivation is simply by using (42) on the left and Proposition 30 on the right.

•• If Γ is disconnected after deleting f, we obtain

$$\frac{1}{|\operatorname{Aut}(\Gamma)|} \sum_{\mathsf{A} \in \mathbb{Z}_{\geq 0}^{\mathsf{F}}} \left(\frac{L^3}{3C_1^2}\right) \frac{\partial \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(f)}{\partial X} \prod_{v \in \mathsf{V}} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(v) \prod_{e \in \mathsf{E}, e \neq f} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(e) = \frac{1}{2} \operatorname{Cont}_{\Gamma_f^1}(H) \operatorname{Cont}_{\Gamma_f^2}(H)$$

by the same method.

By combining the above two equations for all the edges of all the graphs $\Gamma \in \mathsf{G}_q(\mathbb{P}^2)$ and using the vanishing

$$\frac{\partial \mathrm{Cont}_{\Gamma}^{\mathsf{A}}(v)}{\partial X} = 0$$

of Lemma 34, we obtain

(43)
$$\left(\frac{L^3}{3C_1^2}\right)\frac{\partial}{\partial X}\langle\rangle_{g,0}^{\mathsf{SQ}} = \frac{1}{2}\sum_{i=1}^{g-1}\langle H\rangle_{g-i,1}^{\mathsf{SQ}}\langle H\rangle_{i,1}^{\mathsf{SQ}} + \frac{1}{2}\langle H,H\rangle_{g-1,2}^{\mathsf{SQ}}.$$

We have followed here the notation of Section 0.2. The equality (43)

holds in the ring $\mathbb{G}_2[A_2, C_1^{-1}]$. Since $A_2 = \frac{1}{L^3}(3X + 1 - \frac{L^3}{2})$ and $\langle \rangle_{g,0}^{\mathsf{SQ}} = \mathcal{F}_g^{\mathsf{SQ}}$, the left side of (43) is, by the chain rule,

$$\frac{1}{C_1^2} \frac{\partial \mathcal{F}_g^{\mathsf{SQ}}}{\partial A_2} \in \mathbb{G}_2[A_2, C_1^{-1}].$$

On the right side of (43), we have

$$\langle H \rangle_{g-i,1}^{\mathsf{SQ}} = \mathcal{F}_{g-i,1}^{\mathsf{SQ}}(q) = \mathcal{F}_{g-i,1}^{\mathsf{GW}}(Q(q)),$$

where the first equality is by definition and the second is by wallcrossing (4). Then,

$$\mathcal{F}_{g-i,1}^{\mathsf{GW}}(Q(q)) = \frac{\partial \mathcal{F}_{g-i}^{\mathsf{GW}}}{\partial T}(Q(q)) = \frac{\partial \mathcal{F}_{g-i}^{\mathsf{SQ}}}{\partial T}(q)$$

where the first equality is by the divisor equation in Gromov-Witten theory and the second is again by wall-crossing (4), so we conclude

$$\langle H \rangle_{g-i,1}^{\mathsf{SQ}} = \frac{\partial \mathcal{F}_{g-i}^{\mathsf{SQ}}}{\partial T}(q) \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]].$$

Similarly, we obtain

$$\langle H \rangle_{i,1}^{\mathsf{SQ}} = \frac{\partial \mathcal{F}_i^{\mathsf{SQ}}}{\partial T}(q) \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]],$$

$$\langle H, H \rangle_{g-1,2}^{\mathsf{SQ}} = \frac{\partial^2 \mathcal{F}_{g-1}^{\mathsf{SQ}}}{\partial T^2}(q) \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]].$$

Together, the above equations transform (43) into exactly the holomorphic anomaly equation of Theorem 4,

$$\frac{1}{C_1^2} \frac{\partial \mathcal{F}_g^{\mathsf{SQ}}}{\partial A_2}(q) = \frac{1}{2} \sum_{i=1}^{g-1} \frac{\partial \mathcal{F}_{g-i}^{\mathsf{SQ}}}{\partial T}(q) \frac{\partial \mathcal{F}_i^{\mathsf{SQ}}}{\partial T}(q) + \frac{1}{2} \frac{\partial^2 \mathcal{F}_{g-1}^{\mathsf{SQ}}}{\partial T^2}(q)$$

as an equality in $\mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]].$

The series L and A_2 are expected to be algebraically independent. Since we do not have a proof of the independence, to lift holomorphic anomaly equation to the equality

$$\frac{1}{C_1^2} \frac{\partial \mathcal{F}_g^{\mathsf{SQ}}}{\partial A_2} = \frac{1}{2} \sum_{i=1}^{g-1} \frac{\partial \mathcal{F}_{g-i}^{\mathsf{SQ}}}{\partial T} \frac{\partial \mathcal{F}_i^{\mathsf{SQ}}}{\partial T} + \frac{1}{2} \frac{\partial^2 \mathcal{F}_{g-1}^{\mathsf{SQ}}}{\partial T^2}$$

in the ring $\mathbb{G}_2[A_2, C_1^{-1}]$, we must prove the equalities

(44)
$$\langle H \rangle_{g-i,1}^{SQ} = \frac{\partial \mathcal{F}_{g-i}^{SQ}}{\partial T}, \quad \langle H \rangle_{i,1}^{SQ} = \frac{\partial \mathcal{F}_{i}^{SQ}}{\partial T}, \\ \langle H, H \rangle_{g-1,2}^{SQ} = \frac{\partial^2 \mathcal{F}_{g-1}^{SQ}}{\partial T^2}$$

in the ring $\mathbb{G}_2[A_2, C_1^{-1}]$. The lifting follow from the argument in Section 7.3 in [16].

We do not study the genus 1 unpointed series $\mathcal{F}_1^{SQ}(q)$ in the paper, so we take

$$\begin{split} \langle H \rangle_{1,1}^{\mathsf{SQ}} &= \quad \frac{\partial \mathcal{F}_1^{\mathsf{SQ}}}{\partial T} \,, \\ \langle H, H \rangle_{1,2}^{\mathsf{SQ}} &= \quad \frac{\partial^2 \mathcal{F}_1^{\mathsf{SQ}}}{\partial T^2} \end{split}$$

as definitions of the right side in the genus 1 case. There is no difficulty in calculating these series explicitly using Proposition 30.

7. Holomorphic anomaly for $K\mathbb{P}^3$

7.1. **Overview.** We fix a torus action $\mathsf{T} = (\mathbb{C}^*)^4$ on \mathbb{P}^3 with weights¹²

$$-\lambda_0,\ldots,-\lambda_3$$

on the vector space \mathbb{C}^4 . The T-weight on the fiber over p_i of the canonical bundle

$$(45) \qquad \qquad \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathbb{P}^3$$

¹²The associated weights on $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ are $\lambda_0, \ldots, \lambda_3$ and so match the conventions of Section 0.1.

is $-4\lambda_i$. The toric Calabi-Yau $K\mathbb{P}^3$ is the total space of (45). The basic generating series and other essential objects defined in Section 2 – Section 5 can be defined¹³ similarly for $K\mathbb{P}^3$. We will not repeat the definitions of these objects unless necessary.

7.2. I-functions.

7.2.1. Evaluations. Let $\widetilde{H} \in H^*_{\mathsf{T}}([\mathbb{C}^4/\mathbb{C}^*])$ and $H \in H^*_{\mathsf{T}}(\mathbb{P}^3)$ denote the respective hyperplane classes. The I-function of Definition 13 for $K\mathbb{P}^3$ is evaluated in [5].

Proposition 36. For $\mathbf{t} = t\widetilde{H} \in H^*_{\mathsf{T}}([\mathbb{C}^4/\mathbb{C}^*], \mathbb{Q}),$

(46)
$$\mathbb{I}(t) = \sum_{d=0}^{\infty} q^d e^{t(H+dz)/z} \frac{\prod_{k=0}^{4d-1} (-4H-kz)}{\prod_{i=0}^{3} \prod_{k=1}^{d} (H-\lambda_i+kz)}$$

We define the series $I_{i,j}$ by following expansion of the I-function after restriction t = 0,

$$\mathbb{I}|_{t=0} = 1 + \frac{I_{10}H}{z} + \frac{I_{20}H^2 + I_{21}H}{z^2} + \frac{I_{30}H^3 + I_{31}H^2 + I_{32}H}{z^3} + \mathcal{O}(\frac{1}{z^4}) + \mathcal{O}(\frac{1}{z^$$

For example,

$$I_{10}(q) = \sum_{d=1}^{\infty} 4 \frac{(4d-1)!}{(d!)^4} q^d \in \mathbb{C}[[q]],$$

$$I_{20}(q) = \sum_{d=1}^{\infty} 4 \frac{(4d-1)!}{(d!)^4} \Big(4 \operatorname{Har}[4d-1] - 4 \operatorname{Har}[d] \Big) q^d \in \mathbb{C}[[q]],$$

$$I_{21}(q) = \sum_{d=1}^{\infty} 4s_1 \frac{(4d-1)!}{(d!)^4} \operatorname{Har}[d] q^d \in \mathbb{C}(\lambda_0, \dots, \lambda_3)[[q]].$$

Here $\operatorname{Har}[d] := \sum_{k=1}^{d} \frac{1}{k}$.

¹³In fact the contents of Section 2 – 5 can be stated universally for all $K\mathbb{P}^n$.

We return now to the functions $S_i(\gamma)$ defined in Section 2.3. We define the following additional series in q:

$$\begin{split} C_1 &= 1 + \mathsf{D}I_{10} \,, \ J_{10} = \frac{I_{10} + \mathsf{D}I_{20}}{C_1} \,, \ J_{11} = \frac{\mathsf{D}I_{21}}{C_1} \,, \\ J_{20} &= \frac{I_{20} + \mathsf{D}I_{30}}{C_1} \,, \ J_{21} = \frac{I_{21} + \mathsf{D}I_{31}}{C_1} \,, \ J_{22} = \frac{I_{22} + \mathsf{D}I_{32}}{C_1} \,, \\ C_2 &= 1 + \mathsf{D}J_{10} \,, \ K_{10} = \frac{J_{10} + \mathsf{D}J_{20}}{C_2} \,, \\ K_{11} &= \frac{J_{11} + \mathsf{D}J_{21} - (\mathsf{D}J_{11})J_{10}}{C_2} \,, \ K_{12} = \frac{\mathsf{D}J_{22} - (\mathsf{D}J_{11})J_{11}}{C_2} \,, \\ C_3 &= 1 + \mathsf{D}K_{10} \,. \end{split}$$

Here, $\mathsf{D} = q \frac{d}{dq}$. The following relations were proven in [21],

(47)
$$C_2 = C_3,$$

 $C_1^2 C_2^2 = (1 - 4^4 q)^{-1}.$

Using Birkhoff factorization, an evaluation of the series $S(H^j)$ can be obtained from the I-function, see [13]:

$$\begin{split} & \mathbb{S}(1) = \mathbb{I}, \\ (48) & \mathbb{S}(H) = \frac{z \frac{d}{dt} \mathbb{S}(1)}{C_1}, \\ & \mathbb{S}(H^2) = \frac{z \frac{d}{dt} \mathbb{S}(H) - (\mathsf{D}J_{11}) \mathbb{S}(H)}{C_2}, \\ & \mathbb{S}(H^3) = \frac{z \frac{d}{dt} \mathbb{S}(H^2) - (\mathsf{D}K_{11}) \mathbb{S}(H^2) - (\mathsf{D}K_{12}) \mathbb{S}(H)}{C_3}. \end{split}$$

7.2.2. Further calculations. Define small I-function

$$\overline{\mathbb{I}}(q) \in H^*_{\mathsf{T}}(\mathbb{P}^3, \mathbb{Q})[[q]]$$

by the restriction

$$\overline{\mathbb{I}}(q) = \mathbb{I}(q, t)|_{t=0}.$$

Define differential operators

$$\mathsf{D} = q \frac{d}{dq}, \quad M = H + z \mathsf{D}.$$

Applying $z \frac{d}{dt}$ to \mathbb{I} and then restricting to t = 0 has same effect as applying M to $\overline{\mathbb{I}}$

$$\left[\left(z \frac{d}{dt} \right)^k \mathbb{I} \right] \Big|_{t=0} = M^k \,\overline{\mathbb{I}} \,.$$

The function $\overline{\mathbb{I}}$ satisfies following Picard-Fuchs equation

(49)
$$\left(\prod_{j=0}^{3} (M-\lambda_j) - 4qM(4M+z)(4M+2z)(4M+3z)\right)\overline{\mathbb{I}} = 0$$

implied by the Picard-Fuchs equation for I,

$$\left(\prod_{j=0}^{3} \left(z\frac{d}{dt} - \lambda_j\right) - q\prod_{k=0}^{3} \left(4z\frac{d}{dt} + kz\right)\right) \mathbb{I} = 0.$$

The restriction $\overline{\mathbb{I}}|_{H=\lambda_i}$ admits following asymptotic form

(50)
$$\overline{\mathbb{I}}|_{H=\lambda_i} = e^{\mu_i/z} \left(R_{0,i} + R_{1,i}z + R_{2,i}z^2 + \ldots \right)$$

with series $\mu_i, R_{k,i} \in \mathbb{C}(\lambda_0, \ldots, \lambda_d)[[q]].$

A derivation of (50) is obtained in [21] via the Picard-Fuchs equation (49) for $\overline{\mathbb{I}}|_{H=\lambda_i}$. The series μ_i and $R_{k,i}$ are found by solving differential equations obtained from the coefficient of z^k . For example,

$$\lambda_i + \mathsf{D}\mu_i = L_i,$$

$$R_{0,i} = \left(\frac{\lambda_i \prod_{j \neq i} (\lambda_i - \lambda_j)}{f(L_i)}\right)^{\frac{1}{2}}.$$

From the equations (48) and (50), we can show the series

 $\overline{\mathbb{S}}_{i}(1) = \overline{\mathbb{S}}(1)|_{H=\lambda_{i}}, \ \overline{\mathbb{S}}_{i}(H) = \overline{\mathbb{S}}(H)|_{H=\lambda_{i}}, \ \overline{\mathbb{S}}_{i}(H^{2}) = \overline{\mathbb{S}}(H^{2})|_{H=\lambda_{i}}, \ \overline{\mathbb{S}}_{i}(H^{3}) = \overline{\mathbb{S}}(H^{3})|_{H=\lambda_{i}}$ have the following asymptotic expansions:

$$\overline{\mathbb{S}}_{i}(1) = e^{\frac{\mu_{i}}{z}} \left(R_{00,i} + R_{01,i}z + R_{02,i}z^{2} + \dots \right),$$
(51)
$$\overline{\mathbb{S}}_{i}(H) = e^{\frac{\mu_{i}}{z}} \frac{L_{i}}{C_{1}} \left(R_{10,i} + R_{11,i}z + R_{12}z^{2} + \dots \right),$$

$$\overline{\mathbb{S}}_{i}(H^{2}) = e^{\frac{\mu_{i}}{z}} \frac{L_{i}^{2}}{C_{1}C_{2}} \left(R_{20,i} + R_{21,i}z + R_{22,i}z^{2} + \dots \right),$$

$$\overline{\mathbb{S}}_{i}(H^{3}) = e^{\frac{\mu_{i}}{z}} \frac{L_{i}^{3}}{C_{1}C_{2}C_{3}} \left(R_{30,i} + R_{31,i}z + R_{32,i}z^{2} + \dots \right).$$

We follow here the normalization of [21]. Note

$$R_{0k,i} = R_{k,i}$$

As in [21, Theorem 4], we expect the following constraints.

Conjecture 37. For all $k \ge 0$, we have

D D

$$R_{k,i} \in \mathbb{G}_3$$
.

Conjecture 37 is the main obstruction for the proof of Conjecture 5 and 6. By the same argument of Section 7, we obtain the following result.

Theorem 38. Conjecture 37 implies Conjecture 5 and 6.

By applying asymptotic expansions (51) to (48), we obtain the following results.

Lemma 39. We have

$$\begin{split} R_{1\,p+1,i} &= R_{0\,p+1,i} + \frac{\mathsf{D}R_{0\,p,i}}{L_i} \,, \\ R_{2\,p+1,i} &= R_{1\,p+1,i} - E_{11,i}R_{1\,k,i} + \frac{\mathsf{D}R_{1\,p,i}}{L_i} + \left(\frac{\mathsf{D}L_i}{L_i^2} - \frac{A_2}{L_i}\right)R_{1\,p,i} \,, \\ R_{3\,p+1,i} &= R_{2\,p+1,i} - E_{21,i}R_{2\,k,i} - E_{22,i}R_{1\,k,i} + \frac{\mathsf{D}R_{2\,p,i}}{L_i} + \left(2\frac{\mathsf{D}L_i}{L_i^2} - \frac{A_2}{L_i} - \frac{\mathsf{D}C_2}{L_i}\right)R_{1\,p,i} \end{split}$$

with

$$E_{11,i} = \frac{\mathsf{D}J_{11}}{L_i}, \ E_{21,i} = \frac{\mathsf{D}K_{11}}{L_i}, \ E_{22,i} = \frac{C_2}{L_i^2}\mathsf{D}K_{12}.$$

7.3. Determining DA_2 and new series. The following relation was proven in [16].

(52)
$$A_2^2 + (L^4 - 1)A_2 + 2\mathsf{D}A_2 - \frac{3}{16}(L^4 - 1) = 0.$$

By the above result, the differential ring

$$(53) \qquad \qquad \mathbb{G}_3[A_2, \mathsf{D}A_2, \mathsf{D}\mathsf{D}A_2, \ldots]$$

is just the polynomial ring $G[A_2]$. The second equation in (47) yields the following relation.

(54)
$$2A_2 + 2\frac{\mathsf{D}C_2}{C_2} = L^4 - 1.$$

Denote by $\operatorname{Coeff}(x^i y^j)$ the coefficient of $x^i y^j$ in

$$\sum_{k=0}^{3} e^{-\frac{\mu_i}{x} - \frac{\mu_i}{y}} \mathbb{S}_i(\phi_k)|_{z=x} \mathbb{S}_i(\phi^k)|_{z=y}.$$

From (18) and (51), we obtain the following equations.

$$\operatorname{Coeff}(x^2) - \frac{1}{2}\operatorname{Coeff}(xy) = 0,$$

$$\operatorname{Coeff}(x^4) - \operatorname{Coeff}(x^3y) + \frac{1}{2}\operatorname{Coeff}(x^2y^2) = 0$$

Above equations immediately yields the following relations.

(55)
$$E_{11,i} = \frac{E_{21,i}}{2} - \frac{s_1 L^2}{2C_1 L_i} + \frac{s_1 L^4}{2L_i},$$
$$E_{22,i} = \frac{L^4 (s_1^2 (-3 + 2C_1 L^2 + C_1^2 L^4) - 4s_2 (-1 + C_1^2))}{8C_1^2 L_i^2}$$
$$\frac{s_1 (-3L^2 + C_1 L^4)}{4C_1 L_i} E_{21,i} - \frac{3}{8} E_{21}^2.$$

We define the series B_2 and B_4 which appeared in the introduction by

$$B_2 = L_i E_{21,i}$$
$$B_4 = \mathsf{D}B_2.$$

Note that $B_2(q), B_4(q) \in \mathbb{C}[[q]].$

From Lemma 39 with the relations (52), (54) and (55), we obtain results for $\overline{\mathbb{S}}(H)|_{H=\lambda_i}$, $\overline{\mathbb{S}}(H^2)|_{H=\lambda_i}$ and $\overline{\mathbb{S}}(H^3)|_{H=\lambda_i}$.

Lemma 40. Suppose Conjecture 37 is true. Then for all $k \ge 0$, we have for all $k \ge 0$,

$$R_{1k,i}, R_{2k,i}, R_{3k,i} \in \mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}]$$

7.4. Vertex, edge, and leg analysis. By parallel argument as in Section 4, we have decomposition of the contribution to $\Gamma \in \mathsf{G}_{g,k}(\mathbb{P}^3)$ to the stable quotient theory of $K\mathbb{P}^3$ into vertex terms, edge terms and leg terms

$$\operatorname{Cont}_{\Gamma} = \frac{1}{|\operatorname{Aut}(\Gamma)|} \sum_{\mathsf{A} \in \mathbb{Z}_{>0}^{\mathsf{F}}} \prod_{v \in \mathsf{V}} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(v) \prod_{e \in \mathsf{E}} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(e) \prod_{e \in \mathsf{L}} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(l).$$

The following lemmas follow from the argument in Section 5.

Lemma 41. Suppose Conjecture 37 is true. Then we have

$$\operatorname{Cont}_{\Gamma}^{\mathsf{A}}(v) \in \mathbb{G}_3$$

Let $e \in \mathsf{E}$ be an edge connecting the T-fixed points $p_i, p_j \in \mathbb{P}^3$. Let the A-values of the respective half-edges be (k, l).

Lemma 42. Suppose Conjecture 37 is true. Then we have

 $\operatorname{Cont}_{\Gamma}^{\mathsf{A}}(e) \in \mathbb{G}_{3}[A_{2}, B_{2}, B_{4}, C_{1}^{\pm 1}].$

Lemma 43. Suppose Conjecture 37 is true. Then we have

$$Cont_{\Gamma}^{\mathsf{A}}(l) \in \mathbb{G}_{3}[A_{2}, B_{2}, B_{4}, C_{1}^{\pm 1}].$$

7.5. **Proof of Theorem 7.** Conjecture 37 can be proven for the choices of $\lambda_0, \ldots, \lambda_3$ such that

$$\begin{split} \lambda_i &\neq \lambda_j \text{ for } i \neq j ,\\ 4s_2^2 - s_1 s_3 &= 0 ,\\ 2s_2^3 - 27s_1^2 s_4 &= 0 . \end{split}$$

by the argument in Appendix. Hence, statement (i),

$$\mathcal{F}_{g,a+b}^{SQ}[a,b](q) \in \mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}]$$

follows from the arguments in Proposition 29 and Lemmas 41 - 43. Statement (ii), \mathcal{F}_g^{SQ} has at most degree 2(3g-3) with respect to A_2 , holds since a stable graph of genus g has at most 3g-3 edges. Since

$$\frac{\partial}{\partial T} = \frac{q}{C_1} \frac{\partial}{\partial q} \,,$$

statement (iii),

(57)
$$\frac{\partial^k \mathcal{F}_g^{\mathsf{SQ}}}{\partial T^k}(q) \in \mathbb{G}[A_2, B_2, B_4, C_1^{\pm 1}],$$

follows from divisor equation in stable quotient theory and statement (i). $\hfill \Box$

7.6. Proof of Theorem 6: first equation. Let $\Gamma \in \mathsf{G}_g(\mathbb{P}^3)$ be a decorated graph. Let us fix an edge $f \in \mathsf{E}(\Gamma)$:

• if Γ is connected after deleting f, denote the resulting graph by

$$\Gamma_f^0 \in \mathsf{G}_{g-1,2}(\mathbb{P}^3)$$

•• if Γ is disconnected after deleting f, denote the resulting two graphs by

$$\Gamma_f^1 \in \mathsf{G}_{g_1,1}(\mathbb{P}^3) \text{ and } \Gamma_f^2 \in \mathsf{G}_{g_2,1}(\mathbb{P}^3)$$

where $g = g_1 + g_2$.

There is no canonical order for the 2 new markings. We will always sum over the 2 labellings. So more precisely, the graph Γ_f^0 in case • should be viewed as sum of 2 graphs

$$\Gamma^0_{f,(1,2)} + \Gamma^0_{f,(2,1)}$$
.

Similarly, in case $\bullet \bullet$, we will sum over the ordering of g_1 and g_2 . As usual, the summation will be later compensated by a factor of $\frac{1}{2}$ in the formulas.

By the argument in Section 5.3, we have the following formula for the contribution of the graph Γ to the stable quotient theory of $K\mathbb{P}^3$,

$$\operatorname{Cont}_{\Gamma} = \frac{1}{|\operatorname{Aut}(\Gamma)|} \sum_{\mathsf{A} \in \mathbb{Z}_{\geq 0}^{\mathsf{F}}} \prod_{v \in \mathsf{V}} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(v) \prod_{e \in \mathsf{E}} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(e)$$

Let f connect the T-fixed points $p_i, p_j \in \mathbb{P}^3$. Let the A-values of the respective half-edges be (k, l). Denote by \mathbb{D}_1 the differential operator

$$\frac{L^2}{4C_1}\frac{\partial}{\partial A_2} + \frac{-2s_1L^4 - C_1(3B_2L^2 - s_1L^6)}{4C_1^2}\frac{\partial}{\partial B_4}.$$

By Lemma 39 and the explicit formula for $\operatorname{Cont}_{\Gamma}^{\mathsf{A}}(f)$ in Lemma 35¹⁴, we have

(58)

$$\mathbb{D}_1 \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(f) = (-1)^{k+l} \left(\frac{L_i^2 L_j R_{2k-1,i} R_{1l-1,j}}{C_1^2 C_2} + \frac{L_i L_j^2 R_{1k-1,i} R_{2l-1,j}}{C_1^2 C_2} \right).$$

• If Γ is connected after deleting f, we have

$$\frac{1}{|\operatorname{Aut}(\Gamma)|} \sum_{\mathsf{A} \in \mathbb{Z}_{\geq 0}^{\mathsf{F}}} \mathbb{D}_{1} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(f) \prod_{v \in \mathsf{V}} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(v) \prod_{e \in \mathsf{E}, e \neq f} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(e) \\ = \operatorname{Cont}_{\Gamma_{f}^{0}}(H, H^{2}) + \operatorname{Cont}_{\Gamma_{f}^{0}}(H^{2}, H)$$

The derivation is simply by using (58) on the left and the argument in Proposition 30 on the right.

•• If Γ is disconnected after deleting f, we obtain

$$\frac{1}{|\operatorname{Aut}(\Gamma)|} \sum_{\mathsf{A} \in \mathbb{Z}_{\geq 0}^{\mathsf{F}}} \mathbb{D}_{1} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(f) \prod_{v \in \mathsf{V}} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(v) \prod_{e \in \mathsf{E}, e \neq f} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(e) = \operatorname{Cont}_{\Gamma_{f}^{1}}(H) \operatorname{Cont}_{\Gamma_{f}^{2}}(H^{2}) + \operatorname{Cont}_{\Gamma_{f}^{1}}(H^{2}) \operatorname{Cont}_{\Gamma_{f}^{2}}(H)$$

by the same method.

By combining the above two equations for all the edges of all the graphs $\Gamma \in \mathsf{G}_g(\mathbb{P}^3)$ and using the vanishing

$$\frac{\partial \text{Cont}_{\Gamma}^{\mathsf{A}}(v)}{\partial A_2} = 0, \ \frac{\partial \text{Cont}_{\Gamma}^{\mathsf{A}}(v)}{\partial B_4} = 0$$

of Lemma 41, we obtain

(59)
$$\mathbb{D}_1 \langle \rangle_{g,0}^{\mathsf{SQ}} = \sum_{i=1}^{g-1} \langle H \rangle_{g-i,1}^{\mathsf{SQ}} \langle H^2 \rangle_{i,1}^{\mathsf{SQ}} + \langle H, H^2 \rangle_{g-1,2}^{\mathsf{SQ}}$$

¹⁴Lemma 35 is stated for $K\mathbb{P}^2$, but parallel statement holds for $K\mathbb{P}^3$.

We have followed here the notation of Section 0.3. The equality (59) holds in the ring $G_3[A_2, B_2, B_4, C_1^{\pm 1}]$.

7.7. **Proof of Theorem 6: second equation.** By the argument in Section 5.3, we have the following formula for the contribution of the graph Γ to the stable quotient theory of $K\mathbb{P}^3$,

$$\operatorname{Cont}_{\Gamma} = \frac{1}{|\operatorname{Aut}(\Gamma)|} \sum_{\mathsf{A} \in \mathbb{Z}_{\geq 0}^{\mathsf{F}}} \prod_{v \in \mathsf{V}} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(v) \prod_{e \in \mathsf{E}} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(e) \,.$$

Let f connect the T-fixed points $p_i, p_j \in \mathbb{P}^3$. Let the A-values of the respective half-edges be (k, l). Denote by \mathbb{D}_2 the differential operator

$$\frac{2L^2}{C_1(L^4-1-2A_2)}\frac{\partial}{\partial B_2}\,.$$

By Lemma 39 and the explicit formula for $\operatorname{Cont}_{\Gamma}^{\mathsf{A}}(f)$ in Lemma 35¹⁵, we have

(60)
$$\mathbb{D}_2 \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(f) = (-1)^{k+l} \frac{2L_i L_j R_{1\,k-1,i} R_{1\,l-1,j}}{C_1^2}.$$

• If Γ is connected after deleting f, we have

$$\frac{1}{|\operatorname{Aut}(\Gamma)|} \sum_{\mathsf{A} \in \mathbb{Z}_{\geq 0}^{\mathsf{F}}} \mathbb{D}_2 \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(f) \prod_{v \in \mathsf{V}} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(v) \prod_{e \in \mathsf{E}, e \neq f} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(e) = \operatorname{Cont}_{\Gamma_f^0}(H, H).$$

The derivation is simply by using (60) on the left and the arguments in Proposition 30 on the right.

•• If Γ is disconnected after deleting f, we obtain

$$\frac{1}{|\operatorname{Aut}(\Gamma)|} \sum_{\mathsf{A} \in \mathbb{Z}_{\geq 0}^{\mathsf{F}}} \mathbb{D}_{2} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(f) \prod_{v \in \mathsf{V}} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(v) \prod_{e \in \mathsf{E}, e \neq f} \operatorname{Cont}_{\Gamma}^{\mathsf{A}}(e) = \operatorname{Cont}_{\Gamma_{f}^{1}}(H) \operatorname{Cont}_{\Gamma_{f}^{2}}(H)$$

by the same method.

By combining the above two equations for all the edges of all the graphs $\Gamma \in \mathsf{G}_g(\mathbb{P}^3)$ and using the vanishing

$$\frac{\partial \mathrm{Cont}_{\Gamma}^{\mathsf{A}}(v)}{\partial B_2} = 0$$

¹⁵Lemma 35 is stated for $K\mathbb{P}^2$, but parallel statement holds for $K\mathbb{P}^3$.

of Lemma 41, we obtain

(61)
$$\mathbb{D}_2\langle\rangle_{g,0}^{\mathsf{SQ}} = \sum_{i=1}^{g-1} \langle H \rangle_{g-i,1}^{\mathsf{SQ}} \langle H \rangle_{i,1}^{\mathsf{SQ}} + \langle H, H \rangle_{g-1,2}^{\mathsf{SQ}}.$$

We have followed here the notation of Section 0.3. The equality (61) holds in the ring $\mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}]$.

On the right side of (61), we have

$$\langle H \rangle_{g-i,1}^{\mathsf{SQ}} = \mathcal{F}_{g-i,1}^{\mathsf{SQ}}[1,0](q) = \mathcal{F}_{g-i,1}^{\mathsf{GW}}[1,0](Q(q)),$$

where the first equality is by definition and the second is by wallcrossing (7). Then,

$$\mathcal{F}_{g-i,1}^{\mathsf{GW}}[1,0](Q(q)) = \frac{\partial \mathcal{F}_{g-i}^{\mathsf{GW}}}{\partial T}(Q(q)) = \frac{\partial \mathcal{F}_{g-i}^{\mathsf{SQ}}}{\partial T}(q)$$

where the first equality is by the divisor equation in Gromov-Witten theory and the second is again by wall-crossing (7), so we conclude

$$\langle H \rangle_{g-i,1}^{\mathsf{SQ}} = \frac{\partial \mathcal{F}_{g-i}^{\mathsf{SQ}}}{\partial T}(q) \in \mathbb{C}(\lambda_0, \dots, \lambda_3)[[q]].$$

Similarly, we obtain

$$\langle H \rangle_{i,1}^{\mathsf{SQ}} = \frac{\partial \mathcal{F}_i^{\mathsf{SQ}}}{\partial T}(q) \in \mathbb{C}(\lambda_0, \dots, \lambda_3)[[q]], \langle H, H \rangle_{g-1,2}^{\mathsf{SQ}} = \frac{\partial^2 \mathcal{F}_{g-1}^{\mathsf{SQ}}}{\partial T^2}(q) \in \mathbb{C}(\lambda_0, \dots, \lambda_3)[[q]].$$

Together, the above equations transform (61) into exactly the second holomorphic anomaly equation of Theorem 8,

$$\frac{2L^4}{C_1^2(L^4 - 1 - 2A_2)} \frac{\partial \mathcal{F}_g^{\mathsf{SQ}}}{\partial B_2} = \sum_{i=1}^{g-1} \frac{\partial \mathcal{F}_{g-i}^{\mathsf{SQ}}}{\partial T} \frac{\partial \mathcal{F}_i^{\mathsf{SQ}}}{\partial T} + \frac{\partial^2 \mathcal{F}_{g-1}^{\mathsf{SQ}}}{\partial T^2}.$$

as an equality in $\mathbb{C}(\lambda_0, \ldots, \lambda_3)[[q]]$. To lift holomorphic anomaly equation to the equality

$$\frac{2L^4}{C_1^2(L^4 - 1 - 2A_2)} \frac{\partial \mathcal{F}_g^{\mathsf{SQ}}}{\partial B_2} = \sum_{i=1}^{g-1} \frac{\partial \mathcal{F}_{g-i}^{\mathsf{SQ}}}{\partial T} \frac{\partial \mathcal{F}_i^{\mathsf{SQ}}}{\partial T} + \frac{\partial^2 \mathcal{F}_{g-1}^{\mathsf{SQ}}}{\partial T^2}$$

in the ring $\mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}]$, we must prove the equalities

(62)
$$\langle H \rangle_{g-i,1}^{SQ} = \frac{\partial \mathcal{F}_{g-i}^{SQ}}{\partial T}, \quad \langle H \rangle_{i,1}^{SQ} = \frac{\partial \mathcal{F}_{i}^{SQ}}{\partial T}, \\ \langle H, H \rangle_{g-1,2}^{SQ} = \frac{\partial^2 \mathcal{F}_{g-1}^{SQ}}{\partial T^2}$$

in the ring $\mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}]$. The lifting follow from the argument in Section 7.3 in [16].

We do not study the genus 1 unpointed series $\mathcal{F}_1^{SQ}(q)$ in the paper, so we take

$$\langle H \rangle_{1,1}^{\mathsf{SQ}} = \frac{\partial \mathcal{F}_1^{\mathsf{SQ}}}{\partial T} ,$$

$$\langle H, H \rangle_{1,2}^{\mathsf{SQ}} = \frac{\partial^2 \mathcal{F}_1^{\mathsf{SQ}}}{\partial T^2} .$$

as definitions of the right side in the genus 1 case. There is no difficulty in calculating these series explicitly using the argument in Proposition 30.

8. Appendix

8.1. **Overviews.** In section 0.1 the equivariant Gromov-Witten invariants of the local \mathbb{P}^n were defined,

$$N_{g,d}^{\mathsf{GW}} = \int_{[\overline{M}_g(\mathbb{P}^n,d)]^{\mathrm{vir}}} e\Big(-R\pi_*f^*\mathcal{O}_{\mathbb{P}^n}(-n-1)\Big)\,.$$

We associate Gromov-Witten generating series by

$$\mathcal{F}_{g}^{\mathsf{GW},n}(Q) = \sum_{d=0}^{\infty} \widetilde{N}_{g,d}^{\mathsf{GW}} Q^{d} \in \mathbb{C}(\lambda_{0},\ldots,\lambda_{n})[[Q]].$$

Motivated by mirror symmetry ([1, 2, 19]), we can make the following predictions about the genus g generating series $\mathcal{F}_{g}^{\mathsf{GW},n}$.

(A) There exist a finitely generated subring

$$\mathbf{G} \in \mathbb{C}(\lambda_0, \dots, \lambda_n)[[Q]]$$

(B) The series $\mathcal{F}_{g}^{\mathsf{GW},n}$ for all g. (B) The series $\mathcal{F}_{g}^{\mathsf{GW},n}$ satisfy holomorphic anomaly equations, i.e. recursive formulas for the derivative of $\mathcal{F}_{g}^{\mathsf{GW},n}$ with respect to some generators in **G**.

8.1.1. *I*-function. *I*-function defined by

$$I_n = \sum_{d=0}^{\infty} \frac{\prod_{k=1}^{(n+1)d-1} (-(n+1)H - kz)}{\prod_{i=0}^{n} \prod_{k=1}^{d} (H + kz - \lambda_i)} q^d \in H^*_{\mathsf{T}}(\mathbb{P}^n, \mathbb{C})[[q]] \,,$$

is the central object in the study of Gromov-Witten invariants of local \mathbb{P}^n geometry. See [16], [17] for the arguments. Several important properties of the function I_n was studied in [21] after the specialization

(63)
$$\lambda_i = \zeta_{n+1}^i$$

where ζ_{n+1} is primitive (n + 1)-th root of unity. For the study of full equivariant Gromov-Witten theories, we extend the result of [21] without the specialization (63).

8.1.2. *Picard-Fuchs equation and Birkhoff factorization*. Define differential operators

$$\mathsf{D} = q \frac{d}{dq}, \ M = H + z \mathsf{D}.$$

The function I_n satisfies following Picard-Fuchs equation

$$\left(\prod_{i=0}^{n} \left(M - \lambda_{i}\right) - q \prod_{k=0}^{n} \left(-(n+1)M - kz\right)\right) I_{n} = 0.$$

The restriction $I_n|_{H=\lambda_i}$ admits following asymptotic form

(64)
$$I_n|_{H=\lambda_i} = e^{\frac{\mu}{z}} \left(R_{0,i} + R_{1,i}z + R_{2,i}z^2 + \dots \right)$$

with series μ_i , $R_{k,i} \in \mathbb{C}(\lambda_0, \ldots, \lambda_n)[[q]]$.

A derivation of (64) is obtained from [4, Theorem 5.4.1] and the uniqueness lemma in [4, Section 7.7]. The series μ_i and $R_{k,i}$ are found by solving defferential equations obtained from the coefficient of z^k . For example,

$$\lambda_i + \mathsf{D}\mu_i = L_i$$

where $L_i(q)$ is the series in q defined by the root of following degree (n+1) polynomial in \mathcal{L}

$$\prod_{i=0}^{n} (\mathcal{L} - \lambda_i) - (-1)^{n+1} q \mathcal{L}^{n+1}.$$

with initial conditions,

$$\mathcal{L}_i(0) = \lambda_i$$
.

Let f_n be the polynomial of degree n in variable x over $\mathbb{C}(\lambda_0, \ldots, \lambda_n)$ defined by

$$f_n(x) := \sum_{k=0}^n (-1)^k k s_{k+1} x^{n-k} \,,$$

where s_k is k-th elementary symmetric function in $\lambda_0, \ldots, \lambda_n$. The ring

$$\mathbb{G}_n := \mathbb{C}(\lambda_0, \dots, \lambda_n)[L_0^{\pm 1}, \dots, L_n^{\pm 1}, f_n(L_0)^{-\frac{1}{2}}, \dots, f_n(L_n)^{-\frac{1}{2}}]$$

will play a basic role.

The following Conjecture was proven under the specialization (63) in [21, Theorem 4].

Conjecture 44. For all $k \ge 0$, we have

$$R_{k,i} \in \mathbb{G}_n$$
.

Conjecture 44 for the case n = 1 will be proven in Section 8.3. Conjecture 44 for the case n = 2 will be proven in Section 8.4 under the specialization (72). In fact, the argument in Section 8.4 proves Conjecture 44 for all n under the specialization which makes $f_n(x)$ into power of a linear polynomial.

8.2. Admissibility of differential equations. Let R be a commutative ring. Fix a polynomial $f(x) \in R[x]$. We consider a differential operator of *level* n with following forms.

(65)
$$\mathcal{P}(A_{lp}, f)[X_0, \dots, X_{n+1}] = \mathsf{D}X_{n+1} - \sum_{n \ge l \ge 0, p \ge 0} A_{lp} \mathsf{D}^p X_{n-l},$$

where $\mathsf{D} := \frac{d}{dx}$ and $A_{lp} \in \mathsf{R}[x]_f := \mathsf{R}[x][f^{-1}]$. We assume that only finitely many A_{lp} are not zero.

Definition 45. Let R_i be the solutions of the equations for $k \ge 0$,

(66)
$$\mathcal{P}(A_{lp}, f)[X_{k+1}, \dots, X_{k+n}] = 0,$$

with $R_0 = 1$. We use the conventions $X_i = 0$ for i < 0. We say differential equations (66) is admissible if the solutions R_k satisfies for $k \ge 0$,

$$R_k \in \mathsf{R}[x]_f$$

Remark 46. Note that the admissibility of $\mathcal{P}(A_{lp}, f)$ in Definition 45 do not depend on the choice of the solutions R_k .

Lemma 47. Let f be a degree one polynomial in x. Each $A \in \mathsf{R}[x]_f$ can be written uniquely as

$$A = \sum_{i \in \mathbb{Z}} a_i f^i$$

with finitely many non-zero $a_i \in \mathsf{R}$. We define the order Ord(A) of A with respect to f by smallest i such that a_i is not zero. Then

$$\mathcal{P}(A_{lp}, f)[X_0, \dots, X_{n+1}] := \mathsf{D}X_{n+1} - \sum_{n \ge l \ge 0, \, p \ge 0} A_{lp} \mathsf{D}^p X_{n-l} = 0 \,,$$

is admissible if following condition holds:

(67)

$$\begin{array}{l} \operatorname{Ord}(A_{l0}) \leq -2, \\ \operatorname{Ord}(A_{l1}) \leq 0, \\ \operatorname{Ord}(A_{lp}) \leq p+1 \quad \text{for } p \geq 2. \end{array}$$

Proof. The proof follows from simple induction argument.

Lemma 48. Let f be a degree two polynomial in x. Denote by

 R_{f}

the subspace of $\mathsf{R}[x]_f$ generated by f^i for $i \in \mathbb{Z}$. Each $A \in \mathsf{R}_f$ can be written uniquely as

$$A = \sum_{i \in \mathbb{Z}} a_i f^i$$

with finitely many non-zero $a_i \in \mathsf{R}$. We define the order Ord(A) of $A \in \mathsf{R}_f$ with respect to f by smallest i such that a_i is not zero. Then

$$\mathcal{P}(A_{lp}, f)[X_0, \dots, X_{n+1}] := \mathsf{D}X_{n+1} - \sum_{n \ge l \ge 0, \, p \ge 0} A_{lp} \mathsf{D}^p X_{n-l} = 0 \,,$$

is admissible if following condition holds:

$$A_{lp} = B_{lp} \qquad if \ p \ is \ odd ,$$
$$A_{lp} = \frac{df}{dx} \cdot B_{lp} \quad if \ p \ is \ even ,$$

where B_{lp} are elements of R_f with

$$Ord(B_{l0}) \leq -2$$
,
 $Ord(B_{lp}) \leq \left[\frac{p-1}{2}\right] \text{ for } p \geq 1$.

Proof. Since f is degree two polynomial in x, we have

$$\frac{d^2f}{dx^2}, \ (\frac{df}{dx})^2 \in \mathsf{R}_f$$

Then the proof of Lemma follows from simple induction argument. \Box

8.3. Local \mathbb{P}^1 .

8.3.1. Overview. In this section, we prove Conjecture 44 for the case n = 1. Recall the *I*-function for $K\mathbb{P}^1$,

(68)
$$I_1(q) = \sum_{d=0}^{\infty} \frac{\prod_{k=0}^{2d-1} (-2H - kz)}{\prod_{i=0}^{1} \prod_{k=1}^{d} (H - \lambda_i + kz)} q^d.$$

The function I_1 satisfies following Picard-Fuchs equation

(69)
$$((M - \lambda_0)(M - \lambda_1) - 2qM(2M + z))I_1 = 0.$$

Recall the notation used in above equation,

$$\mathsf{D} = q \frac{d}{dq}, \ M = H + z \mathsf{D}.$$

The restriction $I_1|_{H=\lambda_i}$ admits following asymptotic form

(70)
$$I_1|_{H=\lambda_i} = e^{\mu_i/z} \left(R_{0,i} + R_{1,i}z + R_{2,i}z^2 + \ldots \right)$$

with series $\mu_i, R_{k,i} \in \mathbb{C}(\lambda_0, \lambda_1)[[q]]$. The series μ_i and $R_{k,i}$ are found by solving differential equations obtained from the coefficient of z^k in (69). For example, we have for i = 0, 1,

$$\lambda_i + \mathsf{D}\mu_i = L_i,$$

$$R_{0,i} = \left(\frac{\lambda_i \prod_{j \neq i} (\lambda_i - \lambda_j)}{f_1(L_i)}\right)^{\frac{1}{2}},$$

$$\begin{aligned} R_{1,i} &= \left(\frac{\lambda_i \prod_{j \neq i} (\lambda_i - \lambda_j)}{f(L_i)}\right)^{\frac{1}{2}} \cdot \\ &\left(\frac{-16s_1^2 s_2^2 + 88s_2^3 + (27s_1^3 s_2 - 132s_1 s_2^2)L_i + (-12s_1^4 + 54s_1^2 s_2)L_i^2}{24s_1 (L_i s_1 - 2s_2)^3} + \frac{12\lambda_i^2 - 9\lambda_i \lambda_{i+1} + \lambda_{i+1}^2}{24(\lambda_i^3 - \lambda_i \lambda_{i+1}^2)}\right) \end{aligned}$$

Here $s_1 = \lambda_0 + \lambda_1$ and $s_2 = \lambda_0 \lambda_1$. In the above expression of $R_{1,i}$, we used the convention $\lambda_2 = \lambda_0$.

8.3.2. Proof of Conjecture 44. We introduce new differential operator D_i defined by for i = 0, 1,

$$\mathsf{D}_i = (\mathsf{D}L_i)^{-1}\mathsf{D}.$$

By definition, D_i acts on rational functions in L_i as the ordinary derivation with respect to L_i . If we use following normalizations,

$$R_{k,i} = f_1(L_i)^{-\frac{1}{2}} \Phi_{k,i}$$

the Picard-Fuchs equation (73) yields the following differential equations,

(71)
$$\mathsf{D}_{i}\Phi_{p,i} - A_{00,i}\Phi_{p-1,i} - A_{01,i}\mathsf{D}_{i}\Phi_{p-1,i} - A_{02,i}\mathsf{D}_{i}^{2}\Phi_{p-1,i} = 0,$$

with

$$A_{00,i} = \frac{-s_1^2 s_2^2 + (-s_1^3 s_2 + 8s_1 s_2^2) L_i + (2s_1^4 - 9s_1^2 s_2) L_i^2}{4(L_i s_1 - 2s_2)^4},$$

$$A_{01,i} = \frac{2s_1 s_2^2 + (-s_1^2 s_2 - 8s_2^2) L_i + (-s_1^3 + 10s_1 s_2) L_i^2 - s_1^2 L_i^3}{2(L_i s_1 - 2s_2)^3},$$

$$A_{02,i} = \frac{s_2^2 - 2(s_1 s_2) L_i + (s_1^2 + s_2) L_i^2 - s_1 L_i^3}{(L_i s_1 - 2s_2)^2}.$$

Here s_k is the k-th elementary symmetric functions in λ_0, λ_1 . Since the differential equations (71) satisfy the condition (67), we conclude the differential equations (71) is admissible.

8.3.3. *Gomov-Witten series.* By the result of previous subsection, we obtain the following result which verifies the prediction (A) in Section 8.1.

Theorem 49. For the Gromov-Witten series of $K\mathbb{P}^1$, we have

 $\mathcal{F}_g^{\mathsf{GW},1}(Q(q)) \in \mathbb{G}_1\,,$

where Q(q) is the mirror map of $K\mathbb{P}^1$ defined by

$$Q(q) := q \cdot exp\left(2\sum_{d=1}^{\infty} \frac{(2d-1)!}{(d!)^2} q^d\right).$$

Theorem 49 follows from the argument in [16]. The prediction (B) in Section 8.1 is trivial statement for $K\mathbb{P}^1$.

8.4. Local \mathbb{P}^2 .

8.4.1. Overview. In this section, we prove Conjecture 44 for the case n = 2 with following specializations,

$$\lambda_i \neq \lambda_j \text{ for } i \neq j$$
,

(72)
$$(\lambda_0\lambda_1 + \lambda_1\lambda_2 + \lambda_2\lambda_0)^2 - 3\lambda_0\lambda_1\lambda_2(\lambda_0 + \lambda_1 + \lambda_2) = 0.$$

For the rest of the section, the specialization (72) will be imposed. Recall the *I*-function for $K\mathbb{P}^2$.

$$I_2(q) = \sum_{d=0}^{\infty} \frac{\prod_{k=0}^{3d-1} (-3H - kz)}{\prod_{i=0}^2 \prod_{k=1}^d (H - \lambda_i + kz)} q^d.$$

The function I_2 satisfies following Picard-Fuchs equation

(73)

$$\left((M - \lambda_0)(M - \lambda_1)(M - \lambda_2) + 3qM(3M + z)(3M + 2z) \right) I_2 = 0$$

Recall the notation used in above equation,

$$\mathsf{D} = q \frac{d}{dq}, \ M = H + z \mathsf{D}.$$

The restriction $I_2|_{H=\lambda_i}$ admits following asymptotic form

$$I_2|_{H=\lambda_i} = e^{\mu_i/z} \left(R_{0,i} + R_{1,i}z + R_{2,i}z^2 + \ldots \right)$$

with series $\mu_i, R_{k,i} \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]]$. The series μ_i and $R_{k,i}$ are found by solving differential equations obtained from the coefficient of z^k in (73). For example,

$$\begin{split} \lambda_i + \mathsf{D}\mu_i &= L_i \,, \\ R_{0,i} &= \Big(\frac{\lambda_i \prod_{j \neq i} (\lambda_i - \lambda_j)}{f_2(L_i)}\Big)^{\frac{1}{2}} \end{split}$$

8.4.2. Proof of Conjecture 44. We introduce new differential operator D_i defined by

$$\mathsf{D}_i = (\mathsf{D}L_i)^{-1}\mathsf{D}.$$

If we use following normalizations,

$$R_{k,i} = f_2(L_i)^{-\frac{1}{2}} \Phi_{k,i}$$

the Picard-Fuchs equation (73) yields the following differential equations,

(74)
$$\mathsf{D}_{i}\Phi_{p,i} - A_{00,i}\Phi_{p-1,i} - A_{01,i}\mathsf{D}_{i}\Phi_{p-1,i} - A_{02,i}\mathsf{D}_{i}^{2}\Phi_{p-1,i}$$

- $A_{10,i}\Phi_{p-2,i} - A_{11,i}\mathsf{D}_{i}\Phi_{p-2,i} - A_{12,i}\mathsf{D}_{i}^{2}\Phi_{p-2,i} - A_{13,i}\mathsf{D}_{i}^{3}\Phi_{p-2,i} = 0$,

with $A_{jl,i} \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[L_i, f_2(L_i)^{-1}]$. We give the exact values of $A_{jl,i}$ for reader's convinience.

$$A_{00,i} = \frac{s_1}{9(s_1L_i - s_2)^5} \left(s_1 s_2^3 + (-4s_1^2 s_2^2 + 3s_2^3) L_i + (-s_1^3 s_2 + 12s_1 s_2^2) L_i^2 + (11s_1^4 - 36s_1^2 s_2) L_i^3 \right),$$

$$\begin{split} A_{01,i} &= \frac{-s_1}{3(s_1L_i - s_2)^4} \Big(s_2^3 - 4(s_1s_2^2)L_i + (3s_1^2s_2 + 9s_2^2)L_i^2 \\ &\quad + (3s_1^3 - 21s_1s_2)L_i^3 + 3s_1^2L_i^4 \Big) \,, \end{split}$$
$$\begin{aligned} A_{02,i} &= \frac{-1}{3(s_1L_i - s_2)^3} \Big(s_2^3 - 5(s_1s_2^2)L_i + 9s_1^2s_2L_i^2 + (-6s_1^3 - 3s_1s_2)L_i^3 \\ &\quad + 6s_1^2L_i^4 \Big) \,, \end{split}$$

$$\begin{split} A_{10,i} &= \frac{s_1^2 L_i}{27(s_1 L_i - s_2)^9} \Big((8s_1^2 s_2^5 - 21s_2^6) + (-48s_1^3 s_2^4 + 126s_1 s_2^5) L_i + (120s_1^4 s_2^3 \\ &\quad - 315s_1^2 s_2^4) L_i^2 + (-124s_1^5 s_2^2 + 264s_1^3 s_2^3 + 144s_1 s_2^4) L_i^3 \\ &\quad + (12s_1^6 s_2 + 153s_1^4 s_2^2 - 432s_1^2 s_2^3) L_i^4 + (60s_1^7 - 342s_1^5 s_2 \\ &\quad + 432s_1^3 s_2^2) L_i^5 + (-33s_1^6 + 108s_1^4 s_2) L_i^6 \Big) \,, \end{split}$$

$$\begin{split} A_{11,i} &= \frac{-s_1 L_i}{27(s_1 L_i - s_2)^8} \Big((8s_1^2 s_2^5 - 21s_2^6) + (-48s_1^3 s_2^4 + 126s_1 s_2^5) L_i \\ &+ (120s_1^4 s_2^3 - 315s_1^2 s_2^4) L_i^2 + (-124s_1^5 s_2^2 + 264s_1^3 s_2^3 + 144s_1 s_2^4) L_i^3 \\ &+ (12s_1^6 s_2 + 153s_1^4 s_2^2 - 432s_1^2 s_2^3) L_i^4 + (60s_1^7 - 342s_1^5 s_2 \\ &+ 432s_1^3 s_2^2) L_i^5 + (-33s_1^6 + 108s_1^4 s_2) L_i^6 \Big) \end{split}$$

$$\begin{split} A_{12,i} &= \frac{s_1}{9(s_1L_i - s_2)^7} \left(-s_2^6 + 9s_1s_2^5L_i + (-32s_1^2s_2^4 - 9s_2^5)L_i^2 \right. \\ &\quad + (57s_1^3s_2^3 + 60s_1s_2^4)L_i^3 + (-48s_1^4s_2^2 - 171s_1^2s_2^3)L_i^4 \\ &\quad + (9s_1^5s_2 + 237s_1^3s_2^2 + 27s_1s_2^3)L_i^5 + (9s_1^6 - 144s_1^4s_2 - 90s_1^2s_2^2)L_i^6 \\ &\quad + (9s_1^5 + 108s_1^3s_2)L_i^7 - 18s_1^4L_i^8 \right), \end{split}$$

$$A_{13,i} = -\frac{(3L_i^2 s_1^2 - 3L_i s_1 s_2 + s_2^2)(-3L_i^3 s_1 + 3L_i^2 s_1^2 - 3L_i s_1 s_2 + s_2^2)^2}{27(s_1 L_i - s_2)^6}.$$

Here s_k is the k-th elementary symmetric functions in $\lambda_0, \lambda_1, \lambda_2$. Since the differential equations (74) satisfy the condition (67), we conclude that the differential equations (74) is admissible.

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