EQUIVARIANT HOLOMORPHIC ANOMALY EQUATION

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ABSTRACT. In [\[16\]](#page-56-0) the fundamental relationship between stable quotient invariants and the B-model for local \mathbb{P}^2 in all genera was studied under some specialization of equivariant variables. We generalize the argument of [\[16\]](#page-56-0) to full equivariant settings without the specialization. Our main results are the proof of holomorphic anomaly equations for the equivariant Gromov-Witten theories of local \mathbb{P}^2 and local \mathbb{P}^3 . We also state the generalization to full equivariant formal quintic theory of the result in [\[17\]](#page-56-1).

CONTENTS

0. INTRODUCTION

0.1. Equivariant local \mathbb{P}^n theories. Equivariant local \mathbb{P}^n theories can be constructed as follows. Let the algebraic torus

$$
\mathsf{T}_{n+1} = (\mathbb{C}^*)^{n+1}
$$

act with the standard linearization on \mathbb{P}^n with weights $\lambda_0, \ldots, \lambda_n$ on the vector space $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Let $\overline{M}_g(\mathbb{P}^n, d)$ be the moduli space of stable maps to \mathbb{P}^n equipped with the canonical T_{n+1} -action, and let

 $\mathsf{C}\to \overline{M}_g({\mathbb P}^n,d)\,,\,\, f:\mathcal{C}\to {\mathbb P}^n\,,\,\, \mathsf{S}=f^*\mathcal{O}_{{\mathbb P}^n}(-1)\to \mathsf{C}$

.

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be the standard universal structures.

The equivariant Gromov-Witten invariants of the local \mathbb{P}^n are defined via the equivariant integrals

(1)
$$
N_{g,d}^{\mathsf{GW}} = \int_{[\overline{M}_g(\mathbb{P}^n,d)]^{\text{vir}}} e\left(-R\pi_*f^*\mathcal{O}_{\mathbb{P}^n}(-n-1)\right).
$$

The integral [\(1\)](#page-1-0) defines a rational function in λ_i

$$
N_{g,d}^{\mathsf{GW}} \in \mathbb{Q}(\lambda_0,\ldots,\lambda_n).
$$

Over the moduli space of stable quotients, there is a universal curve

(2)
$$
\pi: \mathcal{C} \to \overline{Q}_g(\mathbb{P}^n, d)
$$

with a universal quotient

$$
0 \longrightarrow \mathsf{S} \longrightarrow \mathbb{C}^N \otimes \mathcal{O}_{\mathcal{C}} \xrightarrow{q_U} \mathsf{Q} \longrightarrow 0.
$$

The equivariant stable quotient invariants of the local \mathbb{P}^n are defined via the equivariant integrals

(3)
$$
N_{g,d}^{\mathsf{SQ}} = \int_{[\overline{Q}_g(\mathbb{P}^n,d)]^{\text{vir}}} e\Big(-R\pi_*\mathsf{S}\Big).
$$

The integral [\(3\)](#page-1-1) also defines a rational function in λ_i

$$
N_{g,d}^{\mathsf{SQ}} \in \mathbb{Q}(\lambda_0,\ldots,\lambda_n)\,.
$$

We refer the reader to [\[16,](#page-56-0) Section 1] for a more leisurely treatment of stable quotients.

In [\[16\]](#page-56-0) it was observed that the analysis of I-function in [\[21\]](#page-56-3) plays important role in the study of local \mathbb{P}^n theories. But the result in [\[21\]](#page-56-3) holds only after the specialization to $(n + 1)$ -th root of unity ζ_{n+1} ,

$$
\lambda_i = \zeta_{n+1}^i \, .
$$

In order to generalize the results in [\[16\]](#page-56-0) to full equivariant theories, one needs the analogous generalization of the results in [\[21\]](#page-56-3) to full equivariant settings. This will be studied in Appendix.

0.2. Holomorphic anomaly for $K\mathbb{P}^2$. We state the precise form of the holomorphic anomaly equations for local \mathbb{P}^2 . Denote by $K\mathbb{P}^2$ the total space of the canonical bundle over \mathbb{P}^2 . Let $H \in H^2(K\mathbb{P}^2,\mathbb{Q})$ be the hyperplane class obtained from \mathbb{P}^2 , and let

$$
\mathcal{F}_{g,m}^{\mathsf{GW}}(Q) = \langle H, \dots, H \rangle_{g,m}^{\mathsf{GW}} = \sum_{d=0}^{\infty} Q^d \int_{[\overline{M}_{g,m}(K\mathbb{P}^2,d)]^{\text{vir}}} \prod_{i=1}^m \text{ev}_i^*(H),
$$

$$
\mathcal{F}_{g,m}^{\mathsf{SQ}}(q) = \langle H, \dots, H \rangle_{g,m}^{\mathsf{SQ}} = \sum_{d=0}^{\infty} Q^d \int_{[\overline{Q}_{g,m}(K\mathbb{P}^2,d)]^{\text{vir}}} \prod_{i=1}^m \text{ev}_i^*(H).
$$

be the Gromov-Witten and stable quotient series respectively (involving the evaluation morphisms at the markings). The relationship between the Gromov-Witten and stable quotient invariants of $K\mathbb{P}^2$ is proven in [\[7\]](#page-56-4) in case $2g - 2 + n > 0$:

(4)
$$
\mathcal{F}_{g,m}^{\mathsf{GW}}(Q(q)) = \mathcal{F}_{g,m}^{\mathsf{SQ}}(q) ,
$$

where $Q(q)$ is the mirror map,

$$
I_1^{K\mathbb{P}^2}(q) = \log(q) + 3\sum_{d=1}^{\infty} (-q)^d \frac{(3d-1)!}{(d!)^3},
$$

$$
Q(q) = \exp\left(I_1^{K\mathbb{P}^2}(q)\right) = q \cdot \exp\left(3\sum_{d=1}^{\infty} (-q)^d \frac{(3d-1)!}{(d!)^3}\right)
$$

To state the holomorphic anomaly equations, we need the following additional series in q.

$$
L(q) = (1 + 27q)^{-\frac{1}{3}} = 1 - 9q + 162q^{2} + \dots,
$$

\n
$$
C_1(q) = q \frac{d}{dq} I_1^{KP^2},
$$

\n
$$
A_2(q) = \frac{1}{L^3} \left(3 \frac{q \frac{d}{dq} C_1}{C_1} + 1 - \frac{L^3}{2} \right).
$$

We also need new series $L_i(q)$ defined by roots of following degree 3 polynomial in $\mathcal L$ for $i = 0, 1, 2$:

$$
(1+27q)\mathcal{L}^3-(\lambda_0+\lambda_1+\lambda_2)\mathcal{L}^2+(\lambda_0\lambda_1+\lambda_1\lambda_2+\lambda_2\lambda_0)\mathcal{L}-\lambda_0\lambda_2\lambda_3,
$$

with initial conditions,

$$
L_i(0)=\lambda_i.
$$

Let f_2 be the polynomial of degree 2 in variable x over $\mathbb{C}(\lambda_0, \lambda_1, \lambda_2)$ defined by

$$
f_2(x) := (\lambda_0 + \lambda_1 + \lambda_2)x^2 - 2(\lambda_0\lambda_1 + \lambda_1\lambda_2 + \lambda_2\lambda_0)x + 3\lambda_0\lambda_1\lambda_2.
$$

.

The ring

$$
\mathbb{G}_2 := \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[L_0^{\pm 1}, L_1^{\pm 1}, L_2^{\pm 1}, f(L_0)^{-\frac{1}{2}}, f(L_1)^{-\frac{1}{2}}, f(L_2)^{-\frac{1}{2}}]
$$

will play a basic role in our paper. Consider the free polynomial rings in the variables A_2 and C_1^{-1} over \mathbb{G}_2 ,

(5)
$$
\mathbb{G}_2[A_2], \mathbb{G}_2[A_2, C_1^{-1}].
$$

We have canonical maps

(6)
$$
\mathbb{G}_2[A_2] \to \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]] , \mathbb{G}_2[A_2, C_1^{-1}] \to \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]]
$$

given by assigning the above defined series $A_2(q)$ and C_1^{-1} to the variables A_2 and C_1^{-1} respectively. Therefore we can consider elements of the rings [\(5\)](#page-3-0) either as free polynomials in the variables A_2 and C_1^{-1} or as series in q.

Let $F(q) \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]]$ be a series in q. When we write

$$
F(q) \in \mathbb{G}_2[A_2],
$$

we mean there is a cononical lift $F \in \mathbb{G}_2[A_2]$ for which

$$
F \to F(q) \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]]
$$

under the map (6) . The symbol F without the argument q is the canonical lift. The notation

$$
F(q) \in \mathbb{G}_2[A_2, C_1^{-1}]
$$

is parallel.

Let T be the standard coordinate mirror to $t = \log(q)$,

$$
T = I_1^{K\mathbb{P}^2}(q) \, .
$$

Then $Q(q) = \exp(T)$ is the mirror map.

Conjecture 1. For the stable quotient invariants of $K\mathbb{P}^2$,

- (i) $\mathcal{F}_g^{SQ}(q) \in \mathbb{G}_2[A_2]$ for $g \geq 2$,
- (ii) $\mathcal{F}_g^{\mathsf{SQ}}$ is of degree at most 3g 3 with respect to A_2 ,
- (iii) $\frac{\partial^k \mathcal{F}^{\mathsf{SQ}}}{\partial T^k} (q) \in \mathbb{G}_2[A_2, C_1^{-1}]$ for $g \geq 1$ and $k \geq 1$,
- (iv) $\frac{\partial^k \mathcal{F}^{\mathsf{SQ}_g}}{\partial T^k}$ is homogeneous of degree k with respect to C_1^{-1} .

Here, $\mathcal{F}_g^{\mathsf{SQ}} = \mathcal{F}_{g,0}^{\mathsf{SQ}}$ $g,0$.

Conjecture 2. The holomorphic anomaly equations for the stable quotient invariants of $K\mathbb{P}^2$ hold for $g \geq 2$:

$$
\frac{1}{C_1^2}\frac{\partial \mathcal{F}^{\mathsf{SQ}}_g}{\partial A_2}=\frac{1}{2}\sum_{i=1}^{g-1}\frac{\partial \mathcal{F}^{\mathsf{SQ}}_{g-i}}{\partial T}\frac{\partial \mathcal{F}^{\mathsf{SQ}}_i}{\partial T}+\frac{1}{2}\frac{\partial^2 \mathcal{F}^{\mathsf{SQ}}_{g-1}}{\partial T^2}\,.
$$

The derivative of $\mathcal{F}^{\mathsf{SQ}}_g$ with respect to A_2 in the above equation is well-defined since

$$
\mathcal{F}_g^{\mathsf{SQ}} \in \mathbb{G}_2[A_2]
$$

by part (i) of Conjecture [2.](#page-3-2) By parts (ii) and (iii),

$$
\frac{\partial \mathcal{F}_{g-i}^{\mathsf{SQ}}}{\partial T} \frac{\partial \mathcal{F}_{i}^{\mathsf{SQ}}}{\partial T} \ , \ \frac{\partial^2 \mathcal{F}_{g-1}^{\mathsf{SQ}}}{\partial T^2} \in \mathbb{G}_2[A_2, C_1^{-1}]
$$

are both of degree 2 in C_1^{-1} . Hence, the holomorphic anomaly equation of Conjecture [2](#page-3-2) may be viewed as holding in $\mathbb{G}[A_2]$ since the factors of C_1^{-1} on both sides cancel. If we use the specializations by primitive third root of unity ζ

$$
\lambda_i=\zeta^i\,,
$$

the holomorphic anomaly equations here for $K\mathbb{P}^2$ recover the precise form presented in [\[1,](#page-56-5) (4.27)] via B-model physics.

Conjecture [2](#page-3-2) determine $\mathcal{F}^{\mathsf{SQ}}_g \in \mathbb{G}_2[A_2]$ uniquely as a polynomial in A_2 up to a constant term in \mathbb{G}_2 . The degree of the constant term can be bounded. Therefore Conjecture [2](#page-3-2) determine $\mathcal{F}^{\mathsf{SQ}}_g$ from the lower genus theory together with a finite amount of date.

We will prove the following special cases of the conjectures in Section [6.](#page-35-0)

Theorem 3. Conjecture [1](#page-3-3) holds for the choices of $\lambda_0, \lambda_1, \lambda_2$ such that

$$
\lambda_i \neq \lambda_j \text{ for } i \neq j,
$$

$$
(\lambda_0 \lambda_1 + \lambda_1 \lambda_2 + \lambda_2 \lambda_0)^2 - 3\lambda_0 \lambda_1 \lambda_2 (\lambda_0 + \lambda_1 + \lambda_2) = 0.
$$

Theorem 4. Conjecture [2](#page-3-2) holds for the choices of $\lambda_0, \lambda_1, \lambda_2$ such that

$$
\lambda_i \neq \lambda_j \text{ for } i \neq j,
$$

$$
(\lambda_0 \lambda_1 + \lambda_1 \lambda_2 + \lambda_2 \lambda_0)^2 - 3\lambda_0 \lambda_1 \lambda_2 (\lambda_0 + \lambda_1 + \lambda_2) = 0.
$$

0.3. Holomorphic anomaly equations for $K\mathbb{P}^3$. We state the precise form of the holomorphic anomaly equations for local \mathbb{P}^3 . Since the study of local \mathbb{P}^3 will be parallel to the study of local \mathbb{P}^2 , we will sometime use the same notations for local \mathbb{P}^2 and local \mathbb{P}^3 . Since the study of two theories are logically independent in our paper, the indication of each notation will be clear from the context. Denote by $K\mathbb{P}^3$ the total space of the canonical bundle over \mathbb{P}^3 . Let $H \in H^2(K\mathbb{P}^3,\mathbb{Q})$ be the hyperplane class obtained from \mathbb{P}^3 , and let

$$
\mathcal{F}_{g,m}^{\mathsf{GW}}[a,b](Q) = \langle \tau_0(H)^a \tau_0(H^2)^b \rangle_{g,m}^{\mathsf{GW}} = \sum_{d=0}^{\infty} Q^d \int_{[\overline{M}_{g,m}(K\mathbb{P}^3,d)]^{\text{vir}}} \prod_{i=1}^a \text{ev}_i^*(H) \prod_{i=a+1}^{a+b} \text{ev}_i^*(H^2),
$$

$$
\mathcal{F}_{g,m}^{\mathsf{SQ}}[a,b](q) = \langle \tau_0(H)^a \tau_0(H^2)^b \rangle_{g,m}^{\mathsf{SQ}} = \sum_{d=0}^{\infty} q^d \int_{[\overline{Q}_{g,m}(K\mathbb{P}^3,d)]^{\text{vir}}} \prod_{i=1}^a \text{ev}_i^*(H) \prod_{i=a+1}^{a+b} \text{ev}_i^*(H^2).
$$

be the Gromov-Witten and stable quotient series respectively with $a+$ $b = m$ (involving the evaluation morphisms at the markings). The relationship between the Gromov-Witten and stable quotient invariants of $K\mathbb{P}^3$ is proven in [\[7\]](#page-56-4) in case $2g - 2 + n > 0$:

(7)
$$
\mathcal{F}_{g,m}^{\text{GW}}[a,b](Q(q)) = \mathcal{F}_{g,m}^{\text{SQ}}[a,b](q) ,
$$

where $Q(q)$ is the mirror map,

$$
I_1^{K\mathbb{P}^3}(q) = \log(q) + 4 \sum_{d=1}^{\infty} q^d \frac{(4d-1)!}{(d!)^4},
$$

$$
Q(q) = \exp\left(I_1^{K\mathbb{P}^3}(q)\right) = q \cdot \exp\left(4 \sum_{d=1}^{\infty} q^d \frac{(4d-1)!}{(d!)^4}\right).
$$

To state the holomorphic anomaly equations, we need the following additional series in q.

$$
L(q) = (1 - 4^4 q)^{-\frac{1}{4}} = 1 + 64q + 10240q^2 + \dots,
$$

\n
$$
C_1(q) = q \frac{d}{dq} I_1^{K\mathbb{P}^3},
$$

\n
$$
A_2(q) = \frac{q \frac{d}{dq} C_1}{C_1}.
$$

We will define extra series $B_2(q)$, $B_4(q) \in \mathbb{C}[[q]]$ in [\(56\)](#page-43-0). We also need new series $L_i(q)$ defined by roots of following degree 4 polynomial in $\mathcal L$ for $i = 0, 1, 2, 3$:

$$
(1-4^4q)\mathcal{L}^4 - s_1\mathcal{L}^3 + s_2\mathcal{L}^2 - s_3\mathcal{L} + s_4,
$$

with initial conditions,

$$
L_i(0)=\lambda_i.
$$

Here, s_k is k-th elementary symmetric function in $\lambda_0, \ldots, \lambda_3$. Let f_3 be the polynomial of degree 3 in variable x over $\mathbb{C}(\lambda_0, \ldots, \lambda_3)$ defined by

$$
f_3(x) := s_1 x^3 - 2s_2 x^2 + 3s_3 x - 4s_4.
$$

The ring

$$
\mathbb{G}_3 := \mathbb{C}(\lambda_0, \dots, \lambda_3)[L_0^{\pm 1}, \dots, L_3^{\pm 1}, f(L_0)^{-\frac{1}{2}}, \dots, f(L_3)^{-\frac{1}{2}}]
$$

will play a basic role in our paper. Consider the free polynomial rings in the variables A_2 , B_2 , B_4 and C_1^{-1} over \mathbb{G}_3 ,

(8)
$$
\mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}].
$$

We have canonical map

(9)
$$
\mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}] \to \mathbb{C}(\lambda_0, \dots, \lambda_3)[[q]]
$$

given by assigning the above defined series $A_2(q)$, $B_2(q)$, $B_4(q)$ and $C_1(q)$ to the variables A_2 , B_2 , B_4 and C_1 respectively. Therefore we can consider elements of the rings [\(8\)](#page-6-0) either as free polynomials in the variables A_2 , B_2 , B_4 and C_1 or as series in q.

Let $F(q) \in \mathbb{C}(\lambda_0, \ldots, \lambda_3)[[q]]$ be a series in q. When we write

$$
F(q) \in G_3[A_2, B_2, B_4, C_1^{\pm 1}],
$$

we mean there is a cononical lift $F \in \mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}]$ for which

$$
F \to F(q) \in \mathbb{C}(\lambda_0, \dots, \lambda_2)[[q]]
$$

under the map [\(9\)](#page-6-1). The symbol F without the argument q is the canonical lift. The notation

$$
F(q) \in \mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}]
$$

is parallel.

Let T be the standard coordinate mirror to $t = \log(q)$,

$$
T = I_1^{K\mathbb{P}^3}(q) .
$$

Then $Q(q) = \exp(T)$ is the mirror map.

Conjecture 5. For the stable quotient invariants of $K\mathbb{P}^3$,

- (i) \mathcal{F}_{aa}^{SQ} $g_{g,a+b}^{SQ}[a,b](q) \in \mathbb{G}_3[A_2,B_2,B_4,C_1^{\pm 1}]$ for $g \geq 2$,
- (ii) $\mathcal{F}_g^{\mathsf{SQ}}$ is of degree at most $2(3g-3)$ with respect to A_2 ,
- (iii) $\frac{\partial^k \mathcal{F}^{\mathsf{SQ}}}{\partial T^k}(q) \in \mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}]$ for $g \ge 1$ and $k \ge 1$.

Here, $\mathcal{F}_g^{\mathsf{SQ}} = \mathcal{F}_{g,0}^{\mathsf{SQ}}$ $g_{,0}^{SQ}[0,0].$

Conjecture 6. The holomorphic anomaly equations for the stable quotient invariants of $K\mathbb{P}^3$ hold for $g \geq 2$:

$$
\frac{L^2}{4C_1} \frac{\partial \mathcal{F}_g^{\mathsf{SQ}}}{\partial A_2} + \frac{-2s_1 L^4 - C_1 (3B_2 L^2 - s_1 L^6)}{4C_1^2} \frac{\partial \mathcal{F}_g^{\mathsf{SQ}}}{\partial B_4} = \frac{\sum_{i=1}^{g-1} \mathcal{F}_{g-i,1}^{\mathsf{SQ}}[0,1] \mathcal{F}_{i,1}^{\mathsf{SQ}}[1,0] + \mathcal{F}_{g-1,1}^{\mathsf{SQ}}[1,1],
$$

$$
\frac{2L^4}{C_1^2(L^4 - 1 - 2A_2)} \frac{\partial \mathcal{F}_g^{\mathsf{SQ}}}{\partial B_2} = \sum_{i=1}^{g-1} \frac{\partial \mathcal{F}_{g-i}^{\mathsf{SQ}}}{\partial T} \frac{\partial \mathcal{F}_i^{\mathsf{SQ}}}{\partial T} + \frac{\partial^2 \mathcal{F}_{g-1}^{\mathsf{SQ}}}{\partial T^2}.
$$

The derivative of $\mathcal{F}_g^{\mathsf{SQ}}$ with respect to A_2 , B_2 and B_4 in the above equations is well-defined since

$$
\mathcal{F}_g^{\mathsf{SQ}} \in \mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}]
$$

by part (i) of Conjecture [6.](#page-6-2)

We will prove the following special cases of the conjectures in Section [6.](#page-35-0)

Theorem 7. Conjecture [5](#page-6-3) holds for the choices of $\lambda_0, \ldots, \lambda_3$ such that

$$
\lambda_i \neq \lambda_j \text{ for } i \neq j,
$$

\n $4s_2^2 - s_1s_3 = 0,$
\n $2s_2^3 - 27s_1^2s_4 = 0.$

Theorem 8. Conjecture [6](#page-6-2) holds for the choices of $\lambda_0, \ldots, \lambda_3$ such that

$$
\lambda_i \neq \lambda_j \text{ for } i \neq j,
$$

\n $4s_2^2 - s_1s_3 = 0,$
\n $2s_2^3 - 27s_1^2s_4 = 0.$

For Calabi-Yau 3-folds, holomorphic anomaly equations were first discovered in physics([\[2\]](#page-56-6)). Also there were many studies to understand holomorphic anomaly equations mathematically ([\[12,](#page-56-7) [16,](#page-56-0) [17,](#page-56-1) [19\]](#page-56-8)). But less is known for higher dimensional Calabi-Yau manifolds in physics. It might be interesting question to find physical arguments for the holomorphic anomaly equation for $K\mathbb{P}^3$ proposed in our paper.

0.4. Holomorphic anomaly for equivariant formal quintic invariants. A particular twisted theory on \mathbb{P}^4 related to the quintic 3fold was introduced in [\[17\]](#page-56-1). Let the algebraic torus

$$
\mathsf{T} = (\mathbb{C}^*)^5
$$

act with the standard linearization on \mathbb{P}^4 with weights $\lambda_0, \ldots, \lambda_4$ on the vector space $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)).$

Let

(10)
$$
\mathbf{C} \to \overline{M}_g(\mathbb{P}^4, d)
$$
, $f : \mathbf{C} \to \mathbb{P}^4$, $\mathbf{S} = f^* \mathcal{O}_{\mathbb{P}^4}(-1) \to \mathbf{C}$

be the universal curve, the universal map, and the universal bundle over the moduli space of stable maps — all equipped with canonical

T-actions. We define the *formal quintic invariants* by^{[1](#page-8-0)}

(11)
$$
\widetilde{N}_{g,d}^{\text{GW}} = \int_{[\overline{M}_g(\mathbb{P}^4,d)]^{vir}} e(R\pi_*(\mathsf{S}^{-5})),
$$

where $e(R\pi_*(S^{-5}))$ is the equivariant Euler class defined *after* localization. More precisely, on each T-fixed locus of $\overline{M}_g(\mathbb{P}^4, d)$, both

 $R^0\pi_*(\mathsf{S}^{-5})$ and $R^1\pi_*(\mathsf{S}^{-5})$

are vector bundles with moving weights, so

$$
e(R\pi_*(\mathsf{S}^{-5})) = \frac{c_{\text{top}}(R^0\pi_*(\mathsf{S}^{-5}))}{c_{\text{top}}(R^1\pi_*(\mathsf{S}^{-5}))}
$$

is well-defined. The integral [\(11\)](#page-8-1) is homogeneous of degree 0 in localized equivariant cohomology and defines a rational function in λ_i ,

$$
\widetilde{N}_{g,d}^{\mathsf{GW}} \in \mathbb{C}(\lambda_0,\ldots,\lambda_4).
$$

Let $g \geq 2$. The associated Gromov-Witten generating series is

$$
\widetilde{\mathcal{F}}_g^{\mathsf{GW}}(Q) = \sum_{d=0}^{\infty} \widetilde{N}_{g,d}^{\mathsf{GW}} Q^d \in \mathbb{C}[[Q]]\,.
$$

Let

$$
I_0^{\mathsf{Q}}(q) = \sum_{d=0}^{\infty} q^d \frac{(5d)!}{(d!)^5}, \quad I_1^{\mathsf{Q}}(q) = \log(q) I_0^{\mathsf{Q}}(q) + 5 \sum_{d=1}^{\infty} q^d \frac{(5d)!}{(d!)^5} \left(\sum_{r=d+1}^{5d} \frac{1}{r} \right).
$$

We *define* the generating series of stable quotient invariants for formal quintic theory by the wall-crossing formula for the true quintic theory which has been recently proven by Ciocan-Fontanine and Kim in [\[6\]](#page-56-9),

(12)
$$
\widetilde{\mathcal{F}}_g^{\mathsf{GW}}(Q(q)) = I_0^{\mathsf{Q}}(q)^{2g-2} \cdot \widetilde{\mathcal{F}}_g^{\mathsf{SQ}}(q)
$$

with respect to the true quintic mirror map

$$
Q(q) = \exp\left(\frac{I_1^{\mathsf{Q}}(q)}{I_0^{\mathsf{Q}}(q)}\right) = q \cdot \exp\left(\frac{5 \sum_{d=1}^{\infty} q^d \frac{(5d)!}{(d!)^5} \left(\sum_{r=d+1}^{5d} \frac{1}{r}\right)}{\sum_{d=0}^{\infty} \frac{(5d)!}{(d!)^5}}\right).
$$

Denote the B-model side of [\(12\)](#page-8-2) by

$$
\widetilde{F}_g^{\mathsf{B}}(q) = I_0^{\mathsf{Q}}(q)^{2g-2} \widetilde{\mathcal{F}}_g^{\mathsf{SQ}}(q) .
$$

In order to state the holomorphic anomaly equations, we require several series in q . First, let

$$
L(q) = (1 - 5^5 q)^{-\frac{1}{5}} = 1 + 625q + 117185q^2 + \dots
$$

¹The negative exponent denotes the dual: **S** is a line bundle and $S^{-5} = (S^*)^{\otimes 5}$.

Let $D = q \frac{d}{dq}$, and let

$$
C_0(q) = I_0^{\mathsf{Q}}, \quad C_1(q) = \mathsf{D}\left(\frac{I_1^{\mathsf{Q}}}{I_0^{\mathsf{Q}}}\right).
$$

We define

$$
K_{2}(q) = -\frac{1}{L^{5}} \frac{DC_{0}}{C_{0}},
$$

\n
$$
A_{2}(q) = \frac{1}{L^{5}} \left(-\frac{1}{5} \frac{DC_{1}}{C_{1}} - \frac{2}{5} \frac{DC_{0}}{C_{0}} - \frac{3}{25} \right),
$$

\n
$$
A_{4}(q) = \frac{1}{L^{10}} \left(-\frac{1}{25} \left(\frac{DC_{0}}{C_{0}} \right)^{2} - \frac{1}{25} \left(\frac{DC_{0}}{C_{0}} \right) \left(\frac{DC_{1}}{C_{1}} \right) \right.
$$

\n
$$
+ \frac{1}{25} D \left(\frac{DC_{0}}{C_{0}} \right) + \frac{2}{25^{2}} \right),
$$

\n
$$
A_{6}(q) = \frac{1}{31250L^{15}} \left(4 + 125D \left(\frac{DC_{0}}{C_{0}} \right) + 50 \left(\frac{DC_{0}}{C_{0}} \right) \left(1 + 10D \left(\frac{DC_{0}}{C_{0}} \right) \right) \right.
$$

\n
$$
-5L^{5} \left(1 + 10 \left(\frac{DC_{0}}{C_{0}} \right) + 25 \left(\frac{DC_{0}}{C_{0}} \right)^{2} + 25D \left(\frac{q \frac{d}{dq} C_{0}}{C_{0}} \right) \right)
$$

\n
$$
+125D^{2} \left(\frac{DC_{0}}{C_{0}} \right) - 125 \left(\frac{DC_{0}}{C_{0}} \right)^{2} \left(\left(\frac{DC_{1}}{C_{1}} \right) - 1 \right) \right).
$$

For the full equivariant formal quintic theory, we need extra series $B_1, B_2, B_3, B_4 \in \mathbb{C}[[q]]$ obtained from *I*-function of quintic. We will give the exact definitions of these extra series in forthcoming paper [\[15\]](#page-56-10). These series are closely related to the extra generators in [\[12\]](#page-56-7), where the formal quintic theory was studied in genus 2 case with connections to real quintic theory.

We also need new series $L_i(q)$ defined by roots of following degree 5 polynomial in $\mathcal L$ for $i = 0, \ldots, 4$:

$$
(1-5^5q)\mathcal{L}^5 - s_1\mathcal{L}^4 + s_2\mathcal{L}^3 - s_3\mathcal{L}^2 + s_4\mathcal{L} - s_5,
$$

with initial conditions,

$$
L_i(0)=\lambda_i.
$$

Here s_k is k-th elementary symmetric function in $\lambda_0, \ldots, \lambda_4$. Let f_4 be the polynomial of degree 4 in variable x over $\mathbb{C}(\lambda_0, \ldots, \lambda_4)$ defined by

$$
f_4(x) := s_1 x^4 - 2s_2 x^3 + 3s_3 x^2 - 4s_3 x + 5s_4.
$$

The ring

$$
\mathbb{G}_{\mathsf{Q}} := \mathbb{C}(\lambda_0, \ldots, \lambda_4)[L_0^{\pm 1}, \ldots, L_4^{\pm 1}, f_4(L_0)^{-\frac{1}{2}}, \ldots, f_4(L_4)^{-\frac{1}{2}}]
$$

will play a basic role in formal quintic theory.

Let T be the standard coordinate mirror to $t = \log(q)$,

$$
T = \frac{I_1^{\mathsf{Q}}(q)}{I_0^{\mathsf{Q}}(q)}.
$$

Then $Q(q) = \exp(T)$ is the mirror map. Let

$$
\mathbb{G}_{\mathbb{Q}}[A_2, A_4, A_6, B_1, B_2, B_3, B_4, C_0^{\pm 1}, C_1^{-1}, K_2]
$$

be the free polynomial ring over \mathbb{G}_{Q} .

Conjecture 9. For the series $\mathcal{F}_{g}^{\mathbf{B}}$ associated to the formal quintic,

- (i) $\widetilde{\mathcal{F}}_g^{\mathsf{B}}(q) \in \mathbb{G}_{\mathsf{Q}}[A_2, A_4, A_6, B_1, \ldots, B_4, C_0^{\pm 1}, C_1^{-1}, K_2]$ for $g \geq 2$, (ii) $\frac{\partial^k \tilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial T^k}(q) \in \mathbb{G}_{\mathsf{Q}}[A_2, A_4, A_6, B_1, \ldots, B_4, C_0^{\pm 1}, C_1^{-1}, K_2] \text{ for } g \geq 1,$ $k \geq 1$,
- (iii) $\frac{\partial^k \tilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial T^k}$ is homogeneous with respect to C_1^{-1} of degree k.

Conjecture 10. The series $\tilde{\mathcal{F}}_g^{\mathsf{B}}$ associated to the formal quintic satisfy some holomorphic anomaly equations which specialize to

$$
\begin{split} \frac{1}{C_0^2C_1^2}\frac{\partial \widetilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial A_2}-\frac{1}{5C_0^2C_1^2}\frac{\partial \widetilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial A_4}K_2+\frac{1}{50C_0^2C_1^2}\frac{\partial \widetilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial A_6}K_2^2=\\ \frac{1}{2}\sum_{i=1}^{g-1}\frac{\partial \widetilde{\mathcal{F}}_{g-i}^{\mathsf{B}}}{\partial T}\frac{\partial \widetilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial T}+\frac{1}{2}\frac{\partial^2 \widetilde{\mathcal{F}}_{g-1}^{\mathsf{B}}}{\partial T^2}\,,\\ \frac{\partial \widetilde{\mathcal{F}}_g^{\mathsf{B}}}{\partial K_2}=0\,, \end{split}
$$

with the choice of $\lambda_i = \zeta_5^i$.

We expect holomorphic anomaly equations in Conjecture [10](#page-10-0) to hold in the ring

(13) $G_Q[A_2, A_4, A_6, B_1, B_2, B_3, B_4, C_0^{\pm 1}, C_1^{-1}, K_2].$

Remark 11. If we specialize λ_i to (the power of) primitive fifth root of unity ζ_5 ,

$$
\lambda_i = \zeta_5^i \,,
$$

the expected equations in Conjecture 10 exactly matches^{[2](#page-11-1)} the conjectural holomorphic anomaly equation [\[1,](#page-56-5) (2.52)] for the true quintic theory and this was the main result in [\[17\]](#page-56-1). Also the ring [\(13\)](#page-10-1) can be reduced to the Yamaguchi-Yau ring introduced in [\[20\]](#page-56-11) for the true quintic theory only with the choice of specialization [\(14\)](#page-10-2). This explains why the specialization (14) used in [\[17\]](#page-56-1) is the natural choice.

Theorem 12. Conjecture [9](#page-10-3) holds for the choices of $\lambda_0, \ldots, \lambda_4$ such that

$$
\lambda_i \neq \lambda_j \text{ for } i \neq j,
$$

\n
$$
2s_1s_3 = s_2^2,
$$

\n
$$
8s_1^2s_4 = s_2^3,
$$

\n
$$
80s_1^3s_5 = s_2^4.
$$

where s_k is k-th elementary symmetric function in $\lambda_0, \ldots, \lambda_4$.

Theorems [12](#page-11-2) will be proven and the precise form of holomorphic anomaly equations in Conjecture [10](#page-10-0) will appear in [\[15\]](#page-56-10).

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1. Localization Graph

1.1. **Torus action.** Let $T = (\mathbb{C}^*)^{n+1}$ act diagonally on the vector space \mathbb{C}^{n+1} with weights

$$
-\lambda_0,\ldots,-\lambda_n.
$$

Denote the T-fixed points of the induced T-action on \mathbb{P}^n by

$$
p_0,\ldots,p_n.
$$

The weights of T on the tangent space $T_{p_j}(\mathbb{P}^n)$ are

$$
\lambda_j-\lambda_0,\ldots,\widehat{\lambda_j-\lambda_j},\ldots,\lambda_j-\lambda_n.
$$

²Our functions K_2 and A_{2k} are normalized differently with respect to C_0 and C_1 . The dictionary to exactly match the notation of $[1, (2.52)]$ $[1, (2.52)]$ is to multiply our K_2 by $(C_0C_1)^2$ and our A_{2k} by $(C_0C_1)^{2k}$.

There is an induced T-action on the moduli space $\overline{Q}_{g,k}(\mathbb{P}^n,d)$. The localization formula of [\[11\]](#page-56-12) applied to the virtual fundamental class $[\overline{Q}_{g,k}(\mathbb{P}^n,d)]^{vir}$ will play a fundamental role our paper. The T-fixed loci are represented in terms of dual graphs, and the contributions of the T-fixed loci are given by tautological classes. The formulas here are standard, see [\[13,](#page-56-13) [18\]](#page-56-14).

1.2. Graphs. Let the genus g and the number of markings k for the moduli space be in the stable range

(15)
$$
2g - 2 + k > 0.
$$

We can organize the T-fixed loci of $\overline{Q}_{g,k}(\mathbb{P}^n,d)$ according to decorated graphs. A *decorated graph* $\Gamma \in \mathcal{G}_{g,k}(\mathbb{P}^n)$ consists of the data (V, E, N, g, p) where

- (i) V is the vertex set,
- (ii) E is the edge set (including possible self-edges),
- (iii) $N: \{1, 2, ..., k\} \rightarrow V$ is the marking assignment,
- (iv) $g: V \to \mathbb{Z}_{\geq 0}$ is a genus assignment satisfying

$$
g = \sum_{v \in V} \mathsf{g}(v) + h^1(\Gamma)
$$

and for which (V, E, N, g) is stable graph^{[3](#page-12-0)},

(v) $p: V \to (\mathbb{P}^n)^T$ is an assignment of a T-fixed point $p(v)$ to each vertex $v \in V$.

The markings $L = \{1, \ldots, k\}$ are often called *legs*.

To each decorated graph $\Gamma \in \mathsf{G}_{g,k}(\mathbb{P}^n)$, we associate the set of fixed loci of

$$
\sum_{d\geq 0} \left[\overline{Q}_{g,k}(\mathbb{P}^n,d) \right]^{\text{vir}} q^d
$$

with elements described as follows:

- (a) If $\{v_{i_1}, \ldots, v_{i_j}\} = \{v \mid p(v) = p_i\}$, then $f^{-1}(p_i)$ is a disjoint union of connected stable curves of genera $\mathsf{g}(v_{i_1}), \ldots, \mathsf{g}(v_{i_j})$ and finitely many points.
- (b) There is a bijective correspondence between the connected components of $C \setminus D$ and the set of edges and legs of Γ respecting vertex incidence where C is domain curve and D is union of all subcurves of C which appear in (a) .

We write the localization formula as

$$
\sum_{d\geq 0} \left[\overline{Q}_{g,k}(\mathbb{P}^n,d) \right]^{\text{vir}} q^d = \sum_{\Gamma \in \mathsf{G}_{g,k}(\mathbb{P}^n)} \text{Cont}_{\Gamma}.
$$

³Corresponding to a stratum of the moduli space of stable curves $\overline{M}_{g,n}$.

While $\mathsf{G}_{g,k}(\mathbb{P}^n)$ is a finite set, each contribution Cont_{Γ} is a series in q obtained from an infinite sum over all edge possibilities (b).

1.3. Unstable graphs. The moduli spaces of stable quotients

$$
\overline{Q}_{0,2}({\mathbb P}^n,d)\quad\text{and}\quad \overline{Q}_{1,0}({\mathbb P}^n,d)
$$

for $d > 0$ are the only^{[4](#page-13-1)} cases where the pair (g, k) does not satisfy the Deligne-Mumford stability condition [\(15\)](#page-12-1).

An appropriate set of decorated graphs $\mathsf{G}_{0,2}(\mathbb{P}^n)$ is easily defined: The graphs $\Gamma \in \mathsf{G}_{0,2}(\mathbb{P}^n)$ all have 2 vertices connected by a single edge. Each vertex carries a marking. All of the conditions (i)-(v) of Section [1.2](#page-12-2) are satisfied except for the stability of (V, E, N, γ) . The localization formula holds,

(16)
$$
\sum_{d\geq 1} \left[\overline{Q}_{0,2}(\mathbb{P}^n,d) \right]^{\text{vir}} q^d = \sum_{\Gamma \in \mathsf{G}_{0,2}(\mathbb{P}^n)} \text{Cont}_{\Gamma},
$$

For $\overline{Q}_{1,0}(\mathbb{P}^n,d)$, the matter is more problematic — usually a marking is introduced to break the symmetry.

2. Basic correlators

2.1. Overview. We review here basic generating series in q which arise in the genus 0 theory of quasimap invariants. The series will play a fundamental role in the calculations of Sections [3](#page-21-0) - [6](#page-35-0) related to the holomorphic anomaly equation for $K\mathbb{P}^2$.

We fix a torus action $\mathsf{T} = (\mathbb{C}^*)^3$ on \mathbb{P}^2 with weights^{[5](#page-13-2)}

$$
-\lambda_0, -\lambda_1, -\lambda_2
$$

on the vector space \mathbb{C}^3 . The T-weight on the fiber over p_i of the canonical bundle

$$
(17) \t\t \t\t \mathcal{O}_{\mathbb{P}^2}(-3) \to \mathbb{P}^2
$$

is $-3\lambda_i$. The toric Calabi-Yau $K\mathbb{P}^2$ is the total space of [\(17\)](#page-13-3).

⁴The moduli spaces $\overline{Q}_{0,0}(\mathbb{P}^n,d)$ and $\overline{Q}_{0,1}(\mathbb{P}^n,d)$ are empty by the definition of a stable quotient.

⁵The associated weights on $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ are $\lambda_0, \lambda_1, \lambda_2$ and so match the conventions of Section [0.1.](#page-0-1)

2.2. First correlators. We require several correlators defined via the Euler class of the obstruction bundle,

$$
e(\text{Obs}) = e(R^1 \pi_* \mathsf{S}^3),
$$

associated to the $K\mathbb{P}^2$ geometry on the moduli space $\overline{Q}_{g,n}(\mathbb{P}^2,d)$. The first two are obtained from standard stable quotient invariants. For $\gamma_i \in H^*_{\mathsf{T}}(\mathbb{P}^2)$, let

$$
\left\langle \gamma_1 \psi^{a_1}, \dots, \gamma_n \psi^{a_n} \right\rangle_{g,n,d}^{\mathsf{SQ}} = \int_{[\overline{Q}_{g,n}(\mathbb{P}^2,d)]^{\text{vir}}} e(\text{Obs}) \cdot \prod_{i=1}^n \text{ev}_i^*(\gamma_i) \psi_i^{a_i},
$$

$$
\left\langle \left\langle \gamma_1 \psi^{a_1}, \dots, \gamma_n \psi^{a_n} \right\rangle \right\rangle_{0,n}^{\mathsf{SQ}} = \sum_{d \geq 0} \sum_{k \geq 0} \frac{q^d}{k!} \left\langle \gamma_1 \psi^{a_1}, \dots, \gamma_n \psi^{a_n}, t, \dots, t \right\rangle_{0,n+k,d}^{\mathsf{SQ}},
$$

where, in the second series, $t \in H^*_{\mathsf{T}}(\mathbb{P}^2)$. We will systematically use the quasimap notation 0+ for stable quotients,

$$
\left\langle \gamma_1 \psi^{a_1}, \dots, \gamma_n \psi^{a_n} \right\rangle_{g,n,d}^{0+} = \left\langle \gamma_1 \psi^{a_1}, \dots, \gamma_n \psi^{a_n} \right\rangle_{g,n,d}^{\mathsf{SQ}}
$$

$$
\left\langle \left\langle \gamma_1 \psi^{a_1}, \dots, \gamma_n \psi^{a_n} \right\rangle \right\rangle_{0,n}^{0+} = \left\langle \left\langle \gamma_1 \psi^{a_1}, \dots, \gamma_n \psi^{a_n} \right\rangle \right\rangle_{0,n}^{\mathsf{SQ}}.
$$

2.3. Light markings. Moduli of quasimaps can be considered with n ordinary (weight 1) markings and k light (weight ϵ) markings^{[6](#page-14-0)},

$$
\overline{Q}_{g,n|k}^{0+,0+}(\mathbb{P}^2,d)\,.
$$

Let $\gamma_i \in H^*_{\mathsf{T}}(\mathbb{P}^2)$ be equivariant cohomology classes, and let

$$
\delta_j \in H^*_\mathsf{T}([\mathbb{C}^3/\mathbb{C}^*])
$$

be classes on the stack quotient. Following the notation of [\[13\]](#page-56-13), we define series for the $K\mathbb{P}^2$ geometry,

$$
\left\langle \gamma_1 \psi^{a_1}, \dots, \gamma_n \psi^{a_n}; \delta_1, \dots, \delta_k \right\rangle_{g,n|k,d}^{0+,0+} =
$$
\n
$$
\int_{[\overline{Q}_{g,n|k}^{0+,0+} (\mathbb{P}^2,d)]^{\text{vir}}} e(\text{Obs}) \cdot \prod_{i=1}^n \text{ev}_i^*(\gamma_i) \psi_i^{a_i} \cdot \prod_{j=1}^k \widehat{\text{ev}}_j^*(\delta_j),
$$
\n
$$
\left\langle \left\langle \gamma_1 \psi^{a_1}, \dots, \gamma_n \psi^{a_n} \right\rangle \right\rangle_{0,n}^{0+,0+} =
$$
\n
$$
\sum_{d \geq 0} \sum_{k \geq 0} \frac{q^d}{k!} \left\langle \gamma_1 \psi^{a_1}, \dots, \gamma_n \psi^{a_n}; t, \dots, t \right\rangle_{0,n|k,d}^{0+,0+},
$$

 6 See Sections 2 and 5 of [\[5\]](#page-56-15).

where, in the second series, $t \in H^*_{\mathsf{T}}([\mathbb{C}^3/\mathbb{C}^*]).$

For each T-fixed point $p_i \in \mathbb{P}^2$, let

$$
e_i = e(T_{p_i}(\mathbb{P}^2)) \cdot (-3\lambda_i)
$$

be the equivariant Euler class of the tangent space of $K\mathbb{P}^2$ at p_i . Let

$$
\phi_i = \frac{-3\lambda_i \prod_{j \neq i} (H - \lambda_j)}{e_i}, \quad \phi^i = e_i \phi_i \quad \in H^*_{\mathsf{T}}(\mathbb{P}^2)
$$

be cycle classes. Crucial for us are the series

$$
\mathbb{S}_{i}(\gamma) = e_{i} \left\langle \left\langle \frac{\phi_{i}}{z - \psi}, \gamma \right\rangle \right\rangle_{0,2}^{0+,0+},
$$

$$
\mathbb{V}_{ij} = \left\langle \left\langle \frac{\phi_{i}}{x - \psi}, \frac{\phi_{j}}{y - \psi} \right\rangle \right\rangle_{0,2}^{0+,0+}.
$$

Unstable degree 0 terms are included by hand in the above formulas. For $\mathbb{S}_i(\gamma)$, the unstable degree 0 term is $\gamma|_{p_i}$. For \mathbb{V}_{ij} , the unstable degree 0 term is $\frac{\delta_{ij}}{e_i(x+y)}$.

We also write

$$
\mathbb{S}(\gamma) = \sum_{i=0}^{2} \phi_i \mathbb{S}_i(\gamma) .
$$

The series \mathbb{S}_i and \mathbb{V}_{ij} satisfy the basic relation

(18)
$$
e_i \mathbb{V}_{ij}(x, y) e_j = \frac{\sum_{k=0}^2 \mathbb{S}_i(\phi_k)|_{z=x} \mathbb{S}_j(\phi^k)|_{z=y}}{x+y}
$$

proven^{[7](#page-15-0)} in [\[7\]](#page-56-4).

Associated to each T-fixed point $p_i \in \mathbb{P}^2$, there is a special T-fixed point locus,

(19)
$$
\overline{Q}_{0,k|m}^{0+,0+}(\mathbb{P}^2,d)^{\mathsf{T},p_i} \subset \overline{Q}_{0,k|m}^{0+,0+}(\mathbb{P}^2,d)\,,
$$

where all markings lie on a single connected genus 0 domain component contracted to p_i . Let Nor denote the equivariant normal bundle of $Q_{0,n|k}^{0+,0+}$ $_{0,n|k}^{0+,0+}$ (\mathbb{P}^2 , d)^{T,p_i} with respect to the embedding [\(19\)](#page-15-1). Define

$$
\left\langle \gamma_1 \psi^{a_1}, \dots, \gamma_n \psi^{a_n}; \delta_1, \dots, \delta_k \right\rangle_{0, n|k, d}^{0+, 0+, p_i} =
$$
\n
$$
\int_{[\overline{Q}^{0+,0+}_{0,n|k}(\mathbb{P}^2,d)^{T,p_i}]} \frac{e(\text{Obs})}{e(\text{Nor})} \cdot \prod_{i=1}^n \text{ev}_i^*(\gamma_i) \psi_i^{a_i} \cdot \prod_{j=1}^k \widehat{\text{ev}}_j^*(\delta_j),
$$

⁷In Gromov-Witten theory, a parallel relation is obtained immediately from the WDDV equation and the string equation. Since the map forgetting a point is not always well-defined for quasimaps, a different argument is needed here [\[7\]](#page-56-4)

$$
\left\langle \left\langle \gamma_1 \psi^{a_1}, \dots, \gamma_n \psi^{a_n} \right\rangle \right\rangle_{0,n}^{0+,0+,p_i} =
$$

$$
\sum_{d \geq 0} \sum_{k \geq 0} \frac{q^d}{k!} \left\langle \gamma_1 \psi^{a_1}, \dots, \gamma_n \psi^{a_n}; t, \dots, t \right\rangle_{0,n|k,\beta}^{0+,0+,p_i}
$$

2.4. Graph spaces and I-functions.

2.4.1. Graph spaces. The big I-function is defined in [\[5\]](#page-56-15) via the geometry of weighted quasimap graph spaces. We briefly summarize the constructions of [\[5\]](#page-56-15) in the special case of $(0+, 0+)$ -stability. The more general weightings discussed in [\[5\]](#page-56-15) will not be needed here.

As in Section [2.3,](#page-14-1) we consider the quotient

 $\mathbb{C}^3/\mathbb{C}^*$

associated to \mathbb{P}^2 . Following [\[5\]](#page-56-15), there is a $(0+, 0+)$ -stable quasimap graph space

(20)
$$
\mathsf{QG}_{g,n|k,d}^{0+,0+}([\mathbb{C}^3/\mathbb{C}^*])\,.
$$

A C-point of the graph space is described by data

$$
((C, \mathbf{x}, \mathbf{y}), (f, \varphi) : C \longrightarrow [\mathbb{C}^3/\mathbb{C}^*] \times [\mathbb{C}^2/\mathbb{C}^*]).
$$

By the definition of stability, φ is a regular map to

$$
\mathbb{P}^1=\mathbb{C}^2/\!\!/\mathbb{C}^*
$$

of class 1. Hence, the domain curve C has a distinguished irreducible component C_0 canonically isomorphic to \mathbb{P}^1 via φ . The *standard* \mathbb{C}^* action,

(21)
$$
t \cdot [\xi_0, \xi_1] = [t\xi_0, \xi_1], \text{ for } t \in \mathbb{C}^*, [\xi_0, \xi_1] \in \mathbb{P}^1
$$

induces a C ∗ -action on the graph space.

The \mathbb{C}^* -equivariant cohomology of a point is a free algebra with generator z,

$$
H_{\mathbb{C}^*}^*(\mathrm{Spec}(\mathbb{C})) = \mathbb{Q}[z].
$$

Our convention is to define z as the \mathbb{C}^* -equivariant first Chern class of the tangent line $T_0\mathbb{P}^1$ at $0 \in \mathbb{P}^1$ with respect to the action [\(21\)](#page-16-0),

$$
z=c_1(T_0\mathbb{P}^1).
$$

The T-action on \mathbb{C}^3 lifts to a T-action on the graph space [\(20\)](#page-16-1) which commutes with the C ∗ -action obtained from the distinguished domain

.

component. As a result, we have a $\mathsf{T} \times \mathbb{C}^*$ -action on the graph space and $\overline{T} \times \mathbb{C}^*$ -equivariant evaluation morphisms

$$
\begin{aligned}\n\text{ev}_i: \mathsf{QG}_{g,n|k,\beta}^{0+,0+}([\mathbb{C}^3/\mathbb{C}^*]) &\to \mathbb{P}^2, & i &= 1,\ldots,n, \\
\hat{\text{ev}}_j: \mathsf{QG}_{g,n|k,\beta}^{0+,0+}([\mathbb{C}^3/\mathbb{C}^*]) &\to [\mathbb{C}^3/\mathbb{C}^*], & j &= 1,\ldots,k.\n\end{aligned}
$$

Since a morphism

$$
f:C\to [\mathbb{C}^3/\mathbb{C}^*]
$$

is equivalent to the data of a principal G -bundle P on C and a section u of $P \times_{\mathbb{C}^*} \mathbb{C}^3$, there is a natural morphism

$$
C \to E\mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}^3
$$

and hence a pull-back map

$$
f^*: H^*_{\mathbb{C}^*}([\mathbb{C}^3/\mathbb{C}^*]) \to H^*(C).
$$

The above construction applied to the universal curve over the moduli space and the universal morphism to $[\mathbb{C}^3/\mathbb{C}^*]$ is T-equivariant. Hence, we obtain a pull-back map

$$
\widehat{\text{ev}}_j^*: H^*_{\mathsf{T}}(\mathbb{C}^3, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[z] \to H^*_{\mathsf{T} \times \mathbb{C}^*}(\mathsf{Q} \mathsf{G}^{0+,0+}_{g,n|k,\beta}([\mathbb{C}^3/\mathbb{C}^*]), \mathbb{Q})
$$

associated to the evaluation map $\widehat{\text{ev}}_j$.

2.4.2. I-functions. The description of the fixed loci for the \mathbb{C}^* -action on

$$
\mathsf{QG}_{g,0|k,d}^{0+,0+}([\mathbb{C}^3/\mathbb{C}^*])
$$

is parallel to the description in [\[4,](#page-56-16) §4.1] for the unweighted case. In particular, there is a distinguished subset $\mathsf{M}_{k,d}$ of the \mathbb{C}^* -fixed locus for which all the markings and the entire curve class d lie over $0 \in \mathbb{P}^1$. The locus $M_{k,d}$ comes with a natural *proper* evaluation map ev_{\bullet} obtained from the generic point of \mathbb{P}^1 :

$$
\mathrm{ev}_{\bullet}: \mathsf{M}_{k,d} \to \mathbb{C}^3/\!\!/ \mathbb{C}^* = \mathbb{P}^2.
$$

We can explicitly write

$$
\mathsf{M}_{k,d}\cong \mathsf{M}_d\times 0^k\subset \mathsf{M}_d\times (\mathbb{P}^1)^k,
$$

where M_d is the \mathbb{C}^* -fixed locus in $\mathsf{QG}_{0,0,d}^{0+}([\mathbb{C}^3/\mathbb{C}^*])$ for which the class d is concentrated over $0 \in \mathbb{P}^1$. The locus M_d parameterizes quasimaps of class d,

$$
f: \mathbb{P}^1 \longrightarrow [\mathbb{C}^3/\mathbb{C}^*],
$$

with a base-point of length d at $0 \in \mathbb{P}^1$. The restriction of f to $\mathbb{P}^1 \setminus \{0\}$ is a constant map to \mathbb{P}^2 defining the evaluation map ev_{\bullet} .

As in [\[3,](#page-56-17) [4,](#page-56-16) [8\]](#page-56-18), we define the big I-function as the generating function for the push-forward via ev_{\bullet} of localization residue contributions of $\mathsf{M}_{k,d}$. For $\mathbf{t} \in H^*_{\mathsf{T}}([\mathbb{C}^3/\mathbb{C}^*], \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[z]$, let

$$
\operatorname{Res}_{\mathsf{M}_{k,d}}(\mathbf{t}^{k}) = \prod_{j=1}^{k} \widehat{\mathrm{ev}}_{j}^{*}(\mathbf{t}) \cap \operatorname{Res}_{\mathsf{M}_{k,d}}[\mathsf{Q}\mathsf{G}_{g,0|k,d}^{0+,0+}([\mathbb{C}^{3}/\mathbb{C}^{*}])]^{\operatorname{vir}} \n= \frac{\prod_{j=1}^{k} \widehat{\mathrm{ev}}_{j}^{*}(\mathbf{t}) \cap [\mathsf{M}_{k,d}]^{\operatorname{vir}}}{\mathrm{e}(\mathrm{Nor}_{\mathsf{M}_{k,d}}^{\operatorname{vir}})},
$$

where $\text{Nor}_{\mathsf{M}_{k,d}}^{\text{vir}}$ is the virtual normal bundle.

Definition 13. The big I-function for the $(0+, 0+)$ -stability condition, as a formal function in t, is

$$
\mathbb{I}(q, \mathbf{t}, z) = \sum_{d \geq 0} \sum_{k \geq 0} \frac{q^d}{k!} \text{ev}_{\bullet *} \left(\text{Res}_{\mathsf{M}_{k,d}}(\mathbf{t}^k) \right).
$$

2.4.3. Evaluations. Let $\widetilde{H} \in H^*_{\mathsf{T}}((\mathbb{C}^3/\mathbb{C}^*))$ and $H \in H^*_{\mathsf{T}}(\mathbb{P}^2)$ denote the respective hyperplane classes. The I-function of Definition [13](#page-18-0) is evaluated in [\[5\]](#page-56-15).

Proposition 14. For $\mathbf{t} = t\widetilde{H} \in H^*_{\mathsf{T}}([\mathbb{C}^3/\mathbb{C}^*], \mathbb{Q}),$

(22)
$$
\mathbb{I}(t) = \sum_{d=0}^{\infty} q^d e^{t(H+dz)/z} \frac{\prod_{k=0}^{3d-1} (-3H - kz)}{\prod_{i=0}^{2} \prod_{k=1}^{d} (H - \lambda_i + kz)}.
$$

Observe that the I-function has following expandsion after restriction $t = 0$,

$$
\mathbb{I}|_{t=0} = 1 + \frac{I_1 H}{z} + \frac{I_{2,0} H^2 + I_{2,1} (\lambda_0 + \lambda_1 + \lambda_2) H}{z^2} + \mathcal{O}(\frac{1}{z^3}),
$$

where

$$
I_1(q) = \sum_{d=1}^{\infty} 3 \frac{(3d-1)!}{(d!)^3} (-q)^d,
$$

\n
$$
I_{2,0}(q) = \sum_{d=1}^{\infty} 3 \frac{(3d-1)!}{(d!)^3} \left(3\text{Har}[3d-1] - 3\text{Har}[d] \right) (-q)^d,
$$

\n
$$
I_{2,1}(q) = \sum_{d=1}^{\infty} 3 \frac{(3d-1)!}{(d!)^3} \text{Har}[d] (-q)^d.
$$

Here $\text{Har}[d] := \sum_{k=1}^d$ 1 $\frac{1}{k}$.

We return now to the functions $\mathbb{S}_i(\gamma)$ defined in Section [2.3.](#page-14-1) Using Birkhoff factorization, an evaluation of the series $\mathbb{S}(H^j)$ can be obtained from the I-function, see [\[13\]](#page-56-13):

 $\mathbb{S}(1) = \mathbb{I}$,

(23)
$$
S(H) = \frac{z \frac{d}{dt} S(1)}{z \frac{d}{dt} S(1)|_{t=0, H=1, z=\infty}},
$$

$$
S(H^2) = \frac{z \frac{d}{dt} S(H) - (\lambda_0 + \lambda_1 + \lambda_2) N_2 S(H)}{(z \frac{d}{dt} S(H) - (\lambda_0 + \lambda_1 + \lambda_2) N_2 S(H))|_{t=0, H=1, z=\infty}}
$$

For a series $F \in \mathbb{C}[[\frac{1}{z}]]$, the specialization $F|_{z=\infty}$ denotes constant term of F with respect to $\frac{1}{z}$. Here, N_2 is series in q defined by

.

$$
N_2(q) := d \frac{q}{dq} \left(\frac{q \frac{d}{dq} I_{2,1}}{1 + q \frac{d}{dq} I_{1,0}} \right).
$$

2.4.4. Further calculations. Define small I-function

 $\overline{\mathbb{I}}(q) \in H^*_{\mathsf{T}}(\mathbb{P}^2,\mathbb{Q})[[q]]$

by the restriction

$$
\overline{\mathbb{I}}(q) = \mathbb{I}(q,t)|_{t=0}.
$$

Define differential operators

$$
D = q \frac{d}{dq}, \quad M = H + zD.
$$

Applying $z\frac{d}{dt}$ to I and then restricting to $t = 0$ has same effect as applying M to $\bar{\mathbb{I}}$

$$
\left[\left(z \frac{d}{dt} \right)^k \mathbb{I} \right] \Big|_{t=0} = M^k \overline{\mathbb{I}} \, .
$$

The function $\overline{\mathbb{I}}$ satisfies following Picard-Fuchs equation

$$
(24)\ \left((M-\lambda_0)(M-\lambda_1)(M-\lambda_2)+3qM(3M+z)(3M+2z)\right)\overline{\mathbb{I}}=0
$$

implied by the Picard-Fuchs equation for I,

$$
\left(\prod_{j=0}^{2} \left(z\frac{d}{dt} - \lambda_j\right) + 3q\left(z\frac{d}{dt}\right) \left(3\left(z\frac{d}{dt}\right) + z\right) \left(3\left(z\frac{d}{dt}\right) + 2z\right)\right) \mathbb{I} = 0.
$$

The restriction $\overline{\mathbb{I}}|_{H=\lambda_i}$ admits following asymptotic form

(25)
$$
\overline{\mathbb{I}}|_{H=\lambda_i} = e^{\mu_i/z} \left(R_{0,i} + R_{1,i} z + R_{2,i} z^2 + \ldots \right)
$$

with series $\mu_i, R_{k,i} \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]]$.

A derivation of [\(25\)](#page-19-0) is obtained in [\[21\]](#page-56-3) via the Picard-Fuchs equation [\(24\)](#page-19-1) for $\overline{\mathbb{I}}|_{H=\lambda_i}$. The series μ_i and $R_{k,i}$ are found by solving differential equations obtained from the coefficient of z^k . For example,

$$
\lambda_i + D\mu_i = L_i,
$$

\n
$$
R_{0,i} = \left(\frac{\lambda_i \prod_{j \neq i} (\lambda_i - \lambda_j)}{f(L_i)}\right)^{\frac{1}{2}}.
$$

Define the series C_1 and C_2 by the equations

(26)
$$
C_1 = z \frac{d}{dt} \mathbb{S}(1)|_{z=\infty, t=0, H=1},
$$

$$
C_2 = \left(z \frac{d}{dt} \mathbb{S}(H) - (\lambda_0 + \lambda_1 + \lambda_2) N_2 \mathbb{S}(H) \right)|_{z=\infty, t=0, H=1}.
$$

The following relation was proven in [\[21\]](#page-56-3),

(27)
$$
C_1^2 C_2 = (1 + 27q)^{-1}.
$$

From the equations [\(23\)](#page-19-2) and [\(25\)](#page-19-0), we can show the series

$$
\overline{\mathbb{S}}_i(1) = \overline{\mathbb{S}}(1)|_{H=\lambda_i}, \quad \overline{\mathbb{S}}_i(H) = \overline{\mathbb{S}}(H)|_{H=\lambda_i}, \quad \overline{\mathbb{S}}_i(H^2) = \overline{\mathbb{S}}(H^2)|_{H=\lambda_i}
$$

have the following asymptotic expansions:

(28)
$$
\overline{\mathbf{S}}_i(1) = e^{\frac{\mu_i}{z}} \left(R_{00,i} + R_{01,i} z + R_{02,i} z^2 + \dots \right),
$$

$$
\overline{\mathbf{S}}_i(H) = e^{\frac{\mu_i}{z}} \frac{L_i}{C_1} \left(R_{10,i} + R_{11,i} z + R_{12} z^2 + \dots \right),
$$

$$
\overline{\mathbf{S}}_i(H^2) = e^{\frac{\mu_i}{z}} \frac{L_i^2}{C_1 C_2} \left(R_{20,i} + R_{21,i} z + R_{22,i} z^2 + \dots \right).
$$

We follow here the normalization of [\[21\]](#page-56-3). Note

$$
R_{0k,i} = R_{k,i}.
$$

As in [\[21,](#page-56-3) Theorem 4], we expect the following constraints.

Conjecture 15. For all $k \geq 0$, we have

$$
R_{k,i}\in\mathbb{G}_2.
$$

Conjecture [15](#page-20-0) is the main obstruction for the proof of Conjecture [1](#page-3-3) and [2.](#page-3-2) By the same argument of Section [6,](#page-35-0) we obtain the following result.

Theorem 16. Conjecture [15](#page-20-0) implies Conjecture [1](#page-3-3) and [2.](#page-3-2)

By applying asymptotic expansions [\(28\)](#page-20-1) to [\(23\)](#page-19-2), we obtain the following results.

Lemma 17. We have

$$
R_{1\,p+1,i} = R_{0\,p+1,i} + \frac{\mathsf{D}R_{0\,p,i}}{L_i},
$$
\n
$$
R_{2\,p+1,i} = R_{1\,p+1,i} + \frac{\mathsf{D}R_{1\,p,i}}{L_i} + \left(\frac{\mathsf{D}L_i}{L_i^2} - \frac{X}{L_i}\right) - (\lambda_0 + \lambda_1 + \lambda_2)N_2\frac{R_{1\,k,i}}{L_i},
$$
\nwith $X = \frac{\mathsf{D}C_1}{C_1}$.

From Lemma [17,](#page-21-1) we obtain results for $\overline{S}(H)|_{H=\lambda_i}$ and $\overline{S}(H^2)|_{H=\lambda_i}$.

Lemma 18. Suppose Conjecture [15](#page-20-0) is true. Then for all $k \geq 0$, we have for all $k \geq 0$,

$$
R_{1\,k,i} \in \mathbb{G}_2, R_{2\,k,i} = Q_{2\,k,i} - \frac{R_{1\,k-1,i}}{L}X - (\lambda_0 + \lambda_1 + \lambda_2)N_2 \frac{R_{1\,k,i}}{L_i},
$$

with $Q_{2 k,i} \in \mathbb{G}_2$.

2.5. Determining DX and N_2 . The following relation was proven in [\[16\]](#page-56-0).

(29)
$$
X^2 - (L^3 - 1)X + DX - \frac{2}{9}(L^3 - 1) = 0.
$$

By the above result, the differential ring

(30)
$$
\mathbb{G}_2[X, DX, DDX, \ldots]
$$

is just the polynomial ring $\mathbb{G}_2[X]$. Denote by $\text{Coeff}(x^i y^j)$ the coefficient of $x^i y^j$ in

$$
\sum_{k=0}^{2} e^{-\frac{\mu_{i}}{x} - \frac{\mu_{i}}{y}} \mathbb{S}_{i}(\phi_{k})|_{z=x} \, \mathbb{S}_{i}(\phi^{k})|_{z=y}.
$$

From [\(18\)](#page-15-2) and [\(28\)](#page-20-1), we obtain the following equation.

$$
Coeff(x2) + Coeff(y2) – Coeff(xy) = 0.
$$

Above equation immediately yields the following relation.

(31)
$$
N_2 = -\frac{1}{2}C_2 + \frac{1}{2}L^3.
$$

3. HIGHER GENUS SERIES ON $\overline{M}_{g,n}$

3.1. Intersection theory on $\overline{M}_{g,n}$. We review here the now standard method used by Givental $[9, 10, 14]$ $[9, 10, 14]$ $[9, 10, 14]$ to express genus g descendent correlators in terms of genus 0 data.

Let t_0, t_1, t_2, \ldots be formal variables. The series

$$
T(c) = t_0 + t_1 c + t_2 c^2 + \dots
$$

in the additional variable c plays a basic role. The variable c will later be replaced by the first Chern class ψ_i of a cotangent line over $\overline{M}_{q,n}$,

$$
T(\psi_i)=t_0+t_1\psi_i+t_2\psi_i^2+\ldots,
$$

with the index i depending on the position of the series T in the correlator.

Let $2g - 2 + n > 0$. For $a_i \in \mathbb{Z}_{\geq 0}$ and $\gamma \in H^*(\overline{M}_{g,n})$, define the correlator

$$
\langle \langle \psi^{a_1}, \ldots, \psi^{a_n} | \gamma \rangle \rangle_{g,n} = \sum_{k \geq 0} \frac{1}{k!} \int_{\overline{M}_{g,n+k}} \gamma \psi_1^{a_1} \cdots \psi_n^{a_n} \prod_{i=1}^k T(\psi_{n+i}).
$$

In the above summation, the $k = 0$ term is

$$
\int_{\overline{M}_{g,n}} \gamma \, \psi_1^{a_1} \cdots \psi_n^{a_n} \, .
$$

We also need the following correlator defined for the unstable case,

$$
\langle \langle 1, 1 \rangle \rangle_{0,2} = \sum_{k>0} \frac{1}{k!} \int_{\overline{M}_{0,2+k}} \prod_{i=1}^{k} T(\psi_{2+i}).
$$

For formal variables x_1, \ldots, x_n , we also define the correlator

(32)
$$
\left\langle \left\langle \frac{1}{x_1 - \psi}, \dots, \frac{1}{x_n - \psi} \right| \gamma \right\rangle_{g,n}
$$

in the standard way by expanding $\frac{1}{x_i-\psi}$ as a geometric series. Denote by L the differential operator

$$
\mathbb{L} = \frac{\partial}{\partial t_0} - \sum_{i=1}^{\infty} t_i \frac{\partial}{\partial t_{i-1}} = \frac{\partial}{\partial t_0} - t_1 \frac{\partial}{\partial t_0} - t_2 \frac{\partial}{\partial t_1} - \dots
$$

The string equation yields the following result.

Lemma 19. For $2g - 2 + n > 0$, we have $\mathbb{L}\langle\langle 1, \ldots, 1 | \gamma \rangle\rangle_{g,n} = 0$ and

$$
\mathbb{L}\left\langle \left\langle \frac{1}{x_1 - \psi}, \dots, \frac{1}{x_n - \psi} \middle| \gamma \right\rangle \right\rangle_{g,n} = \left\langle \left(\frac{1}{x_1} + \dots + \frac{1}{x_n} \right) \left\langle \left\langle \frac{1}{x_1 - \psi}, \dots, \frac{1}{x_n - \psi} \middle| \gamma \right\rangle \right\rangle_{g,n}.
$$

After the restriction $t_0 = 0$ and application of the dilaton equation, the correlators are expressed in terms of finitely many integrals (by the dimension constraint). For example,

$$
\langle\langle 1, 1, 1 \rangle\rangle_{0,3} |_{t_0=0} = \frac{1}{1-t_1},
$$

$$
\langle\langle 1, 1, 1, 1 \rangle\rangle_{0,4} |_{t_0=0} = \frac{t_2}{(1-t_1)^3},
$$

$$
\langle\langle 1, 1, 1, 1, 1 \rangle\rangle_{0,5} |_{t_0=0} = \frac{t_3}{(1-t_1)^4} + \frac{3t_2^2}{(1-t_1)^5},
$$

$$
\langle\langle 1, 1, 1, 1, 1, 1 \rangle\rangle_{0,6} |_{t_0=0} = \frac{t_4}{(1-t_1)^5} + \frac{10t_2t_3}{(1-t_1)^6} + \frac{15t_2^3}{(1-t_1)^7}.
$$

We consider $\mathbb{C}(t_1)[t_2, t_3, \ldots]$ as Z-graded ring over $\mathbb{C}(t_1)$ with $\deg(t_i) = i - 1$ for $i \geq 2$.

Define a subspace of homogeneous elements by

$$
\mathbb{C}\left[\frac{1}{1-t_1}\right][t_2,t_3,\ldots]_{\text{Hom}}\subset \mathbb{C}(t_1)[t_2,t_3,\ldots].
$$

We easily see

$$
\langle\langle \psi^{a_1}, \ldots, \psi^{a_n} | \gamma \rangle\rangle_{g,n} |_{t_0=0} \in \mathbb{C} \left[\frac{1}{1-t_1} \right] [t_2, t_3, \ldots]_{\text{Hom}}.
$$

Using the leading terms (of lowest degree in $\frac{1}{(1-t_1)}$), we obtain the following result.

Lemma 20. The set of genus θ correlators

$$
\left\{ \langle \langle 1,\ldots,1\rangle \rangle_{0,n} \,|_{t_0=0} \right\}_{n\geq 4}
$$

freely generate the ring $\mathbb{C}(t_1)[t_2, t_3, \ldots]$ over $\mathbb{C}(t_1)$.

By Lemma [20,](#page-23-0) we can find a unique representation of $\langle \langle \psi^{a_1}, \dots, \psi^{a_n} \rangle \rangle_{g,n}|_{t_0=0}$ in the variables

(33)
$$
\left\{ \langle \langle 1,\ldots,1 \rangle \rangle_{0,n} |_{t_0=0} \right\}_{n\geq 3}.
$$

The $n = 3$ correlator is included in the set [\(33\)](#page-23-1) to capture the variable t_1 . For example, in $g = 1$,

$$
\langle\langle 1,1\rangle\rangle_{1,2}|_{t_0=0} = \frac{1}{24} \left(\frac{\langle\langle 1,1,1,1,1\rangle\rangle_{0,5}|_{t_0=0}}{\langle 1,1,1\rangle\rangle_{0,3}|_{t_0=0}} - \frac{\langle\langle 1,1,1,1\rangle\rangle_{0,4}^2|_{t_0=0}}{\langle\langle 1,1,1\rangle\rangle_{0,3}|_{t_0=0}} \right),
$$

$$
\langle\langle 1\rangle\rangle_{1,1}|_{t_0=0} = \frac{1}{24} \frac{\langle\langle 1,1,1,1\rangle\rangle_{0,4}|_{t_0=0}}{\langle\langle 1,1,1\rangle\rangle_{0,3}|_{t_0=0}}
$$

A more complicated example in $g = 2$ is

$$
\langle \langle \ \rangle \rangle_{2,0}|_{t_0=0} = \frac{1}{1152} \frac{\langle \langle 1, 1, 1, 1, 1, 1 \rangle \rangle_{0,6}|_{t_0=0}}{\langle \langle 1, 1, 1 \rangle \rangle_{0,3}|_{t_0=0}^2} - \frac{7}{1920} \frac{\langle \langle 1, 1, 1, 1, 1 \rangle \rangle_{0,5}|_{t_0=0} \langle \langle 1, 1, 1, 1 \rangle \rangle_{0,4}|_{t_0=0}}{\langle \langle 1, 1, 1 \rangle \rangle_{0,3}|_{t_0=0}^3} + \frac{1}{360} \frac{\langle \langle 1, 1, 1, 1 \rangle \rangle_{0,4}|_{t_0=0}^3}{\langle \langle 1, 1, 1 \rangle \rangle_{0,3}|_{t_0=0}^4}.
$$

Definition 21. For $\gamma \in H^*(\overline{M}_{g,k})$, let

$$
\mathsf{P}^{a_1,\ldots,a_n,\gamma}_{g,n}(s_0,s_1,s_2,\ldots) \in \mathbb{Q}(s_0,s_1,\ldots)
$$

be the unique rational function satisfying the condition

$$
\langle \langle \psi^{a_1}, \ldots, \psi^{a_n} | \gamma \rangle \rangle_{g,n} |_{t_0=0} = \mathsf{P}^{a_1, a_2, \ldots, a_n, \gamma}_{g,n} |_{s_i=\langle \langle 1, \ldots, 1 \rangle \rangle_{0, i+3} |_{t_0=0}}.
$$

Proposition 22. For $2g - 2 + n > 0$, we have

$$
\langle \langle 1,\ldots,1|\,\gamma\,\rangle \rangle_{g,n} = \mathsf{P}^{0,\ldots,0,\gamma}_{g,n}|_{s_i=\langle \langle 1,\ldots,1\rangle \rangle_{0,i+3}}.
$$

Proof. Both sides of the equation satisfy the differential equation

$$
\mathbb{L}=0.
$$

By definition, both sides have the same initial conditions at $t_0 = 0$. \Box Proposition 23. For $2g - 2 + n > 0$,

$$
\left\langle \left\langle \frac{1}{x_1 - \psi_1}, \dots, \frac{1}{x_n - \psi_n} \middle| \gamma \right\rangle \right\rangle_{g,n} =
$$

$$
e^{\langle \langle 1,1 \rangle \rangle_{0,2} (\sum_i \frac{1}{x_i})} \sum_{a_1, \dots, a_n} \frac{P_{g,n}^{a_1, \dots, a_n, \gamma} |_{s_i = \langle \langle 1, \dots, 1 \rangle \rangle_{0, i+3}}}{x_1^{a_1 + 1} \cdots x_n^{a_n + 1}}.
$$

Proof. Both sides of the equation satisfy differential equation

$$
\mathbb{L} - \sum_{i} \frac{1}{x_i} = 0.
$$

Both sides have the same initial conditions at $t_0 = 0$. We use here

$$
\mathbb{L}\langle\langle 1,1\rangle\rangle_{0,2}=1\,,\hspace{0.5cm}\langle\langle 1,1\rangle\rangle_{0,2}|_{t_0=0}=0\,.
$$

There is no conflict here with Lemma [19](#page-22-0) since $(g, n) = (0, 2)$ is not in the stable range. $\hfill \square$

3.2. The unstable case $(0, 2)$. The definition given in (32) of the correlator is valid in the stable range

$$
2g-2+n>0.
$$

The unstable case $(g, n) = (0, 2)$ plays a special role. We define

$$
\left\langle \left\langle \frac{1}{x_1 - \psi_1}, \frac{1}{x_2 - \psi_2} \right\rangle \right\rangle_{0,2}
$$

by adding the degenerate term

$$
\frac{1}{x_1+x_2}
$$

to the terms obtained by the expansion of $\frac{1}{x_i - \psi_i}$ as a geometric series. The degenerate term is associated to the (unstable) moduli space of genus 0 with 2 markings.

Proposition 24. We have

$$
\left\langle \left\langle \frac{1}{x_1 - \psi_1}, \frac{1}{x_2 - \psi_2} \right\rangle \right\rangle_{0,2} = e^{\langle \langle 1, 1 \rangle \rangle_{0,2} \left(\frac{1}{x_1} + \frac{1}{x_2}\right)} \left(\frac{1}{x_1 + x_2}\right).
$$

Proof. Both sides of the equation satisfy differential equation

$$
\mathbb{L} - \sum_{i=1}^{2} \frac{1}{x_i} = 0.
$$

Both sides have the same initial conditions at $t_0 = 0$.

3.3. Local invariants and wall crossing. The torus T acts on the moduli spaces $\overline{M}_{g,n}(\mathbb{P}^2, d)$ and $\overline{Q}_{g,n}(\mathbb{P}^2, d)$. We consider here special localization contributions associated to the fixed points $p_i \in \mathbb{P}^2$.

Consider first the moduli of stable maps. Let

$$
\overline{M}_{g,n}(\mathbb{P}^2,d)^{\mathsf{T},p_i} \subset \overline{M}_{g,n}(\mathbb{P}^2,d)
$$

be the union of T-fixed loci which parameterize stable maps obtained by attaching $\mathsf{T}\text{-fixed rational tails to a genus } q, n\text{-pointed Deligne-}$ Mumford stable curve contracted to the point $p_i \in \mathbb{P}^2$. Similarly, let

$$
\overline{Q}_{g,n}({\mathbb P}^2,d)^{{\sf T},p_i}\subset \overline{Q}_{g,n}({\mathbb P}^2,d)
$$

be the parallel T-fixed locus parameterizing stable quotients obtained by attaching base points to a genus g , *n*-pointed Deligne-Mumford stable curve contracted to the point $p_i \in \mathbb{P}^2$.

Let Λ_i denote the localization of the ring

$$
\mathbb{C}[\lambda_0^{\pm 1},\lambda_1^{\pm 1},\lambda_2^{\pm 1}]
$$

at the three tangent weights at $p_i \in \mathbb{P}^2$. Using the virtual localization formula [\[11\]](#page-56-12), there exist unique series

$$
S_{p_i} \in \Lambda_i[\psi][[Q]]
$$

for which the localization contribution of the T-fixed locus $\overline{M}_{g,n}(\mathbb{P}^2,d)^{\mathsf{T},p_i}$ to the equivariant Gromov-Witten invariants of $K\mathbb{P}^2$ can be written as

$$
\sum_{d=0}^{\infty} Q^d \int_{[\overline{M}_{g,n}(K\mathbb{P}^2,d)^{T,p_i}]^{\text{vir}}} \psi_1^{a_1} \cdots \psi_n^{a_n} =
$$

$$
\sum_{k=0}^{\infty} \frac{1}{k!} \int_{\overline{M}_{g,n+k}} H_g^{p_i} \psi_1^{a_1} \cdots \psi_n^{a_n} \prod_{j=1}^k S_{p_i}(\psi_{n+j}).
$$

Here, $H_g^{p_i}$ is the standard vertex class,

(34)
$$
\frac{e(\mathbb{E}_g^* \otimes T_{p_i}(\mathbb{P}^2))}{e(T_{p_i}(\mathbb{P}^2))} \cdot \frac{e(\mathbb{E}_g^* \otimes (-3\lambda_i))}{(-3\lambda_i)},
$$

obtained the Hodge bundle $\mathbb{E}_g \to \overline{M}_{g,n+k}$.

Similarly, the application of the virtual localization formula to the moduli of stable quotients yields classes

$$
F_{p_i,k} \in H^*(\overline{M}_{g,n|k}) \otimes_{\mathbb{C}} \Lambda_i
$$

for which the contribution of $\overline{Q}_{g,n}(\mathbb{P}^2, d)^{T,p_i}$ is given by

$$
\sum_{d=0}^{\infty} q^d \int_{[\overline{Q}_{g,n}(K\mathbb{P}^2,d)^{\mathsf{T},p_i}]^{\text{vir}}} \psi_1^{a_1} \cdots \psi_n^{a_n} = \sum_{k=0}^{\infty} \frac{q^k}{k!} \int_{\overline{M}_{g,n|k}} \mathsf{H}^{p_i}_g \psi_1^{a_1} \cdots \psi_n^{a_n} F_{p_i,k}.
$$

Here $\overline{M}_{g,n|k}$ is the moduli space of genus g curves with markings

$$
\{p_1,\cdots,p_n\}\cup\{\hat{p}_1\cdots\hat{p}_k\}\in C^{\text{ns}}\subset C
$$

satisfying the conditions

(i) the points p_i are distinct,

(ii) the points \hat{p}_j are distinct from the points p_i ,

with stability given by the ampleness of

$$
\omega_C(\sum_{i=1}^m p_i + \epsilon \sum_{j=1}^k \hat{p}_j)
$$

for every strictly positive $\epsilon \in \mathbb{Q}$.

The Hodge class $H_g^{p_i}$ is given again by formula [\(34\)](#page-26-0) using the Hodge bundle

$$
\mathbb{E}_g \to \overline{M}_{g,n|k}.
$$

Definition 25. For $\gamma \in H^*(\overline{M}_{g,n})$, let

$$
\langle \langle \psi_1^{a_1}, \dots, \psi_n^{a_n} | \gamma \rangle \rangle_{g,n}^{p_i, \infty} = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\overline{M}_{g,n+k}} \gamma \psi_1^{a_1} \cdots \psi_n^{a_n} \prod_{j=1}^k S_{p_i}(\psi_{n+j}),
$$

$$
\langle \langle \psi_1^{a_1}, \dots, \psi_n^{a_n} | \gamma \rangle \rangle_{g,n}^{p_i,0+} = \sum_{k=0}^{\infty} \frac{q^k}{k!} \int_{\overline{M}_{g,n|k}} \gamma \psi_1^{a_1} \cdots \psi_n^{a_n} F_{p_i,k}.
$$

Proposition 26 (Ciocan-Fontanine, Kim [\[7\]](#page-56-4)). For $2g - 2 + n > 0$, we have the wall crossing relation

$$
\langle \langle \psi_1^{a_1}, \dots, \psi_n^{a_n} | \gamma \rangle \rangle_{g,n}^{p_i, \infty}(Q(q)) = \langle \langle \psi_1^{a_1}, \dots, \psi_n^{a_n} | \gamma \rangle \rangle_{g,n}^{p_i, 0+}(q)
$$

where $Q(q)$ is the mirror map

$$
Q(q) = \exp(I_1^{K\mathbb{P}^2}(q)).
$$

Proposition [26](#page-27-0) is a consequence of [\[7,](#page-56-4) Lemma 5.5.1]. The mirror map here is the mirror map for $K\mathbb{P}^2$ discussed in Section [0.2.](#page-1-2) Propositions [22](#page-24-0) and [26](#page-27-0) together yield

$$
\langle\langle 1,\ldots,1|\,\gamma\,\rangle\rangle_{g,n}^{p_i,\infty} = \mathsf{P}_{g,n}^{0,\ldots,0,\gamma}\big(\langle\langle 1,1,1\rangle\rangle_{0,3}^{p_i,\infty},\langle\langle 1,1,1,1\rangle\rangle_{0,4}^{p_i,\infty},\ldots\big),\langle\langle 1,\ldots,1|\,\gamma\,\rangle\rangle_{g,n}^{p_i,0+} = \mathsf{P}_{g,n}^{0,\ldots,0,\gamma}\big(\langle\langle 1,1,1\rangle\rangle_{0,3}^{p_i,0+},\langle\langle 1,1,1,1\rangle\rangle_{0,4}^{p_i,0+},\ldots\big).
$$

Similarly, using Propositions [23](#page-24-1) and [26,](#page-27-0) we obtain

$$
\left\langle \left\langle \frac{1}{x_1 - \psi}, \dots, \frac{1}{x_n - \psi} \middle| \gamma \right\rangle \right\rangle_{g,n}^{p_i, \infty} =
$$
\n
$$
e^{\langle \langle 1,1 \rangle \rangle_{0,2}^{p_i, \infty}} \left(\sum_i \frac{1}{x_i} \right) \sum_{a_1, \dots, a_n} \frac{P_{g,n}^{a_1, \dots, a_n, \gamma} \left(\langle \langle 1,1,1 \rangle \rangle_{0,3}^{p_i, \infty}, \langle \langle 1,1,1,1 \rangle \rangle_{0,4}^{p_i, \infty}, \dots \right)}{x_1^{a_1 + 1} \cdots x_n^{a_n + 1}},
$$

$$
(35) \ \ \left\langle \left\langle \frac{1}{x_1 - \psi}, \dots, \frac{1}{x_n - \psi} \right| \gamma \right\rangle \right\rangle_{g,n}^{p_i,0+} =
$$

$$
e^{\langle \langle 1,1 \rangle \rangle_{0,2}^{p_i,0+} \left(\sum_i \frac{1}{x_i} \right)} \sum_{a_1, \dots, a_n} \frac{P_{g,n}^{a_1, \dots, a_n, \gamma} \left(\langle \langle 1,1,1 \rangle \rangle_{0,3}^{p_i,0+}, \langle \langle 1,1,1,1 \rangle \rangle_{0,4}^{p_i,0+}, \dots \right)}{x_1^{a_1+1} \cdots x_n^{a_n+1}}.
$$

4. HIGHER GENUS SERIES ON $K\mathbb{P}^2$

4.1. Overview. We apply the localization strategy introduced first by Givental [\[9,](#page-56-19) [10,](#page-56-20) [14\]](#page-56-21) for Gromov-Witten theory to the stable quotient invariants of local \mathbb{P}^2 . The contribution $\text{Cont}_{\Gamma}(q)$ discussed in Section [1](#page-11-0) of a graph $\Gamma \in \mathsf{G}_g(\mathbb{P}^2)$ can be separated into vertex and edge contributions. We express the vertex and edge contributions in terms of the series \mathbb{S}_i and \mathbb{V}_{ij} of Section [2.3.](#page-14-1)

4.2. Edge terms. Recall the definition^{[8](#page-28-1)} of V_{ij} given in Section [2.3,](#page-14-1)

(36)
$$
\mathbb{V}_{ij} = \left\langle \left\langle \frac{\phi_i}{x - \psi}, \frac{\phi_j}{y - \psi} \right\rangle \right\rangle_{0,2}^{0+,0+}
$$

Let \overline{V}_{ij} denote the restriction of V_{ij} to $t = 0$. Via formula [\(16\)](#page-13-4), \overline{V}_{ij} is a summation of contributions of fixed loci indexed by a graph Γ consisting of two vertices connected by a unique edge. Let w_1 and w_2 be T-weights. Denote by

.

$$
\overline{\mathbb{V}}^{w_1,w_2}_{ij}
$$

the summation of contributions of T-fixed loci with tangent weights precisely w_1 and w_2 on the first rational components which exit the vertex components over p_i and p_j .

The series $\overline{V}_{ij}^{w_1,w_2}$ includes *both* vertex and edge contributions. By definition [\(36\)](#page-28-2) and the virtual localization formula, we find the following relationship between $\overline{V}_{ij}^{w_1,w_2}$ and the corresponding pure edge contribution $\mathsf{E}_{ij}^{w_1,w_2}$,

$$
e_{i} \overline{V}_{ij}^{w_{1},w_{2}} e_{j} = \left\langle \left\langle \frac{1}{w_{1} - \psi}, \frac{1}{x_{1} - \psi} \right\rangle \right\rangle_{0,2}^{p_{i},0+} E_{ij}^{w_{1},w_{2}} \left\langle \left\langle \frac{1}{w_{2} - \psi}, \frac{1}{x_{2} - \psi} \right\rangle \right\rangle_{0,2}^{p_{j},0+}
$$

\n
$$
= \frac{e^{\frac{\langle \langle 1,1 \rangle \rangle_{0,2}^{p_{i},0+}}{w_{1}} + \frac{\langle \langle 1,1 \rangle \rangle_{0,2}^{p_{j},0+}}{x_{1}}}{w_{1} + x_{1}} E_{ij}^{w_{1},w_{2}} e^{\frac{\langle \langle 1,1 \rangle \rangle_{0,2}^{p_{i},0+}}{w_{2}} + \frac{\langle \langle 1,1 \rangle \rangle_{0,2}^{p_{j},0+}}{x_{2}}} w_{2} + x_{2}
$$

\n
$$
= \sum_{q_{1},q_{2}} e^{\frac{\langle \langle 1,1 \rangle \rangle_{0,2}^{p_{i},0+}}{x_{1}} + \frac{\langle \langle 1,1 \rangle \rangle_{0,2}^{p_{j},0+}}{w_{1}}} e^{\frac{\langle \langle 1,1 \rangle \rangle_{0,2}^{p_{j},0+}}{x_{2}}} + \frac{\langle \langle 1,1 \rangle \rangle_{0,2}^{p_{j},0+}}{w_{2}} (-1)^{a_{1} + a_{2}} \frac{E_{ij}^{w_{1},w_{2}}}{w_{1}^{a_{1}} w_{2}^{a_{2}}} x_{1}^{a_{1} - 1} x_{2}^{a_{2} - 1}.
$$

After summing over all possible weights, we obtain

 a_1,a_2

$$
e_i\left(\overline{\mathbb{V}}_{ij}-\frac{\delta_{ij}}{e_i(x+y)}\right)e_j=\sum_{w_1,w_2}e_i\overline{\mathbb{V}}_{ij}^{w_1,w_2}e_j.
$$

The above calculations immediately yield the following result.

⁸We use the variables x_1 and x_2 here instead of x and y.

Lemma 27. We have

$$
\left[e^{-\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_i,0+}}{x_1}}e^{-\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_j,0+}}{x_2}}e_i\left(\overline{V}_{ij}-\frac{\delta_{ij}}{e_i(x+y)}\right)e_j\right]_{x_1^{a_1-1}x_2^{a_2-1}} = \sum_{w_1,w_2} e^{\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_i,0+}}{w_1}}e^{\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_j,0+}}{w_2}}(-1)^{a_1+a_2}\frac{E_{ij}^{w_1,w_2}}{w_1^{a_1}w_2^{a_2}}
$$

.

The notation $[\ldots]_{x_1^{a_1-1}x_2^{a_2-1}}$ in Lemma [27](#page-29-0) denotes the coefficient of $x_1^{a_1-1}x_2^{a_2-1}$ in the series expansion of the argument.

4.3. A simple graph. Before treating the general case, we present the localization formula for a simple graph^{[9](#page-29-1)}. Let $\Gamma \in \mathsf{G}_g(\mathbb{P}^2)$ consist of two vertices and one edge,

$$
v_1, v_2 \in \Gamma(V) \,, \quad e \in \Gamma(E)
$$

with genus and T-fixed point assignments

$$
\mathsf{g}(v_i) = g_i \,, \quad \mathsf{p}(v_i) = p_i \,.
$$

Let w_1 and w_2 be tangent weights at the vertices p_1 and p_2 respectively. Denote by $\text{Cont}_{\Gamma,w_1,w_2}$ the summation of contributions to

(37)
$$
\sum_{d>0} q^d \left[\overline{Q}_g(K\mathbb{P}^2, d) \right]^{\text{vir}}
$$

of T-fixed loci with tangent weights precisely w_1 and w_2 on the first rational components which exit the vertex components over p_1 and p_2 . We can express the localization formula for [\(37\)](#page-29-2) as

$$
\left\langle \left\langle \frac{1}{w_1 - \psi} \left| \mathsf{H}^{p_1}_{g_1} \right\rangle \right\rangle_{g_1, 1}^{p_1, 0+} \mathsf{E}^{w_1, w_2}_{12} \left\langle \left\langle \frac{1}{w_2 - \psi} \left| \mathsf{H}^{p_2}_{g_2} \right\rangle \right\rangle_{g_2, 1}^{p_2, 0+} \right\rangle \right\langle
$$

which equals

$$
\sum_{a_1,a_2}e^{\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_1,0+}}{w_1}}\frac{\mathsf{P}\left[\psi^{a_1-1}\left|\mathsf{H}^{p_1}_{g_1}\right]^{p_1,0+}}{w_1^{a_1}}\mathsf{E}^{w_1,w_2}_{12}e^{\frac{\langle\langle 1,1\rangle\rangle_{0,2}^{p_2,0+}}{w_2}}\frac{\mathsf{P}\left[\psi^{a_2-1}\left|\mathsf{H}^{p_2}_{g_2}\right]^{p_2,0+}}{w_2^{a_2}}\right]}{w_2^{a_2}}
$$

where $H_{g_i}^{p_i}$ is the Hodge class [\(34\)](#page-26-0). We have used here the notation

$$
P\left[\psi_1^{k_1},\ldots,\psi_n^{k_n}\middle|\mathsf{H}_{h}^{p_i}\right]_{h,n}^{p_i,0+}=\n P_{h,1}^{k_1,\ldots,k_n,\mathsf{H}_{h}^{p_i}}\left(\langle\langle 1,1,1\rangle\rangle_{0,3}^{p_i,0+},\langle\langle 1,1,1,1\rangle\rangle_{0,4}^{p_i,0+},\ldots\right)
$$

and applied [\(35\)](#page-27-1).

⁹We follow here the notation of Section [1.](#page-11-0)

After summing over all possible weights w_1, w_2 and applying Lemma [27,](#page-29-0) we obtain the following result for the full contribution

$$
\text{Cont}_{\Gamma} = \sum_{w_1, w_2} \text{Cont}_{\Gamma, w_1, w_2}
$$

of Γ to $\sum_{d\geq 0} q^d \left[\overline{Q}_g(K\mathbb{P}^2, d) \right]$ ^{vir}.

Proposition 28. We have

$$
Cont_{\Gamma} = \sum_{a_1, a_2 > 0} P\left[\psi^{a_1-1} \middle| H_{g_1}^{p_i}\right]_{g_1, 1}^{p_i, 0+} P\left[\psi^{a_2-1} \middle| H_{g_2}^{p_j}\right]_{g_2, 1}^{p_j, 0+}
$$

$$
\cdot (-1)^{a_1+a_2} \left[e^{-\frac{\langle \langle 1, 1 \rangle \rangle_{0, 2}^{p_i, 0+}}{x_1} e^{-\frac{\langle \langle 1, 1 \rangle \rangle_{0, 2}^{p_j, 0+}}{x_2} e_i} \left(\overline{\nabla}_{ij} - \frac{\delta_{ij}}{e_i(x_1+x_2)}\right) e_j\right]_{x_1^{a_1-1} x_2^{a_2-1}}.
$$

4.4. A general graph. We apply the argument of Section [4.3](#page-29-3) to obtain a contribution formula for a general graph Γ.

Let $\Gamma \in \mathsf{G}_{g,0}(\mathbb{P}^2)$ be a decorated graph as defined in Section [1.](#page-11-0) The flags of Γ are the half-edges^{[10](#page-30-0)}. Let $\overline{\mathsf{F}}$ be the set of flags. Let

$$
w: F \to \mathrm{Hom}(T, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}
$$

be a fixed assignment of T-weights to each flag.

We first consider the contribution $Cont_{\Gamma,w}$ to

$$
\sum_{d\geq 0}q^d\left[\overline{Q}_g(K{\mathbb P}^2,d)\right]^{\operatorname{vir}}
$$

of the T-fixed loci associated Γ satisfying the following property: the tangent weight on the first rational component corresponding to each $f \in F$ is exactly given by $w(f)$. We have

(38)
$$
\text{Cont}_{\Gamma,\mathbf{w}} = \frac{1}{|\text{Aut}(\Gamma)|} \sum_{\mathbf{A} \in \mathbb{Z}_{>0}^{\mathsf{F}}} \prod_{v \in \mathsf{V}} \text{Cont}_{\Gamma,\mathbf{w}}^{\mathbf{A}}(v) \prod_{e \in \mathsf{E}} \text{Cont}_{\Gamma,\mathbf{w}}(e) .
$$

The terms on the right side of [\(38\)](#page-30-1) require definition:

• The sum on the right is over the set $\mathbb{Z}_{>0}^{\mathsf{F}}$ of all maps

$$
A:F\to\mathbb{Z}_{>0}
$$

corresponding to the sum over a_1, a_2 in Proposition [28.](#page-30-2)

¹⁰Flags are either half-edges or markings.

• For $v \in V$ with *n* incident flags with w-values (w_1, \ldots, w_n) and A-values $(a_1, a_2, ..., a_n),$

$$
Cont_{\Gamma,w}^{\mathsf{A}}(v) = \frac{\mathsf{P}\left[\psi_1^{a_1-1}, \dots, \psi_n^{a_n-1} \middle| \mathsf{H}_{\mathsf{g}(v)}^{\mathsf{p}(v)}\right]_{\mathsf{g}(v),n}^{\mathsf{p}(v),0+}}{w_1^{a_1} \cdots w_n^{a_n}}.
$$

• For $e \in \mathsf{E}$ with assignments $(\mathsf{p}(v_1), \mathsf{p}(v_2))$ for the two associated vertices^{[11](#page-31-0)} and w-values (w_1, w_2) for the two associated flags,

$$
\mathrm{Cont}_{\Gamma, \mathsf{w}}(e) = e^{\frac{\langle (1,1) \rangle_{0,2}^{\mathsf{p}(v_1),0+}}{w_1}} e^{\frac{\langle (1,1) \rangle_{0,2}^{\mathsf{p}(v_2),0+}}{w_2}} \mathsf{E}_{\mathsf{p}(v_1),\mathsf{p}(v_2)}^{w_1,w_2}.
$$

The localization formula then yields [\(38\)](#page-30-1) just as in the simple case of Section [4.3.](#page-29-3)

By summing the contribution [\(38\)](#page-30-1) of Γ over all the weight functions w and applying Lemma [27,](#page-29-0) we obtain the following result which generalizes Proposition [28.](#page-30-2)

Proposition 29. We have

$$
\mathrm{Cont}_\Gamma=\frac{1}{|\mathrm{Aut}(\Gamma)|}\sum_{\mathsf{A}\in\mathbb{Z}_{>0}^\mathsf{F}}\prod_{v\in\mathsf{V}}\mathrm{Cont}^\mathsf{A}_\Gamma(v)\prod_{e\in\mathsf{E}}\mathrm{Cont}^\mathsf{A}_\Gamma(e)\,,
$$

where the vertex and edge contributions with incident flag A-values (a_1, \ldots, a_n) and (b_1, b_2) respectively are

$$
\text{Cont}_{\Gamma}^{\mathsf{A}}(v) = \mathsf{P}\left[\psi_1^{a_1-1}, \dots, \psi_n^{a_n-1} \middle| \mathsf{H}_{\mathsf{g}(v)}^{\mathsf{p}(v)}\right]_{\mathsf{g}(v),n}^{\mathsf{p}(v),0+},
$$
\n
$$
\text{Cont}_{\Gamma}^{\mathsf{A}}(e) = (-1)^{b_1+b_2} \left[e^{-\frac{\langle (1,1)\rangle_{0,2}^{\mathsf{p}(v_1),0+}}{x_1}} e^{-\frac{\langle (1,1)\rangle_{0,2}^{\mathsf{p}(v_2),0+}}{x_2}} e_i \left(\overline{\mathbb{V}}_{ij} - \frac{1}{e_i(x+y)}\right) e_j \right]_{x_1^{b_1-1} x_2^{b_2-1}},
$$

where $p(v_1) = p_i$ and $p(v_2) = p_i$ in the second equation.

4.5. Legs. Let $\Gamma \in \mathsf{G}_{g,n}(\mathbb{P}^2)$ be a decorated graph with markings. While no markings are needed to define the stable quotient invariants of $K\mathbb{P}^2$, the contributions of decorated graphs with markings will appear in the proof of the holomorphic anomaly equation. The formula for the contribution $Cont_\Gamma(H, \ldots, H)$ of Γ to

$$
\sum_{d\geq 0} q^d \prod_{j=0}^n \mathrm{ev}^*(H) \cap \left[\overline{Q}_{g,n}(K\mathbb{P}^2, d) \right]^{\mathrm{vir}}
$$

is given by the following result.

 $\overline{^{11} \text{In case } e}$ is self-edge, $v_1 = v_2$.

Proposition 30. We have

$$
Cont_{\Gamma}(H, ..., H) = \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\mathbf{A} \in \mathbb{Z}_{>0}^{\mathsf{F}}} \prod_{v \in \mathsf{V}} Cont_{\Gamma}^{\mathbf{A}}(v) \prod_{e \in \mathsf{E}} Cont_{\Gamma}^{\mathbf{A}}(e) \prod_{l \in \mathsf{L}} Cont_{\Gamma}^{\mathbf{A}}(l),
$$

where the leg contribution is

$$
\text{Cont}_{\Gamma}^{\mathsf{A}}(l) = (-1)^{\mathsf{A}(l)-1} \left[e^{-\frac{\langle (1,1) \rangle_{0,2}^{\mathsf{B}(l),0+}}{z}} \overline{\mathbb{S}}_{\mathsf{p}(l)}(H) \right]_{z^{\mathsf{A}(l)-1}}
$$

The vertex and edge contributions are same as before.

The proof of Proposition [30](#page-32-1) follows the vertex and edge analysis. We leave the details as an exercise for the reader. The parallel statement for Gromov-Witten theory can be found in [\[9,](#page-56-19) [10,](#page-56-20) [14\]](#page-56-21).

5. Vertices, edges, and legs

5.1. Overview. Using the results of Givental [\[9,](#page-56-19) [10,](#page-56-20) [14\]](#page-56-21) combined with wall-crossing [\[7\]](#page-56-4), we calculate here the vertex and edge contributions in terms of the function R_k of Section [2.4.4.](#page-19-3)

5.2. Calculations in genus 0. We follow the notation introduced in Section [3.1.](#page-21-2) Recall the series

$$
T(c) = t_0 + t_1 c + t_2 c^2 + \dots
$$

Proposition 31. (Givental [\[9,](#page-56-19) [10,](#page-56-20) [14\]](#page-56-21)) For $n \geq 3$, we have

$$
\langle \langle 1, \ldots, 1 \rangle \rangle_{0,n}^{p_i, \infty} =
$$

$$
(\sqrt{\Delta_i})^{2g-2+n} \left(\sum_{k \ge 0} \frac{1}{k!} \int_{\overline{M}_{0,n+k}} T(\psi_{n+1}) \cdots T(\psi_{n+k}) \right) \Big|_{t_0 = 0, t_1 = 0, t_j \ge 2} = (-1)^j Q_{j-1,i}
$$

where the functions $\sqrt{\Delta_i}$, $Q_{l,i}$ are defined by

$$
\overline{\mathbb{S}}_i^{\infty}(1) = e_i \left\langle \left\langle \frac{\phi_i}{z - \psi}, 1 \right\rangle \right\rangle_{0,2}^{p_i, \infty} = \frac{e^{\frac{\left\langle (1,1)\right\rangle_{0,2}^{p_i, \infty}}{z}}}{\sqrt{\Delta_i}} \left(1 + \sum_{l=1}^{\infty} Q_{l,i} z^l\right).
$$

The existence of the above asymptotic expansion of $\overline{S}_{i}^{\infty}$ $\sum_{i}^{8}(1)$ can also be proven by the argument of [\[4,](#page-56-16) Theorem 5.4.1]. Similarly, we have an asymptotic expansion of $\overline{S}_i(1)$,

$$
\overline{S}_i(1) = e^{\frac{\langle \langle 1,1 \rangle \rangle_{0,2}^{p_i,0+}}{z}} \left(\sum_{l=0}^{\infty} R_{l,i} z^l \right).
$$

.

By (28) , we have

$$
\langle \langle 1, 1 \rangle \rangle_{0,2}^{p_i, 0+} = \mu_i.
$$

After applying the wall-crossing result of Proposition [26,](#page-27-0) we obtain

$$
\langle\langle 1,\ldots,1\rangle\rangle_{0,n}^{p_i,\infty}(Q(q)) = \langle\langle 1,\ldots,1\rangle\rangle_{0,n}^{p_i,0+}(q),
$$

$$
\overline{\mathbb{S}}_i^{\infty}(1)(Q(q)) = \overline{\mathbb{S}}_i(1)(q),
$$

where $Q(q)$ is mirror map for $K\mathbb{P}^2$ as before. By comparing asymptotic expansions of $\overline{\mathbb{S}}_i^{\infty}$ $\sum_{i=1}^{\infty}$ (1) and \overline{S}_i (1), we get a wall-crossing relation between $Q_{l,i}$ and $R_{l,i}$,

$$
\sqrt{\Delta_i}(Q(q)) = \frac{1}{R_{0,i}(q)},
$$

$$
Q_{l,i}(Q(q)) = \frac{R_{l,i}(q)}{R_{0,i}(q)} \text{ for } l \ge 1.
$$

We have proven the following result.

Proposition 32. For $n \geq 3$, we have

$$
\langle \langle 1, \ldots, 1 \rangle \rangle_{0,n}^{p_i,0+} =
$$

$$
R_{0,i}^{2g-2+n} \left(\sum_{k \geq 0} \frac{1}{k!} \int_{\overline{M}_{0,n+k}} T(\psi_{n+1}) \cdots T(\psi_{n+k}) \right) \Big|_{t_0=0, t_1=0, t_j \geq 2-(-1)^j} \frac{R_{j-1,i}}{R_{0,i}}.
$$

Proposition [32](#page-33-0) immediately implies the evaluation

(39)
$$
\langle \langle 1, 1, 1 \rangle \rangle_{0,3}^{p_i,0+} = \frac{1}{R_{0,i}}.
$$

Another simple consequence of Proposition [32](#page-33-0) is the following basic property.

Corollary 33. For $n \geq 3$, we have

$$
\langle \langle 1, \ldots, 1 \rangle \rangle_{0,n}^{p_i,0+} \in \mathbb{C}[R_{0,i}^{\pm 1}, R_{1,i}, R_{2,i}, \ldots].
$$

5.3. Vertex and edge analysis. By Proposition [29,](#page-31-1) we have decomposition of the contribution to $\Gamma \in \mathsf{G}_g(\mathbb{P}^2)$ to the stable quotient theory of $K\mathbb{P}^2$ into vertex terms and edge terms

$$
\mathrm{Cont}_\Gamma=\frac{1}{|\mathrm{Aut}(\Gamma)|}\sum_{\mathsf{A}\in\mathbb{Z}_{>0}^\mathsf{F}}\prod_{v\in\mathsf{V}}\mathrm{Cont}^\mathsf{A}_\Gamma(v)\prod_{e\in\mathsf{E}}\mathrm{Cont}^\mathsf{A}_\Gamma(e)\,.
$$

Lemma 34. Suppose Conjecture [15](#page-20-0) is true. Then we have

$$
\text{Cont}^{\mathsf{A}}_{\Gamma}(v) \in \mathbb{G}_2 \, .
$$

Proof. By Proposition [29,](#page-31-1)

$$
Cont_{\Gamma}^{\mathsf{A}}(v) = \mathsf{P}\left[\psi_1^{a_1-1}, \ldots, \psi_n^{a_n-1} \, \middle| \, \mathsf{H}_{\mathsf{g}(v)}^{\mathsf{p}(v)}\right]_{\mathsf{g}(v),n}^{\mathsf{p}(v),0+}
$$

The right side of the above formula is a polynomial in the variables

$$
\frac{1}{\langle\langle 1,1,1\rangle\rangle_{0,3}^{\mathsf{p}(v),0+}} \quad \text{and} \quad \left\{ \langle\langle 1,\ldots,1\rangle\rangle_{0,n}^{\mathsf{p}(v),0+} |_{t_0=0} \right\}_{n\geq 4}
$$

with coefficients in $\mathbb{C}(\lambda_0, \lambda_1, \lambda_2)$. The Lemma then follows from the evaluation [\(39\)](#page-33-1), Corollary [33,](#page-33-2) and Conjecture [15.](#page-20-0)

Let $e \in \mathsf{E}$ be an edge connecting the T-fixed points $p_i, p_j \in \mathbb{P}^2$. Let the A-values of the respective half-edges be (k, l) .

Lemma 35. Suppose Conjecture [15](#page-20-0) is true. Then we have $Cont^A_{\Gamma}(e) \in$ $G[X]$ and

- the degree of $\text{Cont}^{\mathsf{A}}_{\Gamma}(e)$ with respect to X is 1,
- the coefficient of X in Cont $_{\Gamma}^{A}(e)$ is

$$
(-1)^{k+l}\frac{3L_iL_jR_{1k-1,i}R_{1l-1,j}}{L^3}.
$$

Proof. By Proposition [29,](#page-31-1)

$$
\text{Cont}_{\Gamma}^{\mathsf{A}}(e) = (-1)^{k+l} \left[e^{-\frac{\mu \lambda_i}{x} - \frac{\mu \lambda_j}{y}} e_i \left(\overline{\nabla}_{ij} - \frac{\delta_{ij}}{e_i (x+y)} \right) e_j \right]_{x^{k-1} y^{l-1}}.
$$

Using also the equation

$$
e_i \overline{\mathbb{V}}_{ij}(x, y) e_j = \frac{\sum_{r=0}^2 \overline{\mathbb{S}}_i(\phi_r)|_{z=x} \overline{\mathbb{S}}_j(\phi^r)|_{z=y}}{x+y},
$$

we write $\text{Cont}_{\Gamma}^{\mathsf{A}}(e)$ as

$$
\left[(-1)^{k+l}e^{-\frac{\mu\lambda_i}{x}-\frac{\mu\lambda_j}{y}}\sum_{r=0}^2\overline{\mathbb{S}}_i(\phi_r)|_{z=x}\overline{\mathbb{S}}_j(\phi^r)|_{z=y}\right]_{x^ky^{l-1}-x^{k+1}y^{l-2}+\ldots+(-1)^{k-1}x^{k+l-1}}
$$

where the subscript signifies a (signed) sum of the respective coefficients. If we substitute the asymptotic expansions [\(28\)](#page-20-1) for

$$
\overline{\mathbb{S}}_i(1)\,,\quad \overline{\mathbb{S}}_i(H)\,,\quad \overline{\mathbb{S}}_i(H^2)
$$

in the above expression, the Lemma follows from Conjecture [15,](#page-20-0) Lemma [18](#page-21-3) and [\(31\)](#page-21-4). \Box

.

5.4. Legs. Using the contribution formula of Proposition [30,](#page-32-1)

$$
\text{Cont}_{\Gamma}^{\mathsf{A}}(l) = (-1)^{\mathsf{A}(l)-1} \left[e^{-\frac{\langle (1,1) \rangle_{0,2}^{\mathsf{P}(l),0+}}{z}} \overline{\mathbb{S}}_{\mathsf{p}(l)}(H) \right]_{z^{\mathsf{A}(l)-1}},
$$

we easily conclude under the assumption of Conjecture [15](#page-20-0)

$$
C_1 \cdot \text{Cont}^{\mathsf{A}}_{\Gamma}(l) \in \mathbb{G}_2 \, .
$$

6. HOLOMORPHIC ANOMALY FOR $K\mathbb{P}^2$

6.1. Proof of Theorem [3.](#page-4-0) By definition, we have

(40)
$$
A_2(q) = \frac{1}{L^3} \left(3X + 1 - \frac{L^3}{2} \right).
$$

Conjecture [15](#page-20-0) was proven in Appendix for the choices of $\lambda_0, \lambda_1, \lambda_2$ such that

$$
(\lambda_0\lambda_1 + \lambda_1\lambda_2 + \lambda_2\lambda_0)^2 - 3\lambda_0\lambda_1\lambda_2(\lambda_0 + \lambda_1 + \lambda_2) = 0.
$$

Hence, statement (i),

$$
\mathcal{F}_g^{\mathsf{SQ}}(q) \in \mathbb{G}_2[A_2],
$$

follows from Proposition [29](#page-31-1) and Lemmas [34](#page-33-3) - [35.](#page-34-0) Statement (ii), $\mathcal{F}_g^{\mathsf{SQ}}$ has at most degree $3g-3$ with respect to A_2 , holds since a stable graph of genus g has at most $3g - 3$ edges. Since

$$
\frac{\partial}{\partial T} = \frac{q}{C_1} \frac{\partial}{\partial q} \,,
$$

statement (iii),

(41)
$$
\frac{\partial^k \mathcal{F}_g^{\mathsf{SQ}}}{\partial T^k}(q) \in \mathbb{G}_2[A_2][C_1^{-1}],
$$

follows since the ring

$$
\mathbb{G}_2[A_2] = \mathbb{G}_2[X]
$$

is closed under the action of the differential operator

$$
\mathsf{D}=q\frac{\partial}{\partial q}
$$

by [\(29\)](#page-21-5). The degree of C_1^{-1} in [\(41\)](#page-35-1) is 1 which yields statement (iv). \Box

6.2. **Proof of Theorem [2.](#page-3-2)** Let $\Gamma \in \mathsf{G}_g(\mathbb{P}^2)$ be a decorated graph. Let us fix an edge $f \in E(\Gamma)$:

• if Γ is connected after deleting f, denote the resulting graph by

$$
\Gamma_f^0 \in \mathsf{G}_{g-1,2}(\mathbb{P}^2) \,,
$$

•• if Γ is disconnected after deleting f, denote the resulting two graphs by

$$
\Gamma_f^1 \in \mathsf{G}_{g_1,1}(\mathbb{P}^2) \quad \text{and} \quad \Gamma_f^2 \in \mathsf{G}_{g_2,1}(\mathbb{P}^2)
$$

where $g = g_1 + g_2$.

There is no canonical order for the 2 new markings. We will always sum over the 2 labellings. So more precisely, the graph Γ_f^0 in case \bullet should be viewed as sum of 2 graphs

$$
\Gamma^0_{f,(1,2)} + \Gamma^0_{f,(2,1)} \, .
$$

Similarly, in case $\bullet\bullet$, we will sum over the ordering of g_1 and g_2 . As usual, the summation will be later compensated by a factor of $\frac{1}{2}$ in the formulas.

By Proposition [29,](#page-31-1) we have the following formula for the contribution of the graph Γ to the stable quotient theory of $K\mathbb{P}^2$,

$$
\mathrm{Cont}_\Gamma=\frac{1}{|\mathrm{Aut}(\Gamma)|}\sum_{\mathsf{A}\in\mathbb{Z}_{\geq 0}^\mathsf{F}}\prod_{v\in\mathsf{V}}\mathrm{Cont}^\mathsf{A}_\Gamma(v)\prod_{e\in\mathsf{E}}\mathrm{Cont}^\mathsf{A}_\Gamma(e)\,.
$$

Let f connect the T-fixed points $p_i, p_j \in \mathbb{P}^2$. Let the A-values of the respective half-edges be (k, l) . By Lemma [35,](#page-34-0) we have

(42)
$$
\frac{\partial \text{Cont}_{\Gamma}^{\mathsf{A}}(f)}{\partial X} = (-1)^{k+l} \frac{3L_i L_j R_{1k-1,i} R_{1l-1,j}}{L^3}.
$$

• If Γ is connected after deleting f, we have

$$
\frac{1}{|\text{Aut}(\Gamma)|} \sum_{\mathsf{A}\in\mathbb{Z}_{\geq 0}^{\mathsf{F}}}\left(\frac{L^3}{3C_1^2}\right) \frac{\partial \text{Cont}_{\Gamma}^{\mathsf{A}}(f)}{\partial X} \prod_{v\in\mathsf{V}} \text{Cont}_{\Gamma}^{\mathsf{A}}(v) \prod_{e\in\mathsf{E}, e\neq f} \text{Cont}_{\Gamma_f^0}^{\mathsf{A}}(e)\\ = \frac{1}{2}\, \text{Cont}_{\Gamma_f^0}(H,H)\,.
$$

The derivation is simply by using [\(42\)](#page-36-0) on the left and Proposition [30](#page-32-1) on the right.

•• If Γ is disconnected after deleting f, we obtain

$$
\frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\mathsf{A}\in\mathbb{Z}_{\geq 0}^{\mathsf{F}}}\left(\frac{L^3}{3C_1^2}\right) \frac{\partial \mathrm{Cont}^{\mathsf{A}}_{\Gamma}(f)}{\partial X} \prod_{v\in\mathsf{V}} \mathrm{Cont}^{\mathsf{A}}_{\Gamma}(v) \prod_{e\in\mathsf{E}, e\neq f} \mathrm{Cont}^{\mathsf{A}}_{\Gamma}(e) \n= \frac{1}{2} \mathrm{Cont}_{\Gamma_f^1}(H) \, \mathrm{Cont}_{\Gamma_f^2}(H)
$$

by the same method.

By combining the above two equations for all the edges of all the graphs $\Gamma \in \mathsf{G}_g(\mathbb{P}^2)$ and using the vanishing

$$
\frac{\partial \text{Cont}^{\mathsf{A}}_{\Gamma}(v)}{\partial X} = 0
$$

of Lemma [34,](#page-33-3) we obtain

(43)
$$
\left(\frac{L^3}{3C_1^2}\right) \frac{\partial}{\partial X} \langle \rangle_{g,0}^{\mathsf{SQ}} = \frac{1}{2} \sum_{i=1}^{g-1} \langle H \rangle_{g-i,1}^{\mathsf{SQ}} \langle H \rangle_{i,1}^{\mathsf{SQ}} + \frac{1}{2} \langle H, H \rangle_{g-1,2}^{\mathsf{SQ}}.
$$

We have followed here the notation of Section [0.2.](#page-1-2) The equality [\(43\)](#page-37-0) holds in the ring $\mathbb{G}_2[A_2, C_1^{-1}]$.

Since $A_2 = \frac{1}{L^3} (3X + 1 - \frac{L^3}{2})$ \mathcal{L}^3_2) and $\langle \ \rangle_{g,0}^{\mathsf{SQ}} = \mathcal{F}_g^{\mathsf{SQ}}$, the left side of [\(43\)](#page-37-0) is, by the chain rule,

$$
\frac{1}{C_1^2} \frac{\partial \mathcal{F}_g^{\mathsf{SQ}}}{\partial A_2} \in \mathbb{G}_2[A_2, C_1^{-1}].
$$

On the right side of [\(43\)](#page-37-0), we have

$$
\langle H \rangle_{g-i,1}^{\mathsf{SQ}} = \mathcal{F}_{g-i,1}^{\mathsf{SQ}}(q) = \mathcal{F}_{g-i,1}^{\mathsf{GW}}(Q(q)),
$$

where the first equality is by definition and the second is by wallcrossing [\(4\)](#page-2-0). Then,

$$
\mathcal{F}_{g-i,1}^{\mathsf{GW}}(Q(q)) = \frac{\partial \mathcal{F}_{g-i}^{\mathsf{GW}}}{\partial T}(Q(q)) = \frac{\partial \mathcal{F}_{g-i}^{\mathsf{SQ}}}{\partial T}(q)
$$

where the first equality is by the divisor equation in Gromov-Witten theory and the second is again by wall-crossing [\(4\)](#page-2-0), so we conclude

$$
\langle H \rangle^{\mathsf{SQ}}_{g-i,1} = \frac{\partial \mathcal{F}_{g-i}^{\mathsf{SQ}}}{\partial T}(q) \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]]\,.
$$

Similarly, we obtain

$$
\langle H \rangle^{\mathsf{SQ}}_{i,1} = \frac{\partial \mathcal{F}^{\mathsf{SQ}}_i}{\partial T}(q) \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]],
$$

$$
\langle H, H \rangle^{\mathsf{SQ}}_{g-1,2} = \frac{\partial^2 \mathcal{F}^{\mathsf{SQ}}_{g-1}}{\partial T^2}(q) \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]].
$$

Together, the above equations transform [\(43\)](#page-37-0) into exactly the holomorphic anomaly equation of Theorem [4,](#page-4-1)

$$
\frac{1}{C_1^2} \frac{\partial \mathcal{F}_g^{\mathsf{SQ}}}{\partial A_2}(q) = \frac{1}{2} \sum_{i=1}^{g-1} \frac{\partial \mathcal{F}_{g-i}^{\mathsf{SQ}}}{\partial T}(q) \frac{\partial \mathcal{F}_i^{\mathsf{SQ}}}{\partial T}(q) + \frac{1}{2} \frac{\partial^2 \mathcal{F}_{g-1}^{\mathsf{SQ}}}{\partial T^2}(q)
$$

as an equality in $\mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]]$.

The series L and A_2 are expected to be algebraically independent. Since we do not have a proof of the independence, to lift holomorphic anomaly equation to the equality

$$
\frac{1}{C_1^2} \frac{\partial \mathcal{F}_g^{\mathsf{SQ}}}{\partial A_2} = \frac{1}{2} \sum_{i=1}^{g-1} \frac{\partial \mathcal{F}_{g-i}^{\mathsf{SQ}}}{\partial T} \frac{\partial \mathcal{F}_i^{\mathsf{SQ}}}{\partial T} + \frac{1}{2} \frac{\partial^2 \mathcal{F}_{g-1}^{\mathsf{SQ}}}{\partial T^2}
$$

in the ring $\mathbb{G}_2[A_2, C_1^{-1}]$, we must prove the equalities

(44)
$$
\langle H \rangle_{g-i,1}^{\mathsf{SQ}} = \frac{\partial \mathcal{F}_{g-i}^{\mathsf{SQ}}}{\partial T}, \quad \langle H \rangle_{i,1}^{\mathsf{SQ}} = \frac{\partial \mathcal{F}_{i}^{\mathsf{SQ}}}{\partial T},
$$

$$
\langle H, H \rangle_{g-1,2}^{\mathsf{SQ}} = \frac{\partial^2 \mathcal{F}_{g-1}^{\mathsf{SQ}}}{\partial T^2}
$$

in the ring $\mathbb{G}_2[A_2, C_1^{-1}]$. The lifting follow from the argument in Section 7.3 in [\[16\]](#page-56-0).

We do not study the genus 1 unpointed series $\mathcal{F}^{\mathsf{SQ}}_1$ $i^{sq}(q)$ in the paper, so we take

$$
\langle H \rangle^{\mathsf{SQ}}_{1,1} = \frac{\partial \mathcal{F}_{1}^{\mathsf{SQ}}}{\partial T},
$$

$$
\langle H, H \rangle^{\mathsf{SQ}}_{1,2} = \frac{\partial^2 \mathcal{F}_{1}^{\mathsf{SQ}}}{\partial T^2}
$$

.

as definitions of the right side in the genus 1 case. There is no difficulty in calculating these series explicitly using Proposition [30.](#page-32-1)

7. HOLOMORPHIC ANOMALY FOR $K\mathbb{P}^3$

7.1. Overview. We fix a torus action $\mathsf{T} = (\mathbb{C}^*)^4$ on \mathbb{P}^3 with weights^{[12](#page-38-1)}

$$
-\lambda_0,\ldots,-\lambda_3
$$

on the vector space \mathbb{C}^4 . The T-weight on the fiber over p_i of the canonical bundle

$$
(45) \t\t \t\t \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathbb{P}^3
$$

¹²The associated weights on $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ are $\lambda_0, \ldots, \lambda_3$ and so match the conventions of Section [0.1.](#page-0-1)

is $-4\lambda_i$. The toric Calabi-Yau $K\mathbb{P}^3$ is the total space of [\(45\)](#page-38-2). The basic generating series and other essential objects defined in Section [2](#page-13-0) - Section [5](#page-32-0) can be defined^{[13](#page-39-0)} similarly for $K\mathbb{P}^3$. We will not repeat the definitions of these objects unless necessary.

7.2. I-functions.

7.2.1. Evaluations. Let $\widetilde{H} \in H^*_{\mathsf{T}}([\mathbb{C}^4/\mathbb{C}^*])$ and $H \in H^*_{\mathsf{T}}(\mathbb{P}^3)$ denote the respective hyperplane classes. The I-function of Definition [13](#page-18-0) for $K\mathbb{P}^3$ is evaluated in [\[5\]](#page-56-15).

Proposition 36. For $\mathbf{t} = t\widetilde{H} \in H^*_{\mathsf{T}}([\mathbb{C}^4/\mathbb{C}^*], \mathbb{Q}),$

(46)
$$
\mathbb{I}(t) = \sum_{d=0}^{\infty} q^d e^{t(H+dz)/z} \frac{\prod_{k=0}^{4d-1} (-4H - kz)}{\prod_{i=0}^{3} \prod_{k=1}^{d} (H - \lambda_i + kz)}.
$$

We define the series $I_{i,j}$ by following expansion of the I-function after restriction $t = 0$,

$$
\mathbb{I}|_{t=0} = 1 + \frac{I_{10}H}{z} + \frac{I_{20}H^2 + I_{21}H}{z^2} + \frac{I_{30}H^3 + I_{31}H^2 + I_{32}H}{z^3} + \mathcal{O}(\frac{1}{z^4}).
$$

For example,

$$
I_{10}(q) = \sum_{d=1}^{\infty} 4 \frac{(4d-1)!}{(d!)^4} q^d \in \mathbb{C}[[q]] ,
$$

\n
$$
I_{20}(q) = \sum_{d=1}^{\infty} 4 \frac{(4d-1)!}{(d!)^4} \left(4\text{Har}[4d-1] - 4\text{Har}[d] \right) q^d \in \mathbb{C}[[q]] ,
$$

\n
$$
I_{21}(q) = \sum_{d=1}^{\infty} 4s_1 \frac{(4d-1)!}{(d!)^4} \text{Har}[d] q^d \in \mathbb{C}(\lambda_0, \dots, \lambda_3)[[q]] .
$$

Here $\text{Har}[d] := \sum_{k=1}^d$ 1 $\frac{1}{k}$.

¹³In fact the contents of Section [2](#page-13-0) – [5](#page-32-0) can be stated universally for all $K\mathbb{P}^n$.

We return now to the functions $\mathbb{S}_i(\gamma)$ defined in Section [2.3.](#page-14-1) We define the following additional series in q :

$$
C_1 = 1 + DI_{10}, J_{10} = \frac{I_{10} + DI_{20}}{C_1}, J_{11} = \frac{DI_{21}}{C_1},
$$

\n
$$
J_{20} = \frac{I_{20} + DI_{30}}{C_1}, J_{21} = \frac{I_{21} + DI_{31}}{C_1}, J_{22} = \frac{I_{22} + DI_{32}}{C_1},
$$

\n
$$
C_2 = 1 + DJ_{10}, K_{10} = \frac{J_{10} + DJ_{20}}{C_2},
$$

\n
$$
K_{11} = \frac{J_{11} + DJ_{21} - (DJ_{11})J_{10}}{C_2}, K_{12} = \frac{DJ_{22} - (DJ_{11})J_{11}}{C_2},
$$

\n
$$
C_3 = 1 + DK_{10}.
$$

Here, $D = q \frac{d}{dq}$. The following relations were proven in [\[21\]](#page-56-3),

(47)
$$
C_2 = C_3,
$$

$$
C_1^2 C_2^2 = (1 - 4^4 q)^{-1}.
$$

Using Birkhoff factorization, an evaluation of the series $S(H^j)$ can be obtained from the I-function, see [\[13\]](#page-56-13):

(48)
$$
S(1) = I,
$$

\n
$$
S(H) = \frac{z \frac{d}{dt} S(1)}{C_1},
$$

\n
$$
S(H^2) = \frac{z \frac{d}{dt} S(H) - (DJ_{11}) S(H)}{C_2},
$$

\n
$$
S(H^3) = \frac{z \frac{d}{dt} S(H^2) - (DK_{11}) S(H^2) - (DK_{12}) S(H)}{C_3}.
$$

7.2.2. Further calculations. Define small I-function

$$
\overline{\mathbb{I}}(q) \in H^*_\mathsf{T}(\mathbb{P}^3,\mathbb{Q})[[q]]
$$

by the restriction

$$
\overline{\mathbb{I}}(q) = \mathbb{I}(q,t)|_{t=0}.
$$

Define differential operators

$$
D = q \frac{d}{dq}, \quad M = H + zD.
$$

Applying $z\frac{d}{dt}$ to I and then restricting to $t = 0$ has same effect as applying M to $\bar{\mathbb{I}}$

$$
\left[\left(z \frac{d}{dt} \right)^k \mathbb{I} \right] \Big|_{t=0} = M^k \overline{\mathbb{I}} \, .
$$

The function $\overline{\mathbb{I}}$ satisfies following Picard-Fuchs equation

(49)
$$
\left(\prod_{j=0}^{3} (M - \lambda_j) - 4qM(4M + z)(4M + 2z)(4M + 3z)\right)\overline{\mathbb{I}} = 0
$$

implied by the Picard-Fuchs equation for I,

 $\overline{2}$

$$
\left(\prod_{j=0}^3 \left(z\frac{d}{dt} - \lambda_j\right) - q \prod_{k=0}^3 \left(4z\frac{d}{dt} + kz\right)\right) \mathbb{I} = 0.
$$

The restriction $\overline{\mathbb{I}}|_{H=\lambda_i}$ admits following asymptotic form

(50)
$$
\overline{\mathbb{I}}|_{H=\lambda_i} = e^{\mu_i/z} \left(R_{0,i} + R_{1,i} z + R_{2,i} z^2 + \ldots \right)
$$

with series $\mu_i, R_{k,i} \in \mathbb{C}(\lambda_0, \ldots, \lambda_d)[[q]]$.

A derivation of [\(50\)](#page-41-0) is obtained in [\[21\]](#page-56-3) via the Picard-Fuchs equation [\(49\)](#page-41-1) for $\overline{\mathbb{I}}|_{H=\lambda_i}$. The series μ_i and $R_{k,i}$ are found by solving differential equations obtained from the coefficient of z^k . For example,

$$
\lambda_i + D\mu_i = L_i,
$$

\n
$$
R_{0,i} = \left(\frac{\lambda_i \prod_{j \neq i} (\lambda_i - \lambda_j)}{f(L_i)}\right)^{\frac{1}{2}}.
$$

From the equations [\(48\)](#page-40-0) and [\(50\)](#page-41-0), we can show the series

 $\overline{S}_i(1) = \overline{S}(1)|_{H=\lambda_i}, \ \ \overline{S}_i(H) = \overline{S}(H)|_{H=\lambda_i}, \ \ \overline{S}_i(H^2) = \overline{S}(H^2)|_{H=\lambda_i}, \ \ \overline{S}_i(H^3) = \overline{S}(H^3)|_{H=\lambda_i}$ have the following asymptotic expansions:

(51)
$$
\overline{\mathbf{S}}_i(1) = e^{\frac{\mu_i}{z}} \Big(R_{00,i} + R_{01,i} z + R_{02,i} z^2 + \dots \Big),
$$

\n
$$
\overline{\mathbf{S}}_i(H) = e^{\frac{\mu_i}{z}} \frac{L_i}{C_1} \Big(R_{10,i} + R_{11,i} z + R_{12} z^2 + \dots \Big),
$$

\n
$$
\overline{\mathbf{S}}_i(H^2) = e^{\frac{\mu_i}{z}} \frac{L_i^2}{C_1 C_2} \Big(R_{20,i} + R_{21,i} z + R_{22,i} z^2 + \dots \Big),
$$

\n
$$
\overline{\mathbf{S}}_i(H^3) = e^{\frac{\mu_i}{z}} \frac{L_i^3}{C_1 C_2 C_3} \Big(R_{30,i} + R_{31,i} z + R_{32,i} z^2 + \dots \Big).
$$

We follow here the normalization of [\[21\]](#page-56-3). Note

$$
R_{0k,i} = R_{k,i}.
$$

As in [\[21,](#page-56-3) Theorem 4], we expect the following constraints.

Conjecture 37. For all $k \geq 0$, we have

$$
R_{k,i}\in\mathbb{G}_3.
$$

Conjecture [37](#page-42-0) is the main obstruction for the proof of Conjecture [5](#page-6-3) and [6.](#page-6-2) By the same argument of Section [7,](#page-38-0) we obtain the following result.

Theorem 38. Conjecture [37](#page-42-0) implies Conjecture [5](#page-6-3) and [6.](#page-6-2)

By applying asymptotic expansions [\(51\)](#page-41-2) to [\(48\)](#page-40-0), we obtain the following results.

Lemma 39. We have

$$
R_{1 p+1,i} = R_{0 p+1,i} + \frac{D R_{0 p,i}}{L_i},
$$

\n
$$
R_{2 p+1,i} = R_{1 p+1,i} - E_{11,i} R_{1 k,i} + \frac{D R_{1 p,i}}{L_i} + \left(\frac{D L_i}{L_i^2} - \frac{A_2}{L_i}\right) R_{1 p,i},
$$

\n
$$
R_{3 p+1,i} = R_{2 p+1,i} - E_{21,i} R_{2 k,i} - E_{22,i} R_{1 k,i} + \frac{D R_{2 p,i}}{L_i} + \left(2 \frac{D L_i}{L_i^2} - \frac{A_2}{L_i} - \frac{\frac{D C_2}{C_2}}{L_i}\right) R_{1 p,i}
$$

\nwith

with

$$
E_{11,i} = \frac{\mathsf{D} J_{11}}{L_i}, \ E_{21,i} = \frac{\mathsf{D} K_{11}}{L_i}, \ E_{22,i} = \frac{C_2}{L_i^2} \mathsf{D} K_{12}.
$$

7.3. Determining $\mathsf{D}A_2$ and new series. The following relation was proven in [\[16\]](#page-56-0).

(52)
$$
A_2^2 + (L^4 - 1)A_2 + 2DA_2 - \frac{3}{16}(L^4 - 1) = 0.
$$

By the above result, the differential ring

(53) G3[A2, DA2, DDA2, . . .]

is just the polynomial ring $G[A_2]$. The second equation in [\(47\)](#page-40-1) yields the following relation.

(54)
$$
2A_2 + 2\frac{DC_2}{C_2} = L^4 - 1.
$$

Denote by $\text{Coeff}(x^i y^j)$ the coefficient of $x^i y^j$ in

$$
\sum_{k=0}^{3} e^{-\frac{\mu_{i}}{x} - \frac{\mu_{i}}{y}} \mathbb{S}_{i}(\phi_{k})|_{z=x} \, \mathbb{S}_{i}(\phi^{k})|_{z=y} \, .
$$

From (18) and (51) , we obtain the following equations.

$$
Coeff(x2) - \frac{1}{2} Coeff(xy) = 0,
$$

\n
$$
Coeff(x4) - Coeff(x3y) + \frac{1}{2} Coeff(x2y2) = 0.
$$

Above equations immediately yields the following relations.

(55)
$$
E_{11,i} = \frac{E_{21,i}}{2} - \frac{s_1 L^2}{2C_1 L_i} + \frac{s_1 L^4}{2L_i},
$$

$$
E_{22,i} = \frac{L^4(s_1^2(-3 + 2C_1 L^2 + C_1^2 L^4) - 4s_2(-1 + C_1^2))}{8C_1^2 L_i^2}
$$

$$
\frac{s_1(-3L^2 + C_1 L^4)}{4C_1 L_i} E_{21,i} - \frac{3}{8} E_{21}^2.
$$

We define the series B_2 and B_4 which appeared in the introduction by

(56)
$$
B_2 = L_i E_{21,i}, B_4 = DB_2.
$$

Note that $B_2(q)$, $B_4(q) \in \mathbb{C}[[q]]$.

From Lemma [39](#page-42-1) with the relations (52) , (54) and (55) , we obtain results for $\overline{S}(H)|_{H=\lambda_i}$, $\overline{S}(H^2)|_{H=\lambda_i}$ and $\overline{S}(H^3)|_{H=\lambda_i}$.

Lemma 40. Suppose Conjecture [37](#page-42-0) is true. Then for all $k \geq 0$, we have for all $k \geq 0$,

$$
R_{1\,k,i},\,R_{2\,k,i}\,,R_{3\,k,i}\in\mathbb{G}_3[A_2,B_2,B_4,C_1^{\pm 1}]\,.
$$

7.4. Vertex, edge, and leg analysis. By parallel argument as in Section [4,](#page-28-0) we have decomposition of the contribution to $\Gamma \in \mathsf{G}_{g,k}(\mathbb{P}^3)$ to the stable quotient theory of $K\mathbb{P}^3$ into vertex terms, edge terms and leg terms

$$
\mathrm{Cont}_\Gamma=\frac{1}{|\mathrm{Aut}(\Gamma)|}\sum_{\mathsf{A}\in\mathbb{Z}_{>0}^\mathsf{F}}\prod_{v\in\mathsf{V}}\mathrm{Cont}^\mathsf{A}_\Gamma(v)\prod_{e\in\mathsf{E}}\mathrm{Cont}^\mathsf{A}_\Gamma(e)\prod_{e\in\mathsf{L}}\mathrm{Cont}^\mathsf{A}_\Gamma(l).
$$

The following lemmas follow from the argument in Section [5.](#page-32-0)

Lemma 41. Suppose Conjecture [37](#page-42-0) is true. Then we have

$$
\text{Cont}_{\Gamma}^{\mathsf{A}}(v) \in \mathbb{G}_3.
$$

Let $e \in \mathsf{E}$ be an edge connecting the T-fixed points $p_i, p_j \in \mathbb{P}^3$. Let the A-values of the respective half-edges be (k, l) .

Lemma 42. Suppose Conjecture [37](#page-42-0) is true. Then we have

Cont_r^A(e) \in G₃[$A_2, B_2, B_4, C_1^{\pm 1}$].

Lemma 43. Suppose Conjecture [37](#page-42-0) is true. Then we have

$$
Cont^{\mathsf{A}}_{\Gamma}(l) \in \mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}].
$$

7.5. Proof of Theorem [7.](#page-7-0) Conjecture [37](#page-42-0) can be proven for the choices of $\lambda_0, \ldots, \lambda_3$ such that

$$
\lambda_i \neq \lambda_j \text{ for } i \neq j,
$$

\n $4s_2^2 - s_1s_3 = 0,$
\n $2s_2^3 - 27s_1^2s_4 = 0.$

by the argument in Appendix. Hence, statement (i),

$$
\mathcal{F}_{g,a+b}^{\mathsf{SQ}}[a,b](q) \in \mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}],
$$

follows from the arguments in Proposition [29](#page-31-1) and Lemmas [41](#page-43-2) - [43.](#page-44-0) Statement (ii), $\mathcal{F}_g^{\mathsf{SQ}}$ has at most degree $2(3g-3)$ with respect to A_2 , holds since a stable graph of genus g has at most $3g - 3$ edges. Since

$$
\frac{\partial}{\partial T} = \frac{q}{C_1} \frac{\partial}{\partial q} \,,
$$

statement (iii),

(57)
$$
\frac{\partial^k \mathcal{F}_g^{\mathsf{SQ}}}{\partial T^k}(q) \in \mathbb{G}[A_2, B_2, B_4, C_1^{\pm 1}],
$$

follows from divisor equation in stable quotient theory and statement (i).

7.6. Proof of Theorem [6:](#page-6-2) first equation. Let $\Gamma \in \mathsf{G}_g(\mathbb{P}^3)$ be a decorated graph. Let us fix an edge $f \in \mathsf{E}(\Gamma)$:

• if Γ is connected after deleting f , denote the resulting graph by

$$
\Gamma_f^0 \in \mathsf{G}_{g-1,2}(\mathbb{P}^3) \,,
$$

•• if Γ is disconnected after deleting f, denote the resulting two graphs by

$$
\Gamma_f^1\in\mathsf{G}_{g_1,1}(\mathbb{P}^3)\quad\text{and}\quad \Gamma_f^2\in\mathsf{G}_{g_2,1}(\mathbb{P}^3)
$$

where $q = q_1 + q_2$.

There is no canonical order for the 2 new markings. We will always sum over the 2 labellings. So more precisely, the graph Γ_f^0 in case \bullet should be viewed as sum of 2 graphs

$$
\Gamma^0_{f,(1,2)} + \Gamma^0_{f,(2,1)} \, .
$$

Similarly, in case $\bullet\bullet$, we will sum over the ordering of g_1 and g_2 . As usual, the summation will be later compensated by a factor of $\frac{1}{2}$ in the formulas.

By the argument in Section [5.3,](#page-33-4) we have the following formula for the contribution of the graph Γ to the stable quotient theory of $K\mathbb{P}^3$,

$$
Cont_{\Gamma} = \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\mathsf{A} \in \mathbb{Z}_{\geq 0}^{\mathsf{F}}} \prod_{v \in \mathsf{V}} \mathrm{Cont}_{\Gamma}^{\mathsf{A}}(v) \prod_{e \in \mathsf{E}} \mathrm{Cont}_{\Gamma}^{\mathsf{A}}(e) .
$$

Let f connect the T-fixed points $p_i, p_j \in \mathbb{P}^3$. Let the A-values of the respective half-edges be (k, l) . Denote by \mathbb{D}_1 the differentail operator

$$
\frac{L^2}{4C_1} \frac{\partial}{\partial A_2} + \frac{-2s_1 L^4 - C_1 (3B_2 L^2 - s_1 L^6)}{4C_1^2} \frac{\partial}{\partial B_4}.
$$

By Lemma [39](#page-42-1) and the explicit formula for $\text{Cont}^{\mathsf{A}}_{\Gamma}(f)$ in Lemma [35](#page-34-0)^{[14](#page-45-0)}, we have

(58)

$$
\mathbb{D}_1 \text{Cont}_{\Gamma}^{\mathsf{A}}(f) = (-1)^{k+l} \Big(\frac{L_i^2 L_j R_{2k-1,i} R_{1l-1,j}}{C_1^2 C_2} + \frac{L_i L_j^2 R_{1k-1,i} R_{2l-1,j}}{C_1^2 C_2} \Big).
$$

• If Γ is connected after deleting f , we have

$$
\frac{1}{|\text{Aut}(\Gamma)|} \sum_{\mathsf{A}\in\mathbb{Z}_{\geq 0}^{\mathsf{F}}} \mathbb{D}_1 \text{Cont}_{\Gamma}^{\mathsf{A}}(f) \prod_{v\in\mathsf{V}} \text{Cont}_{\Gamma}^{\mathsf{A}}(v) \prod_{e\in\mathsf{E}, e\neq f} \text{Cont}_{\Gamma_f^0}(e)
$$

$$
= \text{Cont}_{\Gamma_f^0}(H, H^2) + \text{Cont}_{\Gamma_f^0}(H^2, H) \, .
$$

The derivation is simply by using [\(58\)](#page-45-1) on the left and the argument in Proposition [30](#page-32-1) on the right.

 $\bullet\bullet$ If Γ is disconnected after deleting $f,$ we obtain

$$
\frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\mathbf{A} \in \mathbb{Z}_{\geq 0}^{\mathsf{F}}} \mathbb{D}_1 \text{Cont}_{\Gamma}^{\mathbf{A}}(f) \prod_{v \in \mathbf{V}} \text{Cont}_{\Gamma}^{\mathbf{A}}(v) \prod_{e \in \mathsf{E}, e \neq f} \text{Cont}_{\Gamma}^{\mathbf{A}}(e)
$$

$$
= \text{Cont}_{\Gamma_f^1}(H) \text{Cont}_{\Gamma_f^2}(H^2) + \text{Cont}_{\Gamma_f^1}(H^2) \text{Cont}_{\Gamma_f^2}(H)
$$

by the same method.

By combining the above two equations for all the edges of all the graphs $\Gamma \in \mathsf{G}_g(\mathbb{P}^3)$ and using the vanishing

$$
\frac{\partial \text{Cont}_{\Gamma}^{\mathsf{A}}(v)}{\partial A_2} = 0, \ \frac{\partial \text{Cont}_{\Gamma}^{\mathsf{A}}(v)}{\partial B_4} = 0
$$

.

of Lemma [41,](#page-43-2) we obtain

(59)
$$
\mathbb{D}_1 \langle \rangle_{g,0}^{\mathsf{SQ}} = \sum_{i=1}^{g-1} \langle H \rangle_{g-i,1}^{\mathsf{SQ}} \langle H^2 \rangle_{i,1}^{\mathsf{SQ}} + \langle H, H^2 \rangle_{g-1,2}^{\mathsf{SQ}}
$$

¹⁴Lemma [35](#page-34-0) is stated for $K\mathbb{P}^2$, but parallel statement holds for $K\mathbb{P}^3$.

We have followed here the notation of Section [0.3.](#page-4-2) The equality [\(59\)](#page-45-2) holds in the ring $G_3[A_2, B_2, B_4, C_1^{\pm 1}].$

7.7. Proof of Theorem [6:](#page-6-2) second equation. By the argument in Section [5.3,](#page-33-4) we have the following formula for the contribution of the graph Γ to the stable quotient theory of $K\mathbb{P}^3$,

$$
\mathrm{Cont}_\Gamma=\frac{1}{|\mathrm{Aut}(\Gamma)|}\sum_{\mathsf{A}\in\mathbb{Z}_{\geq 0}^\mathsf{F}}\prod_{v\in\mathsf{V}}\mathrm{Cont}^\mathsf{A}_\Gamma(v)\prod_{e\in\mathsf{E}}\mathrm{Cont}^\mathsf{A}_\Gamma(e)\,.
$$

Let f connect the T-fixed points $p_i, p_j \in \mathbb{P}^3$. Let the A-values of the respective half-edges be (k, l) . Denote by \mathbb{D}_2 the differential operator

$$
\frac{2L^2}{C_1(L^4-1-2A_2)}\frac{\partial}{\partial B_2}.
$$

By Lemma [39](#page-42-1) and the explicit formula for $\text{Cont}^{\mathsf{A}}_{\Gamma}(f)$ in Lemma [35](#page-34-0)^{[15](#page-46-0)}, we have

(60)
$$
\mathbb{D}_2 \text{Cont}_{\Gamma}^{\mathsf{A}}(f) = (-1)^{k+l} \frac{2L_i L_j R_{1 k-1,i} R_{1 l-1,j}}{C_1^2}.
$$

• If Γ is connected after deleting f, we have

$$
\frac{1}{|\text{Aut}(\Gamma)|} \sum_{\mathsf{A} \in \mathbb{Z}_{\geq 0}^{\mathsf{F}}} \mathbb{D}_2 \text{Cont}_{\Gamma}^{\mathsf{A}}(f) \prod_{v \in \mathsf{V}} \text{Cont}_{\Gamma}^{\mathsf{A}}(v) \prod_{e \in \mathsf{E}, e \neq f} \text{Cont}_{\Gamma}^{\mathsf{A}}(e) = \text{Cont}_{\Gamma_f^0}(H, H).
$$

The derivation is simply by using [\(60\)](#page-46-1) on the left and the arguments in Proposition [30](#page-32-1) on the right.

•• If Γ is disconnected after deleting f, we obtain

$$
\frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\mathsf{A} \in \mathbb{Z}_{\geq 0}^{\mathsf{F}}} \mathbb{D}_2 \, \mathrm{Cont}^{\mathsf{A}}_{\Gamma}(f) \prod_{v \in \mathsf{V}} \mathrm{Cont}^{\mathsf{A}}_{\Gamma}(v) \prod_{e \in \mathsf{E}, e \neq f} \mathrm{Cont}^{\mathsf{A}}_{\Gamma}(e) \\ = \mathrm{Cont}_{\Gamma_f^1}(H) \, \mathrm{Cont}_{\Gamma_f^2}(H)
$$

by the same method.

By combining the above two equations for all the edges of all the graphs $\Gamma \in \mathsf{G}_g(\mathbb{P}^3)$ and using the vanishing

$$
\frac{\partial \text{Cont}^{\mathsf{A}}_{\Gamma}(v)}{\partial B_2} = 0
$$

¹⁵Lemma [35](#page-34-0) is stated for $K\mathbb{P}^2$, but parallel statement holds for $K\mathbb{P}^3$.

of Lemma [41,](#page-43-2) we obtain

(61)
$$
\mathbb{D}_2\langle \rangle_{g,0}^{\mathsf{SQ}} = \sum_{i=1}^{g-1} \langle H \rangle_{g-i,1}^{\mathsf{SQ}} \langle H \rangle_{i,1}^{\mathsf{SQ}} + \langle H, H \rangle_{g-1,2}^{\mathsf{SQ}}.
$$

We have followed here the notation of Section [0.3.](#page-4-2) The equality [\(61\)](#page-47-0) holds in the ring $G_3[A_2, B_2, B_4, C_1^{\pm 1}].$

On the right side of [\(61\)](#page-47-0), we have

$$
\langle H \rangle_{g-i,1}^{\mathsf{SQ}} = \mathcal{F}_{g-i,1}^{\mathsf{SQ}}[1,0](q) = \mathcal{F}_{g-i,1}^{\mathsf{GW}}[1,0](Q(q)),
$$

where the first equality is by definition and the second is by wallcrossing [\(7\)](#page-5-0). Then,

$$
\mathcal{F}_{g-i,1}^{\text{GW}}[1,0](Q(q))\ =\ \frac{\partial \mathcal{F}_{g-i}^{\text{GW}}}{\partial T}(Q(q))\ =\ \frac{\partial \mathcal{F}_{g-i}^{\text{SQ}}}{\partial T}(q)
$$

where the first equality is by the divisor equation in Gromov-Witten theory and the second is again by wall-crossing [\(7\)](#page-5-0), so we conclude

$$
\langle H \rangle^{\mathsf{SQ}}_{g-i,1} = \frac{\partial \mathcal{F}_{g-i}^{\mathsf{SQ}}}{\partial T}(q) \in \mathbb{C}(\lambda_0,\ldots,\lambda_3)[[q]]\,.
$$

Similarly, we obtain

$$
\langle H \rangle_{i,1}^{\mathsf{SQ}} = \frac{\partial \mathcal{F}_i^{\mathsf{SQ}}}{\partial T}(q) \in \mathbb{C}(\lambda_0, \dots, \lambda_3)[[q]],
$$

$$
\langle H, H \rangle_{g-1,2}^{\mathsf{SQ}} = \frac{\partial^2 \mathcal{F}_{g-1}^{\mathsf{SQ}}}{\partial T^2}(q) \in \mathbb{C}(\lambda_0, \dots, \lambda_3)[[q]].
$$

Together, the above equations transform [\(61\)](#page-47-0) into exactly the second holomorphic anomaly equation of Theorem [8,](#page-7-1)

$$
\frac{2L^4}{C_1^2(L^4 - 1 - 2A_2)} \frac{\partial \mathcal{F}_g^{\mathsf{SQ}}}{\partial B_2} = \sum_{i=1}^{g-1} \frac{\partial \mathcal{F}_{g-i}^{\mathsf{SQ}}}{\partial T} \frac{\partial \mathcal{F}_i^{\mathsf{SQ}}}{\partial T} + \frac{\partial^2 \mathcal{F}_{g-1}^{\mathsf{SQ}}}{\partial T^2}.
$$

as an equality in $\mathbb{C}(\lambda_0, \ldots, \lambda_3)[[q]]$. To lift holomorphic anomaly equation to the equality

$$
\frac{2L^4}{C_1^2(L^4 - 1 - 2A_2)} \frac{\partial \mathcal{F}_g^{\mathsf{SQ}}}{\partial B_2} = \sum_{i=1}^{g-1} \frac{\partial \mathcal{F}_{g-i}^{\mathsf{SQ}}}{\partial T} \frac{\partial \mathcal{F}_i^{\mathsf{SQ}}}{\partial T} + \frac{\partial^2 \mathcal{F}_{g-1}^{\mathsf{SQ}}}{\partial T^2}
$$

in the ring $\mathbb{G}_3[A_2, B_2, B_4, C_1^{\pm 1}]$, we must prove the equalities

(62)
$$
\langle H \rangle_{g-i,1}^{\mathsf{SQ}} = \frac{\partial \mathcal{F}_{g-i}^{\mathsf{SQ}}}{\partial T}, \quad \langle H \rangle_{i,1}^{\mathsf{SQ}} = \frac{\partial \mathcal{F}_{i}^{\mathsf{SQ}}}{\partial T},
$$

$$
\langle H, H \rangle_{g-1,2}^{\mathsf{SQ}} = \frac{\partial^2 \mathcal{F}_{g-1}^{\mathsf{SQ}}}{\partial T^2}
$$

in the ring $G_3[A_2, B_2, B_4, C_1^{\pm 1}]$. The lifting follow from the argument in Section 7.3 in [\[16\]](#page-56-0).

We do not study the genus 1 unpointed series $\mathcal{F}^{\mathsf{SQ}}_1$ $i^{SQ}(q)$ in the paper, so we take

$$
\langle H \rangle^{\mathsf{SQ}}_{1,1} = \frac{\partial \mathcal{F}_{1}^{\mathsf{SQ}}}{\partial T},
$$

$$
\langle H, H \rangle^{\mathsf{SQ}}_{1,2} = \frac{\partial^2 \mathcal{F}_{1}^{\mathsf{SQ}}}{\partial T^2}.
$$

as definitions of the right side in the genus 1 case. There is no difficulty in calculating these series explicitly using the argument in Proposition [30.](#page-32-1)

8. Appendix

8.1. Overviews. In section [0.1](#page-0-1) the equivariant Gromov-Witten invariants of the local \mathbb{P}^n were defined,

$$
N_{g,d}^{\mathsf{GW}} = \int_{[\overline{M}_g(\mathbb{P}^n,d)]^{\text{vir}}} e\Big(-R\pi_*f^*\mathcal{O}_{\mathbb{P}^n}(-n-1)\Big).
$$

We associate Gromov-Witten generating series by

$$
\mathcal{F}_g^{\mathsf{GW},n}(Q) = \sum_{d=0}^{\infty} \widetilde{N}_{g,d}^{\mathsf{GW}} Q^d \in \mathbb{C}(\lambda_0,\ldots,\lambda_n)[[Q]]\,.
$$

Motivated by mirror symmetry([\[1,](#page-56-5) [2,](#page-56-6) [19\]](#page-56-8)), we can make the following predictions about the genus g generating series $\mathcal{F}_g^{\mathsf{GW},n}$.

(A) There exist a finitely generated subring

$$
\mathbf{G} \in \mathbb{C}(\lambda_0, \dots, \lambda_n)[[Q]]
$$

which contains $\mathcal{F}_g^{\mathsf{GW},n}$ for all g.

(B) The series $\mathcal{F}_g^{\mathsf{GW},n}$ satisfy *holomorphic anomaly equations*, i.e. recursive formulas for the derivative of $\mathcal{F}_g^{\mathsf{GW},n}$ with respect to some generators in G.

8.1.1. I-function. I-fucntion defined by

$$
I_n = \sum_{d=0}^{\infty} \frac{\prod_{k=1}^{(n+1)d-1} (-(n+1)H - kz)}{\prod_{i=0}^{n} \prod_{k}^{d} (H + kz - \lambda_i)} q^d \in H_{\mathsf{T}}^{*}(\mathbb{P}^n, \mathbb{C})[[q]]\,,
$$

is the central object in the study of Gromov-Witten invariants of local \mathbb{P}^n geometry. See [\[16\]](#page-56-0), [\[17\]](#page-56-1) for the arguments. Several important properties of the function I_n was studied in [\[21\]](#page-56-3) after the specialization

$$
\lambda_i = \zeta_{n+1}^i
$$

where ζ_{n+1} is primitive $(n + 1)$ -th root of unity. For the study of full equivariant Gromov-Witten theories, we extend the result of [\[21\]](#page-56-3) without the specialization [\(63\)](#page-48-1).

8.1.2. Picard-Fuchs equation and Birkhoff factorization. Define differential operators

$$
\mathsf{D} = q \frac{d}{dq} \,, \ \ M = H + z \mathsf{D} \,.
$$

The function I_n satisfies following Picard-Fuchs equation

$$
\Big(\prod_{i=0}^n \Big(M-\lambda_i\Big)-q\prod_{k=0}^n \Big(-(n+1)M-kz\Big)\Big)I_n=0\,.
$$

The restriction $I_n|_{H=\lambda_i}$ admits following asymptotic form

(64)
$$
I_n|_{H=\lambda_i} = e^{\frac{\mu}{z}} \Big(R_{0,i} + R_{1,i} z + R_{2,i} z^2 + \dots \Big)
$$

with series μ_i , $R_{k,i} \in \mathbb{C}(\lambda_0, \ldots, \lambda_n)[[q]]$.

A derivation of [\(64\)](#page-49-0) is obtained from [\[4,](#page-56-16) Theorem 5.4.1] and the uniqueness lemma in [\[4,](#page-56-16) Section 7.7]. The series μ_i and $R_{k,i}$ are found by solving defferential equations obtained from the coefficient of z^k . For example,

$$
\lambda_i + \mathsf{D}\mu_i = L_i \,,
$$

where $L_i(q)$ is the series in q defined by the root of following degree $(n+1)$ polynomial in $\mathcal L$

$$
\prod_{i=0}^n (\mathcal{L} - \lambda_i) - (-1)^{n+1} q \mathcal{L}^{n+1}.
$$

with initial conditions,

$$
\mathcal{L}_i(0)=\lambda_i.
$$

Let f_n be the polynomial of degree n in variable x over $\mathbb{C}(\lambda_0, \ldots, \lambda_n)$ defined by

$$
f_n(x) := \sum_{k=0}^n (-1)^k k s_{k+1} x^{n-k},
$$

where s_k is k-th elementary symmetric function in $\lambda_0, \ldots, \lambda_n$. The ring

$$
\mathbb{G}_n := \mathbb{C}(\lambda_0, \dots, \lambda_n)[L_0^{\pm 1}, \dots, L_n^{\pm 1}, f_n(L_0)^{-\frac{1}{2}}, \dots, f_n(L_n)^{-\frac{1}{2}}]
$$

will play a basic role.

The following Conjecture was proven under the specializaiton [\(63\)](#page-48-1) in [\[21,](#page-56-3) Theorem 4].

Conjecture 44. For all $k \geq 0$, we have

$$
R_{k,i}\in\mathbb{G}_n\,.
$$

Conjecture [44](#page-50-0) for the case $n = 1$ will be proven in Section [8.3.](#page-51-0) Conjecture [44](#page-50-0) for the case $n = 2$ will be proven in Section [8.4](#page-53-0) under the specialization [\(72\)](#page-53-1). In fact, the argument in Section [8.4](#page-53-0) proves Conjecture [44](#page-50-0) for all *n* under the specialization which makes $f_n(x)$ into power of a linear polynomial.

8.2. Admissibility of differential equations. Let R be a commutative ring. Fix a polynomial $f(x) \in R[x]$. We consider a differential operator of level n with following forms.

(65)
$$
\mathcal{P}(A_{lp},f)[X_0,\ldots,X_{n+1}]=DX_{n+1}-\sum_{n\geq l\geq 0,\,p\geq 0}A_{lp}D^pX_{n-l},
$$

where $D := \frac{d}{dx}$ and $A_{lp} \in R[x]_f := R[x][f^{-1}]$. We assume that only finitely many A_{lp} are not zero.

Definition 45. Let R_i be the solutions of the equations for $k \geq 0$,

(66)
$$
\mathcal{P}(A_{lp},f)[X_{k+1},\ldots,X_{k+n}]=0,
$$

with $R_0 = 1$. We use the conventions $X_i = 0$ for $i < 0$. We say differential equations [\(66\)](#page-50-1) is admissible if the solutions R_k satisfies for $k \geq 0$,

$$
R_k \in \mathsf{R}[x]_f.
$$

Remark 46. Note that the admissibility of $\mathcal{P}(A_{lp}, f)$ in Definition [45](#page-50-2) do not depend on the choice of the solutions R_k .

Lemma 47. Let f be a degree one polynomial in x. Each $A \in \mathbb{R}[x]_f$ can be written uniquely as

$$
A = \sum_{i \in \mathbb{Z}} a_i f^i
$$

with finitely many non-zero $a_i \in \mathsf{R}$. We define the order $Ord(A)$ of A with respect to f by smallest i such that a_i is not zero. Then

$$
\mathcal{P}(A_{lp},f)[X_0,\ldots,X_{n+1}]:=DX_{n+1}-\sum_{n\geq l\geq 0,\,p\geq 0}A_{lp}D^pX_{n-l}=0\,,
$$

is admissible if following condition holds:

(67)
$$
Ord(A_{l0}) \leq -2,
$$

$$
Ord(A_{l1}) \leq 0,
$$

$$
Ord(A_{lp}) \leq p+1 \quad for \ p \geq 2.
$$

Proof. The proof follows from simple induction argument. \Box

Lemma 48. Let f be a degree two polynomial in x . Denote by

 R_f

the subspace of $R[x]_f$ generated by f^i for $i \in \mathbb{Z}$. Each $A \in R_f$ can be written uniquely as

$$
A = \sum_{i \in \mathbb{Z}} a_i f^i
$$

with finitely many non-zero $a_i \in \mathsf{R}$. We define the order $Ord(A)$ of $A \in \mathsf{R}_f$ with respect to f by smallest i such that a_i is not zero. Then

$$
\mathcal{P}(A_{lp},f)[X_0,\ldots,X_{n+1}]:=DX_{n+1}-\sum_{n\geq l\geq 0,\,p\geq 0}A_{lp}D^pX_{n-l}=0\,,
$$

is admissible if following condition holds:

$$
A_{lp} = B_{lp} \t\t if p is odd,A_{lp} = \frac{df}{dx} \cdot B_{lp} \t if p is even,
$$

where B_{lp} are elements of R_f with

$$
Ord(B_{l0}) \leq -2,
$$

$$
Ord(B_{lp}) \leq \left[\frac{p-1}{2}\right] \text{ for } p \geq 1.
$$

Proof. Since f is degree two polynomial in x , we have

$$
\frac{d^2f}{dx^2}\, ,\, (\frac{df}{dx})^2\in {\sf R}_f\, .
$$

Then the proof of Lemma follows from simple induction argument. \Box

8.3. Local \mathbb{P}^1 .

8.3.1. Overview. In this section, we prove Conjecture [44](#page-50-0) for the case $n=1$. Recall the *I*-function for $K\mathbb{P}^1$,

(68)
$$
I_1(q) = \sum_{d=0}^{\infty} \frac{\prod_{k=0}^{2d-1} (-2H - kz)}{\prod_{i=0}^{1} \prod_{k=1}^{d} (H - \lambda_i + kz)} q^d.
$$

The function I_1 satisfies following Picard-Fuchs equation

(69)
$$
\left((M - \lambda_0)(M - \lambda_1) - 2qM(2M + z) \right) I_1 = 0.
$$

Recall the notation used in above equation,

$$
\mathsf{D} = q \frac{d}{dq} \,, \ \ M = H + z \mathsf{D} \,.
$$

The restriction $I_1|_{H=\lambda_i}$ admits following asymptotic form

(70)
$$
I_1|_{H=\lambda_i}=e^{\mu_i/z}\left(R_{0,i}+R_{1,i}z+R_{2,i}z^2+\ldots\right)
$$

with series $\mu_i, R_{k,i} \in \mathbb{C}(\lambda_0, \lambda_1)[[q]]$. The series μ_i and $R_{k,i}$ are found by solving differential equations obtained from the coefficient of z^k in [\(69\)](#page-52-0). For example, we have for $i = 0, 1$,

$$
\lambda_i + D\mu_i = L_i,
$$

\n
$$
R_{0,i} = \left(\frac{\lambda_i \prod_{j \neq i} (\lambda_i - \lambda_j)}{f_1(L_i)}\right)^{\frac{1}{2}},
$$

$$
R_{1,i} = \left(\frac{\lambda_i \prod_{j \neq i} (\lambda_i - \lambda_j)}{f(L_i)}\right)^{\frac{1}{2}}.
$$

$$
\left(\frac{-16s_1^2 s_2^2 + 88s_2^3 + (27s_1^3 s_2 - 132s_1 s_2^2)L_i + (-12s_1^4 + 54s_1^2 s_2)L_i^2}{24s_1(L_i s_1 - 2s_2)^3} + \frac{12\lambda_i^2 - 9\lambda_i \lambda_{i+1} + \lambda_{i+1}^2}{24(\lambda_i^3 - \lambda_i \lambda_{i+1}^2)}\right)
$$

Here $s_1 = \lambda_0 + \lambda_1$ and $s_2 = \lambda_0 \lambda_1$. In the above expression of $R_{1,i}$, we used the convention $\lambda_2 = \lambda_0$.

8.3.2. Proof of Conjecture [44.](#page-50-0) We introduce new differential operator D_i defined by for $i = 0, 1$,

$$
D_i = (DL_i)^{-1}D.
$$

By definition, D_i acts on rational functions in L_i as the ordinary derivation with respect to L_i . If we use following normalizations,

$$
R_{k,i} = f_1(L_i)^{-\frac{1}{2}} \Phi_{k,i}
$$

.

the Picard-Fuchs equation [\(73\)](#page-54-0) yields the following differential equations,

(71)
$$
D_i \Phi_{p,i} - A_{00,i} \Phi_{p-1,i} - A_{01,i} D_i \Phi_{p-1,i} - A_{02,i} D_i^2 \Phi_{p-1,i} = 0,
$$

with

$$
A_{00,i} = \frac{-s_1^2 s_2^2 + (-s_1^3 s_2 + 8s_1 s_2^2) L_i + (2s_1^4 - 9s_1^2 s_2) L_i^2}{4(L_i s_1 - 2s_2)^4},
$$

\n
$$
A_{01,i} = \frac{2s_1 s_2^2 + (-s_1^2 s_2 - 8s_2^2) L_i + (-s_1^3 + 10s_1 s_2) L_i^2 - s_1^2 L_i^3}{2(L_i s_1 - 2s_2)^3},
$$

\n
$$
A_{02,i} = \frac{s_2^2 - 2(s_1 s_2) L_i + (s_1^2 + s_2) L_i^2 - s_1 L_i^3}{(L_i s_1 - 2s_2)^2}.
$$

Here s_k is the k-th elementary symmetric functions in λ_0, λ_1 . Since the differential equations [\(71\)](#page-53-2) satisfy the condition [\(67\)](#page-51-1), we conclude the differential equations [\(71\)](#page-53-2) is admissible.

8.3.3. Gomov-Witten series. By the result of previous subsection, we obtain the following result which verifies the prediction (A) in Section [8.1.](#page-48-2)

Theorem 49. For the Gromov-Witten series of $K\mathbb{P}^1$, we have

 $\mathcal{F}_g^{\mathsf{GW},1}(Q(q)) \in \mathbb{G}_1$,

where $Q(q)$ is the mirror map of $K\mathbb{P}^1$ defined by

$$
Q(q) := q \cdot exp\left(2\sum_{d=1}^{\infty} \frac{(2d-1)!}{(d!)^2} q^d\right).
$$

Theorem [49](#page-53-3) follows from the argument in [\[16\]](#page-56-0). The prediction (B) in Section [8.1](#page-48-2) is trivial statement for $K\mathbb{P}^1$.

8.4. Local \mathbb{P}^2 .

8.4.1. Overview. In this section, we prove Conjecture [44](#page-50-0) for the case $n = 2$ with following specializations,

$$
\lambda_i \neq \lambda_j \text{ for } i \neq j \,,
$$

(72)
$$
(\lambda_0 \lambda_1 + \lambda_1 \lambda_2 + \lambda_2 \lambda_0)^2 - 3\lambda_0 \lambda_1 \lambda_2 (\lambda_0 + \lambda_1 + \lambda_2) = 0.
$$

For the rest of the section, the specialization [\(72\)](#page-53-1) will be imposed. Recall the *I*-function for $K\mathbb{P}^2$.

$$
I_2(q) = \sum_{d=0}^{\infty} \frac{\prod_{k=0}^{3d-1} (-3H - kz)}{\prod_{i=0}^{2} \prod_{k=1}^{d} (H - \lambda_i + kz)} q^d.
$$

The function I_2 satisfies following Picard-Fuchs equation

(73)

$$
((M - \lambda_0)(M - \lambda_1)(M - \lambda_2) + 3qM(3M + z)(3M + 2z))I_2 = 0
$$

Recall the notation used in above equation,

$$
D = q \frac{d}{dq}, \quad M = H + zD.
$$

The restriction $I_2|_{H=\lambda_i}$ admits following asymptotic form

$$
I_2|_{H=\lambda_i}=e^{\mu_i/z}\left(R_{0,i}+R_{1,i}z+R_{2,i}z^2+\ldots\right)
$$

with series $\mu_i, R_{k,i} \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]]$. The series μ_i and $R_{k,i}$ are found by solving differential equations obtained from the coefficient of z^k in [\(73\)](#page-54-0). For example,

$$
\lambda_i + D\mu_i = L_i,
$$

\n
$$
R_{0,i} = \left(\frac{\lambda_i \prod_{j \neq i} (\lambda_i - \lambda_j)}{f_2(L_i)}\right)^{\frac{1}{2}}
$$

.

8.4.2. Proof of Conjecture [44.](#page-50-0) We introduce new differential operator D_i defined by

$$
D_i = (DL_i)^{-1}D.
$$

If we use following normalizations,

$$
R_{k,i} = f_2(L_i)^{-\frac{1}{2}} \Phi_{k,i}
$$

the Picard-Fuchs equation [\(73\)](#page-54-0) yields the following differential equations,

(74)
$$
D_i \Phi_{p,i} - A_{00,i} \Phi_{p-1,i} - A_{01,i} D_i \Phi_{p-1,i} - A_{02,i} D_i^2 \Phi_{p-1,i} - A_{10,i} \Phi_{p-2,i} - A_{11,i} D_i \Phi_{p-2,i} - A_{12,i} D_i^2 \Phi_{p-2,i} - A_{13,i} D_i^3 \Phi_{p-2,i} = 0,
$$

with $A_{jl,i} \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[L_i, f_2(L_i)^{-1}]$. We give the exact values of $A_{jl,i}$ for reader's convinience.

$$
A_{00,i} = \frac{s_1}{9(s_1L_i - s_2)^5} \left(s_1s_2^3 + (-4s_1^2s_2^2 + 3s_2^3)L_i + (-s_1^3s_2 + 12s_1s_2^2)L_i^2 + (11s_1^4 - 36s_1^2s_2)L_i^3 \right),
$$

$$
A_{01,i} = \frac{-s_1}{3(s_1L_i - s_2)^4} \left(s_2^3 - 4(s_1s_2^2)L_i + (3s_1^2s_2 + 9s_2^2)L_i^2 + (3s_1^3 - 21s_1s_2)L_i^3 + 3s_1^2L_i^4 \right),
$$

$$
A_{02,i} = \frac{-1}{3(s_1^3 - 5(s_1s_2^2)L_i + 9s_1^2s_2L_i^2 + (-6s_1^3 - 3s_1s_2)L_i^3}
$$

$$
A_{02,i} = \frac{1}{3(s_1L_i - s_2)^3} \left(s_2^3 - 5(s_1s_2^2)L_i + 9s_1^2s_2L_i^2 + (-6s_1^3 - 3s_1s_2)L_i^3 + 6s_1^2L_i^4 \right),
$$

$$
A_{10,i} = \frac{s_1^2 L_i}{27(s_1 L_i - s_2)^9} \Big((8s_1^2 s_2^5 - 21s_2^6) + (-48s_1^3 s_2^4 + 126s_1 s_2^5) L_i + (120s_1^4 s_2^3 - 315s_1^2 s_2^4) L_i^2 + (-124s_1^5 s_2^2 + 264s_1^3 s_2^3 + 144s_1 s_2^4) L_i^3 + (12s_1^6 s_2 + 153s_1^4 s_2^2 - 432s_1^2 s_2^3) L_i^4 + (60s_1^7 - 342s_1^5 s_2 + 432s_1^3 s_2^2) L_i^5 + (-33s_1^6 + 108s_1^4 s_2) L_i^6 \Big),
$$

$$
A_{11,i} = \frac{-s_1 L_i}{27(s_1 L_i - s_2)^8} \Big((8s_1^2 s_2^5 - 21s_2^6) + (-48s_1^3 s_2^4 + 126s_1 s_2^5) L_i + (120s_1^4 s_2^3 - 315s_1^2 s_2^4) L_i^2 + (-124s_1^5 s_2^2 + 264s_1^3 s_2^3 + 144s_1 s_2^4) L_i^3 + (12s_1^6 s_2 + 153s_1^4 s_2^2 - 432s_1^2 s_2^3) L_i^4 + (60s_1^7 - 342s_1^5 s_2 + 432s_1^3 s_2^2) L_i^5 + (-33s_1^6 + 108s_1^4 s_2) L_i^6 \Big)
$$

$$
A_{12,i} = \frac{s_1}{9(s_1L_i - s_2)^7} \left(-s_2^6 + 9s_1s_2^5L_i + (-32s_1^2s_2^4 - 9s_2^5)L_i^2 + (57s_1^3s_2^3 + 60s_1s_2^4)L_i^3 + (-48s_1^4s_2^2 - 171s_1^2s_2^3)L_i^4 + (9s_1^5s_2 + 237s_1^3s_2^2 + 27s_1s_2^3)L_i^5 + (9s_1^6 - 144s_1^4s_2 - 90s_1^2s_2^2)L_i^6 + (9s_1^5 + 108s_1^3s_2)L_i^7 - 18s_1^4L_i^8 \right),
$$

$$
A_{13,i} = -\frac{(3L_i^2s_1^2 - 3L_is_1s_2 + s_2^2)(-3L_i^3s_1 + 3L_i^2s_1^2 - 3L_is_1s_2 + s_2^2)^2}{27(s_1L_i - s_2)^6}.
$$

Here s_k is the k-th elementary symmetric functions in $\lambda_0, \lambda_1, \lambda_2$. Since the differential equations [\(74\)](#page-54-1) satisfy the condition [\(67\)](#page-51-1), we conclude that the differential equations [\(74\)](#page-54-1) is admissible.

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