ON FINITE POLYNOMIAL MAPPINGS

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Abstract. Let $X \subset \mathbb{C}^n$ be a smooth irreducible affine variety of dimension k and let $F : X \to \mathbb{C}^m$ be a polynomial mapping. We prove that if $m \geq k$, then there is a Zariski open dense subset U in the space of linear mappings $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$ such that for every $L \in U$ the mapping $F + L$ is a finite mapping. Moreover, we can choose U in this way, that all mappings $F + L; L \in U$ are topologically equivalent.

1. Introduction

Assume that we have an algebraic family $\mathcal{F}$ of polynomial generically-finite mappings $f_p : X \to \mathbb{C}^m$; $p \in \mathcal{F}$, where X is a smooth irreducible affine variety. It is important to know the behavior of proper mappings in a such family. In general, proper mappings does not form an algebraic subset of $\mathcal{F}$, but only constructible one. However we show in this note that we have some regular behavior in such family.

As an application we show that if $X \subset \mathbb{C}^n$ is a smooth irreducible affine variety of dimension k and let $F : X \to \mathbb{C}^m$ be a polynomial mapping. If $m \geq k$, then there exists a Zariski open dense subset U in the space of linear mappings $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$ such that:

a) for every $L \in U$ the mapping $F + L$ is a finite mapping.

b) all mappings $F + L, L \in U$ are topologically equivalent.

Let us recall that mappings $f, g : X \to Y$ are topologically equivalent, if there exist homeomorphisms $\phi : X \to X$ and $\psi : Y \to Y$ such that $f = \psi \circ g \circ \phi$.

2. Main results

Let us start with the following:

Theorem 2.1. *Let P, X, Y be smooth irreducible affine algebraic varieties and let $F : P \times X \to P \times Y$ be a generically finite mapping. The mapping F induces a family $\mathcal{F} = \{f_p(\cdot) = F(p, \cdot),\ p \in P\}$. Then either there exists a Zariski open dense subset $U \subset P$ such that for every $p \in P$ a mapping f_p is proper, or there exists a Zariski open dense subset $V \subset P$ such that for every $p \in P$ a mapping f_p is not proper.*

In the first case we have:

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a) foe every non-proper mappings f_p in the family $\mathcal{F}$ we have $\mu(f_p) < \mu(F)$, where $\mu(f)$ denotes the geometric degree of f,

b) all generic mappings f_p are topologically equivalent, i.e., there exists a Zariski open dense subset $W \subset P$, such that for every $p, q \in W$ mappings f_p and f_q are topologically equivalent.

Proof. First of all note that for every $(p, x) \in P \times X$ we have $\mu_{(p,x)}(F) = \mu_x(f_p)$ (here $\mu_x(f)$ denotes the local multiplicity of f in x). In the sequel we use the fact that a mapping $g : X \to Y$ is proper over a point $y \in Y$ if and only if $\sum_{g(x)=y} \mu_x(g) = \mu(g)$ (see [1], [2]).

Let S be the non-properness set of F (see e.g. [1], [2]). If $S = \emptyset$, then all mappings f_p are proper. Hence we can assume that $S \neq \emptyset$. Consider the canonical projection $\pi : S \to P$. We have two possibilities:

(1) $\pi(S)$ is dense in P.

(2) $\pi(S)$ is not dense in P.

In the case a) a generic mapping f_p is not proper. In the second case note that S has dimension dim P + dim $X - 1$ and the fiber of π has dimension at most dim X. This immediately implies that the set $\overline{\pi(S)}$ is a hypersurface in M. Moreover, fibers of π are the whole space X. This means that for all $p \in \pi(S)$ we have $\mu(f_p) < \mu(F)$. Of course outside $\pi(S)$ mappings f_p are proper. Two such a generic mappings are topologically equivalent by [3], Theorem 4.3. □

Now we state our main result:

Theorem 2.2. *Let $X \subset \mathbb{C}^n$ be a smooth irreducible affine variety of dimension k and let $F : X \to \mathbb{C}^m$ be a polynomial mapping. If $m \geq k$, then there existss a Zariski open dense subset U in the space of linear mappings $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$ such that:*

a) for every $L \in U$ the mapping $F + L$ is a finite mapping.

b) all mappings $F + L, L \in U$ are topologically equivalent.

Proof. Let $G : X \ni x \mapsto (x, F(x)) \in X \times \mathbb{C}^m$ and $\tilde{X} = graph(G) \cong X$. Since $m \geq \dim \tilde{X}$ a generic linear projection $\pi : \tilde{X} \to \mathbb{C}^m$ is a proper mapping. Hence also the mapping $\pi \circ G$ is proper. Consequently we get that for a general matrix $A \in GL(m, m)$ and general linear mapping $L \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$ the mapping $H(A, L) = A(F_1, ..., F_m)^T + L$ is proper. Hence also the mapping $A^{-1} \circ H(A, L)$ is proper. This means that the mapping $F + A^{-1}(l_1, ..., l_m)^T$ (where $L = (l_1, ..., l_m)$) is proper. But we can specialize the matrix A to the identity and the mapping L to a given linear mapping $L_0 \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)\}$. Hence we see that there is at least dense subset of linear mappings $L \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$ such that the mapping $F + L : X \to \mathbb{C}^m$ is proper. Consider the algebraic family $\mathcal{F} = \{F + L, L \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)\}$. By Theorem 2.1 we see that there exists a Zariski dense open subset $U \subset \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$ such that every mapping $F + L;\ L \in U$ is proper and all these mappings are topologically equivalent. □

References

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