ALGEBRAIC VECTOR BUNDLES ON THE 2-SPHERE AND SMOOTH RATIONAL VARIETIES WITH INFINITELY MANY REAL FORMS

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In memory of Mariusz Koras

Abstract. We construct smooth rational real algebraic varieties of every dimension ≥ 4 which admit infinitely many pairwise non-isomorphic real forms.

Introduction

A classical problem in real algebraic geometry is the classification of real forms of a given real algebraic variety X, that is, real algebraic varieties Y non isomorphic to X but whose complexifications $Y_{\mathbb{C}}$ are isomorphic to $X_{\mathbb{C}}$ as complex algebraic varieties. For example, the smooth real affine algebraic surfaces $\mathbb{S}^2 = \left\{x^2+y^2+z^2=1\right\}$ and $\mathcal{D} = \left\{uv+z^2=1\right\}$ in $\mathbb{A}^3_{\mathbb{R}}$ have isomorphic complexifications, an explicit isomorphism being simply given by the linear change of complex coordinates $u = x+iy$ and $v = x-iy$, but are non isomorphic. This follows for instance from the fact that the set of real points of $\mathbb{S}^2$ is the usual euclidean 2-sphere $S^2 \subset \mathbb{R}^3$ whereas the set of real points of $\mathcal{D}$ is not compact for the Euclidean topology.

Examples of smooth real projective varieties admitting infinitely many pairwise non-isomorphic real forms were only found very recently successively by Lesieutre [12] in dimension ≥ 6 and by Dinh-Oguiso [4] in every dimension ≥ 2. These are obtained as a by-product of clever constructions of smooth complex projective algebraic varieties defined over $\mathbb{R}$ with discrete but non finitely generated automorphism groups containing infinitely many conjugacy classes algebraic involutions. All their examples are non geometrically rational and to our knowledge, the question of existence of rational real algebraic varieties, projective or not, with infinitely many real forms was left open. Our first main result explicitly fills this gap for smooth real affine fourfolds:

Theorem 1. *The smooth rational real affine fourfold $\mathbb{S}^2 \times \mathbb{A}^2_{\mathbb{R}}$ has at least countably infinitely many pairwise non-isomorphic real forms.*

In contrast with the examples found by Lesieutre and Dinh-Oguiso, which rely on constructions of special classes of complex projective varieties by techniques of birational geometry, ours are inspired by basic results on the classification of topological vector bundles on the real sphere $S^2 \subset \mathbb{R}^3$. Our construction can indeed be interpreted as a sort of "algebraization" of the property that the complexification $E \otimes_{\mathbb{R}} \mathbb{C}$ of any topological real vector bundle $\pi : E \to S^2$ of rank 2 on S^2 is isomorphic, as a topological real vector bundle of rank 4, to the trivial bundle $S^2 \times \mathbb{R}^4$. More precisely, we show that the topological real vector bundles of rank 2 on S^2, which are nothing but the underlying real vector bundles of the complex line bundles $\mathcal{O}_{\mathbb{CP}^1}(n)$, $n \geq 0$, over $\mathbb{CP}^1 \simeq S^2$, admit algebraic models in the form of algebraic vector bundles $p_n : V_n \to \mathbb{S}^2$ of rank 2 on $\mathbb{S}^2$ with pairwise non-isomorphic total spaces, whose complexifications $p_{\mathbb{C}} : V_{n,\mathbb{C}} \to \mathbb{S}^2_{\mathbb{C}}$ are all isomorphic to the trivial bundle $\mathbb{S}^2_{\mathbb{C}} \times \mathbb{A}^2_{\mathbb{C}}$.

It is worth noticing that by a result of Kambayashi [11], $\mathbb{A}^2_{\mathbb{R}}$ has no nontrivial real form. One can check along the same lines using the fact that similarly as to $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$, the automorphism group $\mathrm{Aut}(\mathbb{S}^2_{\mathbb{C}})$ of $\mathbb{S}^2_{\mathbb{C}} \simeq \mathcal{D}_{\mathbb{C}}$ has a structure of a free product of two subgroups amalgamated along their intersection [2, 13], that $\mathcal{D}$ is the unique nontrivial real form of $\mathbb{S}^2$. So while $\mathbb{A}^2_{\mathbb{R}}$ and $\mathbb{S}^2$ both have finitely many real forms, the total spaces of the algebraic vector bundles $p_n : V_n \to \mathbb{S}^2$ provide an infinite countable family of real forms of $\mathbb{S}^2 \times \mathbb{A}^2_{\mathbb{R}}$ which are by construction pairwise locally isomorphic over $\mathbb{S}^2$, but globally pairwise non-isomorphic as real

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algebraic varieties. In contrast, reminiscent of the fact that for every $r \geq 3$ there exists a unique nontrivial topological real vector bundle of rank r on S^2, it turns out that the varieties $V_n \times \mathbb{A}_{\mathbb{R}}^{r-2}$, $n \geq 0$, give rise to a unique class of nontrivial real form of $\mathbb{S}^2 \times \mathbb{A}_{\mathbb{R}}^r$ (see Corollary 12 below).

Our construction thus does not directly yield higher dimensional families of examples by simply taking product with affine spaces. Nevertheless, a suitable adaptation of the technique used by Dinh-Oguiso [4], consisting in our situation of taking products of the V_n with well-chosen real rational affine varieties of log-general type, allows us to derive the following general existence result:

Theorem 2. *For every $d \geq 4$, there exist smooth rational real affine varieties of dimension d which have at least countably infinitely many pairwise non-isomorphic real forms.*

The article is organized as follows. The first section contains a short review of the classical correspondence between quasi-projective real algebraic varieties and quasi-projective complex varieties endowed with a real structure as well as a recollection on Euclidean topologies of real and complex algebraic varieties. Section 2 is devoted to the construction of algebraic models of topological real vector bundles over the 2-sphere $S^2 \subset \mathbb{R}^3$. The existence of such models was known after successive works of Fossum [6] and Moore [14] and, later on, of Swan [17], but we give a new geometric construction in the framework of complex varieties with real structures which we find more transparent. Theorem 1 is then established in Section 3. Section 4 contains the proof of Theorem 2 and a complement to Theorem 1 consisting of explicit formulas for the real structures on $\mathbb{S}_{\mathbb{C}}^2 \times \mathbb{A}_{\mathbb{C}}^2$ corresponding to the real algebraic vector bundles $p_n : V_n \to \mathbb{S}^2$.

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1. Preliminaries

In this article, the term **k**-variety will always refer to a geometrically integral quasi-projective scheme X of finite type over a base field **k** of characteristic zero. A morphism of **k**-varieties is a morphism of **k**-schemes. In the sequel, **k** will be equal to either $\mathbb{R}$ or $\mathbb{C}$, and we will say that X is a real, respectively complex, algebraic variety. To fix the notation, we let $c : \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$ be the étale double cover induced by the inclusion $\mathbb{R} \hookrightarrow \mathbb{C} = \mathbb{R}[i]/(i^2+1)$ and we let $\tau : \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{C})$, $i \mapsto -i$ be the usual complex conjugation.

1.1. **Complex varieties with real structures.** Recall [3] and [7, Exposé VIII] that étale descent for the Galois cover $c : \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$ provides an equivalence between the category of quasi-projective real algebraic varieties and the category of complex algebraic varieties equipped with a descent datum with respect to c. Such a descent datum on a quasi-projective complex algebraic variety $f : V \to \mathrm{Spec}(\mathbb{C})$ is in turn uniquely determined by an isomorphism of $\mathbb{R}$-schemes $\sigma : V \to V$ such $f \circ \sigma = \tau \circ f$ and that satisfies the cocycle relation $\sigma^2 = \mathrm{id}_V$. In other words, σ is an anti-regular involution of V, usually referred to as a *real structure* on V.

For every real algebraic variety X, the *complexification* $X_{\mathbb{C}} = X \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C})$ of X is canonically endowed with a real structure $\sigma_X = \mathrm{id}_X \times \tau$. Conversely, for every complex variety $f : V \to \mathrm{Spec}(\mathbb{C})$ endowed with a real stucture σ, the "quotient" $q : V \to V/\langle\sigma\rangle$ exists in the category of schemes and the structure morphism $f : V \to \mathrm{Spec}(\mathbb{C})$ descends to a morphism $\overline{f} : V/\langle\sigma\rangle \to \mathrm{Spec}(\mathbb{R}) = \mathrm{Spec}(\mathbb{C})/\langle\tau\rangle$ making $V/\langle\sigma\rangle$ into a real algebraic variety X such that $V \simeq X_{\mathbb{C}}$.

Two real structures σ and σ' on a same complex algebraic variety $f : V \to \mathrm{Spec}(\mathbb{C})$ are called equivalent if the associated real algebraic varieties $V/\langle\sigma\rangle$ and $V/\langle\sigma'\rangle$ are isomorphic, which holds if and only if there exists an automorphism of complex algebraic varieties $h : V \to V$ such that $\sigma' \circ h = h \circ \sigma$. A *real form* of a real algebraic variety X is a real algebraic variety X' such that the complex varieties $X_{\mathbb{C}}$ and $X'_{\mathbb{C}}$ are isomorphic. Galois descent then provides a one-to-one correspondence between isomorphism classes of real forms of a given real variety X and equivalence classes of real structures on its complexification $X_{\mathbb{C}}$.

1.2. **Galois descent for vector bundles.** Given a real algebraic variety X, étale descent for the Galois cover $c : \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$ also provides an equivalence between the category of quasi-coherent $\mathcal{O}_X$-modules and the category of pairs $(\mathcal{F}, \varphi)$ consisting of a quasi-coherent $\mathcal{O}_{X_{\mathbb{C}}}$-module $\mathcal{F}$ and an isomorphism $\varphi : \mathcal{F} \xrightarrow{\sim} \sigma_X^* \mathcal{F}$ of $\mathcal{O}_{X_{\mathbb{C}}}$-modules such that $\varphi^2 = (\sigma_X^* \varphi) \circ \varphi$.

Since σ_X is an involution, for every quasi-coherent $\mathcal{O}_{X_{\mathbb{C}}}$-module $\mathcal{E}$, we have $\sigma_X^*(\sigma_X^*\mathcal{E}) \simeq \mathcal{E}$. Letting $\mathcal{F} = \mathcal{E} \oplus \sigma_X^*\mathcal{E}$, the isomorphism $\varphi : \mathcal{F} = \mathcal{E} \oplus \sigma_X^*\mathcal{E} \to \sigma_X^*\mathcal{F} = \sigma_X^*\mathcal{E} \oplus \mathcal{E}$ exchanging the two factors of the direct sum satisfies $\varphi^2 = (\sigma_X^*\varphi) \circ \varphi$. We denote the corresponding quasi-coherent $\mathcal{O}_X$-module by $\mathcal{E}_{\mathbb{R}}$.

In the sequel, we essentially use this construction in the special case where $\mathcal{E}$ is the locally free $\mathcal{O}_{X_{\mathbb{C}}}$-module of germs of sections of a vector bundle $p : E = \mathrm{Spec}(\mathrm{Sym}^{\cdot}\mathcal{E}^{\vee}) \to X_{\mathbb{C}}$ on $X_{\mathbb{C}}$ of finite rank r. In this geometric context, the isomorphism φ can be interpreted as endowing the rank $2r$ vector bundle $\rho = p \oplus \sigma_X^* p : E \oplus \sigma_X^* E \to X_{\mathbb{C}}$ with a lift of σ_X to a real structure $\tilde{\sigma} : E \oplus \sigma_X^* E \to E \oplus \sigma_X^* E$ which is linear on the fibers of ρ, in such a way that ρ descends to a vector bundle

$$p_{\mathbb{R}} : E_{\mathbb{R}} := \mathrm{Spec}(\mathrm{Sym}^{\cdot}\mathcal{E}_{\mathbb{R}}^{\vee}) \simeq (E \oplus \sigma_X^* E)/\langle\tilde{\sigma}\rangle \to X \simeq X_{\mathbb{C}}/\langle\sigma_X\rangle$$

of rank $2r$ on X.

1.3. **Euclidean topologies.** Recall that the set $X(\mathbb{R})$ of real points of a real algebraic variety X is endowed in a natural way with the Euclidean topology, locally induced on each affine open subset by the usual Euclidean topology on the set $\mathbb{A}^n_{\mathbb{R}}(\mathbb{R}) \simeq \mathbb{R}^n$. The so-constructed topology on $X(\mathbb{R})$ is well-defined and independent of the choices made [16, Lemme 1 and Proposition 2]. Similarly, the set of complex points $V(\mathbb{C})$ of a complex algebraic variety V is endowed with the Euclidean topology locally induced by that on $\mathbb{A}^n_{\mathbb{C}}(\mathbb{C}) \simeq \mathbb{C}^n \simeq \mathbb{R}^{2n}$. If X (resp. V) is smooth, then $X(\mathbb{R})$ (resp. $V(\mathbb{C})$) can be further equipped with a natural structure of smooth manifold locally inherited from that on $\mathbb{R}^n$ (resp. $\mathbb{R}^{2n}$). Every morphism $h\colon X \to Y$ of smooth real algebraic varieties induces a continuous map $h(\mathbb{R})\colon X(\mathbb{R}) \to Y(\mathbb{R})$ for the Euclidean topologies, which is a diffeomorphism when f is an isomorphism. Similarly, a morphism of complex varieties $h : V \to W$ induces a continuous map $h(\mathbb{C}) : V(\mathbb{C}) \to W(\mathbb{C})$ which is a diffeomorphism when f is an isomorphism.

If V is a smooth complex variety equipped with a real structure σ, then σ induces a smooth involution of $V(\mathbb{C})$. The set $V(\mathbb{C})^{\sigma}$ of fixed points of σ is called the *real locus* of (V,σ). The quotient map $q : V \to X = V/\langle\sigma\rangle$ restricts to a diffeomorphism between $V(\mathbb{C})^{\sigma}$ endowed with the induced smooth structure and the set of real points $X(\mathbb{R})$ of X endowed with its smooth structure.

Let X be a smooth real algebraic variety, let $p : E \to X_{\mathbb{C}}$ be a vector bundle of rank r on $X_{\mathbb{C}}$ and let $p_{\mathbb{R}} : E_{\mathbb{R}} \to X$ be the vector bundle of rank $2r$ on X descended from the the rank $2r$ vector bundle $p \oplus \sigma_X^* p : E \oplus \sigma^* E \to X_{\mathbb{C}}$ as in §1.2. Then $p_{\mathbb{R}}(\mathbb{R}) : E_{\mathbb{R}}(\mathbb{R}) \to X(\mathbb{R})$ is a topological real vector bundle of rank $2r$ on the smooth manifold $X(\mathbb{R})$. On the other hand, the restriction of $p(\mathbb{C}) : E(\mathbb{C}) \to X_{\mathbb{C}}(\mathbb{C})$ to the real locus $X_{\mathbb{C}}(\mathbb{C})^{\sigma_X}$ of $X_{\mathbb{C}}(\mathbb{C})$ defines through the diffeomorphism $X_{\mathbb{C}}(\mathbb{C})^{\sigma_X} \xrightarrow{\sim} X(\mathbb{R})$ a topological complex vector bundle $\tilde{p} : \tilde{E} \to X(\mathbb{R})$ of rank r on $X(\mathbb{R})$, hence, forgetting about the complex structure, a topological real vector bundle of rank $2r$.

Lemma 3. *With the notation above, $\tilde{p} : \tilde{E} \to X(\mathbb{R})$ and $p_{\mathbb{R}}(\mathbb{R}) : E_{\mathbb{R}}(\mathbb{R}) \to X(\mathbb{R})$ are isomorphic topological real vector bundles of rank $2r$ on $X(\mathbb{R})$.*

Proof. Let $\tilde{\sigma}$ be the lift of σ_X to a real structure $\tilde{\sigma} : E \oplus \sigma_X^* E \to E \oplus \sigma_X^* E$ as in §1.2. Since σ_X acts trivially on the real locus $X_{\mathbb{C}}(\mathbb{C})^{\sigma_X} \simeq X(\mathbb{R})$ of $X_{\mathbb{C}}$, the restriction of $E(\mathbb{C}) \oplus \sigma^* E(\mathbb{C})$ to $X_{\mathbb{C}}(\mathbb{C})^{\sigma_X}$ is equal to $\tilde{E} \oplus \tilde{E}$ on which the restriction of $\tilde{\sigma}$ acts by the involution j exchanging the two factors. By construction $p_{\mathbb{R}}(\mathbb{R}) : E_{\mathbb{R}}(\mathbb{R}) \to X(\mathbb{R})$ is isomorphic to the quotient bundle $(\tilde{E} \oplus \tilde{E})/\langle j\rangle \to X(\mathbb{R})$, and the composition of the diagonal embedding $\tilde{E} \to \tilde{E} \oplus \tilde{E}$ with the quotient morphism $\tilde{E} \oplus \tilde{E} \to (\tilde{E} \oplus \tilde{E})/\langle j\rangle$ induces an isomorphism of topological real vector bundle between $\tilde{p} : \tilde{E} \to X(\mathbb{R})$ and $p_{\mathbb{R}}(\mathbb{R}) : E_{\mathbb{R}}(\mathbb{R}) \to X(\mathbb{R})$. □

2. Algebraic models of topological vector bundles on the 2-sphere

The real 2-sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3,\ x^2 + y^2 + z^2 = 1\}$ equipped with its usual structure of smooth manifold induced by the standard smooth structure on $\mathbb{R}^3$ is diffeomorphic to set of real points $\mathbb{Q}^2(\mathbb{R})$ of the smooth projective quadric surface $\mathbb{Q}^2 \subset \mathbb{P}^3_{\mathbb{R}} = \mathrm{Proj}_{\mathbb{R}}(\mathbb{R}[X,Y,Z,T])$ defined by the equation $X^2 + Y^2 + Z^2 - T^2 = 0$, endowed with its Euclidean topology. The complement of the hyperplane section $H = \{T = 0\}$ of $\mathbb{Q}^2$ is isomorphic to the smooth real affine quadric surface $\mathbb{S}^2 = \mathrm{Spec}(\mathbb{R}[x,y,z]/(x^2+y^2+z^2-1))$. The divisor class group of $\mathbb{Q}^2$ is isomorphic to $\mathbb{Z}$, generated by the class of H, from which it follows that the divisor class group of $\mathbb{S}^2$ is trivial. Furthermore, since H is a conic without real point, the inclusion $\mathbb{S}^2 \hookrightarrow \mathbb{Q}^2$ induces a diffeomorphism $\mathbb{S}^2(\mathbb{R}) \xrightarrow{\simeq} \mathbb{Q}^2(\mathbb{R}) \simeq S^2$.

Every real algebraic vector bundle $F \to \mathbb{S}^2$ gives rise to a topological real vector bundle of the same rank $F(\mathbb{R}) \to \mathbb{S}^2(\mathbb{R})$ on $\mathbb{S}^2(\mathbb{R}) \simeq S^2$. It was shown by Moore [14] (see also Fossum [6]) that every topological real vector bundle $\pi : E \to S^2$ on S^2 is isomorphic to one obtained in this way. In other words, every topological real vector bundle $\pi : E \to S^2$ admits an algebraic model in the form of an algebraic vector bundle on $\mathbb{S}^2$. Later on, Barge and Ojanguren [1] established the surprising much stronger fact that two algebraic vector bundles on $\mathbb{S}^2$ are isomorphic as algebraic vector bundles if and only if their associated topological vector bundles on S^2 are isomorphic as topological vector bundles. Summing up:

Proposition 4. *The map which associates to a real algebraic vector bundle $p : F \to \mathbb{S}^2$ on $\mathbb{S}^2$ the topological vector bundle $p(\mathbb{R}) : \mathbb{F}(\mathbb{R}) \to \mathbb{S}^2(\mathbb{R})$ on $\mathbb{S}^2(\mathbb{R}) \simeq S^2$ induces a one-to-one correspondence between isomorphism classes of algebraic vector bundles on $\mathbb{S}^2$ and isomorphism classes of topological real vector bundles on S^2.*

In the next paragraphs, we review briefly the classification of topological real vector bundles on S^2 and give a new construction of corresponding algebraic models in the framework of complex varieties with real structure.

2.1. **Recollection on topological real vector bundles on S^2.** Every topological real vector bundle on S^2 is orientable, and there exists a bijection

$$\theta : [S^1, \mathrm{GL}_r^+(\mathbb{R})] \to \mathrm{Vect}_r^+(S^2)$$

between the set of homotopy classes of continuous map from the circle S^1 to the group $\mathrm{GL}_r^+(\mathbb{R})$ of invertible matrices of rank r with positive determinant, and the set of isomorphism classes of oriented topological real vector bundles of rank r on S^2. This bijection can be explicitly realized via the so-called clutching construction. Namely, viewing S^2 as the union of its closed lower and upper hemispheres $S^2_{z\leq 0}$ and $S^2_{z\geq 0}$ with common boundary $\partial S^2_{z\leq 0} = \partial S^2_{z\geq 0} = \{z = 0\} \simeq S^1$, a continuous map $f : S^1 \to \mathrm{GL}_r^+(\mathbb{R})$ determines a real vector bundle $\pi : E_f \to S^2$ of rank r obtained as the quotient of $S^2_{z\leq 0} \times \mathbb{R}^r \sqcup S^2_{z\geq 0} \times \mathbb{R}^r$ by identifying $(x, v) \in \partial S^2_{z\leq 0} \times \mathbb{R}^r$ with $(x, f(x) \cdot v) \in \partial S^2_{z\geq 0} \times \mathbb{R}^r$. The isomorphism class of E_f depends only on the homotopy class of f, and the bijection θ is defined by sending a clutching map $f : S^1 \to \mathrm{GL}_r^+(\mathbb{R})$ to the vector bundle E_f it determines (see e.g. [8, Proposition 1.11]).

Noting that $\mathrm{GL}_r^+(\mathbb{R})$ retracts onto the special orthogonal group SO_r, we get that $\mathrm{Vect}_1^+(S^2)$ consists of the trivial line bundle only, and that $\mathrm{Vect}_2^+(S^2)$ is isomorphic to $\pi_1(\mathrm{SO}_2) \simeq \mathbb{Z}$. Identifying S^1 and SO_2 with the set of complex numbers $\alpha = x + iy$ of modulus one, a corresponding collection of clutching maps $f_n : S^1 \to \mathrm{SO}_2$ is simply given by $\alpha \mapsto \alpha^n$, $n \in \mathbb{Z}$. Writing $\alpha = \exp(i\theta)$, $\theta \in \mathbb{R}$, these correspond equivalently to the rotation matrices

$$M_2(n) = \exp(i\theta)^n = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}.$$

The real vector bundle corresponding to $M_2(n) \in \mathrm{SO}_2$ coincides with the image of the underlying real vector bundle of the complex line bundle $\mathcal{O}_{\mathbb{CP}^1}(n)$ on $\mathbb{CP}^1$ via the usual diffeomorphism $\mathbb{CP}^1 \to S^2 = \mathbb{C} \cup \{\infty\}$ mapping $[z_0 : z_1]$ to z_0/z_1. For instance, the tangent bundle $TS^2 \to S^2$ coincides with the image of underlying real vector bundle of $\mathcal{O}_{\mathbb{CP}^1}(2)$. Note that the underlying real vector bundle of $\mathcal{O}_{\mathbb{CP}^1}(-n)$ endowed with the orientation inherited from the complex structure is equal to the underlying real vector bundle of $\mathcal{O}_{\mathbb{CP}^1}(n)$ but equipped with the opposite orientation.

For every $r \geq 3$, $\mathrm{Vect}_r^+(S^2)$ is isomorphic to $\pi_1(\mathrm{SO}_r) \simeq \mathbb{Z}/2\mathbb{Z}$, a corresponding clutching map being given by the matrix

$$\mathrm{diag}((\exp(i\theta), I_{r-2}) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & I_{r-2} \end{pmatrix}$$

where I_{r-2} denote the $(r-2) \times (r-2)$ identity matrix. In other words, for every $r \geq 3$, the unique nontrivial real topological vector bundle of rank r on S^2 is the direct sum of the rank 2 vector bundle $\pi_{f_1} : E_{f_1} \to S^2$ corresponding to $M_2(1)$ and of the trivial vector bundle of rank $r - 2$. It also follows from this description that E_{f_n} is either 1-stably trivial if n is even or 1-stably isomorphic to E_{f_1} is n is odd.

2.2. **Algebraic models as vector bundles on the projective quadric.** In view of the description of isomorphism classes of topological real vector bundles on S^2 recalled in §2.1, to show that every real topological vector bundle $\pi : E \to S^2$ admits an algebraic model, it is enough to show that for every $n \geq 1$, the real topological vector bundle $\pi_{f_n} : E_{f_n} \to S^2$ corresponding to the underlying real vector bundle of the complex line bundle $\mathcal{O}_{\mathbb{CP}^1}(n)$ on $\mathbb{CP}^1$ admits such a model. Models for these bundles were constructed by Moore [14] and Swan [17] in the form of certain projective modules on the coordinate ring of the affine surface $\mathbb{S}^2$. The construction we give below is in contrast of geometric nature, providing models of these bundles in the form of restrictions to $\mathbb{S}^2$ of natural algebraic vector bundles on the real projective quadric $\mathbb{Q}^2$.

The closed embedding $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}\to\mathbb{P}^3_{\mathbb{C}}$ defined by

$$([x_0:x_1][y_0:y_1])\mapsto[X:Y:Z:T]=[x_0y_1+x_1y_0:i(x_1y_0-x_0y_1):x_0y_0-x_1y_1:x_0y_0+x_1y_1]$$

induces an isomorphism $\psi:\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}\xrightarrow{\simeq}\mathbb{Q}^2_{\mathbb{C}}$. The pull-back $\psi^*\sigma_{\mathbb{Q}^2}$ of the canonical real structure $\sigma_{\mathbb{Q}^2}$ on the complexification $\mathbb{Q}^2_{\mathbb{C}}$ of $\mathbb{Q}^2$ is the real structure $\sigma=s_\Delta\circ(\sigma_{\mathbb{P}^1_{\mathbb{R}}}\times\sigma_{\mathbb{P}^1_{\mathbb{R}}})$ on $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$, where s_Δ is the algebraic involution which exchanges the two factors and $\sigma_{\mathbb{P}^1_{\mathbb{R}}}$ is the canonical real structure on $\mathbb{P}^1_{\mathbb{C}}=(\mathbb{P}^1_{\mathbb{R}})_{\mathbb{C}}$. Applying the construction explained in §1.2 to the line bundles

$$p_n:L_n=\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n)\to\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}},\quad n\geq 0$$

on $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$, we obtain a collection of algebraic vector bundles $p_{n,\mathbb{R}}:L_{n,\mathbb{R}}\to\mathbb{Q}^2$ of rank 2 on $\mathbb{Q}^2$.

Lemma 5. *For every $n\geq 0$, the following hold:*

a) The complexification $p_{n,\mathbb{C}}:(L_{n,\mathbb{R}})_{\mathbb{C}}\to\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$ is isomorphic to $\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n)\oplus\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n)$,

b) The topological vector bundle $p_{n,\mathbb{R}}(\mathbb{R}):L_{n,\mathbb{R}}(\mathbb{R})\to\mathbb{Q}^2(\mathbb{R})=\mathbb{S}(\mathbb{R})$ is isomorphic to $\pi_{f_n}:E_{f_n}\to S^2$.

Proof. By construction, $(L_{n,\mathbb{R}})_{\mathbb{C}}$ is isomorphic to $L_n\oplus\sigma^*L_n$. Assertion a) then follows from the the identity

$$\sigma^*(\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n))=(\sigma_{\mathbb{P}^1_{\mathbb{R}}}\times\sigma_{\mathbb{P}^1_{\mathbb{R}}})^*(s_\Delta^*(\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n)))\simeq\mathrm{pr}_2^*(\sigma^*_{\mathbb{P}^1_{\mathbb{R}}}(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n)))$$

and the fact that $\sigma^*_{\mathbb{P}^1_{\mathbb{R}}}(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n))\simeq\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n)$ as line bundles on $\mathbb{P}^1_{\mathbb{C}}$.

The map $\xi=(\mathrm{id}\times\sigma_{\mathbb{P}^2_{\mathbb{R}}})\circ\Delta:\mathbb{P}^1_{\mathbb{C}}\to\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$, where Δ denotes the diagonal embedding, induces a diffeomorphism between $\mathbb{P}^1_{\mathbb{C}}(\mathbb{C})=\mathbb{CP}^1\simeq S^2$ and the real locus of $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$ endowed with the real structure σ. Assertion b) then follows from Lemma 3 and the identity

$$\xi^*(L_n)(\mathbb{C})=((\mathrm{id}\times\sigma_{\mathbb{P}^2_{\mathbb{R}}})\circ\Delta)^*(\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n))(\mathbb{C})\simeq\mathcal{O}_{\mathbb{CP}^1}(n)$$

which holds by construction of L_n. □

Remark 6. Via the isomorphism $\psi:\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}\xrightarrow{\simeq}\mathbb{Q}^2_{\mathbb{C}}$, the line bundle $L_n=\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n)$ coincides with the line bundle on $\mathbb{Q}^2_{\mathbb{C}}$ associated to the Cartier divisor nC, where C is the irreducible and reduced curve $\{X+iY=T-Z=0\}$ on $\mathbb{Q}^2_{\mathbb{C}}$. The $\Gamma(\mathbb{S}^2,\mathcal{O}_{\mathbb{S}^2})$-module of global sections $\Gamma(\mathbb{S}^2,L_{n,\mathbb{R}})$ of the restriction of $L_{n,\mathbb{R}}$ to $\mathbb{S}^2=\mathbb{Q}^2\setminus\{T=0\}$ then coincides with the invertible $\Gamma(\mathbb{S}^2_{\mathbb{C}},\mathcal{O}_{\mathbb{S}^2_{\mathbb{C}}})$-module $\mathfrak{p}^n=(x+iy,1-z)^n\subset\Gamma(\mathbb{S}^2_{\mathbb{C}},\mathcal{O}_{\mathbb{S}^2_{\mathbb{C}}})$, viewed as a projective $\Gamma(\mathbb{S}^2,\mathcal{O}_{\mathbb{S}^2})$-module of rank 2 via the inclusion $\Gamma(\mathbb{S}^2,\mathcal{O}_{\mathbb{S}^2})\hookrightarrow\Gamma(\mathbb{S}^2_{\mathbb{C}},\mathcal{O}_{\mathbb{S}^2_{\mathbb{C}}})$. We thus recover geometrically the construction given by Swan in [17].

Note also that since the inverse image of $\mathbb{Q}^2_{\mathbb{C}}\setminus\mathbb{S}^2_{\mathbb{C}}$ by ψ is the irreducible curve $\Gamma=\{x_0y_0+x_1y_1=0\}$ of type $(1,1)$ in the divisor class group of $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$, the restriction of $(L_{n,\mathbb{R}})_{\mathbb{C}}\simeq\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n)\oplus\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n)$ to $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}\setminus\Gamma\simeq\mathbb{S}^2_{\mathbb{C}}$ is isomorphic to that of $\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n)\oplus\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-n)\simeq L_n\oplus L_n^\vee$, where $L_n^\vee$ denotes the dual of L_n.

3. Real forms of the trivial bundle $\mathbb{S}^2\times\mathbb{A}^2_{\mathbb{R}}$

Notation 7. For every $n\geq 0$, we let $q_n:V_n\to\mathbb{S}^2$ be the restriction to $\mathbb{S}^2\subset\mathbb{Q}^2$ of the rank 2 vector bundle $p_{n,\mathbb{R}}:L_{n,\mathbb{R}}\to\mathbb{Q}^2$ constructed in §2.2.

3.1. **Proof of Theorem 1.** The following proposition implies Theorem 1:

Proposition 8. *The real algebraic varieties V_n, $n\geq 0$, are pairwise non isomorphic real forms of $V_0=\mathbb{S}^2\times\mathbb{A}^2_{\mathbb{R}}$.*

The proof is a combination of Lemma 9 and Lemma 10 below which show that the real algebraic varieties V_n, $n\geq 0$, are pairwise non isomorphic with isomorphic complexifications $V_{n,\mathbb{C}}\simeq\mathbb{S}^2_{\mathbb{C}}\times\mathbb{A}^2_{\mathbb{C}}$.

By construction, the rank 2 vector bundles $p_{n,\mathbb{R}}:L_{n,\mathbb{R}}\to\mathbb{Q}^2$, $n\geq 0$, on $\mathbb{Q}^2$ have pairwise non-isomorphic complexifications $(L_{n,\mathbb{R}})_{\mathbb{C}}\simeq\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n)\oplus\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n)\to\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$. The next lemma shows in contrast that their restrictions to $\mathbb{S}^2\subset\mathbb{Q}^2$ all have isomorphic complexifications:

Lemma 9. *For every $n \geq 0$, the complexification $q_{n,\mathbb{C}} : V_{n,\mathbb{C}} \to \mathbb{S}^2_{\mathbb{C}}$ of $q_n : V_n \to \mathbb{S}^2$ is isomorphic to the trivial vector bundle* $\mathrm{pr}_1 : \mathbb{S}^2_{\mathbb{C}} \times \mathbb{A}^2_{\mathbb{C}} \to \mathbb{S}^2_{\mathbb{C}}$.

Proof. Since the Picard group of $\mathbb{S}^2$ is equal to its divisor class group which is trivial, the determinant $\det(V_n)$ of V_n is isomorphic to the trivial line bundle on $\mathbb{S}^2$. This implies in turn that $\det(V_{n,\mathbb{C}})$ is the trivial line bundle on $\mathbb{S}^2_{\mathbb{C}}$, a fact which also follows more concretely from the observation made in Remark 6 that $V_{n,\mathbb{C}}$ is isomorphic to the direct sum of a line bundle and its dual. Since by a general result of Murthy [15], every algebraic vector bundle of rank 2 on $\mathbb{S}^2_{\mathbb{C}}$ splits a trivial factor, hence is isomorphic to the direct sum of its determinant and a trivial line bundle, we conclude that for every $n \geq 0$, $V_{n,\mathbb{C}}$ is isomorphic to the trivial vector bundle of rank 2 on $\mathbb{S}^2_{\mathbb{C}}$ (see §4.1 below for the construction of explicit isomorphisms $V_{n,\mathbb{C}} \simeq \mathbb{S}^2_{\mathbb{C}} \times \mathbb{A}^2_{\mathbb{C}}$). □

By § 2.1 and Lemma 5, the algebraic vectors bundles $q_n : V_n \to \mathbb{S}^2$, $n \geq 0$, are pairwise non-isomorphic as vector bundles over $\mathbb{S}^2$. The following result then implies the stronger fact that their total spaces are pairwise non isomorphic as abstract algebraic varieties:

Lemma 10. *The total spaces of two algebraic vector bundles $q : V \to \mathbb{S}^2$ and $q' : V' \to \mathbb{S}^2$ are isomorphic as abstract real algebraic varieties if and only if $q : V \to \mathbb{S}^2$ and $q' : V' \to \mathbb{S}^2$ are isomorphic as vector bundles.*

Proof. Let $\Psi : V \to V'$ be an isomorphism of abstract real algebraic varieties. First note that every morphism $f : \mathbb{A}^1_{\mathbb{R}} \to \mathbb{S}^2$ is constant. Indeed, otherwise, since $\mathbb{S}^2$ is affine hence does not contain complete curves, f would extend to a nonconstant morphism $\overline{f} : \mathbb{P}^1_{\mathbb{R}} \to \mathbb{Q}^2$ mapping the real point $\mathbb{P}^1_{\mathbb{R}} \setminus \mathbb{A}^1_{\mathbb{R}}$ to a point of $\mathbb{Q}^2 \setminus \mathbb{S}^2$. But this is impossible since the latter is a conic without real point. The restriction of $q' \circ \Psi : V \to V'$ to every fiber of q over a real point of $\mathbb{S}^2$ is thus constant. Since the set of points s of $\mathbb{S}^2$ such that $\dim((q' \circ \Psi)(q^{-1}(s))) = 0$ is closed in $\mathbb{S}^2$ and $\mathbb{S}^2(\mathbb{R})$ is Zariski dense in $\mathbb{S}^2$, it follows that $q' \circ \Psi$ is constant on the fibers of q, hence descends to a unique automorphism ψ of $\mathbb{S}^2$ such that $q' \circ \Psi = \psi \circ q$. This implies in turn that Ψ induces an isomorphism $\tilde{\Psi} : V \to \tilde{V} = \psi^* V'$ of schemes over $\mathbb{S}^2$. Now it follows from [5, Lemma 1.3] that $p : V \to \mathbb{S}^2$ and $\tilde{p} = p' \circ \tilde{\Psi} : \tilde{V} \to \mathbb{S}^2$ are isomorphic as algebraic vector bundles over $\mathbb{S}^2$. Let us briefly recall the argument for the sake of completeness: since V and $\tilde{V}$ are vector bundles, their relative tangent bundles $T_{V/\mathbb{S}^2}$ and $T_{\tilde{V}/\mathbb{S}^2}$ are isomorphic to p^*V and $\tilde{p}^*\tilde{V}$ respectively. Letting $\alpha : \mathbb{S}^2 \to V$ be any section of p, the composition $\tilde{\alpha} = \tilde{\Psi} \circ \alpha$ is a section of $\tilde{p}$, and the relative differential $d\tilde{\Psi}_{/\mathbb{S}^2} : T_{V/\mathbb{S}^2} \to \tilde{\Psi}^* T_{\tilde{V}/\mathbb{S}^2}$ of $\tilde{\Psi}$ over $\mathbb{S}^2$ then induces an isomorphism

$$\alpha^* d\tilde{\Psi}_{/\mathbb{S}^2} : V \simeq \alpha^* T_{V/\mathbb{S}^2} \xrightarrow{\simeq} \alpha^* \tilde{\Psi}^* T_{\tilde{V}/\mathbb{S}^2} = \tilde{\alpha}^* T_{\tilde{V}/\mathbb{S}^2} \simeq \tilde{V}$$

of algebraic vector bundles over $\mathbb{S}^2$. To complete the proof, it thus remains to show that $\psi^* V'$ is isomorphic to V' as algebraic vector bundles over $\mathbb{S}^2$. By virtue of Proposition 4, it suffices to show that the pull-back of $V'(\mathbb{R})$ by the induced diffeomorphism $\psi(\mathbb{R})$ of $\mathbb{S}^2(\mathbb{R}) \simeq S^2$ is isomorphic to $V'(\mathbb{R})$ as a topological real vector bundle. Since the mapping class group of S^2 is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, $\psi(\mathbb{R})^* V'(\mathbb{R})$ is either isomorphic to $V'(\mathbb{R})$ if $\psi(\mathbb{R})$ is orientation preserving, or to $V'(\mathbb{R})$ but endowed with the opposite orientation otherwise. So in each case $\psi(\mathbb{R})^* V'(\mathbb{R}) \simeq V'(\mathbb{R})$ and the assertion follows. □

Remark 11. In the special case of the rank 2 vector bundles $q_n : V_n \to \mathbb{S}^2$, $n \geq 0$, it is well-known that the associated real topological fourfolds $V_n(\mathbb{R}) \simeq \mathcal{O}_{\mathbb{CP}^1}(n)$ are actually even pairwise non-homeomorphic. This can be seen by comparing their respective first homology groups at infinity $H_1^{\infty}(\mathcal{O}_{\mathbb{CP}^1}(n); \mathbb{Z})$, defined as the limit over exhaustions of $\mathcal{O}_{\mathbb{CP}^1}(n)$ by compact subsets K_i of the homology groups $H_1(\mathcal{O}_{\mathbb{CP}^1}(n) \setminus K_i; \mathbb{Z})$. Since $\mathcal{O}_{\mathbb{CP}^1}(n)$ is homeomorphic to the complement in the Hirzebruch surface $\beta_n : \mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{CP}^1}(-n) \oplus \mathcal{O}_{\mathbb{CP}^1}) \to \mathbb{CP}^1$ of a section $H_{0,n}$ of β_n with self-intersection $-n$, which is unique if $n \neq 0$, it follows by excision that $H_1^{\infty}(\mathcal{O}_{\mathbb{CP}^1}(n); \mathbb{Z})$ is isomorphic to the first homology group of a pointed tubular neighborhood $T_*(H_{0,n})$ in $\mathbb{F}_n$, i.e. a tubular neighborhood of $H_{0;n}$ in $\mathbb{F}_n$ with $H_{0;n}$ removed from it. We conclude that

$$H_1^{\infty}(\mathcal{O}_{\mathbb{CP}^1}(n); \mathbb{Z}) \simeq H_1(T_*(H_{0,n}); \mathbb{Z}) \simeq \mathbb{Z}/\deg \mathcal{N}_{H_{0,n}/\mathbb{F}_n} \mathbb{Z} \simeq \mathbb{Z}/n\mathbb{Z}$$

where $\mathcal{N}_{H_{0,n}/\mathbb{F}_n} \simeq \mathcal{O}_{\mathbb{CP}^1}(-n)$ denotes the normal bundle of $H_{0,n}$ in $\mathbb{F}_n$.

For every $n \geq 0$, the real algebraic variety $V_n \times \mathbb{A}^1_{\mathbb{R}}$ is the total space of an algebraic vector bundle $q_n \circ \mathrm{pr}_1 : \mathrm{V_n} \times \mathbb{A}^1_{\mathbb{R}} \to \mathbb{S}^2$ of rank 3 on $\mathbb{S}^2$. By combining the classification of topological real vector bundles on S^2 given in §2.1 with Proposition 4 and Lemma 10, we obtain the following generalization of Hochster's counter-example to the Zariski Cancellation Problem [9] which, in our notation, corresponds to the case of the vector bundle $p_2 : V_2 \to \mathbb{S}^2$, isomorphic to the tangent bundle $T\mathbb{S}^2 \to \mathbb{S}^2$ of $\mathbb{S}^2$.

Corollary 12. *The real algebraic variety $V_n \times \mathbb{A}^1_{\mathbb{R}}$ is isomorphic to $V_0 \times \mathbb{A}^1_{\mathbb{R}}$ if n is even or to $V_1 \times \mathbb{A}^1_{\mathbb{R}}$ if n is odd. As a consequence, the real algebraic varieties V_{2p} (resp. V_{2p+1}), $p \geq 0$, form a family of pairwise non isomorphic rational factorial real algebraic varieties with isomorphic cylinders $V_{2p} \times \mathbb{A}^1_{\mathbb{R}}$ (resp. $V_{2p+1} \times \mathbb{A}^1_{\mathbb{R}}$).*

4. Examples and applications

4.1. **Explicit family of non-equivalent real structures on $\mathbb{S}^2_{\mathbb{C}} \times \mathbb{A}^2_{\mathbb{C}}$.** By Lemma 9, the complexifications $q_{n,\mathbb{C}} : V_{n,\mathbb{C}} \to \mathbb{S}^2_{\mathbb{C}}$ of the rank 2 vector bundles $q_n : V_n \to \mathbb{S}^2$, $n \geq 0$, are all isomorphic to the trivial vector bundle $\mathrm{pr}_1 : \mathbb{S}^2_{\mathbb{C}} \times \mathbb{A}^2_{\mathbb{C}} \to \mathbb{S}^2_{\mathbb{C}}$. In fact, we have the following more explicit description:

Proposition 13. *For $n \geq 1$, let $P_n, Q_n \in \mathbb{R}[z] \subset \Gamma(\mathbb{S}^2_{\mathbb{C}}, \mathcal{O}_{\mathbb{S}^2_{\mathbb{C}}})$ be any polynomials such that*

$$(1+z)^n P_n(z) + (1-z)^n Q_n(z) = 1.$$

Then the following hold:

a) The composition of the canonical product real structure $\Sigma_0 = \sigma_{\mathbb{S}^2} \times \sigma_{\mathbb{A}^2_{\mathbb{R}}}$ on $\mathbb{S}^2_{\mathbb{C}} \times \mathbb{A}^2_{\mathbb{C}}$ with the automorphism of the trivial bundle $\mathbb{S}^2_{\mathbb{C}} \times \mathbb{A}^2_{\mathbb{C}}$ defined by the matrix

$$A_n = \begin{pmatrix} (x-iy)^n(P_n+Q_n) & (1-z)^n - (1+z)^n \\ -(1+z)^n P_n^2 + (1-z)^n Q_n^2 & -(x+iy)^n(P_n+Q_n) \end{pmatrix} \in \mathrm{GL}_2(\Gamma(\mathbb{S}^2_{\mathbb{C}}, \mathcal{O}_{\mathbb{S}^2_{\mathbb{C}}}))$$

defines a real structure Σ_n on $\mathbb{S}^2_{\mathbb{C}} \times \mathbb{A}^2_{\mathbb{C}}$.

b) There exists an isomorphism $\Theta_n : V_{n,\mathbb{C}} \xrightarrow{\simeq} \mathbb{S}^2_{\mathbb{C}} \times \mathbb{A}^2_{\mathbb{C}}$ of vector bundles over $\mathbb{S}^2_{\mathbb{C}}$ such that $\Sigma_n \circ \Theta_n = \Theta_n \circ \sigma_{V_n}$, where σ_{V_n} denotes the canonical real structure on $V_{n,\mathbb{C}}$.

Proof. The fact that A_n defines an automorphism j_n of the trivial bundle $\mathbb{S}^2_{\mathbb{C}} \times \mathbb{A}^2_{\mathbb{C}}$ such that $(j_n \circ \Sigma_0)^2 = \mathrm{id}_{\mathbb{S}^2_{\mathbb{C}} \times \mathbb{A}^2_{\mathbb{C}}}$ follows from a direct calculation. To construct the isomorphism Θ_n, we recall from Remark 6 that the vector bundle $q_{n,\mathbb{C}} : V_{n,\mathbb{C}} \to \mathbb{S}^2_{\mathbb{C}}$ is isomorphic to the direct sum of the line bundle $E_n \to \mathbb{S}^2_{\mathbb{C}}$ associated to the locally free sheaf $\mathcal{O}_{\mathbb{S}^2_{\mathbb{C}}}(nC)$, where $C = \{x+iy = 1-z = 0\}$ and of the line bundle $\sigma^*_{\mathbb{S}^2} E_n$ associated to the locally free sheaf $\mathcal{O}_{\mathbb{S}^2_{\mathbb{C}}}(n\sigma^{-1}_{\mathbb{S}^2}(C))$, where $\sigma^{-1}_{\mathbb{S}^2}(C) = \{x-iy = 1-z = 0\}$. The canonical real structure σ_{V_n} on $V_{n,\mathbb{C}}$ coincides via this isomorphism with the natural lift of $\sigma_{\mathbb{S}^2}$ to a real structure $\tilde{\sigma}_n$ on $E_n \oplus \sigma^*_{\mathbb{S}^2} E_n$ defined in §1.2.

The surface $\mathbb{S}^2_{\mathbb{C}}$ is covered by the $\sigma_{\mathbb{S}^2}$-invariant principal affine open subsets $U_{\pm} = \mathbb{S}^2_{\mathbb{C}} \setminus \{1 \pm z \neq 0\}$. By definition of C and $\sigma^{-1}_{\mathbb{S}^2}(C)$, we have $C \cap U_- = \sigma^{-1}_{\mathbb{S}^2}(C) \cap U_- = \emptyset$, whereas $C \cap U_+$ and $\sigma^{-1}_{\mathbb{S}^2}(C) \cap U_+$ are principal divisors due to the relation $(1-z) = (1+z)^{-1}(x-iy)(x+iy)$ which holds in the coordinate ring of U_+. The choice of local equations $\{1, x+iy\}$ and $\{1-z, (1+z)^{-1}(x-iy)\}$ for C and $\sigma^{-1}_{\mathbb{S}^2}(C)$ on U_- and U_+ induces local trivializations

$$\gamma_{n,\pm} : E_n \oplus \sigma^*_{\mathbb{S}^2} E_n|_{U_{\pm 1}} \xrightarrow{\simeq} U_{\pm} \times \mathrm{Spec}(\mathbb{C}[t_{n,\pm}, t'_{n,\pm}])$$

for which the isomorphism $\psi_n = \gamma_{n,+} \circ \gamma^{-1}_{n,-}|_{U_+ \cap U_-}$ is given by the matrix

$$D_n = \begin{pmatrix} (x+iy)^n & 0 \\ 0 & (x+iy)^{-n} \end{pmatrix} \in \mathrm{SL}_2(\Gamma(U_+ \cap U_-, \mathcal{O}_{\mathbb{S}^2_{\mathbb{C}}})).$$

A direct calculation using the relation $(1+z)^n P_n(z) + (1-z)^n Q_n(z) = 1$ then shows that $D_n = M^{-1}_{n,+} M_{n,-}$, where

$$M_{n,+} = \begin{pmatrix} \left(\frac{x-iy}{1+z}\right)^n & (1+z)^n \\ -P_n & (x+iy)^n Q_n \end{pmatrix} \quad \text{and} \quad M_{n,-} = \begin{pmatrix} (1-z)^n & \left(\frac{x-iy}{1-z}\right)^n \\ -(x+iy)^n P_n & Q_n \end{pmatrix}$$

are elements of $\mathrm{SL}_2(\Gamma(U_+, \mathcal{O}_{\mathbb{S}^2_{\mathbb{C}}}))$ and $\mathrm{SL}_2(\Gamma(U_-, \mathcal{O}_{\mathbb{S}^2_{\mathbb{C}}}))$ respectively. It follows that the local trivilizations

$$M_{n,\pm} \circ \gamma_{n,\pm} : E_n \oplus \sigma^*_{\mathbb{S}^2} E_n|_{U_{\pm}} \xrightarrow{\simeq} U_{\pm} \times \mathbb{A}^2_{\mathbb{C}}$$

glue to global one $\Theta_n : E_n \oplus \sigma^*_{\mathbb{S}^2} E_n \xrightarrow{\simeq} \mathbb{S}^2_{\mathbb{C}} \times \mathbb{A}^2_{\mathbb{C}}$.

With our choice of local generators, the images of the restrictions of the real structure $\tilde{\sigma}_n$ under the local trivializations $\gamma_{n,\pm}$ are given locally on the open cover $\{U_{\pm}\}$ by the composition of $\sigma_{\mathbb{S}^2} \times \sigma_{\mathbb{A}^2_{\mathbb{R}}}|_{U_{\pm 1} \times \mathbb{A}^2_{\mathbb{C}}}$ with the involutions of the trivial bundles $U_{\pm} \times \mathbb{A}^2_{\mathbb{C}}$ with respective matrices

$$J_{n,\pm} = \begin{pmatrix} 0 & (1 \pm z)^{-n} \\ (1 \pm z)^n & 0 \end{pmatrix}.$$

A direct computation then confirms that the local real structures given by the compositions

$$M_{n,\pm} \circ J_{n,\pm}(\sigma_{\mathbb{S}^2} \times \sigma_{\mathbb{A}^2_{\mathbb{R}}}|_{U_{\pm 1}}) \circ M^{-1}_{n,\pm}|_{U_\pm \times \mathbb{A}^2_{\mathbb{C}}}$$

glue to a global one on $\mathbb{S}^2_{\mathbb{C}} \times \mathbb{A}^2_{\mathbb{C}}$ equal to Σ_n, for which we have by construction $\Theta_n \circ \tilde{\sigma}_n = \Sigma_n \circ \Theta_n$. □

Example 14. For $n = 1$ and 2, one can choose for instance $P_1 = Q_1 = 1/2$, $P_2 = (2-z)/4$ and $Q_2 = (2+z)/4$ to obtain respectively

$$A_1 = \begin{pmatrix} x - iy & -2z \\ -\frac{1}{2}z & -x - iy \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} (x-iy)^2 & -4z \\ \frac{1}{4}z(z^2-2) & -(x+iy)^2 \end{pmatrix}.$$

Corollary 15. *With the notation of Proposition 13, the following hold:*

a) The real structures Σ_n, $n \geq 0$ on $\mathbb{S}^2_{\mathbb{C}} \times \mathbb{A}^2_{\mathbb{C}}$ are pairwise non-equivalent,

b) The real structure $\Sigma_n \times \sigma_{\mathbb{A}^1_{\mathbb{R}}}$ on $\mathbb{S}^2_{\mathbb{C}} \times \mathbb{A}^3_{\mathbb{C}}$ is equivalent to $\Sigma_0 \times \sigma_{\mathbb{A}^1_{\mathbb{R}}} = \sigma_{\mathbb{S}^2} \times \sigma_{\mathbb{A}^3_{\mathbb{R}}}$ if n is even and to $\Sigma_1 \times \sigma_{\mathbb{A}^1_{\mathbb{R}}}$ if n is odd.

Proof. The first assertion follows from Proposition 8 and Proposition 13 since by construction $\mathbb{S}^2_{\mathbb{C}} \times \mathbb{A}^2_{\mathbb{C}}$ endowed with the real structure Σ_n corresponds to the algebraic vector bundle $q_n : V_n \to \mathbb{S}^2$. Since the variety $(\mathbb{S}^2_{\mathbb{C}} \times \mathbb{A}^2_{\mathbb{C}}) \times \mathbb{A}^1_{\mathbb{C}}$ endowed with the real structure $\Sigma_n \times \sigma_{\mathbb{A}^1_{\mathbb{R}}}$ corresponds in turn to the algebraic vector bundle $V_n \times \mathbb{A}^1_{\mathbb{R}}$ on $\mathbb{S}^2$, the second assertion follows from Corollary 12. □

Example 16. Corresponding to the classical fact that the tangent bundle $T\mathbb{S}^2 \to \mathbb{S}^2$ is 1-stably trivial, the real structure $\Sigma_2 \times \sigma_{\mathbb{A}^1_{\mathbb{R}}}$, defined as the composition of the automorphism $\hat{j}_2$ of $\mathbb{S}^2_{\mathbb{C}} \times \mathbb{A}^3_{\mathbb{C}}$ defined by the matrix

$$\hat{A}_2 = \begin{pmatrix} A_2 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_3(\Gamma(\mathbb{S}^2_{\mathbb{C}}, \mathcal{O}_{\mathbb{S}^2_{\mathbb{C}}})),$$

where A_2 is the matrix in Example 14 with the canonical real structure $\Sigma_0 \times \sigma_{\mathbb{A}^1_{\mathbb{R}}}$, is equivalent to $\Sigma_0 \times \sigma_{\mathbb{A}^1_{\mathbb{R}}}$. By definition, this amounts to the identity $\psi \circ (\Sigma_2 \times \sigma_{\mathbb{A}^1_{\mathbb{R}}}) = (\Sigma_0 \times \sigma_{\mathbb{A}^1_{\mathbb{R}}}) \circ \psi$ for some automorphism ψ of the trivial bundle $\mathbb{S}^2_{\mathbb{C}} \times \mathbb{A}^3_{\mathbb{C}}$. Rewriting this identity in the form

$$(\hat{j}_2 \times \mathrm{id}) = \psi^{-1} \circ (\Sigma_0 \times \sigma_{\mathbb{A}^1_{\mathbb{R}}}) \circ \psi \circ (\Sigma_0 \times \sigma_{\mathbb{A}^1_{\mathbb{R}}})^{-1},$$

we see that it holds for instance for the automorphism ψ defined by the following matrix

$$C = \begin{pmatrix} \frac{1}{2}y(x+iy) + \frac{i}{4}z^2 & iz & x \\ -\frac{1}{2}x(x+iy) - \frac{1}{4}z^2 & z & y \\ \frac{1}{4}z(y-ix) & -(y+ix) & z \end{pmatrix} \in \mathrm{GL}_3(\Gamma(\mathbb{S}^2_{\mathbb{C}}, \mathcal{O}_{\mathbb{S}^2_{\mathbb{C}}})).$$

Indeed, a direct computation shows that $(\Sigma_0 \times \sigma_{\mathbb{A}^1_{\mathbb{R}}}) \circ \psi \circ (\Sigma_0 \times \sigma_{\mathbb{A}^1_{\mathbb{R}}})^{-1}$ is defined by the matrix

$$\overline{C} = \begin{pmatrix} \frac{1}{2}y(x-iy) - \frac{i}{4}z^2 & -iz & x \\ -\frac{1}{2}x(x-iy) - \frac{1}{4}z^2 & z & y \\ \frac{1}{4}z(y+ix) & -(y-ix) & z \end{pmatrix} \in \mathrm{GL}_3(\Gamma(\mathbb{S}^2_{\mathbb{C}}, \mathcal{O}_{\mathbb{S}^2_{\mathbb{C}}}))$$

and that $\hat{A}_2 = C^{-1} \cdot \overline{C}$.

4.2. **Proof of Theorem 2.**

Lemma 17. *For every $n \geq 1$, there exists a smooth rational real affine variety X of dimension n such that $X(\mathbb{R}) \neq \emptyset$ and such that $X_{\mathbb{C}}$ is a of log-general type, with trivial automorphism group* $\mathrm{Aut}(X_{\mathbb{C}})$.

Proof. Indeed, it suffices to take for X the complement in $\mathbb{P}^n_{\mathbb{R}}$ of smooth real hypersurface of degree $d > n+1$ (non-connected in the case $n = 1$). □

Theorem 2 is then a consequence of the following more precise result:

Proposition 18. *Let X be a smooth rational real affine variety X as in Lemma 17. Then the rational real affine variety $\mathbb{S}^2 \times \mathbb{A}^2_{\mathbb{R}} \times X$ has at least countably infinitely many pairwise distinct real forms.*

Proof. With the notation 7, it suffices to check that the varieties $V_n \times X$, $n \geq 0$, are pairwise non isomorphic since their complexifications are all isomorphic to $\mathbb{S}^2_{\mathbb{C}} \times \mathbb{A}^2_{\mathbb{C}} \times X_{\mathbb{C}}$ by Lemma 9. So suppose given an isomorphism of abstract real algebraic varieties $h : V_n \times X \to V_m \times X$ for some $n, m \geq 0$. The complexification $h_{\mathbb{C}}$ of h is then an automorphism of $\mathbb{S}^2_{\mathbb{C}} \times \mathbb{A}^2_{\mathbb{C}} \times X_{\mathbb{C}}$. Since $\mathbb{S}^2_{\mathbb{C}} \times \mathbb{A}^2_{\mathbb{C}}$ is $\mathbb{A}^1$-ruled whereas $X_{\mathbb{C}}$ is of log-general type by hypothesis, it follows from Iitaka-Fujita strong cancellation theorem [10] that there exists a unique automorphism ξ of $X_{\mathbb{C}}$ such that $\mathrm{pr}_{X_{\mathbb{C}}} \circ h_{\mathbb{C}} = \xi \circ \mathrm{pr}_{X_{\mathbb{C}}}$. By hypothesis, $\xi = \mathrm{id}_{X_{\mathbb{C}}}$ and so, we conclude that h is actually an isomorphism of schemes over X. Since $X(\mathbb{R}) \neq \emptyset$, the restriction of h over any real point x of X is then an isomorphism of real algebraic varieties $V_n \xrightarrow{\sim} V_m$, which implies that $m = n$ by Lemma 10. □

References

1. J. Barge and M. Ojanguren, *Fibrés algébriques sur une surface rélle*, Comment. Math. Helv., 62 (1987), 616-629.
2. J. Blanc and A. Dubouloz, *Automorphisms of $\mathbb{A}^1$-fibered surfaces*, Trans. Amer. Math. Soc. 363 (2011), 5887-5924.
3. A. Borel and J.-P.Serre, *Théorèmes de finitude en cohomologie galoisienne*, Comment. Math. Helv., 39 (1964), 111-164.
4. T.-C. Dinh and K. Oguiso, *A surface with discrete and non-finitely generated automorphism group*, arXiv:1710.07019.
5. P. Eakin and W. Heinzer, *A cancellation problem for rings*, Conference on Commutative Algebra, Lectures Notes in Math., vol. 311, Springer-Verlag, 1973, pp. 61-77.
6. R. Fossum, *Vector bundles over spheres are algebraic*, Invent. Math. 8 (1969), 222-225.
7. *Revêtements étales et groupe fondamental*, Séminaire de Géométrie Algébrique du Bois Marie 1960-1961 (SGA 1), Dirigé par Alexandre Grothendieck. Augmenté de deux exposés de M. Raynaud, Lecture Notes in Mathematics, Vol. 224. Springer-Verlag, Berlin-New York, 1971.
8. A. Hatcher, *Vector Bundles and K-Theory*, https://www.math.cornell.edu/ hatcher/VBKT/VBpage.html
9. M. Hochster, *Nonuniqueness of coefficient rings in a polynomial ring*, Proc. Amer. Math. Soc. 34 (1972), 81-82.
10. S. Iitaka, T. Fujita, *Cancellation theorem for algebraic varieties*, J. Fac. Sci. Univ. Tokyo, Sect.IA, 24 (1977), 123-127.
11. T. Kambayashi, *On the absence of nontrivial separable forms of the affine plane*, J. of Algebra 35 (1975), 449-456.
12. J. Lesieutre *Projective variety with discrete, non-finitely generated automorphism group*, to appear in Inventiones Math. (arXiv:1609.06391).
13. L. Makar-Limanov, *On groups of automorphisms of a class of surfaces*, Israel J. Math. 69 (1990), no. 2, 250-256.
14. N. Moore, *Algebraic vector bundles over the 2-sphere*, Invent. Math. 14 (1971), 167-172.
15. M.P. Murthy, *Vector bundles over affine surfaces birationally equivalent to a ruled surface*, Ann. of Math. (2) 89 (1969) 242-253.
16. J.-P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier, Grenoble 6 (1955), 1-42.
17. G. Swan, *Algebraic vector bundles on the 2-sphere*, Rocky Mountain J. Math. 23 (1993), no. 4, 1443-1469.

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