

# DIFFERENCE BETWEEN FAMILIES OF WEAKLY AND STRONGLY MAXIMAL INTEGRAL LATTICE-FREE POLYTOPES

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## Abstract

A  $d$ -dimensional closed convex set  $K$  in  $\mathbb{R}^d$  is said to be lattice-free if the interior of  $K$  is disjoint with  $\mathbb{Z}^d$ . We consider the following two families of lattice-free polytopes: the family  $\mathcal{L}^d$  of integral lattice-free polytopes in  $\mathbb{R}^d$  that are not properly contained in another integral lattice-free polytope and its subfamily  $\mathcal{M}^d$  consisting of integral lattice-free polytopes in  $\mathbb{R}^d$  which are not properly contained in another lattice-free set. It is known that  $\mathcal{M}^d = \mathcal{L}^d$  holds for  $d \leq 3$  and, for each  $d \geq 4$ ,  $\mathcal{M}^d$  is a proper subfamily of  $\mathcal{L}^d$ . We derive a super-exponential lower bound on the number of polytopes in  $\mathcal{L}^d \setminus \mathcal{M}^d$  (with standard identification of integral polytopes up to affine unimodular transformations).

## 1 Introduction

By  $|X|$  we denote the cardinality of a finite set  $X$ . Let  $\mathbb{N}$  be the set of all positive integers and let  $d \in \mathbb{N}$  be the dimension. We call elements of  $\mathbb{Z}^d$  are called *integral points* or *integral vectors*. We call a polyhedron  $P \subseteq \mathbb{R}^d$  *integral* if  $P$  is the convex hull of  $P \cap \mathbb{Z}^d$ . Let  $\text{Aff}(\mathbb{Z}^d)$  be the group of affine transformations  $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$  with  $A(\mathbb{Z}^d) = \mathbb{Z}^d$ . We call elements of  $\text{Aff}(\mathbb{Z}^d)$  *affine unimodular transformations*. For a family  $\mathcal{X}$  of subsets of  $\mathbb{R}^d$ , we consider the family of equivalence classes

$$\mathcal{X} / \text{Aff}(\mathbb{Z}^d) := \left\{ \left\{ A(X) : A \in \text{Aff}(\mathbb{Z}^d) \right\} : X \in \mathcal{X} \right\}$$

with respect to identification of the elements of  $\mathcal{X}$  up to affine unimodular transformations. A subset  $K$  of  $\mathbb{R}^d$  is called *lattice-free* if  $K$  is closed, convex,  $d$ -dimensional and the interior of  $K$  contains no points from  $\mathbb{Z}^d$ . A set  $K$  is called *maximal lattice-free* if  $K$  is lattice-free and is not a proper subset of another lattice-free set.

Our objective is to study the relationship between the following two families of integral lattice-free polytopes:

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- The family  $\mathcal{L}^d$  of integral lattice-free polytopes  $P$  in  $\mathbb{R}^d$  such that there exists no integral lattice-free polytope properly containing  $P$ . We call elements of  $\mathcal{L}^d$  *weakly maximal* integral lattice-free polytopes.
- The family  $\mathcal{M}^d$  of integral lattice-free polytopes  $P$  in  $\mathbb{R}^d$  such that there exists no lattice-free set properly containing  $P$ . We call the elements of  $\mathcal{L}^d$  *strongly maximal* integral lattice-free polytopes.

The family  $\mathcal{L}^d$  has applications in mixed-integer optimization, algebra and algebraic geometry; see [DPW16, AKW17], [BHHS16] and [Tre10], respectively. In [AWW11, NZ11] it was shown that  $\mathcal{L}^d$  is finite up to affine unimodular transformations:

**Theorem 1.** ([AWW11, Theorem 2.1], [NZ11, Corollary 1.3])  $\mathcal{L}^d / \text{Aff}(\mathbb{Z}^d)$  is finite.

Several groups of researchers are interested in enumeration of  $\mathcal{L}^d$ , up to affine unimodular transformations, in fixed dimensions. This requires understanding geometric properties of  $\mathcal{L}^d$ . Currently, no explicit description of  $\mathcal{L}^d$  is available for dimensions  $d \geq 4$  and, moreover, it is even extremely hard to decide if a given polytope belongs to  $\mathcal{L}^d$ . A brute-force algorithm based on volume bounds for  $\mathcal{L}^d$  (provided in [NZ11]) would have doubly exponential running time in  $d$ . In contrast to  $\mathcal{L}^d$ , its subfamily  $\mathcal{M}^d$  is easier to deal with. Lovász's characterization [Lov89, Proposition 3.3] of maximal lattice-free sets leads to a straightforward geometric description of polytopes belong to  $\mathcal{M}^d$ . This characterization can be used to decide whether a given polytope is an element of  $\mathcal{M}^d$  in only exponential time in  $d$ . Thus, while enumeration of  $\mathcal{M}^d$  in fixed dimensions is a hard task, too, enumeration of  $\mathcal{L}^d$  is even more challenging.

For a given dimension  $d$ , it is a priori not clear whether or not  $\mathcal{M}^d$  is a proper subset of  $\mathcal{L}^d$ . Recently, it has been shown that the inequality  $\mathcal{M}^d = \mathcal{L}^d$  holds if and only if  $d \leq 3$ . The equality  $\mathcal{M}^d = \mathcal{L}^d$  is rather obvious for  $d \in \{1, 2\}$ , as it is not hard to enumerate  $\mathcal{L}^d$  in these very small dimensions and to check that every element of  $\mathcal{L}^d$  belongs to  $\mathcal{M}^d$ . Starting from dimension three, the problem gets very difficult. Results in [AWW11] and [AKW17] establish the equality  $\mathcal{M}^3 = \mathcal{L}^3$  and enumerate  $\mathcal{L}^3$ , up to affine unimodular transformations. As a complement, in [NZ11, Theorem 1.4] it was shown that for all  $d \geq 4$  there exists a polytope belonging to  $\mathcal{L}^d$  but not to  $\mathcal{M}^d$ .

While Theorem 1.4 in [NZ11] shows that  $\mathcal{L}^d$  and  $\mathcal{M}^d$  are two different families, it does not provide information on the number of polytopes in  $\mathcal{L}^d$  that do not belong to  $\mathcal{M}^d$ . Relying on a result of Konyagin [Kon14], we will show that, asymptotically, the gap between  $\mathcal{L}^d$  and  $\mathcal{M}^d$  is very large.

For  $a_1, \dots, a_d > 0$ , we introduce

$$\kappa(a) := \kappa(a_1, \dots, a_d) = \frac{1}{a_1} + \dots + \frac{1}{a_d}. \quad (1.1)$$

Reciprocals of positive integers are sometimes called Egyptian fractions. Thus, if  $a \in \mathbb{N}^d$ , then  $\kappa(a)$  is a sum of  $d$  Egyptian fractions. We consider the set

$$\mathcal{A}_d := \left\{ (a_1, \dots, a_d) \in \mathbb{N}^d : a_1 \leq \dots \leq a_d, \kappa(a_1, \dots, a_d) = 1 \right\} \quad (1.2)$$

of all different solutions of the Diophantine equation

$$\kappa(x_1, \dots, x_d) = 1$$

in the unknowns  $x_1, \dots, x_d \in \mathbb{N}$ . The set  $\mathcal{A}_d$  represents possible ways to write 1 as a sum of  $d$  Egyptian fractions. It is known that  $\mathcal{A}_d$  is finite. Our main result allows is a lower bound on the cardinality of  $(\mathcal{L}^d \setminus \mathcal{M}^d) / \text{Aff}(\mathbb{Z}^d)$ :

**Theorem 2.**  $|(\mathcal{L}^{d+5} \setminus \mathcal{M}^{d+5}) / \text{Aff}(\mathbb{Z}^{d+5})| \geq |\mathcal{A}_d|$ .

The proof of Theorem 2 is constructive. This means that, for every  $a \in \mathcal{A}_d$ , we generate an element in  $P_a \in \mathcal{L}^{d+5} \setminus \mathcal{M}^{d+5}$  such that for two different elements  $a$  and  $b$  of  $\mathcal{A}_d$ , the respective polytopes  $P_a$  and  $P_b$  do not coincide up to affine unimodular transformations. The proof of Theorem 2 is inspired by the construction in [NZ11]. Using lower bounds on  $|\mathcal{A}_d|$  from [Kon14], we obtain the following asymptotic estimate:

**Corollary 3.**  $\ln \ln |(\mathcal{L}^d \setminus \mathcal{M}^d) / \text{Aff}(\mathbb{Z}^d)| = \Omega\left(\frac{d}{\ln d}\right)$ , as  $d \rightarrow \infty$ .

**Notation.** We view the elements of  $\mathbb{R}^d$  as columns. By  $o$  we denote the zero vector and by  $e_1, \dots, e_d$  the standard basis of  $\mathbb{R}^d$ . If  $x \in \mathbb{R}^d$  and  $i \in \{1, \dots, d\}$ , then  $x_i$  denotes the  $i$ -th component of  $x$ . The relation  $a \leq b$  for  $a, b \in \mathbb{R}^d$  means  $a_i \leq b_i$  for every  $i \in \{1, \dots, d\}$ . The relations  $\geq, >$  and  $<$  on  $\mathbb{R}^d$  are introduced analogously. The abbreviations  $\text{aff}, \text{conv}, \text{int}$  and  $\text{relint}$  stand for the affine hull, convex hull, interior and relative interior, respectively.

## 2 An approach to construction of polytopes in $\mathcal{L}^d \setminus \mathcal{M}^d$

We will present a systematic approach to construction of polytopes in  $\mathcal{L}^d \setminus \mathcal{M}^d$ , but first we discuss general maximal lattice-free sets.

**Definition 4.** Let  $P$  be a lattice-free polyhedron in  $\mathbb{R}^d$ . We say that a facet  $F$  of  $P$  is *blocked* if the relative interior of  $F$  contains an integral point.

Maximal lattice-free sets can be characterized as follows:

**Proposition 5.** ([Lov89, Proposition 3.3].) *Let  $K$  be a  $d$ -dimensional closed convex subset of  $\mathbb{R}^d$ . Then the following conditions are equivalent.*

- (i)  $K$  is maximal lattice-free,
- (ii)  $K$  is a lattice-free polyhedron such that every facet of  $K$  is blocked.

It can happen that some facets of a maximal lattice-free polyhedron are more than just blocked. We introduce a respective notion. Recall that the *integer hull*  $K_I$  of a compact convex set  $K$  in  $\mathbb{R}^d$  is defined by

$$K_I := \text{conv}(K \cap \mathbb{Z}^d).$$

**Definition 6.** Let  $P$  be a  $d$ -dimensional lattice-free polyhedron in  $\mathbb{R}^d$ . A facet  $F$  of  $P$  is called *strongly blocked* if  $F_I$  is  $(d-1)$ -dimensional and  $\mathbb{Z}^d \cap \text{relint } F_I \neq \emptyset$ . The polyhedron  $P$  is called *strongly blocked* if all facets of  $P$  are strongly blocked.

The following proposition extracts the geometric principle behind the construction from [NZ11, Section 3]. (Note that arguments in [NZ11, Section 3] use an algebraic language.)

**Proposition 7.** *Let  $P$  be a strongly blocked lattice-free polytope in  $\mathbb{R}^d$ . Then  $P_I \in \mathcal{L}^d$ . Furthermore, if  $P_I$  is not integral, then  $P_I \notin \mathcal{M}^d$ .*

*Proof.* In order to show  $P_I \in \mathcal{L}^d$  it suffices to verify that, for every  $z \in \mathbb{Z}^d$  such that  $\text{conv}(P_I \cup \{z\})$  is lattice-free, one necessarily has  $z \in P_I$ . If  $z \notin P_I$ , then  $z \notin P$  and so, for some facet  $F$  of  $P$ , the point  $z$  and the polytope  $P$  lie on different sides of the hyperplane  $\text{aff } F$ . Then  $\emptyset \neq \mathbb{Z}^d \cap \text{relint } F_I \subseteq \text{int}(\text{conv}(P \cup \{z\}))$ , yielding a contradiction to the choice of  $z$ . Thus, for every facet  $F$  of  $P$ ,  $z$  and  $P$  lie on the same side of  $\text{aff } F$ . It follows  $z \in P$ . Hence  $z \in P \cap \mathbb{Z}^d \subseteq P_I$ .

If  $P$  is not integral, then  $P_I \notin \mathcal{M}^d$  since  $P_I \subsetneq P$  and  $P$  is lattice-free.  $\square$

### 3 Lattice-free axis-aligned simplices

For  $a \in \mathbb{R}_{>0}^d$ , the  $d$ -dimensional simplex

$$T(a) := \text{conv}\{o, a_1 e_1, \dots, a_d e_d\}.$$

is called *axis-aligned*. The proof of the following proposition is straightforward.

**Proposition 8.** *For  $a \in \mathbb{R}_{>0}^d$ , the following statements hold.*

- I. *The simplex  $T(a)$  is a lattice-free set if and only if  $\kappa(a) \geq 1$ .*
- II. *The simplex  $T(a)$  is a maximal lattice-free set if and only if  $\kappa(a) = 1$ .*

We introduce transformations which preserve the values of  $\kappa$ . The transformations arise from the following trivial identities for  $t > 0$ :

$$\frac{1}{t} = \frac{1}{t+1} + \frac{1}{t(t+1)}, \tag{3.1}$$

$$\frac{1}{t} = \frac{1}{t+2} + \frac{1}{t(t+2)} + \frac{1}{t(t+2)}, \tag{3.2}$$

$$\frac{1}{t} = \frac{2}{3t} + \frac{1}{3t}. \tag{3.3}$$

Consider a vector  $a \in \mathbb{R}_{>0}^d$ . By (3.1), if  $t$  is a component of  $a$ , we can replace this component with two new components  $t+1$  and  $t(t+1)$  to generate a vector  $b \in \mathbb{R}_{>0}^{d+1}$  satisfying  $\kappa(b) = \kappa(a)$ . Identities (3.2) and (3.3) can be applied in a similar fashion. For

every  $d \in \mathbb{N}$ , with the help of (3.1)–(3.3), we introduce the following maps:

$$\phi_d : \mathbb{R}_{>0}^d \rightarrow \mathbb{R}_{>0}^{d+1}, \quad \phi_d(a) := \begin{pmatrix} a_1 \\ \vdots \\ a_{d-1} \\ a_d + 1 \\ a_d(a_d + 1) \end{pmatrix}, \quad (3.4)$$

$$\psi_d : \mathbb{R}_{>0}^d \rightarrow \mathbb{R}_{>0}^{d+3}, \quad \psi_d(a) := \begin{pmatrix} a_1 \\ \vdots \\ a_{d-1} \\ a_d + 3 \\ a_d(a_d + 1) \\ (a_d + 1)(a_d + 3) \\ (a_d + 1)(a_d + 3) \end{pmatrix}, \quad (3.5)$$

$$\xi_d : \mathbb{R}_{>0}^d \rightarrow \mathbb{R}_{>0}^{d+1}, \quad \xi_d(a) := \begin{pmatrix} a_1 \\ \vdots \\ a_{d-1} \\ \frac{3}{2}a_d \\ 3a_d \end{pmatrix}. \quad (3.6)$$

The map  $\phi_d$  replaces the component  $a_d$  by two other components based on (3.1), while  $\xi_d$  replaces  $a_d$  based on (3.3). The map  $\psi_d$  acts by replacing the component  $a_d$  based on (3.1) and then replacing the component  $a_d + 1$  based on (3.2). Identities (3.1)–(3.3) imply

$$\kappa(\phi_d(a)) = \kappa(\psi_d(a)) = \kappa(\xi_d(a)) = \kappa(a). \quad (3.7)$$

**Lemma 9.** *Let  $P = T(\xi_d(a))$ , where  $a \in \mathcal{A}_d$  and  $d \geq 2$ . Then  $P$  is a strongly blocked lattice-free  $(d + 1)$ -dimensional polytope. Furthermore, if  $a_d$  is odd,  $P$  is not integral.*

*Proof.* In this proof, we use the *all-ones vector*

$$\mathbb{1}_d := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^d.$$

For the sake of brevity we introduce the notation  $t := a_d$ . One has  $1 = \kappa(a) = \sum_{i=1}^d \frac{1}{a_i} \geq \sum_{i=1}^d \frac{1}{t} = \frac{d}{t}$ , which implies  $t \geq d \geq 2$ . By (3.7), one has  $\kappa(\xi_d(a)) = 1$  and so, by Proposition 8,  $P$  is maximal lattice-free.

If  $t$  is even, the polytope  $P$  is integral and hence every facet of  $P$ . In view of Proposition 5, integral maximal lattice-free polytopes are strongly blocked, and so we conclude that  $P$  is strongly blocked.

Assume that  $t$  is odd, then the polytope  $P$  has one non-integral vertex. In this case, we need to look at facets of  $P$  more closely, to verify that  $P$  is strongly blocked. We consider all facets of  $P$ .

1. The facet  $F = \text{conv}\{o, a_1e_1, \dots, a_{d-1}e_{d-1}, 3te_{d+1}\}$  is a  $d$ -dimensional integral integral axis-aligned simplex. Since

$$\kappa(a_1, \dots, a_{d-1}, 3t) < 1,$$

the integral point  $e_1 + \dots + e_{d-1} + e_{d+1}$  is in the relative interior of  $F$ . Hence,  $F$  is strongly blocked.

2. The facet  $F = \text{conv}\left\{o, a_1e_1, \dots, a_{d-1}e_{d-1}, \frac{3}{2}te_d\right\}$  contains the  $d$ -dimensional integral axis-aligned simplex

$$G := \text{conv}\left\{o, a_1e_1, \dots, a_{d-1}e_{d-1}, \frac{3t-1}{2}e_d\right\},$$

as a subset. In view of  $t \geq 2$ , we have

$$\kappa\left(a_1, \dots, a_{d-1}, \frac{3t-1}{2}\right) < 1,$$

which implies that the integral point  $e_1 + \dots + e_d$  is in the relative interior of  $G$ . It follows that  $F$  is strongly blocked.

3. The facet  $F := \text{conv}\left\{a_1e_1, \dots, a_{d-1}e_{d-1}, \frac{3}{2}te_d, 3te_{d+1}\right\}$  contains the integral  $d$ -dimensional simplex

$$G := \text{conv}\left\{a_1e_1, \dots, a_{d-1}e_{d-1}, \frac{3t-1}{2}e_d + e_{d+1}, 3te_{d+1}\right\}.$$

as a subset. It turns out that  $\mathbb{1}_{d+1}$  is the relative interior of  $G$ , because  $\mathbb{1}_{d+1}$  is a convex combination of the vertices of  $\text{relint } G$ , with positive coefficients. Indeed, the equality

$$\mathbb{1}_{d+1} = \sum_{i=1}^{d-1} \frac{1}{a_i}(a_i e_i) + \lambda \left(\frac{3t-1}{2}e_d + e_{d+1}\right) + \mu(3te_{d+1})$$

holds for  $\lambda = \frac{2}{3t-1}$  and  $\mu = \frac{t-1}{t(3t-1)}$ , where

$$\sum_{i=1}^{d-1} \frac{1}{a_i} + \lambda + \mu = 1.$$

4. It remains to consider faces  $F$  with the vertex set  $\{o, a_1e_1, \dots, a_d e_d, \frac{3}{2}te_d, 3te_{d+1}\} \setminus \{a_i e_i\}$ , where  $i \in \{1, \dots, d+1\}$ . Without loss of generality, let  $i = 1$  so that  $F = \text{conv}\left\{o, a_2e_2, \dots, \frac{3}{2}te_d, 3te_{d+1}\right\}$ . This facet contains the integral  $d$ -dimensional simplex

$$G := \text{conv}\left\{o, a_2e_2, \dots, a_{d-1}e_{d-1}, \frac{3t-1}{2}e_d + e_{d+1}, 3te_{d+1}\right\}.$$

Similarly to the previous case, one can check that  $e_2 + \dots + e_{d+1}$  is an integral point in the relative interior of  $G$ . Consequently,  $F$  is strongly blocked.

□

## 4 Proof of the main result

For  $d \geq 4$ , Nill and Ziegler [LZ91] construct one vector  $a \in \mathbb{R}_{>0}^d$  with  $T(a)_I \in \mathcal{L}^d \setminus \mathcal{M}^d$ . We generalize this construction and provide many further vectors  $a$  with the above properties. We will also need to verify that for different choices of  $a$ , we get essentially different polytopes  $T(a)_I$ .

**Lemma 10.** *Let  $P$  and  $Q$  be  $d$ -dimensional strongly blocked lattice-free polytopes such that for their integral hulls the equality  $Q_I = A(P_I)$  holds for some  $A \in \text{Aff}(\mathbb{Z}^d)$ . Then  $Q = A(P)$ .*

*Proof.* Since  $A$  is an affine transformation, we have

$$A(P_I) = A(\text{conv}(P \cap \mathbb{Z}^d)) = \text{conv } A(P \cap \mathbb{Z}^d).$$

Using  $A \in \text{Aff}(\mathbb{Z}^d)$ , it is straightforward to check the equality  $A(P \cap \mathbb{Z}^d) = A(P) \cap \mathbb{Z}^d$ . We thus conclude that  $A(P_I) = A(P)_I$ . The assumption  $Q_I = A(P_I)$  yields  $Q_I = A(P)_I$ . Since  $P$  is strongly blocked lattice-free,  $A(P)$  too is strongly blocked lattice-free. We thus have the equality  $Q_I = A(P)_I$  for strongly blocked lattice-free polytopes  $Q$  and  $A(P)$ . To verify the assertion, it suffices to show that a strongly blocked lattice-free polytope  $Q$  is uniquely determined by the knowledge of its integer hull  $Q_I$ . This is quite easy to see. For every strongly blocked facet  $G$  of  $Q_I$ , the affine hull of  $G$  contains a facet of  $Q$ . Conversely, if  $F$  is an arbitrary facet of  $Q$ , then  $G = F_I$  is a strongly blocked facet of  $Q_I$ . Thus, the knowledge of  $Q_I$  allows to determine affine hulls of all facets of  $Q$ . In other words,  $Q_I$  uniquely determines a hyperplane description of  $Q$ .  $\square$

**Lemma 11.** *Let  $a, b \in \mathbb{R}_{>0}^d$  be such that the equality  $T(b) = A(T(a))$  holds for some  $A \in \text{Aff}(\mathbb{Z}^d)$ . Then  $a$  and  $b$  coincide up to permutation of components.*

*Proof.* We use induction on  $d$ . For  $d = 1$ , the assertion is trivial. Let  $d \geq 2$ . One of the  $d$  facets of  $T(a)$  containing  $o$  is mapped by  $A$  to a facet of  $T(b)$  that contains  $o$ . Without loss of generality we can assume that the facet  $T(a_1, \dots, a_{d-1}) \times \{0\}$  of  $T(a)$  is mapped to the facet  $T(b_1, \dots, b_{d-1}) \times \{0\}$  of  $T(b)$ . By the inductive assumption,  $(a_1, \dots, a_{d-1})$  and  $(b_1, \dots, b_{d-1})$  coincide up to permutation of components. Since unimodular transformations preserve the volume,  $T(a)$  and  $T(b)$  have the same volume. This means,  $\prod_{i=1}^d a_i = \prod_{i=1}^d b_i$ . Consequently,  $a_d = b_d$  and we conclude that  $a$  and  $b$  coincide up to permutation of components.  $\square$

*Proof of Theorem 2.* For every  $a \in \mathcal{A}_d$ , we introduce the  $(d + 5)$ -dimensional integral lattice-free polytope

$$P_a := T(\eta(a))_I,$$

where

$$\eta(x) := \xi_{d+4}(\psi_{d+1}(\phi_d(x)))$$

and the functions  $\xi_{d+4}$ ,  $\psi_{d+1}$  and  $\phi_d$  are defined by (3.4)–(3.6).

By (3.7) for each  $a \in \mathcal{A}_d$ , we have  $\kappa(\eta(a)) = 1$ . For  $a \in \mathcal{A}_d$  the last component of  $\phi_d(a)$  is even. This implies that the last component of  $\psi_{d+1}(\phi_d(a))$  is odd. Thus, by Lemma 9,  $T(\eta(a))$  is strongly blocked lattice-free polytope which is not integral.

Let  $a, b \in \mathcal{A}_d$  be such that the polytopes  $P_a$  and  $P_b$  coincide up to affine unimodular transformations. Then, by Lemma 10,  $T(\eta(a))$  and  $T(\eta(b))$  coincide up to affine unimodular transformations. But then, by Lemma 11,  $\eta(a)$  and  $\eta(b)$  coincide up to permutations. Since the components of  $a$  and  $b$  are sorted in the ascending order, the components of  $\eta(a)$  and  $\beta(b)$  too are sorted in the ascending order. Thus, we arrive at the equality  $\eta(a) = \eta(b)$ , which implies  $a = b$ .

In view of Proposition 7, each  $P_a$  with  $a \in \mathcal{A}_d$  belongs to  $\mathcal{L}^d$  but not to  $\mathcal{M}^d$ . Thus, the equivalence classes of the polytopes  $P_a$  with  $a \in \mathcal{A}_d$  with respect to identification up to affine unimodular transformations form a subset of  $(\mathcal{L}^{d+5} \setminus \mathcal{M}^{d+5}) / \text{Aff}(\mathbb{Z}^{d+5})$  of cardinality  $|\mathcal{A}_d|$ . This yields the desired assertion.  $\square$

*Proof of Corollary 3.* The assertion is a direct consequence of Theorem 2 and the asymptotic estimate

$$\ln \ln |\mathcal{A}_d| = \Omega\left(\frac{d}{\ln d}\right)$$

of Konyagin [Kon14, Theorem 1].  $\square$

**Remark 12.** *In view of the upper bound  $\ln \ln |\mathcal{A}_d| = O(d)$  by Sándor [Sán03, Theorem 2], the lower bound of Konyagin is optimal up to the logarithmic factor in the denominator.*

Since all known elements of  $\mathcal{L}^d$  are of the form  $P_I$ , for some strongly blocked lattice-free polytope  $P$ , we ask the following

**Question 13.** *Do there exist polytopes  $L \in \mathcal{L}^d$  which cannot be represented as  $L = P_I$  for any strongly blocked lattice-free polytope  $P$ ?*

If there is a gap between the families  $\mathcal{L}^d$  and the family

$$\left\{ P_I : P \subseteq \mathbb{R}^d \text{ strongly blocked lattice-free polytope} \right\},$$

then it would be interesting to understand how irregular the polytopes from this gap can be. For example, one can ask the following

**Question 14.** *Do there exist polytopes  $L \in \mathcal{L}^d$  with the property that no facet of  $L$  is blocked?*

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