ON THE RELATIVE TWIST FORMULA OF ℓ-ADIC SHEAVES

ENLIN YANG AND YIGENG ZHAO

Abstract. We propose a conjecture on the relative twist formula of ℓ-adic sheaves, which can be viewed as a generalization of Kato-Saito's conjecture. We verify this conjecture under some transversal assumptions.

We also define a relative cohomological characteristic class and prove that its formation is compatible with proper push-forward. A conjectural relation is also given between the relative twist formula and the relative cohomological characteristic class.

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1. Introduction

As an analogy of the theory of D-modules, Beilinson [Bei16] and T. Saito [Sai17a] define the singular support and the characteristic cycle of an ℓ-adic sheaf on a smooth variety respectively. As an application of their theory, we prove a twist formula of epsilon factors in [UYZ], which is a modification of a conjecture due to Kato and T. Saito[KS08, Conjecture 4.3.11].

1.1. **Kato-Saito's conjecture.**

1.1.1. Let X be a smooth projective scheme purely of dimension d over a finite field k of characteristic p. Let Λ be a finite field of characteristic $\ell \neq p$ or $\Lambda = \overline{\mathbb{Q}}_\ell$. Let $\mathcal{F} \in D_c^b(X, \Lambda)$ and $\chi(X_{\bar{k}}, \mathcal{F})$ be the Euler-Poincaré characteristic of $\mathcal{F}$. The Grothendieck L-function $L(X, \mathcal{F}, t)$ satisfies the following functional equation

$$L(X,\mathcal{F},t) = \varepsilon(X,\mathcal{F}) \cdot t^{-\chi(X_{\bar{k}},\mathcal{F})} \cdot L(X, D(\mathcal{F}), t^{-1}), \tag{1.1.1.1}$$

where $D(\mathcal{F})$ is the Verdier dual $R\mathcal{H}om(\mathcal{F}, Rf^!\Lambda)$ of $\mathcal{F}$, $f \colon X \to \mathrm{Spec} k$ is the structure morphism and

$$\varepsilon(X,\mathcal{F}) = \det(-\mathrm{Frob}_k; R\Gamma(X_{\bar{k}}, \mathcal{F}))^{-1} \tag{1.1.1.2}$$

is the epsilon factor (the constant term of the functional equation (1.1.1.1)) and Frob_k is the geometric Frobenius (the inverse of the Frobenius substitution).

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1.1.2. In (1.1.1.1), both $\chi(X_{\bar{k}},\mathcal{F})$ and $\varepsilon(X,\mathcal{F})$ are related to ramification theory. Let $cc_{X/k}(\mathcal{F}) = 0_X^!(CC(\mathcal{F},X/k)) \in CH_0(X)$ be the characteristic class of $\mathcal{F}$ (cf. [Sai17a, Definition 5.7]), where $0_X\colon X \to T^*X$ is the zero section and $CC(\mathcal{F},X/k)$ is the characteristic cycle of $\mathcal{F}$. Then $\chi(X_{\bar{k}},\mathcal{F}) = \deg(cc_{X/k}(\mathcal{F}))$ by the index formula [Sai17a, Theorem 7.13]. The following theorem proved in [UYZ] gives a relation between $\varepsilon(X,\mathcal{F})$ and $cc_{X/k}(\mathcal{F})$, which is a modified version of the formula conjectured by Kato and T. Saito in [KS08, Conjecture 4.3.11].

Theorem 1.1.3 (Twist formula, [UYZ, Theorem 1.5])**.** *We have*

$$\varepsilon(X,\mathcal{F}\otimes\mathcal{G}) = \varepsilon(X,\mathcal{F})^{\mathrm{rank}\mathcal{G}}\cdot \det\mathcal{G}(\rho_X(-cc_{X/k}(\mathcal{F}))) \qquad \text{in } \Lambda^\times, \tag{1.1.3.1}$$

where $\rho_X\colon CH_0(X)\to\pi_1^{\mathrm{ab}}(X)$ is the reciprocity map defined by sending the class $[s]$ of a closed point $s\in X$ to the geometric Frobenius Frob_s and $\det\mathcal{G}\colon \pi_1^{\mathrm{ab}}(X)\to\Lambda^\times$ is the representation associated to the smooth sheaf $\det\mathcal{G}$ of rank 1.

When $\mathcal{F}$ is the constant sheaf Λ, this is proved by S. Saito [SS84]. If $\mathcal{F}$ is a smooth sheaf on an open dense subscheme U of X such that $\mathcal{F}$ is tamely ramified along $D = X\backslash U$, then Theorem 1.1.3 is a consequence of [Sai93, Theorem 1]. In [Vi09a, Vi09b], Vidal proves a similar result on a proper smooth surface over a finite field of characteristic $p > 2$ under certain technical assumptions. Our proof of Theorem 1.1.3 is based on the following theories: one is the theory of singular support [Bei16] and characteristic cycle [Sai17a], and another is Laumon's product formula [Lau87].

1.2. ε-factorization.

1.2.1. Now we assume that X is a smooth projective geometrically connected curve of genus g over a finite field k of characteristic p. Let ω be a non-zero rational 1-form on X and $\mathcal{F}$ an ℓ-adic sheaf on X. The following formula is conjectured by Deligne and proved by Laumon [Lau87, 3.2.1.1]:

$$\varepsilon(X,\mathcal{F}) = p^{[k:\mathbb{F}_p](1-g)\mathrm{rank}(\mathcal{F})}\prod_{v\in|X|}\varepsilon_v(\mathcal{F}|_{X_{(v)}},\omega). \tag{1.2.1.1}$$

For higher dimensional smooth scheme X over k, it is still an open question whether there is an ε-factorization formula (respectively a geometric ε-factorization formula) for $\varepsilon(X,\mathcal{F})$ (respectively $\det R\Gamma(X,\mathcal{F})$).

1.2.2. In [Bei07], Beilinson develops the theory of topological epsilon factors using K-theory spectrum and he asks whether his construction admits a motivic (ℓ-adic or de Rham) counterpart. For de Rham cohomology, such a construction is given by Patel in [Pat12]. Based on [Pat12], Abe and Patel prove a similar twist formula in [AP17] for global de Rham epsilon factors in the classical setting of $\mathcal{D}_X$-modules on smooth projective varieties over a field of characteristic zero. In the ℓ-adic situation, such a geometric ε-factorization formula is still open even if X is a curve. Since the classical local ε-factors depend on an additive character of the base field, a satisfied geometric ε-factorization theory will lie in an appropriate gerbe rather than be a super graded line (cf. [Bei07, Pat12]).

1.2.3. More generally, we could also ask similar questions in a relative situation. Now let $f\colon X\to S$ be a proper morphism between smooth schemes over k. Let $\mathcal{F}$ be an ℓ-adic sheaf on X such that f is universally locally acyclic relatively to $\mathcal{F}$. Under these assumptions, we know that $Rf_*\mathcal{F}$ is locally constant on S. Now we can ask if there is an analogue geometric ε-factorization for the determinant $\det Rf_*\mathcal{F}$. This problem is far beyond the authors' reach at this moment. But, similar to (1.1.3.1), we may consider twist formulas for $\det Rf_*\mathcal{F}$. One of the purposes of this paper is to formulate such a twist formula and prove it under certain assumptions.

1.2.4. *Relative twist formula.* Let S be a regular Noetherian scheme over $\mathbb{Z}[1/\ell]$ and $f\colon X \to S$ a proper smooth morphism purely of relative dimension n. Let $\mathcal{F} \in D_c^b(X, \Lambda)$ such that f is universally locally acyclic relatively to $\mathcal{F}$. Then we conjecture that (see Conjecture 2.1.4) there exists a unique cycle class $cc_{X/S}(\mathcal{F}) \in \mathrm{CH}^n(X)$ such that for any locally constant and constructible sheaf $\mathcal{G}$ of Λ-modules on X, we have an isomorphism of smooth sheaves of rank 1 on S

$$\det Rf_*(\mathcal{F} \otimes \mathcal{G}) \cong (\det Rf_*\mathcal{F})^{\otimes \mathrm{rank}\mathcal{G}} \otimes \det \mathcal{G}(cc_{X/S}(\mathcal{F})) \tag{1.2.4.1}$$

where $\det \mathcal{G}(cc_{X/S}(\mathcal{F}))$ is a smooth sheaf of rank 1 on S (see 2.1.3 for the definition). We call (1.2.4.1) the relative twist formula. As an evidence, we prove a special case of the above conjecture in Theorem 2.4.4. It is also interesting to consider a similar relative twist formula for de Rham epsilon factors in the sense of [AP17]. We will pursue this question elsewhere.

1.2.5. If S is moreover a smooth connected scheme of dimension r over a perfect field k, we construct a candidate for $cc_{X/S}(\mathcal{F})$ in Definition 2.4.3. We also relate the relative characteristic class $cc_{X/S}(\mathcal{F})$ to the total characteristic class of $\mathcal{F}$. Let $K_0(X, \Lambda)$ be the Grothendieck group of $D_c^b(X, \Lambda)$. In [Sai17a, Definition 6.7.2], T. Saito defines the following morphism

$$cc_{X,\bullet}\colon K_0(X, \Lambda) \to \mathrm{CH}_\bullet(X) = \bigoplus_{i=0}^{r+n} \mathrm{CH}_i(X), \tag{1.2.5.1}$$

which sends $\mathcal{F} \in D_c^b(X, \Lambda)$ to the total characteristic class $cc_{X,\bullet}(\mathcal{F})$ of $\mathcal{F}$. Under the assumption that $f\colon X \to S$ is $SS(\mathcal{F}, X/k)$-transversal, we show that $(-1)^r \cdot cc_{X/S}(\mathcal{F}) = cc_{X,r}(\mathcal{F})$ in Proposition 2.5.2.

1.2.6. Following Grothendieck [SGA5], it's natural to ask whether the following diagram

$$\begin{array}{ccc} K_0(X,\Lambda) & \xrightarrow{cc_{X,\bullet}} & CH_\bullet(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ K_0(Y,\Lambda) & \xrightarrow{cc_{Y,\bullet}} & CH_\bullet(Y) \end{array} \tag{1.2.6.1}$$

is commutative or not for any proper map $f\colon X \to Y$ between smooth schemes over k. If $k = \mathbb{C}$, the diagram (1.2.6.1) is commutative by [Gin86, Theorem A.6]. By the philosophy of Grothendieck, the answer is no in general if $\mathrm{char}(k) > 0$ (cf. [Sai17a, Example 6.10]). If k is a finite field and if $f\colon X \to Y$ is moreover projective, as a corollary of Theorem 1.1.3, we prove in [UYZ, Corollary 5.26] that the degree zero part of (1.2.6.1) commutes. In general, motivated by the conjectural formula (1.2.4.1), we propose the following question. Let $f\colon X \to S$ and $g\colon Y \to S$ be smooth morphisms. Let $D_c^b(X/S, \Lambda)$ be the thick subcategory of $D_c^b(X, \Lambda)$ consists of $\mathcal{F} \in D_c^b(X, \Lambda)$ such that f is $SS(\mathcal{F}, X/k)$-transversal. Let $K_0(X/S, \Lambda)$ be the Grothendieck group of $D_c^b(X/S, \Lambda)$. Then for any proper morphism $h\colon X \to Y$ over S, we conjecture that the following diagram commutes (see Conjecture 2.5.4)

$$\begin{array}{ccc} K_0(X/S,\Lambda) & \xrightarrow{cc_{X,r}} & CH_r(X) \\ {\scriptstyle h_*}\downarrow & & \downarrow{\scriptstyle h_*} \\ K_0(Y/S,\Lambda) & \xrightarrow{cc_{Y,r}} & CH_r(Y). \end{array} \tag{1.2.6.2}$$

1.2.7. As an evidence for (1.2.6.2), we construct a relative cohomological characteristic class

$$ccc_{X/S}(\mathcal{F}) \in H^{2n}(X, \Lambda(n)) \tag{1.2.7.1}$$

in Definition 3.2.4 if $X \to S$ is smooth and $SS(\mathcal{F}, X/k)$-transversal. We prove that the formation of $ccc_{X/S}(\mathcal{F})$ is compatible with proper push-forward (see Corollary 3.3.4 for a precise statement). Similar to [Sai17a, Conjecture 6.8.1], we conjecture that we have the following equality (see Conjecture 3.2.6)

$$\mathrm{cl}(cc_{X/S}(\mathcal{F})) = ccc_{X/S}(\mathcal{F}) \quad \text{in} \quad H^{2n}(X, \Lambda(n)) \tag{1.2.7.2}$$

where $\mathrm{cl}\colon \mathrm{CH}^n(X) \to H^{2n}(X, \Lambda(n))$ is the cycle class map.

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Notation and Conventions.

(1) Let p be a prime number and Λ be a finite field of characteristic $\ell \neq p$ or $\Lambda = \overline{\mathbb{Q}}_\ell$.
(2) We say that a complex $\mathcal{F}$ of étale sheaves of Λ-modules on a scheme X over $\mathbb{Z}[1/\ell]$ is *constructible* (respectively *smooth*) if the cohomology sheaf $\mathcal{H}^q(\mathcal{F})$ is constructible for every q and if $\mathcal{H}^q(\mathcal{F}) = 0$ except finitely many q (respectively moreover $\mathcal{H}^q(\mathcal{F})$ is locally constant for all q).
(3) For a scheme S over $\mathbb{Z}[1/\ell]$, let $D^b_c(S, \Lambda)$ be the triangulated category of bounded complexes of Λ-modules with constructible cohomology groups on S and let $K_0(S, \Lambda)$ be the Grothendieck group of $D^b_c(S, \Lambda)$.
(4) For a scheme X, we denote by $|X|$ the set of closed points of X.
(5) For any smooth morphism $X \to S$, we denote by $T^*_X(X/S) \subseteq T^*(X/S)$ the zero section of the relative cotangent bundle $T^*(X/S)$ of X over S. If S is the spectrum of a field, we simply denote $T^*(X/S)$ by T^*X.

2. Relative twist formula

2.1. Reciprocity map.

2.1.1. For a smooth proper variety X purely of dimension n over a finite field k of characteristic p, the reciprocity map $\rho_X\colon \mathrm{CH}^n(X) \to \pi_1^{\mathrm{ab}}(X)$ is given by sending the class $[s]$ of closed point $s \in X$ to the geometric Frobenius Frob_s at s. The map ρ_X is injective with dense image [KS83].

2.1.2. Let S be a regular Noetherian scheme over $\mathbb{Z}[1/\ell]$ and X a smooth proper scheme purely of relative dimension n over S. By [Sai94, Proposition 1], there exists a unique way to attach a pairing

$$\mathrm{CH}^n(X) \times \pi_1^{\mathrm{ab}}(S) \to \pi_1^{\mathrm{ab}}(X) \tag{2.1.2.1}$$

satisfying the following two conditions:

(1) When $S = \mathrm{Spec} k$ is a point, for a closed point $x \in X$, the pairing with the class $[x]$ coincides with the inseparable degree times the Galois transfer $\mathrm{tran}_{k(x)/k}$ (cf.[Tat79, 1]) followed by i_{x*} for $i_x\colon x \to X$

$$\mathrm{Gal}(k^{\mathrm{ab}}/k) \xrightarrow{\mathrm{tran}_{k(x)/k} \times [k(x)\colon k]_i} \mathrm{Gal}(k(x)^{\mathrm{ab}}/k(x)) \xrightarrow{i_{x*}} \pi_1^{\mathrm{ab}}(X).$$

(2) For any point $s \in S$, the following diagram commutes

$$\begin{array}{ccccc}
\mathrm{CH}^n(X) & \times & \pi_1^{\mathrm{ab}}(S) & \longrightarrow & \pi_1^{\mathrm{ab}}(X) \\
\downarrow & & \uparrow & & \uparrow \\
\mathrm{CH}^n(X_s) & \times & \pi_1^{\mathrm{ab}}(s) & \longrightarrow & \pi_1^{\mathrm{ab}}(X_s).
\end{array}$$

2.1.3. For any locally constant and constructible sheaf $\mathcal{G}$ of Λ-modules on X and any $z \in \mathrm{CH}^n(X)$, we have a map

$$\pi_1^{\mathrm{ab}}(S) \xrightarrow{(z,\bullet)} \pi_1^{\mathrm{ab}}(X) \xrightarrow{\det \mathcal{G}} \Lambda^\times \tag{2.1.3.1}$$

where $(z, \bullet)$ is the map determined by the paring (2.1.2.1) and $\det \mathcal{G}$ is the representation associated to the locally constant sheaf $\det \mathcal{G}$ of rank 1. The composition $\det \mathcal{G} \circ (z, \bullet)\colon \pi_1^{\mathrm{ab}}(S) \to \Lambda^\times$ determines a locally constant and constructible sheaf of rank 1 on S, which we simply denote by $\det \mathcal{G}(z)$. Now we propose the following conjecture.

Conjecture 2.1.4 (Relative twist formula)**.** *Let S be a regular Noetherian scheme over $\mathbb{Z}[1/\ell]$ and $f\colon X \to S$ a smooth proper morphism purely of relative dimension n. Let $\mathcal{F} \in D_c^b(X, \Lambda)$ such that f is universally locally acyclic relatively to $\mathcal{F}$. Then there exists a unique cycle class $cc_{X/S}(\mathcal{F}) \in \mathrm{CH}^n(X)$ such that for any locally constant and constructible sheaf $\mathcal{G}$ of Λ-modules on X, we have an isomorphism*

$$\det Rf_*(\mathcal{F} \otimes \mathcal{G}) \cong (\det Rf_*\mathcal{F})^{\otimes \mathrm{rank}\mathcal{G}} \otimes \det \mathcal{G}(cc_{X/S}(\mathcal{F})) \quad \text{in } K_0(S, \Lambda), \tag{2.1.4.1}$$

where $K_0(S, \Lambda)$ is the Grothendieck group of $D_c^b(S, \Lambda)$.

We call this cycle class $cc_{X/S}(\mathcal{F}) \in \mathrm{CH}^n(X)$ *the relative characteristic class of $\mathcal{F}$* if it exists. If S is a smooth scheme over a perfect field k, we construct a candidate for $cc_{X/S}(\mathcal{F})$ in Definition 2.4.3.

As an evidence, we prove a special case of the above conjecture in Theorem 2.4.4. In order to construct a cycle class $cc_{X/S}(\mathcal{F})$ satisfying (2.1.4.1), we use the theory of singular support and characteristic cycle.

2.2. **Transversal condition and singular support.**

2.2.1. Let $f\colon X \to S$ be a smooth morphism of Noetherian schemes over $\mathbb{Z}[1/\ell]$. We denote by $T^*(X/S)$ the vector bundle $\mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_X}(\Omega^1_{X/S})^\vee)$ on X and call it *the relative cotangent bundle on X with respect to S*. We denote by $T^*_X(X/S) = X$ the zero-section of $T^*(X/S)$. A constructible subset C of $T^*(X/S)$ is called *conical* if C is invariant under the canonical $\mathbb{G}_m$-action on $T^*(X/S)$.

Definition 2.2.2 ([Bei16, §1.2] and [HY17, §2])**.** Let $f\colon X \to S$ be a smooth morphism of Noetherian schemes over $\mathbb{Z}[1/\ell]$ and C a closed conical subset of $T^*(X/S)$. Let Y be a Noetherian scheme smooth over S and $h\colon Y \to X$ an S-morphism.

(1) We say that $h\colon Y \to X$ is *C-transversal relatively to S at a geometric point $\bar{y} \to Y$* if for every non-zero vector $\mu \in C_{h(\bar{y})} = C \times_X \bar{y}$, the image $dh_{\bar{y}}(\mu) \in T^*_{\bar{y}}(Y/S) := T^*(Y/S) \times_Y \bar{y}$ is not zero, where $dh_{\bar{y}}\colon T^*_{h(\bar{y})}(X/S) \to T^*_{\bar{y}}(Y/S)$ is the canonical map. We say that $h\colon Y \to X$ is *C-transversal relatively to S* if it is C-transversal relatively to S at every geometric point of Y. If $h : Y \to X$ is C-transversal relatively to S, we put $h^\circ C = dh(C \times_X Y)$ where $dh : T^*(X/S) \times_X Y \to T^*(Y/S)$ is the canonical map induced by h. By the same argument of [Bei16, Lemma 1.1], $h^\circ C$ is a conical closed subset of $T^*(Y/S)$.

(2) Let Z be a Noetherian scheme smooth over S and $g\colon X \to Z$ an S-morphism. We say that $g\colon X \to Z$ is *C-transversal relatively to S at a geometric point $\bar{x} \to X$* if for every non-zero vector $\nu \in T^*_{g(\bar{x})}(Z/S)$, we have $dg_{\bar{x}}(\nu) \notin C_{\bar{x}}$, where $dg_{\bar{x}}\colon T^*_{g(\bar{x})}(Z/S) \to T^*_{\bar{x}}(X/S)$ is the canonical map. We say that $g\colon X \to Z$ is *C-transversal relatively to S* if it is C-transversal relatively to S at all geometric points of X. If the base $B(C) := C \cap T^*_X(X/S)$ of C is proper over Z, we put $g_\circ C := \mathrm{pr}_1(dg^{-1}(C))$, where $\mathrm{pr}_1\colon T^*(Z/S) \times_Z X \to T^*(Z/S)$ denotes the first projection and $dg : T^*(Z/S) \times_Z X \to T^*(X/S)$ is the canonical map. It is a closed conical subset of $T^*(Z/S)$.

(3) A *test pair of X relative to S* is a pair of S-morphisms $(g, h) : Y \leftarrow U \to X$ such that U and Y are Noetherian schemes smooth over S. We say that (g, h) is *C-transversal relatively to*

S if $h: U \to X$ is C-transversal relatively to S and $g: U \to Y$ is $h^{\circ}C$-transversal relatively to S.

Definition 2.2.3 ([Bei16, §1.3] and [HY17, §4])**.** Let $f\colon X \to S$ be a smooth morphism of Noetherian schemes over $\mathbb{Z}[1/\ell]$. Let $\mathcal{F}$ be an object in $D_c^b(X, \Lambda)$.

(1) We say that a test pair $(g,h): Y \leftarrow U \to X$ relative to S is $\mathcal{F}$-*acyclic* if $g: U \to Y$ is universally locally acyclic relatively to $h^*\mathcal{F}$.

(2) For a closed conical subset C of $T^*(X/S)$, we say that $\mathcal{F}$ is *micro-supported on* C *relatively to* S if every C-transversal test pair of X relative to S is $\mathcal{F}$-acyclic.

(3) Let $\mathcal{C}(\mathcal{F}, X/S)$ be the set of all closed conical subsets $C' \subseteq T^*(X/S)$ such that $\mathcal{F}$ is micro-supported on C' relatively to S. Note that $\mathcal{C}(\mathcal{F}, X/S)$ is non-empty if $f: X \to S$ is universally locally acyclic relatively to $\mathcal{F}$. If $\mathcal{C}(\mathcal{F}, X/S)$ has a smallest element, we denote it by $SS(\mathcal{F}, X/S)$ and call it the *singular support* of $\mathcal{F}$ *relative to* S.

Theorem 2.2.4 (Beilinson)**.** *Let $f: X \to S$ be a smooth morphism between Noetherian schemes over $\mathbb{Z}[1/\ell]$ and $\mathcal{F}$ an object of $D_c^b(X, \Lambda)$.*

(1) ([HY17, Theorem 5.2]) *If we further assume that $f: X \to S$ is projective and universally locally acyclic relatively to $\mathcal{F}$, the singular support $SS(\mathcal{F}, X/S)$ exists.*

(2) ([HY17, Theorem 5.2 and Theorem 5.3]) *In general, after replacing S by a Zariski open dense subscheme, the singular support $SS(\mathcal{F}, X/S)$ exists, and for any $s \in S$, we have*

$$SS(\mathcal{F}|_{X_s}, X_s/s) = SS(\mathcal{F}, X/S) \times_S s. \tag{2.2.4.1}$$

(3) ([Bei16, Theorem 1.3]) *If $S = \mathrm{Spec}\, k$ for a field k and if X is purely of dimension d, then $SS(\mathcal{F}, X/S)$ is purely of dimension d.*

2.3. Characteristic cycle and index formula.

2.3.1. Let k be a perfect field of characteristic p. Let X be a smooth scheme purely of dimension n over k, let C be a closed conical subset of T^*X and $f: X \to \mathbb{A}_k^1$ a k-morphism. A closed point $v \in X$ is called *at most an isolated C-characteristic point of $f: X \to \mathbb{A}_k^1$* if there is an open neighborhood $V \subseteq X$ of v such that $f: V - \{v\} \to \mathbb{A}_k^1$ is C-transversal. A closed point $v \in X$ is called an *isolated C-characteristic point* if v is at most an isolated C-characteristic point of $f: X \to \mathbb{A}_k^1$ but $f: X \to \mathbb{A}_k^1$ is not C-transversal at v.

Theorem 2.3.2 (T. Saito, [Sai17a, Theorem 5.9])**.** *Let X be a smooth scheme purely of dimension n over a perfect field k of characteristic p. Let $\mathcal{F}$ be an object of $D_c^b(X, \Lambda)$ and $\{C_\alpha\}_{\alpha \in I}$ the set of irreducible components of $SS(\mathcal{F}, X/k)$. There exists a unique n-cycle $CC(\mathcal{F}, X/k) = \sum_{\alpha \in I} m_\alpha [C_\alpha]$ ($m_\alpha \in \mathbb{Z}$) of T^*X supported on $SS(\mathcal{F}, X/k)$, satisfying the following Milnor formula (2.3.2.1):*

For any étale morphism $g: V \to X$, any morphism $f: V \to \mathbb{A}_k^1$, any isolated $g^{\circ}SS(\mathcal{F}, X/k)$-characteristic point $v \in V$ of $f: V \to \mathbb{A}_k^1$ and any geometric point $\bar{v}$ of V above v, we have

$$-\,\mathrm{dimtot}\ \mathrm{R}\Phi_{\bar{v}}(g^*\mathcal{F}, f) = (g^*CC(\mathcal{F}, X/k), df)_{T^*V, v}, \tag{2.3.2.1}$$

where $\mathrm{R}\Phi_{\bar{v}}(g^*\mathcal{F}, f)$ denotes the stalk at $\bar{v}$ of the vanishing cycle complex of $g^*\mathcal{F}$ relative to f, $\mathrm{dimtot}\ \mathrm{R}\Phi_{\bar{v}}(g^*\mathcal{F}, f)$ is the total dimension of $\mathrm{R}\Phi_{\bar{v}}(g^*\mathcal{F}, f)$ and $g^*CC(\mathcal{F}, X/k)$ is the pull-back of $CC(\mathcal{F}, X/k)$ to T^*V.

We call $CC(\mathcal{F}, X/k)$ the *characteristic cycle of* $\mathcal{F}$. It satisfies the following index formula.

Theorem 2.3.3 (T. Saito, [Sai17a, Theorem 7.13])**.** *Let $\bar{k}$ be an algebraic closure of a perfect field k of characteristic p, X a smooth projective scheme over k and $\mathcal{F} \in D_c^b(X, \Lambda)$. Then, we have*

$$\chi(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}}) = \deg(CC(\mathcal{F}, X/k), T_X^*X)_{T^*X}, \tag{2.3.3.1}$$

where $\chi(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$ denotes the Euler-Poincaré characteristic of $\mathcal{F}|_{X_{\bar{k}}}$.

We give a generalization in Theorem 2.3.5. For a smooth scheme $\pi\colon X \to \mathrm{Spec}k$, and two objects $\mathcal{F}_1$ and $\mathcal{F}_2$ in $D_c^b(X,\Lambda)$, we denote $\mathcal{F}_1 \boxtimes_k^L \mathcal{F}_2 \coloneqq \mathrm{pr}_1^*\mathcal{F}_1 \otimes^L \mathrm{pr}_2^*\mathcal{F}_2 \in D_c^b(X\times X,\Lambda)$, where $\mathrm{pr}_i : X\times X \to X$ is the ith projection, for $i=1,2$. We also denote $D_X(\mathcal{F}_1) = R\mathcal{H}om(\mathcal{F}_1,\mathcal{K}_X)$, where $\mathcal{K}_X = \mathrm{R}\pi^!\Lambda$.

Lemma 2.3.4. *Let X be a smooth variety purely of dimension n over a perfect field k of characteristic p. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two objects in $D_c^b(X,\Lambda)$. Then the diagonal map $\delta\colon \Delta = X \hookrightarrow X\times X$ is $SS(\mathcal{F}_2 \boxtimes^L D_X\mathcal{F}_1, X\times X/k)$-transversal if and only if $SS(\mathcal{F}_2\boxtimes_k^L D_X\mathcal{F}_1, X\times X/k) \subseteq T_\Delta^*(X\times X)$. If we are in this case, then the canonical map*

$$R\mathcal{H}om(\mathcal{F}_1,\Lambda)\otimes^L \mathcal{F}_2 \xrightarrow{\cong} R\mathcal{H}om(\mathcal{F}_1,\mathcal{F}_2)$$

is an isomorphism.

Proof. The first assertion follows from the short exact sequence of vector bundles on X associated to $\delta\colon \Delta = X \hookrightarrow X\times X$:

$$0 \to T_\Delta^*(X\times X) \to T^*(X\times X)\times_{X\times X}\Delta \xrightarrow{d\delta} T^*X \to 0.$$

For the second claim, we have the following canonical isomorphisms

$$\begin{aligned}
R\mathcal{H}om(\mathcal{F}_1,\Lambda)\otimes^L \mathcal{F}_2 &\cong R\mathcal{H}om(\mathcal{F}_1,\Lambda(n)[2n])\otimes^L \Lambda(-n)[-2n]\otimes^L \mathcal{F}_2 \overset{(1)}{\cong} D_X\mathcal{F}_1\otimes^L R\delta^!\Lambda\otimes^L \mathcal{F}_2\\
&\cong \delta^*(\mathcal{F}_2\boxtimes_k^L D_X\mathcal{F}_1)\otimes^L R\delta^!\Lambda \overset{(2)}{\cong} R\delta^!(\mathcal{F}_2\boxtimes_k^L D_X\mathcal{F}_1)\\
&\overset{(3)}{\cong} R\delta^!(R\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}_1, R\mathrm{pr}_1^!\mathcal{F}_2)) \cong R\mathcal{H}om(\delta^*\mathrm{pr}_2^*\mathcal{F}_1, R\delta^! R\mathrm{pr}_1^!\mathcal{F}_2)\\
&\cong R\mathcal{H}om(\mathcal{F}_1,\mathcal{F}_2),
\end{aligned} \tag{2.3.4.1}$$

where

(1) follows from the purity for the closed immersion δ [ILO14, XVI, Théorème 3.1.1];

(2) follows from the assumption that δ is $SS(\mathcal{F}_2\boxtimes_k^L D_X\mathcal{F}_1)$-transversal by [Sai17a, Proposition 8.13 and Definition 8.5];

(3) follows from the Künneth formula [SGA5, Exposé III, (3.1.1)]. □

Theorem 2.3.5. *Let X be a smooth projective variety purely of dimension n over an algebraically closed field k of characteristic p. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two objects in $D_c^b(X,\Lambda)$ such that the diagonal map $\delta\colon \Delta = X\hookrightarrow X\times X$ is properly $SS(\mathcal{F}_2\boxtimes_k^L D_X\mathcal{F}_1, X\times X/k)$-transversal. Then we have*

$$(-1)^n\cdot \dim_\Lambda \mathrm{Ext}(\mathcal{F}_1,\mathcal{F}_2) = \deg\left(CC(\mathcal{F}_1,X/k), CC(\mathcal{F}_2,X/k)\right)_{T^*X} \tag{2.3.5.1}$$

where $\dim_\Lambda \mathrm{Ext}(\mathcal{F}_1,\mathcal{F}_2) = \sum_i (-1)^i \dim_\Lambda \mathrm{Ext}^i_{D_c^b(X,\Lambda)}(\mathcal{F}_1,\mathcal{F}_2)$.

Proof. By the isomorphisms (2.3.4.1), the left hand side of (2.3.5.1) equals to

$$\begin{aligned}
(-1)^n\cdot\chi(X, R\delta^!(\mathcal{F}_2\boxtimes_k^L D_X\mathcal{F}_1)) &= (-1)^n\cdot\chi(X,\delta^*(\mathcal{F}_2\boxtimes_k^L D_X\mathcal{F}_1)\\
&= (-1)^n\cdot\deg(CC(\delta^*(\mathcal{F}_2\boxtimes_k^L D_X(\mathcal{F}_1), X/k), T_X^*X)_{T^*X}.
\end{aligned} \tag{2.3.5.2}$$

Since $\delta\colon X\to X\times X$ is properly $SS(\mathcal{F}_2\boxtimes_k^L D_X\mathcal{F}_1, X\times X/k)$-transversal, we have

$$CC(\delta^*(\mathcal{F}_2\boxtimes_k^L D_X\mathcal{F}_1), X/k) = (-1)^n\delta^* CC(D(\mathcal{F}_2\boxtimes_k^L D_X\mathcal{F}_1, X\times X/k)) \tag{2.3.5.3}$$

$$= (-1)^n\delta^*(CC(\mathcal{F}_2,X/k)\times CC(\mathcal{F}_1,X/k)). \tag{2.3.5.4}$$

where the equality (2.3.5.3) follows from [Sai17a, Theroem 7.6], and (2.3.5.4) follows from [Sai17b, Theorem 2.2.2]. Consider the following commutative diagram

$$\begin{array}{ccccccc} T^*X\times T^*X & = & T^*(X\times X) & \xleftarrow{\mathrm{pr}} & T^*(X\times X)\times_{X\times X}\Delta & \xrightarrow{d\delta} & T^*X \\ & \nwarrow^{\mathrm{diag}} & \uparrow & & \uparrow & \square & \uparrow \\ & & T^*X & \xrightarrow{\cong} & T^*_{\Delta}(X\times X) & \longrightarrow & X. \end{array}$$

We have $\delta^*(CC(\mathcal{F}_2, X/k)\times CC(\mathcal{F}_1, X/k)) = d\delta_*\mathrm{pr}^!(CC(\mathcal{F}_2, X/k)\times CC(\mathcal{F}_1, X/k))$ and

$$\deg(\delta^*(CC(\mathcal{F}_2, X/k)\times CC(\mathcal{F}_1, X/k)), T^*_XX)_{T^*X} = \deg\left(CC(\mathcal{F}_1, X/k), CC(\mathcal{F}_2, X/k)\right)_{T^*X}.$$

Then (2.3.5.1) follows from the above formula and (2.3.5.2). □

Remark 2.3.6. If $\mathcal{F}_1$ is the constant sheaf Λ, then Theorem 2.3.5 is the index formula (2.3.3.1). Theorem 2.3.5 can be viewed as the ℓ-adic version of the global index formula in the setting of $\mathcal{D}_X$-modules (cf. [Gin86, Theorem 11.4.1]).

2.4. **Relative twist formula.**

2.4.1. Let S be a Noetherian scheme over $\mathbb{Z}[1/\ell]$, $f: X\to S$ a smooth morphism of finite type and $\mathcal{F}$ an object of $D^b_c(X,\Lambda)$. Assume that the relative singular support $SS(\mathcal{F}, X/S)$ exists. A cycle $B=\sum_{i\in I} m_i[B_i]$ in $T^*(X/S)$ is called the *characteristic cycle of* $\mathcal{F}$ *relative to* S if each B_i is a subset of $SS(\mathcal{F}, X/S)$, each $B_i\to S$ is open and equidimensional and if, for any algebraic geometric point $\bar{s}$ of S, we have

$$B_{\bar{s}} = \sum_{i\in I} m_i[(B_i)_{\bar{s}}] = CC(\mathcal{F}|_{X_{\bar{s}}}, X_{\bar{s}}/\bar{s}). \tag{2.4.1.1}$$

We denote by $CC(\mathcal{F}, X/S)$ the characteristic cycle of $\mathcal{F}$ on X relative to S. Notice that relative characteristic cycles may not exist in general.

Proposition 2.4.2 (T. Saito, [HY17, Proposition 6.5])**.** *Let k be a perfect field of characteristic p. Let S be a smooth connected scheme of dimension r over k, $f: X\to S$ a smooth morphism of finite type and $\mathcal{F}$ an object of $D^b_c(X,\Lambda)$. Assume that $f: X\to S$ is $SS(\mathcal{F}, X/k)$-transversal and that each irreducible component of $SS(\mathcal{F}, X/k)$ is open and equidimensional over S. Then the relative singular support $SS(\mathcal{F}, X/S)$ and the relative characteristic cycle $CC(\mathcal{F}, X/S)$ exist, and we have*

$$SS(\mathcal{F}, X/S) = \theta(SS(\mathcal{F}, X/k)), \tag{2.4.2.1}$$

$$CC(\mathcal{F}, X/S) = (-1)^r\theta_*(CC(\mathcal{F}, X/k)), \tag{2.4.2.2}$$

where $\theta: T^*X\to T^*(X/S)$ denotes the projection induced by the canonical map $\Omega^1_{X/k}\to\Omega^1_{X/S}$.

Definition 2.4.3. Let k be a perfect field of characteristic p and S a smooth connected scheme of dimension r over k. Let $f: X\to S$ be a smooth morphism purely of relative dimension n and $\mathcal{F}$ an object of $D^b_c(X,\Lambda)$. Assume that $f: X\to S$ is $SS(\mathcal{F}, X/k)$-transversal. Consider the following cartesian diagram

$$\begin{array}{ccc} T^*S\times_S X & \longrightarrow & T^*X \\ \downarrow & & \downarrow \\ X & \xrightarrow{0_{X/S}} & T^*(X/S) \end{array} \tag{2.4.3.1}$$

where $0_{X/S}\colon X\to T^*(X/S)$ is the zero section. Since $f: X\to S$ is $SS(\mathcal{F}, X/k)$-transversal, the refined Gysin pull-back $0^!_{X/S}(CC(\mathcal{F}, X/k))$ of $CC(\mathcal{F}, X/k)$ is a r-cycle class supported on X. We define the *relative characteristic class* of $\mathcal{F}$ to be

$$cc_{X/S}(\mathcal{F}) = (-1)^r\cdot 0^!_{X/S}(CC(\mathcal{F}, X/k)) \quad \textit{in} \quad \mathrm{CH}^n(X). \tag{2.4.3.2}$$

Now we prove a special case of Conjecture 2.1.4.

Theorem 2.4.4 (Relative twist formula)**.** *Let S be a smooth connected scheme of dimension r over a finite field k of characteristic p. Let $f\colon X\to S$ be a smooth projective morphism of relative dimension n. Let $\mathcal{F}\in D^b_c(X,\Lambda)$ and $\mathcal{G}$ a locally constant and constructible sheaf of Λ-modules on X. Assume that f is properly $SS(\mathcal{F},X/k)$-transversal. Then there is an isomorphism*

$$\det Rf_*(\mathcal{F}\otimes\mathcal{G})\cong(\det Rf_*\mathcal{F})^{\otimes \mathrm{rank}\mathcal{G}}\otimes\det\mathcal{G}(cc_{X/S}(\mathcal{F}))\quad \text{in } K_0(S,\Lambda). \tag{2.4.4.1}$$

Note that we also have $cc_{X/S}(\mathcal{F})=(CC(\mathcal{F},X/S),T^*_XX)_{T^*(X/S)}\in \mathrm{CH}^n(X)$.

Proof. We may assume $\mathcal{G}\neq 0$. Since $\mathcal{G}$ is a smooth sheaf, we have $SS(\mathcal{F},X/k)=SS(\mathcal{F}\otimes\mathcal{G},X/k)$. Since f is proper and $SS(\mathcal{F},X/k)$-transversal, by [Sai17a, Lemma 4.3.4], $Rf_*\mathcal{F}$ and $Rf_*(\mathcal{F}\otimes\mathcal{G})$ are smooth sheaves on S. For any closed point $s\in S$, we have the following commutative diagram

$$\begin{array}{ccccc}
T^*X\times_X X_s & \xrightarrow{\theta_s} & T^*X_s\cong T^*(X/S)\times_X X_s & \xleftarrow{0_{X_s}} & X_s\\
\downarrow{\scriptstyle \mathrm{pr}} & \square & \downarrow{\scriptstyle \mathrm{pr}} & \square & \downarrow{\scriptstyle i}\\
\mathbb{T}^*X & \xrightarrow{\theta} & T^*(X/S) & \xleftarrow{0_{X/S}} & X
\end{array}$$

where $0_{X/S}$ and 0_{X_s} are the zero sections. Hence we have

$$\begin{aligned}
cc_{X_s}(\mathcal{F}|_{X_s}) &= (CC(\mathcal{F}|_{X_s},X_s/s),X_s)_{T^*X_s}=0^!_{X_s}CC(\mathcal{F}|_{X_s},X_s/s)\overset{(a)}{=}0^!_{X_s}i^!CC(\mathcal{F},X/k)\\
&=(-1)^r0^!_{X_s}i^*CC(\mathcal{F},X/k)=(-1)^r0^!_{X_s}\theta_{s*}\mathrm{pr}^!CC(\mathcal{F},X/k)\\
&=(-1)^r0^!_{X_s}\mathrm{pr}^!\theta_*CC(\mathcal{F},X/k)=(-1)^r0^!_{X_s}\mathrm{pr}^!((-1)^rCC(\mathcal{F},X/S))\\
&=0^!_{X_s}\mathrm{pr}^!CC(\mathcal{F},X/S)=i^!0^!_{X/S}CC(\mathcal{F},X/S)=i^!cc_{X/S}(\mathcal{F}),
\end{aligned}\tag{2.4.4.2}$$

where the equality (a) follows from [Sai17a, Theorem 7.6] since f is properly $SS(\mathcal{F},X/k)$-transversal.

By Chebotarev density (cf. [Lau87, Théorème 1.1.2]), we may assume that S is the spectrum of a finite field. Then it is sufficient to compare the Frobenius action. Then one use (2.4.4.2) and Theorem 1.1.3. □

Example 2.4.5. Let S be a smooth projective connected scheme over a finite field k of characteristic $p>2$. Let $f\colon X\to S$ be a smooth projective morphism of relative dimension n, $\chi=\mathrm{rank}Rf_*\overline{\mathbb{Q}}_\ell$ the Euler-Poincaré number of the fibers and let $\mathcal{F}$ be a constructible étale sheaf of Λ-modules on S. Then by the projection formula, we have $Rf_*f^*\mathcal{F}\cong\mathcal{F}\otimes Rf_*\overline{\mathbb{Q}}_\ell$. Since f is projective and smooth, $Rf_*\overline{\mathbb{Q}}_\ell$ is a smooth sheaf on S. Using Theorem 1.1.3, we get

$$\varepsilon(S,Rf_*f^*\mathcal{F})=\varepsilon(S,\mathcal{F})^{\chi}\cdot\det Rf_*\overline{\mathbb{Q}}_\ell(-cc_{Y/k}(\mathcal{F})). \tag{2.4.5.1}$$

By [Sai94, Theorem 2], $\det Rf_*\overline{\mathbb{Q}}_\ell=\kappa_{X/S}(-\frac{1}{2}n\chi)$, where $\kappa_{X/S}$ is a character of order at most 2 and is determined by the following way:

(1) If n is odd, then $\kappa_{X/S}$ is trivial.

(2) If $n=2m$ is even, then $\kappa_{X/S}$ is the quadratic character defined by the square root of $(-1)^{\frac{\chi(\chi-1)}{2}}\cdot\delta_{\mathrm{dR},X/S}$, where $\delta_{\mathrm{dR},X/S}\colon(\det H_{\mathrm{dR}}(X/S))^{\otimes 2}\xrightarrow{\simeq}\mathcal{O}_S$ is the de Rham discriminant defined by the non-degenerate symmetric bilinear form $H_{\mathrm{dR}}(X/S)\otimes^L H_{\mathrm{dR}}(X/S)\to\mathcal{O}_S[-2n]$, and $H_{\mathrm{dR}}(X/S)=Rf_*\Omega^\bullet_{X/S}$ is the perfect complex of $\mathcal{O}_S$-modules whose cohomology computes the relative de Rham cohomology of X/S.

Similarly, if $\mathcal{F}$ is a locally constant and constructible étale sheaf of Λ-modules on S, then

$$\begin{aligned}
\det Rf_*f^*\mathcal{F}\cong\det(\mathcal{F}\otimes Rf_*\overline{\mathbb{Q}}_\ell)&\cong(\det\mathcal{F})^{\otimes\chi}\otimes(\det Rf_*\overline{\mathbb{Q}}_\ell)^{\otimes\mathrm{rank}\mathcal{F}}\\
&\cong(\det\mathcal{F})^{\otimes\chi}\otimes(\kappa_{X/S}(-\frac{1}{2}n\chi))^{\otimes\mathrm{rank}\mathcal{F}}.
\end{aligned}\tag{2.4.5.2}$$

2.5. **Total characteristic class.**

2.5.1. In the rest of this section, we relate the relative characteristic class $cc_{X/S}(\mathcal{F})$ to the total characteristic class of $\mathcal{F}$. Let X be a smooth scheme purely of dimension d over a perfect field k of characteristic p. In [Sai17a, Definition 6.7.2], T. Saito defines the following morphism

$$cc_{X,\bullet}\colon K_0(X,\Lambda) \to \mathrm{CH}_\bullet(X) = \bigoplus_{i=0}^{d} \mathrm{CH}_i(X), \tag{2.5.1.1}$$

which sends $\mathcal{F} \in D_c^b(X,\Lambda)$ to the total characteristic class $cc_{X,\bullet}(\mathcal{F})$ of $\mathcal{F}$. For our convenience, for any integer n we put

$$cc_X^n(\mathcal{F}) \coloneqq cc_{X,d-n}(\mathcal{F}) \quad \text{in} \quad \mathrm{CH}^n(X). \tag{2.5.1.2}$$

By [Sai17a, Lemma 6.9], for any $\mathcal{F} \in D_c^b(X,\Lambda)$, we have

$$cc_X^d(\mathcal{F}) = cc_{X,0}(\mathcal{F}) = (CC(\mathcal{F}, X/k), T_X^*X)_{T^*X} \quad \text{in} \quad \mathrm{CH}_0(X), \tag{2.5.1.3}$$

$$cc_X^0(\mathcal{F}) = cc_{X,d}(\mathcal{F}) = (-1)^d \cdot \mathrm{rank}\mathcal{F} \cdot [X] \quad \text{in} \quad \mathrm{CH}_d(X) = \mathbb{Z}. \tag{2.5.1.4}$$

The following proposition gives a computation of $cc_X^n\mathcal{F}$ for any n.

Proposition 2.5.2. *Let S be a smooth connected scheme of dimension r over a perfect field k of characteristic p. Let $f\colon X \to S$ be a smooth morphism purely of relative dimension n. Assume that f is $SS(\mathcal{F}, X/k)$-transversal. Then we have*

$$cc_X^n(\mathcal{F}) = (-1)^r \cdot cc_{X/S}(\mathcal{F}) \quad \text{in} \quad \mathrm{CH}^n(X) \tag{2.5.2.1}$$

where $cc_{X/S}(\mathcal{F})$ is defined in Definition 2.4.3.

Proof. We use the notation of [Sai17a, Lemma 6.2]. We put $F = (T^*S \times_S X) \oplus \mathbb{A}_X^1$ and $E = T^*X \oplus \mathbb{A}_X^1$. We have a canonical injection $i\colon F \to E$ of vector bundles on X induced by $df\colon T^*S \times_S X \to T^*X$. Let $\bar{i}\colon \mathbb{P}(F) \to \mathbb{P}(E)$ be the canonical map induced by $i\colon F \to E$. By [Sai17a, Lemma 6.1.2 and Lemma 6.2.1], we have a commutative diagram:

$$\begin{array}{ccccc}
\mathrm{CH}_r(\mathbb{P}(F)) & & \xleftarrow{\quad \bar{i}^* \quad} & & \mathrm{CH}_{n+r}(\mathbb{P}(E)) \\
\simeq \uparrow & & & & \simeq \uparrow \\
\bigoplus_{q=0}^{r} \mathrm{CH}_q(X) & & \xleftarrow{\quad \text{can} \quad} & & \bigoplus_{q=0}^{n+r} \mathrm{CH}_q(X) \\
\text{can} \downarrow & & & & \text{can} \downarrow \\
\mathrm{CH}_r(X) & = & \mathrm{CH}^n(X) & = & \mathrm{CH}_r(X).
\end{array} \tag{2.5.2.2}$$

Since f is smooth and $SS(\mathcal{F}, X/k)$-transversal, the intersection $SS(\mathcal{F}, X/k) \cap (T^*S \times_S X)$ is contained in the zero section of $T^*S \times_S X$. Thus the Gysin pull-back $i^*(CC(\mathcal{F}, X/k))$ is supported on the zero section of $T^*S \times_S X$. Let $\overline{CC(\mathcal{F}, X/k)}$ be any extension of $CC(\mathcal{F}, X/k)$ to $\mathbb{P}(E)$ (cf. [Sai17a, Definition 6.7.2]). Then $\bar{i}^*(\overline{CC(\mathcal{F}, X/k)})$ is an extension of $i^*(CC(\mathcal{F}, X/k))$ to $\mathbb{P}(F)$. By [Sai17a, Definition 6.7.2], the image of $\overline{CC(\mathcal{F}, X/k)}$ in $\mathrm{CH}^n(X)$ by the right vertical map of (2.5.2.2) equals to $cc_X^n(\mathcal{F}) = cc_{X,r}(\mathcal{F})$. The image of $\bar{i}^*(\overline{CC(\mathcal{F}, X/k)})$ in $\mathrm{CH}^n(X)$ by the left vertical map of (2.5.2.2) equals to $(-1)^r \cdot cc_{X/S}(\mathcal{F})$ (cf. (2.4.3.2)). Now the equality (2.5.2.1) follows from the commutativity of (2.5.2.2). □

2.5.3. Following Grothendieck [SGA5], it's natural to ask the following question: is the diagram

(2.5.3.1)
$$\begin{CD} K_0(X,\Lambda) @>{cc_{X,\bullet}}>> CH_\bullet(X) \\ @V{f_*}VV @VV{f_*}V \\ K_0(Y,\Lambda) @>{cc_{Y,\bullet}}>> CH_\bullet(Y) \end{CD}$$

commutative for any proper map $f\colon X \to Y$ between smooth schemes over a perfect field k? If $k = \mathbb{C}$, the diagram (2.5.3.1) is commutative by [Gin86, Theorem A.6]. By the philosophy of Grothendieck, the answer is no in general if $\mathrm{char}(k) > 0$ (cf. [Sai17a, Example 6.10]). However, in [UYZ, Corollary 1.9], we prove that the degree zero part of the diagram (2.5.3.1) is commutative, i.e., if $f\colon X \to Y$ is a proper map between smooth projective schemes over a finite field k of characteristic p, then we have the following commutative diagram

(2.5.3.2)
$$\begin{CD} K_0(X,\Lambda) @>{cc_{X,0}}>> CH_0(X) \\ @V{f_*}VV @VV{f_*}V \\ K_0(Y,\Lambda) @>{cc_{Y,0}}>> CH_0(Y). \end{CD}$$

Now we propose the following:

Conjecture 2.5.4. *Let S be a smooth connected scheme over a perfect field k of characteristic p. Let $f\colon X \to S$ be a smooth morphism purely of relative dimension n and $g\colon Y \to S$ a smooth morphism purely of relative dimension m. Let $D_c^b(X/S,\Lambda)$ be the thick subcategory of $D_c^b(X,\Lambda)$ consists of $\mathcal{F} \in D_c^b(X,\Lambda)$ such that $f\colon X \to S$ is $SS(\mathcal{F}, X/k)$-transversal. Let $K_0(X/S,\Lambda)$ be the Grothendieck group of $D_c^b(X/S,\Lambda)$. Then for any proper morphism $h\colon X \to Y$ over S,*

(2.5.4.1)
$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & S & \end{array}$$

the following diagram commutes

(2.5.4.2)
$$\begin{CD} K_0(X/S,\Lambda) @>{cc_X^n}>> CH^n(X) \\ @V{h_*}VV @VV{h_*}V \\ K_0(Y/S,\Lambda) @>{cc_Y^m}>> CH^m(Y). \end{CD}$$

That is to say, for any $\mathcal{F} \in D_c^b(X,\Lambda)$, if f is $SS(\mathcal{F}, X/k)$-transversal, then we have

(2.5.4.3)
$$h_*(cc_X^n(\mathcal{F})) = cc_Y^m(Rh_*\mathcal{F}) \quad \text{in} \quad CH^m(Y).$$

Remark 2.5.5. If f is $SS(\mathcal{F}, X/k)$-transversal, by [Sai17a, Lemma 3.8 and Lemma 4.2.6], the morphism $g\colon Y \to S$ is $SS(Rh_*\mathcal{F}, Y/k)$-transversal. Thus we have a well-defined map $h_*\colon K_0(X/S,\Lambda) \to K_0(Y/S,\Lambda)$.

In next section, we formulate and prove a cohomological version of Conjecture 2.5.4 (cf. Corollary 3.3.4).

3. Relative cohomological characteristic class

In this section, we assume that S is a smooth connected scheme over a perfect field k of characteristic p and Λ is a finite field of characteristic ℓ. To simplify our notations, we omit to write R or L to denote the derived functors unless otherwise stated explicitly or for $R\mathcal{H}om$.

We briefly recall the content of this section. Let $X \to S$ be a smooth morphism purely of relative dimension n and $\mathcal{F} \in D_c^b(X, \Lambda)$. If $X \to S$ is $SS(\mathcal{F}, X/k)$-transversal, we construct a relative cohomological characteristic class $ccc_{X/S}(\mathcal{F}) \in H^{2n}(X, \Lambda(n))$ following the method of [AS07, SGA5]. We conjecture that the image of the cycle class $cc_{X/S}(\mathcal{F})$ by the cycle class map $\mathrm{cl} : \mathrm{CH}^n(X) \to H^{2n}(X, \Lambda(n))$ is $ccc_{X/S}(\mathcal{F})$ (cf. Conjecture 2.1.4). In Corollary 3.3.4, we prove that the formation of $ccc_{X/S}\mathcal{F}$ is compatible with proper push-forward.

3.1. **Relative cohomological correspondence.**

3.1.1. Let $\pi_1 \colon X_1 \to S$ and $\pi_2 \colon X_2 \to S$ be smooth morphisms purely of relative dimension n_1 and n_2 respectively. We put $X \coloneqq X_1 \times_S X_2$ and consider the following cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{pr}_2} & X_2 \\ {\scriptstyle \mathrm{pr}_1}\downarrow & \square & \downarrow{\scriptstyle \pi_2} \\ X_1 & \xrightarrow[\pi_1]{} & S. \end{array} \tag{3.1.1.1}$$

Let $\mathcal{E}_i$ and $\mathcal{F}_i$ be objects of $D_c^b(X_i, \Lambda)$ for $i = 1, 2$. We put

$$\mathcal{F} \coloneqq \mathcal{F}_1 \boxtimes_S^L \mathcal{F}_2 \coloneqq \mathrm{pr}_1^* \mathcal{F}_1 \otimes^L \mathrm{pr}_2^* \mathcal{F}_2, \tag{3.1.1.2}$$

$$\mathcal{E} \coloneqq \mathcal{E}_1 \boxtimes_S^L \mathcal{E}_2 \coloneqq \mathrm{pr}_1^* \mathcal{E}_1 \otimes^L \mathrm{pr}_2^* \mathcal{E}_2, \tag{3.1.1.3}$$

which are objects of $D_c^b(X, \Lambda)$. Similarly, we can define $\mathcal{F}_1 \boxtimes_k^L \mathcal{F}_2$, which is an object of $D_c^b(X_1 \times_k X_2, \Lambda)$. We first compare $SS(\mathcal{F}_1 \boxtimes_S^L \mathcal{F}_2, X_1 \times_S X_1/k)$ and $SS(\mathcal{F}_1 \boxtimes_k^L \mathcal{F}_2, X_1 \times_k X_1/k)$.

Lemma 3.1.2. *Assume that $\pi_1 \colon X_1 \to S$ is $SS(\mathcal{F}_1, X_1/k)$-transversal. Then we have*

$$SS(\mathrm{pr}_1^* \mathcal{F}_1, X/k) \cap SS(\mathrm{pr}_2^* \mathcal{F}_2, X/k) \subseteq T_X^* X. \tag{3.1.2.1}$$

Moreover, the closed immersion $i \colon X_1 \times_S X_2 \hookrightarrow X_1 \times_k X_2$ is $SS(\mathcal{F}_1 \boxtimes_k^L \mathcal{F}_2, X_1 \times_k X_2/k)$-transversal and

$$SS(\mathcal{F}_1 \boxtimes_S^L \mathcal{F}_2, X_1 \times_S X_2/k) \subseteq i^\circ(SS(\mathcal{F}_1 \boxtimes_k^L \mathcal{F}_2, X_1 \times_k X_2/k)). \tag{3.1.2.2}$$

Proof. We first prove (3.1.2.1). Since $X_i \to S$ is smooth, we obtain an exact sequence of vector bundles on X_i for $i = 1, 2$

$$0 \to T^*S \times_S X_i \xrightarrow{d\pi_i} T^*X_i \to T^*(X_i/S) \to 0. \tag{3.1.2.3}$$

Since $\pi_1 \colon X_1 \to S$ is $SS(\mathcal{F}_1, X_1/k)$-transversal, we have

$$SS(\mathcal{F}_1, X_1/k) \cap (T^*S \times_S X_1) \subseteq T_S^*S \times_S X_1. \tag{3.1.2.4}$$

Consider the following diagram with exact rows and exact columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \uparrow & & \uparrow & & & & \\ & & T^*(X_2/S) \times_{X_2} X & \xrightarrow{\cong} & T^*(X/X_1) & & & & \\ & & \uparrow & & \uparrow & & & & \\ 0 & \longrightarrow & T^*X_2 \times_{X_2} X & \longrightarrow & T^*X & \longrightarrow & T^*(X/X_2) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & T^*S \times_S X & \longrightarrow & T^*X_1 \times_{X_1} X & \longrightarrow & T^*(X_1/S) \times_{X_1} X & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & & & \\ & & 0 & & 0 & & & & \end{array} \tag{3.1.2.5}$$

Since pr_i is smooth, by [Sai17a, Corollary 8.15], we have

$$SS(\mathrm{pr}_i^* \mathcal{F}_i, X/k) = \mathrm{pr}_i^\circ SS(\mathcal{F}_i, X_i/k) = SS(\mathcal{F}_i, X_i/k) \times_{X_i} X.$$

It follows from (3.1.2.4) and (3.1.2.5) that $\mathrm{pr}_1^\circ SS(\mathcal{F}_1, X_1/k) \cap \mathrm{pr}_2^\circ SS(\mathcal{F}_2, X_2/k) \subseteq T^*_X X$. Thus $SS(\mathrm{pr}_1^*\mathcal{F}_1, X/k) \cap SS(\mathrm{pr}_2^*\mathcal{F}_2, X/k) \subseteq T^*_X X$. This proves (3.1.2.1).

Now we consider the cartesian diagram

$$\begin{array}{ccc} X = X_1 \times_S X_2 & \xrightarrow{i} & X_1 \times_k X_2 \\ \downarrow & \square & \downarrow \\ S & \xrightarrow{\delta} & S \times_k S \end{array} \tag{3.1.2.6}$$

where $\delta\colon S \to S \times_k S$ is the diagonal. We get the following commutative diagram of vector bundles on X with exact rows:

$$\begin{array}{ccccccccc} & & & & T^*X_1 \times_S T^*X_2 & & & & \\ & & & & \| & & & & \\ 0 & \to & \mathcal{N}_{X/(X_1\times_k X_2)} & \to & T^*(X_1 \times_k X_2) \times_{X_1\times_k X_2} X & \xrightarrow{di} & T^*X & \to & 0 \\ & & \uparrow\cong & & \uparrow & & \uparrow & & \\ 0 & \to & \mathcal{N}_{S/(S\times_k S)} \times_S X & \to & T^*(S \times_k S) \times_{S\times_k S} X & \xrightarrow{d\delta} & T^*S \times_S X & \to & 0 \\ & & \| & & \| & & & & \\ & & T^*S \times_S X & \to & (T^*S \times_S X_1) \times_S (T^*S \times_S X_2) & & & & \end{array}$$

where $\mathcal{N}_{S/(S\times_k S)}$ is the conormal bundle associated to $\delta\colon S \to S \times_k S$. By [Sai17b, Theorem 2.2.3], we have $SS(\mathcal{F}_1 \boxtimes_k^L \mathcal{F}_2, X_1 \times_k X_2/k) = SS(\mathcal{F}_1, X_1/k) \times SS(\mathcal{F}_2, X_2/k)$. Therefore by (3.1.2.4), $\mathcal{N}_{X/(X_1\times_k X_2)} \cap SS(\mathcal{F}_1 \boxtimes_k^L \mathcal{F}_2, X_1 \times_k X_2/k)$ is contained in the zero section of $\mathcal{N}_{X/(X_1\times_k X_2)}$. Hence $i\colon X \hookrightarrow X_1 \times_k X_2$ is $SS(\mathcal{F}_1 \boxtimes_k^L \mathcal{F}_2, X_1 \times_k X_2/k)$-transversal. Now the assertion (3.1.2.2) follows from [Sai17a, Lemma 4.2.4]. □

Proposition 3.1.3. *Under the notation in 3.1.1, we assume that*

(1) $SS(\mathcal{E}_i, X_i/k) \cap SS(\mathcal{F}_i, X_i/k) \subseteq T^*_{X_i} X_i$ *for all* $i = 1, 2$;
(2) $\pi_1\colon X_1 \to S$ *is* $SS(\mathcal{E}_1, X_1/k)$*-transversal or* $\pi_2\colon X_2 \to S$ *is* $SS(\mathcal{F}_2, X_2/k)$*-transversal;*
(3) $\pi_1\colon X_1 \to S$ *is* $SS(\mathcal{F}_1, X_1/k)$*-transversal or* $\pi_2\colon X_2 \to S$ *is* $SS(\mathcal{E}_2, X_2/k)$*-transversal.*

Then the following canonical map (*cf.* [Zh15, (7.2.2)] *and* [SGA5, Exposé III, (2.2.4)])

$$R\mathcal{H}om(\mathcal{E}_1, \mathcal{F}_1) \boxtimes_S^L R\mathcal{H}om(\mathcal{E}_2, \mathcal{F}_2) \to R\mathcal{H}om(\mathcal{E}, \mathcal{F}). \tag{3.1.3.1}$$

is an isomorphism.

If S is the spectrum of a field, then the above result is proved in [SGA5, Exposé III, Proposition 2.3]. Our proof below is different from that of *loc.cit.* and is based on [Sai17a].

Proof. In the following, we put $\mathcal{E}_i^\vee := R\mathcal{H}om(\mathcal{E}_i, \Lambda)$. Since $SS(\mathcal{E}_i, X_i/k) \cap SS(\mathcal{F}_i, X_i/k) \subseteq T^*_{X_i} X_i$, Lemma 2.3.4 implies that

$$\mathcal{F}_i \otimes^L \mathcal{E}_i^\vee = \mathcal{F}_i \otimes^L R\mathcal{H}om(\mathcal{E}_i, \Lambda) \xrightarrow{\cong} R\mathcal{H}om(\mathcal{E}_i, \mathcal{F}_i), \text{ for all } i = 1, 2, \tag{3.1.3.2}$$

Hence we have

$$\begin{aligned} R\mathcal{H}om(\mathcal{E}_1, \mathcal{F}_1) \boxtimes_S^L R\mathcal{H}om(\mathcal{E}_2, \mathcal{F}_2) &\cong (\mathcal{F}_1 \otimes^L \mathcal{E}_1^\vee) \boxtimes_S^L (\mathcal{F}_2 \otimes^L \mathcal{E}_2^\vee) \\ &\cong (\mathcal{F}_1 \boxtimes_S^L \mathcal{F}_2) \otimes^L (\mathcal{E}_1^\vee \boxtimes_S^L \mathcal{E}_2^\vee). \end{aligned} \tag{3.1.3.3}$$

Note that we also have

$$\begin{aligned} \mathcal{E}_1^\vee \boxtimes_S^L \mathcal{E}_2^\vee &= \mathrm{pr}_1^* R\mathcal{H}om(\mathcal{E}_1, \Lambda) \otimes^L \mathrm{pr}_2^* R\mathcal{H}om(\mathcal{E}_2, \Lambda) \\ &\cong R\mathcal{H}om(\mathrm{pr}_1^*\mathcal{E}_1, \Lambda) \otimes^L R\mathcal{H}om(\mathrm{pr}_2^*\mathcal{E}_2, \Lambda) \\ &\overset{(a)}{\cong} R\mathcal{H}om(\mathrm{pr}_1^*\mathcal{E}_1, R\mathcal{H}om(\mathrm{pr}_2^*\mathcal{E}_2, \Lambda)) \\ &\cong R\mathcal{H}om(\mathrm{pr}_1^*\mathcal{E}_1 \otimes^L \mathrm{pr}_2^*\mathcal{E}_2, \Lambda) = \mathcal{E}^\vee, \end{aligned} \tag{3.1.3.4}$$

where the isomorphism (a) follows from Lemma 2.3.4 by the fact that (cf. Lemma 3.1.2)

$$SS(\mathrm{pr}_1^*\mathcal{E}_1, X/k)\cap SS(\mathrm{pr}_2^*\mathcal{E}_2, X/k)\subseteq T^*_X X.$$

By Lemma 3.1.2, we have

$$\begin{aligned}&SS(\mathcal{E}, X/k)\cap SS(\mathcal{F}, X/k)\\ &\subseteq i^\circ(SS(\mathcal{E}_1\boxtimes_k^L\mathcal{E}_2, X_1\times_k X_2/k))\cap i^\circ(SS(\mathcal{F}_1\boxtimes_k^L\mathcal{F}_2, X_1\times_k X_2/k))\\ &\overset{(b)}{=} i^\circ(SS(\mathcal{E}_1, X_1)\times SS(\mathcal{E}_2, X_2))\cap i^\circ(SS(\mathcal{F}_1, X_1)\times SS(\mathcal{F}_2, X_2))\\ &\overset{(c)}{\subseteq} T^*_X X,\end{aligned}$$

where the equality (b) follows from [Sai17b, Theorem 2.2.3], and (c) follows from the assumptions (2) and (3) (cf. [Sai17b, Lemma 2.7.2]). Thus by Lemma 2.3.4, we have

$$\mathcal{F}\otimes^L\mathcal{E}^\vee\cong R\mathcal{H}om(\mathcal{E},\mathcal{F}). \tag{3.1.3.5}$$

Combining (3.1.3.3), (3.1.3.4) and (3.1.3.5), we get

$$R\mathcal{H}om(\mathcal{E}_1,\mathcal{F}_1)\boxtimes_S^L R\mathcal{H}om(\mathcal{E}_2,\mathcal{F}_2)\cong\mathcal{F}\otimes^L\mathcal{E}^\vee\cong R\mathcal{H}om(\mathcal{E},\mathcal{F}). \tag{3.1.3.6}$$

This finishes the proof. □

3.1.4. *Künneth formula.* We have the following canonical morphism

$$\mathcal{F}_1\boxtimes_S^L R\mathcal{H}om(\mathcal{F}_2,\pi_2^!\Lambda_S)\to R\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}_2,\mathrm{pr}_1^!\mathcal{F}_1), \tag{3.1.4.1}$$

by taking the adjunction of the following composition map

$$\begin{aligned}\mathrm{pr}_1^*\mathcal{F}_1\otimes\mathrm{pr}_2^*R\mathcal{H}om(\mathcal{F}_2,\pi_2^!\Lambda_S)\otimes\mathrm{pr}_2^*\mathcal{F}_2 &\to\mathrm{pr}_1^*\mathcal{F}_1\otimes\mathrm{pr}_2^*(\mathcal{F}_2\otimes R\mathcal{H}om(\mathcal{F}_2,\pi_2^!\Lambda_S))\\ &\xrightarrow{\text{evaluation}}\mathrm{pr}_1^*\mathcal{F}_1\otimes\mathrm{pr}_2^*\pi_2^!\Lambda_S\to\mathrm{pr}_1^*\mathcal{F}_1\otimes\mathrm{pr}_1^!\Lambda_{X_1}\to\mathrm{pr}_1^!\mathcal{F}_1.\end{aligned}$$

Corollary 3.1.5. *Assume that $\pi_1\colon X_1\to S$ is $SS(\mathcal{F}_1, X_1/k)$-transversal or $\pi_2\colon X_2\to S$ is $SS(\mathcal{F}_2, X_2/k)$-transversal. Then the canonical map* (3.1.4.1) *is an isomorphism.*

If S is the spectrum of a field, then the above result is proved in [SGA5, Exposé III, (3.1.1)]. Our proof below is different from that of *loc.cit.*

Proof. By Proposition 3.1.3, we have the following isomorphisms

$$\begin{aligned}\mathcal{F}_1\boxtimes_S^L R\mathcal{H}om(\mathcal{F}_2,\pi_2^!\Lambda_S) &\overset{Prop.3.1.3}{\cong} R\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}_2,\mathrm{pr}_1^*\mathcal{F}_1\otimes\mathrm{pr}_1^!\Lambda_S)\\ &\overset{(a)}{\cong} R\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}_2,\mathrm{pr}_1^!\mathcal{F}_1),\end{aligned}$$

where (a) follows from the fact that pr_1 is smooth (cf. [ILO14, XVI, Théorème 3.1.1] and [SGA4, XVIII, Theoréme 3.2.5]). □

Definition 3.1.6. Let $X_i,\mathcal{F}_i$ be as in 3.1.1 for $i=1,2$. A *relative correspondence* between X_1 and X_2 is a scheme C over S with morphisms $c_1\colon C\to X_1$ and $c_2\colon C\to X_2$ over S. We put $c=(c_1,c_2)\colon C\to X_1\times_S X_2$ the corresponding morphism. A morphism $u\colon c_2^*\mathcal{F}_2\to c_1^!\mathcal{F}_1$ is called a *relative cohomological correspondence* from $\mathcal{F}_2$ to $\mathcal{F}_1$ on C.

3.1.7. Given a correspondence C as above, we recall that there is a canonical isomorphism [SGA4, XVIII, 3.1.12.2]

$$R\mathcal{H}om(c_2^*\mathcal{F}_2, c_1^!\mathcal{F}_1)\xrightarrow{\cong} c^!R\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}_2,\mathrm{pr}_1^!\mathcal{F}_1). \tag{3.1.7.1}$$

3.1.8. For $i = 1, 2$, consider the following diagram of S-morphisms

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & Y_i \\ & {\scriptstyle \pi_i}\searrow \quad \swarrow{\scriptstyle q_i} & \\ & S, & \end{array}$$

where π_i and q_i are smooth morphisms. We put $X := X_1 \times_S X_2$, $Y := Y_1 \times_S Y_2$ and $f := f_1 \times_S f_2 \colon X \to Y$. Let $\mathcal{M}_i \in D_c^b(Y_i, \Lambda)$ for $i = 1, 2$. We have a canonical map (cf. [Zh15, Construction 7.4] and [SGA5, Exposé III, (1.7.3)])

$$f_1^! \mathcal{M}_1 \boxtimes_S^L f_2^! \mathcal{M}_2 \to f^!(\mathcal{M}_1 \boxtimes_S^L \mathcal{M}_2) \tag{3.1.8.1}$$

which is adjoint to the composite

$$f_!(f_1^! \mathcal{M}_1 \boxtimes_S^L f_2^! \mathcal{M}_2) \xrightarrow[(a)]{\simeq} f_{1!} f_1^! \mathcal{M}_1 \boxtimes_S^L f_{2!} f_2^! \mathcal{M}_2 \xrightarrow{\mathrm{adj} \boxtimes \mathrm{adj}} \mathcal{M}_1 \boxtimes_S^L \mathcal{M}_2 \tag{3.1.8.2}$$

where (a) is the Künneth isomorphism [SGA4, XVII, Théorème 5.4.3].

Proposition 3.1.9. *If $q_2 \colon Y_2 \to S$ is $SS(\mathcal{M}_2, Y_2/k)$-transversal, then the map (3.1.8.1) is an isomorphism.*

If S is the spectrum of a field, the above result is proved in [SGA5, Exposé III, Proposition 1.7.4].

Proof. Consider the following cartesian diagrams

$$\begin{array}{ccccc} X_1 \times_S X_2 & \xrightarrow{f_1 \times \mathrm{id}} & Y_1 \times_S X_2 & \longrightarrow & X_2 \\ \downarrow{\scriptstyle \mathrm{id} \times f_2} & \searrow^{f} & \downarrow{\scriptstyle \mathrm{id} \times f_2} & & \downarrow{\scriptstyle f_2} \\ X_1 \times_S Y_2 & \xrightarrow{f_1 \times \mathrm{id}} & Y_1 \times_S Y_2 & \xrightarrow{\mathrm{pr}_2} & Y_2 \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle q_2} \\ X_1 & \xrightarrow{f_1} & Y_1 & \xrightarrow{q_1} & S \\ & \searrow_{\pi_1} & \downarrow{\scriptstyle q_1} & & \\ & & S. & & \end{array}$$

We may assume that $X_2 = Y_2$ and $f_2 = \mathrm{id}$, i.e., it suffices to show that the canonical map

$$f_1^! \mathcal{M}_1 \boxtimes_S^L \mathcal{M}_2 \xrightarrow{\cong} (f_1 \times \mathrm{id})^! (\mathcal{M}_1 \boxtimes_S^L \mathcal{M}_2). \tag{3.1.9.1}$$

is an isomorphism. Since we have

$$\begin{aligned} \mathcal{M}_2 \cong D_{Y_2} D_{Y_2} \mathcal{M}_2 &\cong R\mathcal{H}om(D_{Y_2} \mathcal{M}_2, \mathcal{K}_{Y_2}) \\ &\cong R\mathcal{H}om(D_{Y_2}(\mathcal{M}_2)(-\mathrm{dim} S)[-2\mathrm{dim} S], q_2^! \Lambda_S), \end{aligned}$$

we may assume $\mathcal{M}_2 = R\mathcal{H}om(\mathcal{L}_2, q_2^! \Lambda_S)$ for some $\mathcal{L}_2 \in D_c^b(Y_2, \Lambda)$. By [Sai17a, Corollary 4.9], we have $SS(\mathcal{M}_2, Y_2/k) = SS(\mathcal{L}_2, Y_2/k)$. Thus by assumption, the morphism $q_2 \colon Y_2 \to S$ is $SS(\mathcal{L}_2, Y_2/k)$-transversal. By Corollary 3.1.5, we have an isomorphism

$$\mathcal{M}_1 \boxtimes_S^L R\mathcal{H}om(\mathcal{L}_2, q_2^! \Lambda_S) \cong R\mathcal{H}om(\mathrm{pr}_2^* \mathcal{L}_2, \mathrm{pr}_1^! \mathcal{M}_1) \quad \text{in } D_c^b(Y_1 \times_S Y_2, \Lambda), \tag{3.1.9.2}$$

$$f_1^! \mathcal{M}_1 \boxtimes_S^L R\mathcal{H}om(\mathcal{L}_2, q_2^! \Lambda_S) \cong R\mathcal{H}om((f_1 \times \mathrm{id})^* \mathrm{pr}_2^* \mathcal{L}_2, \mathrm{pr}_1^! f_1^! \mathcal{M}_1) \text{ in } D_c^b(X_1 \times_S Y_2, \Lambda). \tag{3.1.9.3}$$

We have

$$
\begin{aligned}
(f_1\times \mathrm{id})^!(\mathcal{M}_1\boxtimes^L_S \mathcal{M}_2) &= (f_1\times \mathrm{id})^!(\mathcal{M}_1\boxtimes^L_S R\mathcal{H}om(\mathcal{L}_2, q_2^!\Lambda_S))\\
&\overset{(3.1.9.2)}{\cong} (f_1\times \mathrm{id})^!(R\mathcal{H}om(\mathrm{pr}_2^*\mathcal{L}_2, \mathrm{pr}_1^!\mathcal{M}_1))\\
&\overset{(3.1.7.1)}{\cong} R\mathcal{H}om((f_1\times\mathrm{id})^*\mathrm{pr}_2^*\mathcal{L}_2, (f_1\times \mathrm{id})^!\mathrm{pr}_1^!\mathcal{M}_1)\\
&\cong R\mathcal{H}om((f_1\times\mathrm{id})^*\mathrm{pr}_2^*\mathcal{L}_2, \mathrm{pr}_1^! f_1^!\mathcal{M}_1)\\
&\overset{(3.1.9.3)}{\cong} f_1^!\mathcal{M}_1\boxtimes^L_S R\mathcal{H}om(\mathcal{L}_2, q_2^!\Lambda_S)\cong f_1^!\mathcal{M}_1\boxtimes^L_S\mathcal{M}_2.
\end{aligned}
\tag{3.1.9.4}
$$

This finishes the proof. □

3.2. **Relative cohomological characteristic class.**

3.2.1. We introduce some notation for convenience. For any commutative diagram

$$
\begin{array}{ccc}
W & \xrightarrow{\quad h\quad} & V\\
& {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\
& \mathrm{Spec}k &
\end{array}
$$

of schemes, we put

$$\mathcal{K}_W := Rf^!\Lambda, \tag{3.2.1.1}$$

$$\mathcal{K}_{W/V} := Rh^!\Lambda_V. \tag{3.2.1.2}$$

Under the notation in 3.1.1, by Proposition 3.1.9, we have an isomorphism

$$\mathcal{K}_{X_1/S}\boxtimes^L_S \mathcal{K}_{X_2/S}\simeq \mathcal{K}_{X/S}. \tag{3.2.1.3}$$

3.2.2. Consider a cartesian diagram

$$
\begin{array}{ccc}
E & \longrightarrow & D\\
\downarrow & \overset{e}{\searrow} & \downarrow{\scriptstyle d}\\
C & \xrightarrow{\ c\ } & X
\end{array}
\tag{3.2.2.1}
$$

of schemes over k. Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be objects of $D^b_c(X,\Lambda)$ and $\mathcal{F}\otimes\mathcal{G}\to\mathcal{H}$ any morphism. By the Künneth isomorphism [SGA4, XVII, Théorème 5.4.3] and adjunction, we have

$$e_!(c^!\mathcal{F}\boxtimes^L_X d^!\mathcal{G})\xrightarrow{\simeq} c_!c^!\mathcal{F}\otimes^L d_!d^!\mathcal{G}\to\mathcal{F}\otimes\mathcal{G}\to\mathcal{H}.$$

By adjunction, we get a morphism

$$c^!\mathcal{F}\boxtimes^L_X d^!\mathcal{G}\to e^!\mathcal{H}. \tag{3.2.2.2}$$

Thus we get a pairing

$$\langle,\rangle : H^0(C, c^!\mathcal{F})\otimes H^0(D, d^!\mathcal{G})\to H^0(E, e^!\mathcal{H}). \tag{3.2.2.3}$$

3.2.3. Now we define the relative Verdier pairing by applying the map (3.2.2.3) to relative cohomological correspondences. Let $\pi_1\colon X_1\to S$ and $\pi_2\colon X_2\to S$ be smooth morphisms. Consider a cartesian diagram

$$
\begin{array}{ccc}
E & \longrightarrow & D\\
\downarrow & \overset{e}{\searrow} & \downarrow{\scriptstyle d=(d_1,d_2)}\\
C & \xrightarrow[c=(c_1,c_2)]{} & X = X_1\times_S X_2
\end{array}
\tag{3.2.3.1}
$$

of schemes over S. Let $\mathcal{F}_1 \in D_c^b(X_1, \Lambda)$ and $\mathcal{F}_2 \in D_c^b(X_2, \Lambda)$. Assume that one of the following conditions holds:

(1) $\pi_1 \colon X_1 \to S$ is $SS(\mathcal{F}_1, X_1/k)$-transversal;
(2) $\pi_2 \colon X_2 \to S$ is $SS(\mathcal{F}_2, X_2/k)$-transversal.

By Corollary 3.1.5, we have

$$\begin{aligned} & R\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}_2, \mathrm{pr}_1^!\mathcal{F}_1) \otimes^L R\mathcal{H}om(\mathrm{pr}_1^*\mathcal{F}_1, \mathrm{pr}_2^!\mathcal{F}_2) \\ & \xrightarrow{\simeq} (\mathcal{F}_1 \boxtimes_S^L R\mathcal{H}om(\mathcal{F}_2, \pi_2^!\Lambda_S)) \otimes^L (R\mathcal{H}om(\mathcal{F}_1, \pi_1^!\Lambda_S) \boxtimes_S^L \mathcal{F}_2) \\ & \xrightarrow{\text{evaluation}} \pi_1^!\Lambda_S \boxtimes_S^L \pi_2^!\Lambda_S \overset{(3.2.1.3)}{\cong} \mathcal{K}_{X/S}. \end{aligned} \tag{3.2.3.2}$$

By (3.1.7.1), (3.2.2.2), (3.2.2.3) and (3.2.3.2), we get the following pairings

$$c_! R\mathcal{H}om(c_2^*\mathcal{F}_2, c_1^!\mathcal{F}_1) \otimes^L d_! R\mathcal{H}om(d_1^*\mathcal{F}_1, d_2^!\mathcal{F}_2) \to e_!\mathcal{K}_{E/S}, \tag{3.2.3.3}$$

$$\langle , \rangle : Hom(c_2^*\mathcal{F}_2, c_1^!\mathcal{F}_1) \otimes Hom(d_1^*\mathcal{F}_1, d_2^!\mathcal{F}_2) \to H^0(E, e^!(\mathcal{K}_{X/S})) = H^0(E, \mathcal{K}_{E/S}). \tag{3.2.3.4}$$

The pairing (3.2.3.4) is called the *relative Verdier pairing* (cf. [SGA5, Exposé III (4.2.5)]).

Definition 3.2.4. Let $f \colon X \to S$ be a smooth morphism purely of relative dimension n and $\mathcal{F} \in D_c^b(X, \Lambda)$. We assume that f is $SS(\mathcal{F}, X/k)$-transversal. Let $c = (c_1, c_2) \colon C \to X \times_S X$ be a closed immersion and $u \colon c_2^*\mathcal{F} \to c_1^!\mathcal{F}$ be a relative cohomological correspondence on C. We define the *relative cohomological characteristic class* $ccc_{X/S}(u)$ of u to be the cohomology class $\langle u, 1\rangle \in H^0_{C\cap X}(X, \mathcal{K}_{X/S})$ defined by the pairing (3.2.3.4).

In particular, if $C = X$ and $c \colon C \to X \times_S X$ is the diagonal and if $u \colon \mathcal{F} \to \mathcal{F}$ is the identity, we write

$$ccc_{X/S}(\mathcal{F}) = \langle 1, 1\rangle \quad \text{in} \quad H^{2n}(X, \Lambda(n))$$

and call it the *relative cohomological characteristic class* of $\mathcal{F}$.

If S is the spectrum of a perfect field, then the above definition is [AS07, Definition 2.1.1].

Example 3.2.5. If $\mathcal{F}$ is a locally constant and constructible sheaf of Λ-modules on X, then we have $ccc_{X/S}\mathcal{F} = \mathrm{rank}\mathcal{F} \cdot c_n(\Omega^\vee_{X/S}) \cap [X] \in CH^n(X)$.

Conjecture 3.2.6. *Let S be a smooth connected scheme over a perfect field k of characteristic p. Let $f \colon X \to S$ be a smooth morphism purely of relatively dimension n and $\mathcal{F} \in D_c^b(X, \Lambda)$. Assume that f is $SS(\mathcal{F}, X/k)$-transversal. Let $\mathrm{cl} \colon \mathrm{CH}^n(X) \to H^{2n}(X, \Lambda(n))$ be the cycle class map. Then we have*

$$\mathrm{cl}(cc_{X/S}(\mathcal{F})) = ccc_{X/S}(\mathcal{F}) \quad \text{in} \quad H^{2n}(X, \Lambda(n)), \tag{3.2.6.1}$$

where $cc_{X/S}(\mathcal{F})$ is the relative characteristic class defined in Definition 2.4.3.

If S is the spectrum of a perfect field, then the above conjecture is [Sai17a, Conjecture 6.8.1].

3.3. **Proper push-forward of relative cohomological characteristic class.**

3.3.1. For $i = 1, 2$, let $f_i \colon X_i \to Y_i$ be a proper morphism between smooth schemes over S. Let $X := X_1 \times_S X_2$, $Y := Y_1 \times_S Y_2$ and $f := f_1 \times_S f_2$. Let $p_i \colon X \to X_i$ and $q_i \colon Y \to Y_i$ be the canonical projections for $i = 1, 2$. Consider a commutative diagram

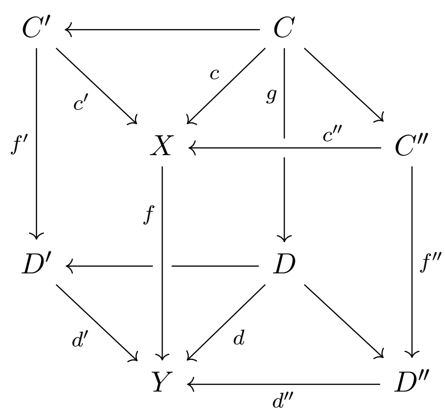
$$\begin{array}{ccc} X & \xleftarrow{c} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xleftarrow{d} & D \end{array} \tag{3.3.1.1}$$

of schemes over S. Assume that c is proper. Put $c_i = p_i c$ and $d_i = q_i d$. By [Zh15, Construction 7.17], we have the following push-forward maps for cohomological correspondence (see also [SGA5, Exposé III, (3.7.6)] if S is the spectrum of a field):

(3.3.1.2) $$f_*\colon Hom(c_2^*\mathcal{L}_2, c_1^!\mathcal{L}_1) \to Hom(d_2^*(f_{2!}\mathcal{L}_2), d_1^!(f_{1*}\mathcal{L}_1)),$$

(3.3.1.3) $$f_*\colon g_*R\mathcal{H}om(c_2^*\mathcal{L}_2, c_1^!\mathcal{L}_1) \to R\mathcal{H}om(d_2^*(f_{2!}\mathcal{L}_2), d_1^!(f_{1*}\mathcal{L}_1)).$$

Theorem 3.3.2 ([SGA5, Théorème 4.4])**.** *For $i = 1,2$, let $f_i\colon X_i \to Y_i$ be a proper morphism between smooth schemes over S. Let $X \coloneqq X_1 \times_S X_2$, $Y \coloneqq Y_1 \times_S Y_2$ and $f \coloneqq f_1 \times_S f_2$. Let $p_i\colon X \to X_i$ and $q_i\colon Y \to Y_i$ be the canonical projections for $i = 1,2$. Consider the following commutative diagram with cartesian horizontal faces*



where c', c'', d' and d'' are proper morphisms between smooth schemes over S. Let $c_i' = p_i c', c_i'' = p_i c'', d_i' = q_i d', d_i'' = q_i d''$ for $i = 1,2$. Let $\mathcal{L}_i \in D_c^b(X_i, \Lambda)$ and we put $\mathcal{M}_i = f_{i*}\mathcal{L}_i$ for $i = 1,2$. Assume that one of the following conditions holds:

1. *$X_1 \to S$ is $SS(\mathcal{L}_1, X_1/k)$-transversal;*
2. *$X_2 \to S$ is $SS(\mathcal{L}_2, X_2/k)$-transversal.*

Then we have the following commutative diagram

(3.3.2.1) $$\begin{array}{ccc} f_*c'_*R\mathcal{H}om(c_2'^*\mathcal{L}_2, c_1'^!\mathcal{L}_1) \otimes^L f_*c''_*R\mathcal{H}om(c_1''^*\mathcal{L}_1, c_2''^!\mathcal{L}_2) & \xrightarrow{(1)} & f_*c_*\mathcal{K}_{C/S} \\ \downarrow{\scriptstyle (2)} & & \downarrow{\scriptstyle (4)} \\ d'_*R\mathcal{H}om(d_2'^*\mathcal{M}_2, d_1'^!\mathcal{M}_1) \otimes^L d''_*R\mathcal{H}om(d_1''^*\mathcal{M}_1, d_2''^!\mathcal{M}_2) & \xrightarrow{(3)} & d_*\mathcal{K}_{D/S} \end{array}$$

where (3) is given by (3.2.3.3), (1) is the composition of $f_*((3.2.3.3))$ with the canonical map $f_*c'_* \otimes^L f_*c''_* \to f_*(c'_* \otimes c''_*)$, (2) is induced from (3.3.1.3), and (4) is defined by

(3.3.2.2) $$f_*c_*\mathcal{K}_{C/S} \simeq d_*g_*\mathcal{K}_{C/S} = d_*g_!g^!\mathcal{K}_{D/S} \xrightarrow{\text{adj}} d_*\mathcal{K}_{D/S}.$$

If S is the spectrum of a field, this is proved in [SGA5, Théoreème 4.4]. We use the same notation as *loc.cit.*

Proof. By [Sai17a, Lemma 3.8 and Lemma 4.2.6] and the assumption, one of the following conditions holds:

(a1) $Y_1 \to S$ is $SS(\mathcal{M}_1, Y_1/k)$-transversal;
(a2) $Y_2 \to S$ is $SS(\mathcal{M}_2, Y_2/k)$-transversal.

Now we can use the same proof of [SGA5, Théorème 4.4]. We only sketch the main step. Put

(3.3.2.3) $$\mathcal{P} = \mathcal{L}_1 \boxtimes_S^L R\mathcal{H}om(\mathcal{L}_2, \mathcal{K}_{X_2/S}), \quad \mathcal{Q} = R\mathcal{H}om(\mathcal{L}_1, \mathcal{K}_{X_1/S}) \boxtimes_S^L \mathcal{L}_2$$

(3.3.2.4) $$\mathcal{E} = \mathcal{M}_1 \boxtimes_S^L R\mathcal{H}om(\mathcal{M}_2, \mathcal{K}_{Y_2/S}), \quad \mathcal{F} = R\mathcal{H}om(\mathcal{M}_1, \mathcal{K}_{Y_1/S}) \boxtimes_S^L \mathcal{M}_2.$$

Then the theorem follows from the following commutative diagram

$$\begin{array}{ccccccc}
f_*c'_*c'^!\mathcal{P}\otimes^L f_*c''_*c''^!\mathcal{Q} & & \longrightarrow & & f_*c_*c^!(\mathcal{P}\otimes^L\mathcal{Q}) & \longrightarrow & f_*c_*c^!\mathcal{K}_{X/S} \\
\downarrow & & & & \downarrow & & \downarrow \\
d'_*d'^!f_*\mathcal{P}\otimes^L d''_*d''^!f_*\mathcal{Q} & \longrightarrow & d_*d^!(f_*\mathcal{P}\otimes^L f_*\mathcal{Q}) & \longrightarrow & d_*d^!f_*(\mathcal{P}\otimes^L\mathcal{Q}) & \longrightarrow & d_*d^!f_*\mathcal{K}_{X/S} \\
\downarrow & & \downarrow & & & & \downarrow \\
d'_*d'^!\mathcal{E}\otimes^L d''_*d''^!\mathcal{F} & \longrightarrow & d_*d^!(\mathcal{E}\otimes^L\mathcal{F}) & & \longrightarrow & & d_*d^!\mathcal{K}_{Y/S}
\end{array}$$

where commutativity can be verified following the same argument of [SGA5, Théorème 4.4]. □

Corollary 3.3.3 ([SGA5, Corollaire 4.5])**.** *Under the assumptions of Theorem 3.3.2, we have a commutative diagram*

(3.3.3.1)
$$\begin{array}{ccc}
Hom(c_2'^*\mathcal{L}_2, c_1'^!\mathcal{L}_1)\otimes Hom(c_1''^*\mathcal{L}_1, c_2''^!\mathcal{L}_2) & \longrightarrow & H^0(C,\mathcal{K}_{C/S}) \\
\Big\downarrow{\scriptstyle (3.3.1.2)\otimes(3.3.1.2)} & & \Big\downarrow{\scriptstyle g_*} \\
Hom(d_2'^*f_{2*}\mathcal{L}_2, d_1'^!f_{1*}\mathcal{L}_1)\otimes Hom(d_1''^*f_{1*}\mathcal{L}_1, d_2''^!f_{2*}\mathcal{L}_2) & \longrightarrow & H^0(D,\mathcal{K}_{D/S}).
\end{array}$$

Corollary 3.3.4. *Let S be a smooth connected scheme over a perfect field k of characteristic p. Let $f\colon X\to S$ be a smooth morphism purely of relative dimension n and $g\colon Y\to S$ a smooth morphism purely of relative dimension m. Assume that f is $SS(\mathcal{F},X/k)$-transversal. Then for any proper morphism $h\colon X\to Y$ over S,*

(3.3.4.1)
$$\begin{array}{ccc}
X & \xrightarrow{\ h\ } & Y \\
& {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\
& S &
\end{array}$$

we have

(3.3.4.2)
$$f_*ccc_{X/S}(\mathcal{F}) = ccc_{Y/S}(Rf_*\mathcal{F}) \quad \text{in} \quad H^{2m}(Y,\Lambda(m)).$$

Proof. This follows from Corollary 3.3.3 and Definition 3.2.4. □

References

[AP17] T. Abe and D. Patel, *On a localization formula of epsilon factors via microlocal geometry*, to appear in Annals of K-theory.

[AS07] A. Abbes and T. Saito, *The characteristic class and ramification of an ℓ-adic étale sheaf*, Inventiones Mathematicae, 168 (2007):567-612.

[SGA4] M. Artin, A. Grothendieck, and J.-L. Verdier, *Théorie des topos et cohomologie étale des schémas*, Lecture Notes in Mathematics, vol. 269,270,305, Springer-Verlag, 1972-1973, Séminaire de Géométrie Algébrique du Bois-Marie 1963-64 (SGA4).

[Bei16] A. Beilinson, *Constructible sheaves are holonomic*, Selecta Mathematica, 22, Issue 4 (2016):1797-1819.

[Bei07] A. Beilinson, *Topological E-factors*, Pure Appl. Math. Q., 3(1, part 3) (2007): 357-39.

[Del72] P. Deligne, *Les constantes des equations fonctionnelles des fonctions L*, Modular Functions of One Variables II, Lecture Notes in Math. 349, Springer, Berlin-Heidelberg-New York, 1972.

[Del80] P. Deligne, *La conjecture de Weil, II*, Inst. Hautes Études Sci. Publ. Math. 52 (1980): 137-252.

[DH81] P. Deligne and G. Henniart, *Sur la variation, par torsion, des constantes locales d'équations fonctionnelles de fonctions L*, Inventiones Mathematicae 64 (1981): 89-118.

[Gin86] V. Ginsburg, *Characteristic varieties and vanishing cycles*, Inventiones Mathematicae 84 (1986): 327-402.

[SGA5] A. Grothendieck *et al.*, *Cohomologie ℓ-adique et fonctions L.* Séminaire de Géométrie Algébrique du Bois-Marie 1965–1966 (SGA 5). dirigé par A. Grothendieck avec la collaboration de I. Bucur, C. Houzel, L. Illusie, J.-P. Jouanolou et J-P. Serre. Lecture Notes in Mathematics 589, Springer-verlag, Berlin-New York, (1977).

[HY17] H. Hu and E. Yang, *Relative singular support and the semi-continuity of characteristic cycles for étale sheaves*, Selecta Mathematica, Volume 24, Issue 3 (2018):2235-2273.

[ILO14] L. Illusie, Y. Lazslo, and F. Orgogozo, *Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents,* Astérisque, vol. 363-364, Soc. Math. France, 2014, Séminaire à l'École Polytechnique 2006-2008.

[Lau87] G. Laumon, *Transformation de Fourier, constantes d'équations fonctionnelles et conjecture de Weil*, Publications Mathématiques de l'IHÉS 65 (1987): 131-210.

[KS13] K. Kato and T. Saito, *Ramification theory for varieties over a local field*, Publications Mathématiques de l'IHÉS 117 (2013): 1-178.

[KS08] K. Kato and T. Saito, *Ramification theory for varieties over a perfect field*, Annals of Mathematics, 168 (2008): 33-96.

[KS83] K. Kato and S. Saito, *Unramified class field theory of arithmetical surfaces*, Annals of Mathematics, 118(1983): 241-275.

[Kat88] N. M. Katz, *Gauss sums, Kloosterman sums, and monodromy groups*, volume 116 of Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 1988.

[KeS16] M. Kerz and S. Saito, *Chow group of 0-cycles with modulus and higher dimensional class field theory*, Duke Mathematical Journal 165(15) (2016): 2811-2897.

[Pat12] D. Patel, *De Rham ε-factors*, Inventiones Mathematicae, Volume 190, Number 2, (2012): 299-355.

[Sai93] T. Saito, *ε-factor of a tamely ramified sheaf on a variety*, Inventiones Mathematicae 113 (1993): 389-417.

[Sai94] T. Saito, *Jacobi sum Hecke characters, de Rham discriminant, and the determinant of ℓ-adic cohomologies*, Journal of Algebraic Geometry, 3 (1994): 411-434.

[Sai17a] T. Saito, *The characteristic cycle and the singular support of a constructible sheaf*, Inventiones Mathematicae, 207 (2017): 597-695.

[Sai17b] T. Saito, *Characteristic cycle of the external product of constructible sheaves*, Manuscripta Mathematica, Volume 154, Issue 12 (2017): 1-12.

[SY16] T. Saito and Y. Yatagawa, *Wild ramification determines the characteristic cycle*, Annales Scientifiques de l'ENS 50, fascicule 4 (2017): 1065-1079.

[SS84] S. Saito, *Functional equations of L-functions of varieties over finite fields*, J. Fac. Sci. Univ. of Tokyo, Sec. IA 31 (1984): 287-296.

[Tat79] J. Tate, *Number theoretic background*, in Automorphic forms, representations and L-functions, PSPM, vol 33, part II. A. M. S., Providence (1979): 3-26.

[UYZ] N. Umezaki, E. Yang and Y. Zhao, *Characteristic class and the ε-factor of an étale sheaf*, arXiv:1701.02841, submitted.

[Vi09a] I. Vidal, *Formule du conducteur pour un caractère l-adique*, Compositio Mathematica, 145 (2009): 687-717.

[Vi09b] I. Vidal, *Formule de torsion pour le facteur epsilon d'un caractère sur une surface*, Manuscripta Mathematica, 130 (2009): 21-44.

[Zh15] W. Zheng, *Six operations and Lefschetz-Verdier formula for Deligne-Mumford stacks*, Science China Mathematics 58 (2015), no. 3 (special issue for Algebraic Geometry in East Asia 2013), pp. 565-632.

Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany
E-mail address: `enlin.yang@mathematik.uni-regensburg.de, yangenlin0727@126.com`

Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany
E-mail address: `yigeng.zhao@mathematik.uni-regensburg.de`