THE Hilb/Sym CORRESPONDENCE FOR $\mathbb{C}^2$: DESCENDENTS AND FOURIER-MUKAI

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ABSTRACT. We study here the crepant resolution correspondence for the T-equivariant descendent Gromov-Witten theories of $\mathsf{Hilb}^n(\mathbb{C}^2)$ and $\mathsf{Sym}^n(\mathbb{C}^2)$. The descendent correspondence is obtained from our previous matching of the associated CohFTs by applying Givental's quantization formula to a specific symplectic transformation K. The first result of the paper is an explicit computation of K. Our main result then establishes a fundamental relationship between the Fourier-Mukai equivalence of the associated derived categories (by Bridgeland, King, and Reid) and the symplectic transformation K via Iritani's integral structure. The results use Haiman's Fourier-Mukai calculations and are exactly aligned with Iritani's point of view on crepant resolution.

CONTENTS

0. INTRODUCTION

0.1. **Overview.** The diagonal action on $\mathbb{C}^2$ of the torus $\mathsf{T} = (\mathbb{C}^*)^2$ lifts canonically to the Hilbert scheme of n points $\mathsf{Hilb}^n(\mathbb{C}^2)$ and the orbifold symmetric product

$$\mathsf{Sym}^n(\mathbb{C}^2) = \left[(\mathbb{C}^2)^n/\Sigma_n\right].$$

Both the Hilbert-Chow morphism

$$\mathsf{Hilb}^n(\mathbb{C}^2) \to (\mathbb{C}^2)^n/\Sigma_n \tag{0.1}$$

and the coarsification morphism

$$\mathsf{Sym}^n(\mathbb{C}^2) \to (\mathbb{C}^2)^n/\Sigma_n \tag{0.2}$$

are T-equivariant crepant resolutions of the singular quotient variety $(\mathbb{C}^2)^n/\Sigma_n$.

The geometries of the two crepant resolutions $\mathsf{Hilb}^n(\mathbb{C}^2)$ and $\mathsf{Sym}^n(\mathbb{C}^2)$ are connected in many beautiful ways. The classical McKay correspondence [19] provides an isomorphism on the level

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of T-equivariant cohomology: T-equivariant singular cohomology for $\mathsf{Hilb}^n(\mathbb{C}^2)$ and T-equivariant Chen-Ruan orbifold cohomology for $\mathsf{Sym}^n(\mathbb{C}^2)$. A lift of the McKay correspondence to an equivalence of T-equivariant derived categories was proven by Bridgeland, King, and Reid [4] using a Fourier-Mukai transformation.

Quantum cohomology provides a different enrichment of the McKay correspondence. For the crepant resolutions $\mathsf{Hilb}^n(\mathbb{C}^2)$ and $\mathsf{Sym}^n(\mathbb{C}^2)$, the genus 0 equivalence of the T-equivariant Gromov-Witten theories was proven in [5] using [6, 22]. Going further, the crepant resolution correspondence in all genera was proven in [25] by matching the associated R-matrices and Cohomological Field Theories (CohFTs), see [24, Section 4] for a survey.

The results of [5, 25] concern the T-equivariant Gromov-Witten theory with *primary* insertions. However, following a remarkable proposal of Iritani, to see the connection between the Fourier-Mukai transformation of [4] and the crepant resolution correspondence for Gromov-Witten theory, *descendent* insertions are required. Our first result here is a determination of the crepant resolution correspondence for the T-equivariant Gromov-Witten theories of $\mathsf{Hilb}^n(\mathbb{C}^2)$ and $\mathsf{Sym}^n(\mathbb{C}^2)$ with descendent insertions via a symplectic transformation K which we compute explicitly. The main result of the paper is a proof of a fundamental relationship between the Fourier-Mukai equivalence of the associated derived categories [4] and the symplectic transformation K via Iritani's integral structure. The results use Haiman's Fourier-Mukai calculations [12, 13] and are exactly aligned with Iritani's point of view on crepant resolutions [16, 17].

0.2. **Descendent correspondence.** The descendent correspondence for the T-equivariant Gromov-Witten theories of $\mathsf{Hilb}^n(\mathbb{C}^2)$ and $\mathsf{Sym}^n(\mathbb{C}^2)$ is obtained from the CohFT matching of [25] together with the quantization formula of Givental [11]. Our first result is a formula for the symplectic transformation

$$\mathsf{K} \in \mathrm{Id} + z^{-1} \cdot \mathrm{End}(H^*_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2)))[[z^{-1}]]$$

defining the descendent correspondence.[1]

The formula for K is best described in terms of the Fock space $\mathcal{F}$ which is freely generated over $\mathbb{C}$ by commuting creation operators α_{-k} for $k \in \mathbb{Z}_{>0}$ acting on the vacuum vector $v_\emptyset$. The annihilation operators $\alpha_k, k \in \mathbb{Z}_{>0}$ satisfy

$$\alpha_k \cdot v_\emptyset = 0\,, \quad k > 0$$

and commutation relations

$$[\alpha_k, \alpha_l] = k\delta_{k+l}\,.$$

The Fock space $\mathcal{F}$ admits an additive basis

$$|\mu\rangle = \frac{1}{\mathfrak{z}(\mu)} \prod_i \alpha_{-\mu_i} v_\emptyset\,, \quad \mathfrak{z}(\mu) = |\mathrm{Aut}(\mu)| \prod_i \mu_i\,,$$

indexed by partitions $\mu = (\mu_1, \mu_2, ...)$.

An additive isomorphism

$$\mathcal{F} \otimes_{\mathbb{C}} \mathbb{C}[t_1, t_2] \stackrel{\sim}{=} \bigoplus_{n \geq 0} H^*_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2))\,, \tag{0.3}$$

[1] Cohomology will always be taken here with $\mathbb{C}$-coefficients.

is given by identifying $|\mu\rangle$ on the left with the corresponding Nakajima basis elements on the right. The intersection pairing $(-,-)^{\text{Hilb}}$ on the T-equivariant cohomology of $\mathsf{Hilb}^n(\mathbb{C}^2)$ induces a pairing on Fock space,

$$\eta(\mu,\nu) = \frac{(-1)^{|\mu|-\ell(\mu)}}{(t_1t_2)^{\ell(\mu)}}\frac{\delta_{\mu\nu}}{\mathfrak{z}(\mu)}\,.$$

In the following result, we write the formula for K in terms of the Fock space,

$$\mathsf{K} \in \text{Id} + z^{-1}\cdot \text{End}(\mathcal{F}\otimes_{\mathbb{C}} \mathbb{C}[t_1,t_2])[[z^{-1}]],,$$

using (0.3).

Theorem 1. *The descendent correspondence is determined by the symplectic transformation* K *given by the formula*

$$\mathsf{K}\left(\mathsf{J}^\lambda\right) = \frac{z^{|\lambda|}}{(2\pi\sqrt{-1})^{|\lambda|}}\left(\prod_{\mathsf{w}:\mathsf{T}\text{-weights of } Tan_\lambda \mathsf{Hilb}^n(\mathbb{C}^2)} \Gamma(\mathsf{w}/z+1)\right)\spadesuit \mathsf{H}^\lambda_z\,.$$

Here, J^λ is the Jack symmetric function defined by equation (1.5) of Section 1, and H^λ_z is the Macdonald polynomial[2], see [12, 18, 23]. The linear operator

$$\spadesuit : \mathcal{F}\to\mathcal{F}$$

is defined by

$$\spadesuit|\mu\rangle = z^{\ell(\mu)}\frac{(2\pi\sqrt{-1})^{\ell(\mu)}}{\prod_i \mu_i}\prod_i \frac{\mu_i^{\mu_i t_1/z}\mu_i^{\mu_i t_2/z}}{\Gamma(\mu_i t_1/z)\Gamma(\mu_i t_2/z)}|\mu\rangle\,.$$

The descendent correspondence in genus 0, expressed in terms of Givental's Lagrangian cones, is explained[3] in Theorem 10 of Section 3.2,

$$\mathcal{L}^{\text{Sym}} = \mathsf{C}\mathsf{K}q^{-D/z}\mathcal{L}^{\text{Hilb}}\,,$$

where $D = -|(2,1^{n-2})\rangle$ is the T-equivariant first Chern class of the tautological vector bundle on $\mathsf{Hilb}^n(\mathbb{C}^2)$. The descendent correspondence for all g, formulated in terms of generating series,

$$e^{-F_1^{\text{Sym}}(\tilde{t})}\mathcal{D}^{\text{Sym}} = \widehat{\mathsf{C}}\,\widehat{\mathsf{K}}\,\widehat{q^{-D/z}}\,\left(e^{-F_1^{\text{Hilb}}(t_D)}\mathcal{D}^{\text{Hilb}}\right)\,,$$

is discussed in Theorem 11 of Section 3.3.

For toric crepant resolutions, the symplectic transformation underlying the descendent correspondence is constructed in [9] by using explicit slices of Givental's Lagrangian cones constructed via the Toric Mirror Theorem [7, 10]. We proceed differently here. The symplectic transformation K is constructed by comparing the two fundamental solutions S^{Hilb} and S^{Sym} of the QDE given by descendent Gromov-Witten invariants of $\mathsf{Hilb}^n(\mathbb{C}^2)$ and $\mathsf{Sym}^n(\mathbb{C}^2)$ respectively. Via the Hilb/Sym correspondence in genus 0, Theorem 1 is then simply a reformulation of the calculation of the connection matrix in [23, Theorem 4].

[2]The footnote z indicates a rescaling of the parameters, $\mathsf{H}^\lambda_z = \mathsf{H}^\lambda(\frac{t_1}{z},\frac{t_2}{z})$.
[3]See for (2.5) the definition of the symplectic isomorphism C.

0.3. **Fourier-Mukai.** An equivalence of T-equivariant derived categories

$$\mathbb{FM} : D^b_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2)) \to D^b_{\mathsf{T}}(\mathsf{Sym}^n(\mathbb{C}^2))$$

is constructed by Bridgeland, King, and Reid in [4] via a tautological Fourier-Mukai kernel. We also denote by $\mathbb{FM}$ the induced isomorphism on T-equivariant K-groups,

$$\mathbb{FM} : K_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2)) \to K_{\mathsf{T}}(\mathsf{Sym}^n(\mathbb{C}^2))\,. \tag{0.4}$$

Iritani [16] has proposed a beautiful framework for the crepant resolution correspondence. In the case of $\mathsf{Hilb}^n(\mathbb{C}^2)$ and $\mathsf{Sym}^n(\mathbb{C}^2)$, the isomorphism (0.4) on K-theory should be related to a symplectic transformation

$$\mathcal{H}^{\text{Hilb}} \to \mathcal{H}^{\text{Sym}}$$

via Iritani's integral structure. The Givental spaces $\mathcal{H}^{\text{Hilb}}$ and $\mathcal{H}^{\text{Sym}}$ will be defined below (in a multivalued form). A discussion of Iritani's perspective can be found in [17]. Our main result is a formulation and proof of Iritani's proposal for the crepant resolutions $\mathsf{Hilb}^n(\mathbb{C}^2)$ and $\mathsf{Sym}^n(\mathbb{C}^2)$. For the precise statement, further definitions are required.

- Define the operators $\deg_0^{\text{Hilb}}$, ρ^{Hilb}, and μ^{Hilb} on $H^*_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2))$ as follows. For $\phi \in H^k_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2))$,

$$\begin{aligned} \deg_0^{\text{Hilb}}(\phi) &= k\phi\,, \\ \mu^{\text{Hilb}}(\phi) &= \left(\frac{k}{2} - \frac{2n}{2}\right)\phi\,, \\ \rho^{\text{Hilb}}(\phi) &= c_1^{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2)) \cup \phi\,. \end{aligned}$$

The *multi-valued Givental space* $\widetilde{\mathcal{H}}^{\text{Hilb}}$ for $\mathsf{Hilb}^n(\mathbb{C}^2)$ is defined by

$$\widetilde{\mathcal{H}}^{\text{Hilb}} = H^*_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2), \mathbb{C}) \otimes_{\mathbb{C}[t_1,t_2]} \mathbb{C}(t_1,t_2)[[\log(z)]]((z^{-1}))\,.$$

Definition 2. *Let* $\Psi^{\text{Hilb}} : K_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2)) \to \widetilde{\mathcal{H}}^{\text{Hilb}}$ *be defined by*

$$\Psi^{\text{Hilb}}(E) = z^{-\mu^{\text{Hilb}}} z^{\rho^{\text{Hilb}}} \left(\Gamma_{\text{Hilb}} \cup (2\pi\sqrt{-1})^{\frac{\deg_0^{\text{Hilb}}}{2}} \text{ch}(E)\right)\,,$$

where $\text{ch}(-)$ *is the* T*-equivariant Chern character,* $\Gamma_{Hilb} \in H^*_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2))$ *is the* T*-equivariant Gamma class of* $\mathsf{Hilb}^n(\mathbb{C}^2)$ *of* [9, Section 3.1]*, and the operators*

$$z^{-\mu^{\text{Hilb}}} : \widetilde{\mathcal{H}}^{\text{Hilb}} \to \widetilde{\mathcal{H}}^{\text{Hilb}}\,, \qquad z^{\rho^{\text{Hilb}}} : \widetilde{\mathcal{H}}^{\text{Hilb}} \to \widetilde{\mathcal{H}}^{\text{Hilb}}$$

are defined by

$$z^{-\mu^{\text{Hilb}}} = \sum_{k\geq 0} \frac{\left(-\mu^{\text{Hilb}}\log z\right)^k}{k!}\,, \qquad z^{\rho^{Hilb}} = \sum_{k\geq 0} \frac{\left(\rho^{Hilb}\log z\right)^k}{k!}\,.$$

Since $|\mu\rangle$ is identified with the corresponding Nakajima basis element, we have

$$\deg_0^{\text{Hilb}}|\mu\rangle = 2(n - \ell(\mu))|\mu\rangle\,.$$

Also, since t_1, t_2 both have degree 2, we have

$$\deg_0^{\text{Hilb}} t_1 = 2 = \deg_0^{\text{Hilb}} t_2\,.$$

- Define the operators[4] $\deg_0^{\mathrm{Sym}}$, ρ^{Sym}, and μ^{Sym} on $H^*_{\mathsf{T}}(I\mathrm{Sym}^n(\mathbb{C}^2))$ as follows. For $\phi \in H^k_{\mathsf{T}}(I\mathrm{Sym}^n(\mathbb{C}^2))$,

$$\deg_0^{\mathrm{Sym}}(\phi) = k\phi\,,$$
$$\mu^{\mathrm{Sym}}(\phi) = \left(\frac{\deg_{\mathrm{CR}}(\phi)}{2} - \frac{2n}{2}\right)\phi\,,$$
$$\rho^{\mathrm{Sym}}(\phi) = c_1^{\mathsf{T}}(\mathrm{Sym}^n(\mathbb{C}^2)) \cup_{\mathrm{CR}} \phi\,.$$

There are *two* degree operators here: $\deg_0^{\mathrm{Sym}}$ extracts the usual degree of a cohomology class on the inertia orbifold, and $\deg_{\mathrm{CR}}$ extracts the age-shifted degree. Also, we have

$$\deg_{\mathrm{CR}} t_1 = \deg_0^{\mathrm{Sym}} t_1 = 2 = \deg_{\mathrm{CR}} t_2 = \deg_0^{\mathrm{Sym}} t_2\,.$$

The multi-valued Givental space $\widetilde{\mathcal{H}}^{\mathrm{Sym}}$ for $\mathrm{Sym}^n(\mathbb{C}^2)$ is defined by

$$\widetilde{\mathcal{H}}^{\mathrm{Sym}} = H^*_{\mathsf{T}}(I\mathrm{Sym}^n(\mathbb{C}^2)) \otimes_{\mathbb{C}[t_1,t_2]} \mathbb{C}(t_1,t_2)[[\log z]]((z^{-1}))\,.$$

Definition 3. *Let* $\Psi^{\mathrm{Sym}} : K_{\mathsf{T}}(\mathrm{Sym}^n(\mathbb{C}^2)) \to \widetilde{\mathcal{H}}^{\mathrm{Sym}}$ *be defined by*

$$\Psi^{\mathrm{Sym}}(E) = z^{-\mu^{\mathrm{Sym}}} z^{\rho^{\mathrm{Sym}}} \left(\Gamma_{\mathrm{Sym}} \cup (2\pi\sqrt{-1})^{\frac{\deg_0^{\mathrm{Sym}}}{2}} \widetilde{\mathrm{ch}}(E)\right)\,,$$

where $\widetilde{\mathrm{ch}}(-)$ *is the* T*-equivariant orbifold Chern character,* $\Gamma_{\mathrm{Sym}} \in H^*_{\mathsf{T}}(I\mathrm{Sym}^n(\mathbb{C}^2))$ *is the* T*-equivariant Gamma class of* $\mathrm{Sym}^n(\mathbb{C}^2)$ *of* [9, Section 3.1]*, and the operators*

$$z^{-\mu^{\mathrm{Sym}}} : \widetilde{\mathcal{H}}^{\mathrm{Sym}} \to \widetilde{\mathcal{H}}^{\mathrm{Sym}}\,, \qquad z^{\rho^{\mathrm{Sym}}} : \widetilde{\mathcal{H}}^{\mathrm{Sym}} \to \widetilde{\mathcal{H}}^{\mathrm{Sym}}$$

are defined by

$$z^{-\mu^{\mathrm{Sym}}} = \sum_{k\geq 0} \frac{(-\mu^{\mathrm{Sym}} \log z)^k}{k!}\,, \qquad z^{\rho^{\mathrm{Sym}}} = \sum_{k\geq 0} \frac{(\rho^{\mathrm{Sym}} \log z)^k}{k!}\,.$$

The precise relationship between $\mathbb{FM}$ and K via Iritani's integral structure is the central result of the paper.

Theorem 4. *The following diagram is commutative*[5]*:*

$$\begin{array}{ccc} K_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2)) & \xrightarrow{\mathbb{FM}} & K_{\mathsf{T}}(\mathrm{Sym}^n(\mathbb{C}^2)) \\ \Big\downarrow{\scriptstyle \Psi^{\mathrm{Hilb}}} & & \Big\downarrow{\scriptstyle \Psi^{\mathrm{Sym}}} \\ \widetilde{\mathcal{H}}^{\mathrm{Hilb}} & \xrightarrow{\mathsf{CK}|_{z\mapsto -z}} & \widetilde{\mathcal{H}}^{\mathrm{Sym}}. \end{array}$$

The bottom row of the diagram of Theorem 4 is determined by the analytic continuation of solutions of the quantum differential equation of $\mathsf{Hilb}^n(\mathbb{C}^2)$ along the ray from 0 to -1 in the q-plane [23, Theorem 4]. A lifting of monodromies of the quantum differential equation of $\mathsf{Hilb}^n(\mathbb{C}^2)$ to autoequivalences of $D^b_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2))$ has been announced by Bezrukavnikov and Okounkov in [20, Sections 3.2.8 and 5.2.7] and [21, Section 3.2]. In their upcoming paper [2], commutative diagrams

[4] In the definition of ρ^{Sym} we denote by $\cup_{\mathrm{CR}}$ the Chen-Ruan cup product on cohomology of the inertia stack.

[5] Our variable z corresponds to $-z$ in [9] as can be seen by the difference in the quantum differential equation (2.2) here and the quantum differential equation [9, equation (2.5)]. After the substitution $z \mapsto -z$ in K, Theorem 4 matches the conventions of Iritani's framework in [9].

parallel to Theorem 4 are constructed in cases of *flops* of holomorphic symplectic manifolds.[6] Theorem 4 fits into the framework of [2] if the relationship between $\mathsf{Hilb}^n(\mathbb{C}^2)$ and $\mathsf{Sym}^n(\mathbb{C}^2)$ is viewed morally as a flop in their sense.

A special aspect of the ray from 0 to -1 is the identification of the end result of the analytic continuation (the right side of the diagram) with the orbifold geometry $\mathsf{Sym}^n(\mathbb{C}^2)$. The identification of the end results of other paths from 0 to -1 with geometric theories is an interesting direction of study. Are there twisted orbifold theories which realize these analytic continuations?

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1. Quantum differential equations

1.1. **The differential equation.** We recall the quantum differential equation for $\mathsf{Hilb}^n(\mathbb{C}^2)$ calculated in [22] and further studied in [23]. We follow here the exposition [22, 23].

The quantum differential equation (QDE) for the Hilbert schemes of points on $\mathbb{C}^2$ is given by

$$q\frac{d}{dq}\Phi = \mathsf{M}_D\Phi\,, \quad \Phi \in \mathcal{F}\otimes_{\mathbb{C}}\mathbb{C}(t_1,t_2), \tag{1.1}$$

where M_D is the operator of quantum multiplication by $D = -|2,1^{n-2}\rangle$,

$$\begin{aligned}\mathsf{M}_D = (t_1+t_2)\sum_{k>0}\frac{k}{2}\frac{(-q)^k+1}{(-q)^k-1}\alpha_{-k}\alpha_k \;-\; &\frac{t_1+t_2}{2}\frac{(-q)+1}{(-q)-1}|\cdot| \\ &+\frac{1}{2}\sum_{k,l>0}\Big[t_1t_2\alpha_{k+l}\alpha_{-k}\alpha_{-l}-\alpha_{-k-l}\alpha_k\alpha_l\Big].\end{aligned} \tag{1.2}$$

Here $|\cdot| = \sum_{k>0}\alpha_{-k}\alpha_k$ is the energy operator.

While the quantum differential equation (1.1) has a regular singular point at $q=0$, the point $q=-1$ is regular.

[6] In fact, the study of commutative diagrams connecting derived equivalences and the solutions of the quantum differential equation has old roots in the subject. See, for example, [3, 14]. These papers refer to talks of Kontsevich on homological mirror symmetry in the 1990s for the first formulations.

The quantum differential equation considered in Givental's theory contains a parameter z. In the case of the Hilbert schemes of points on $\mathbb{C}^2$, the QDE with parameter z is

$$zq\frac{d}{dq}\Phi = \mathsf{M}_D\Phi, \quad \Phi \in \mathcal{F}\otimes_{\mathbb{C}}\mathbb{C}(t_1,t_2)\,. \tag{1.3}$$

For $\Phi \in \mathcal{F}\otimes_{\mathbb{C}}\mathbb{C}(t_1,t_2)$, define

$$\Phi_z = \Phi\left(\frac{t_1}{z},\frac{t_2}{z},q\right). \tag{1.4}$$

Define $\Theta \in \mathrm{Aut}(\mathcal{F})$ by

$$\Theta|\mu\rangle = z^{\ell(\mu)}|\mu\rangle\,.$$

The following Proposition allows us to use the results in [23].

Proposition 5. *If Φ is a solution of (1.1), then $\Theta\Phi_z$ is a solution of (1.3).*

Proposition 5 follow immediately from the following direct computation.

Lemma 6. *For $k > 0$, we have $\Theta\alpha_k = \frac{1}{z}\alpha_k\Theta$ and $\Theta\alpha_{-k} = z\alpha_{-k}\Theta$.*

1.2. **Solutions.** We recall the solution of QDE (1.1) constructed in [23]. Let

$$J_\lambda \in \mathcal{F}\otimes_{\mathbb{C}}\mathbb{C}(t_1,t_2)$$

be the integral form of the Jack symmetric function depending on the parameter $\alpha = 1/\theta$ of [18, 23]. Then

$$\mathsf{J}^\lambda = t_2^{|\lambda|}t_1^{\ell(\cdot)}J_\lambda|_{\alpha=-t_1/t_2} \tag{1.5}$$

is an eigenfunction of $\mathsf{M}_D(0)$ with eigenvalue $-c(\lambda;t_1,t_2) := -\sum_{(i,j)\in\lambda}[(j-1)t_1+(i-1)t_2]$. The coefficient of

$$|\mu\rangle \in \mathcal{F}\otimes_{\mathbb{C}}\mathbb{C}(t_1,t_2)$$

in the expansion of J^λ is $(t_1t_2)^{\ell(\mu)}$ times a polynomial in t_1 and t_2 of degree $|\lambda|-\ell(\mu)$.

The paper [23] also uses a Hermitian pairing $\langle -,-\rangle_H$ on the Fock space $\mathcal{F}$ defined by the three following properties

- $\langle\mu|\nu\rangle_H = \frac{1}{(t_1t_2)^{\ell(\mu)}}\frac{\delta_{\mu\nu}}{\mathfrak{z}(\mu)}$,
- $\langle af,g\rangle_H = a\langle f,g\rangle_H, \quad a\in\mathbb{C}(t_1,t_2)$,
- $\langle f,g\rangle_H = \overline{\langle g,f\rangle}_H$, where $\overline{a(t_1,t_2)} = a(-t_1,-t_2)$.

By a direct calculation, we find

$$\left\langle \mathsf{J}^\lambda,\mathsf{J}^\mu\right\rangle_H = \eta(\mathsf{J}^\lambda,\mathsf{J}^\mu)\,, \tag{1.6}$$

where η is the T-equivariant pairing on $\mathrm{Hilb}^n(\mathbb{C}^2)$. Since J^λ corresponds to the T-equivariant class of the T-fixed point of $\mathrm{Hilb}^n(\mathbb{C}^2)$ associated to λ,

$$||\mathsf{J}^\lambda||^2 = ||\mathsf{J}^\lambda||_H^2 = \prod_{\text{w: tangent weights at }\lambda}\mathsf{w} \tag{1.7}$$

see [23].

There are solutions to (1.1) of the form

$$\mathsf{Y}^\lambda(q) q^{-c(\lambda;t_1,t_2)}, \quad \mathsf{Y}^\lambda(q) \in \mathcal{F} \otimes_{\mathbb{C}} \mathbb{C}(t_1,t_2)[[q]],$$

which converge for $|q| < 1$ and satisfy $\mathsf{Y}^\lambda(0) = \mathsf{J}^\lambda$. We refer to [15, Chapter XIX] for a discussion of how these solutions are constructed.

By [23, Corollary 1],

$$\langle \mathsf{Y}^\lambda(q), \mathsf{Y}^\mu(q)\rangle_H = \delta_{\lambda\mu} ||\mathsf{J}^\lambda||_H^2 = \langle \mathsf{J}^\lambda, \mathsf{J}^\mu\rangle_H. \tag{1.8}$$

As in [23, Section 3.1.3], let Y be the matrix whose column vectors are Y^λ. Fix an auxiliary basis $\{e_\lambda\}$ of $\mathcal{F}$. We then view Y as the matrix representation[7] of an operator such that $\mathsf{Y}(e_\lambda) = \mathsf{Y}^\lambda$.

Define the following further diagonal matrices in the basis $\{e_\lambda\}$:

Matrix	Eigenvalues
L	$z^{-\|\lambda\|} \prod_{\mathsf{w}\text{: tangent weights at } \lambda} \mathsf{w}^{1/2}$
L_0	$q^{-c(\lambda;t_1,t_2)/z}$

Define

$$\mathsf{Y}_z = \mathsf{Y}\left(\frac{t_1}{z}, \frac{t_2}{z}, q\right).$$

Consider the following solution to (1.3),

$$\mathsf{S} = \Theta \mathsf{Y}_z L^{-1} L_0\,. \tag{1.9}$$

We may view S as the matrix representation of an operator where in the domain we use the basis $\{e_\lambda\}$ while in the range we use the basis $\{|\mu\rangle\}$.

Proposition 7. $\Theta\mathsf{Y}_z L^{-1}$ *can be expanded into a convergent power series in* $1/z$ *with coefficients* *End*$(\mathcal{F})$*-valued analytic functions in* q, t_1, t_2.

Proof. Let Φ^λ be the column of $\Theta\mathsf{Y}_z L^{-1}$ indexed by λ. By construction of Y,

$$\Theta \mathsf{Y}_z L^{-1}\Big|_{q=0} = \Theta \mathsf{J}_z L^{-1},$$

hence $\Phi^\lambda\Big|_{q=0} = \Theta \mathsf{J}_z^\lambda z^{|\lambda|} \prod_{\mathsf{w}\text{: tangent weights at } \lambda} \mathsf{w}^{-1/2}$. Write $\mathsf{J}^\lambda = \sum_\epsilon \mathsf{J}_\epsilon^\lambda(t_1,t_2)|\epsilon\rangle$. Then we have

$$\begin{aligned}\Theta \mathsf{J}_z^\lambda z^{|\lambda|} &= \sum_\epsilon \mathsf{J}_\epsilon^\lambda(t_1/z, t_2/z) z^{\ell(\epsilon)} z^{|\lambda|}|\epsilon\rangle \\ &= \sum_\epsilon \mathsf{J}_\epsilon^\lambda(t_1,t_2) z^{-2\ell(\epsilon)} z^{\ell(\epsilon)-|\lambda|} z^{\ell(\epsilon)} z^{|\lambda|}|\epsilon\rangle = \mathsf{J}^\lambda.\end{aligned}$$

Together with (1.7), we find $\Phi^\lambda\Big|_{q=0} = \mathsf{J}^\lambda/||\mathsf{J}^\lambda||$.

Since S is a solution to (1.3), Φ^λ is a solution to the differential equation

$$zq\frac{d}{dq}\Phi^\lambda = (\mathsf{M}_D + c(\lambda;t_1,t_2))\Phi^\lambda. \tag{1.10}$$

[7] In the domain of Y we use the basis $\{e_\lambda\}$, while in the range of Y we use the basis $\{|\mu\rangle\}$.

By uniqueness of solutions to (1.10) with given initial conditions, Φ^λ can also be constructed using the Peano-Baker series (see [1]) with the initial condition

$$\Phi^\lambda\Big|_{q=0} = \mathsf{J}^\lambda/||\mathsf{J}^\lambda|| \,.$$

As the Peano-Baker series is manifestly a power series in z^{-1} with analytic coefficients, the Proposition follows. □

2. Descendent Gromov-Witten theory

2.1. **Hilbert schemes.** Let $\mathsf{S}^{\mathrm{Hilb}}(q,t_D)$ be the generating series of genus 0 descendent Gromov-Witten invariants of $\mathsf{Hilb}^n(\mathbb{C}^2)$,

$$\eta(a,\mathsf{S}^{\mathrm{Hilb}}(q,t_D)b) = \eta(a,b) + \sum_{k\geq 0} z^{-1-k} \sum_{m,d} \frac{q^d}{m!} \langle a, \underbrace{t_D D, ..., t_D D}_{m}, b\psi^k_{m+2}\rangle_{0,d}^{\mathsf{Hilb}^n(\mathbb{C}^2)} \tag{2.1}$$

By definition, $\mathsf{S}^{\mathrm{Hilb}}$ is a formal power series in $1/z$ whose coefficients are in $\mathrm{End}(\mathcal{F})[t_D][[q]]$, written in the basis $\{|\mu\rangle\}$. $\mathsf{S}^{\mathrm{Hilb}}(q,t_D)$ satisfies the following two differential equations:

$$z\frac{\partial}{\partial t_D}\mathsf{S}^{\mathrm{Hilb}}(q,t_D) = (D\star_{t_D})\mathsf{S}^{\mathrm{Hilb}}(q,t_D), \tag{2.2}$$

$$zq\frac{\partial}{\partial q}\mathsf{S}^{\mathrm{Hilb}}(q,t_D) - z\frac{\partial}{\partial t_D}\mathsf{S}^{\mathrm{Hilb}}(q,t_D) = -\mathsf{S}^{\mathrm{Hilb}}(q,t_D)(D\cdot). \tag{2.3}$$

Here $(D\star_{t_D}) = (D\star_{t_D D})$ is the operator of quantum multiplication by the divisor D at the point[8] $t_D D$,

$$\eta((D\star_{t_D})a,b) = \sum_{m\geq 0, d\geq 0} \frac{q^d}{m!}\langle D, a, \underbrace{t_D D, ..., t_D D}_{m}, b\rangle_{0,d}^{\mathsf{Hilb}^n(\mathbb{C}^2)},$$

and $(D\cdot)$ is the operator of classical cup product by D. In particular,

$$(D\star_{t_D})\Big|_{t_D=0} = \mathsf{M}_D(q), \quad (D\cdot) = (D\cdot)\Big|_{t_D=0} = \mathsf{M}_D(0)\,. \tag{2.4}$$

Equation (2.2) follows from the topological recursion relations in genus 0. Equation (2.3) follows from the divisor equations for *descendent* Gromov-Witten invariants.

We first determine $\mathsf{S}^{\mathrm{Hilb}}\Big|_{t_D=0}$. Combining (2.2) and (2.3) and setting $t_D = 0$, we find

$$zq\frac{\partial}{\partial q}\left(\mathsf{S}^{\mathrm{Hilb}}\Big|_{t_D=0}\right) = \mathsf{M}_D(q)\left(\mathsf{S}^{\mathrm{Hilb}}\Big|_{t_D=0}\right) - \left(\mathsf{S}^{\mathrm{Hilb}}\Big|_{t_D=0}\right)\mathsf{M}_D(0)\,.$$

So, we see

$$\begin{aligned} zq\frac{\partial}{\partial q}\left(\mathsf{S}^{\mathrm{Hilb}}\Big|_{t_D=0}\mathsf{J}^\lambda/||\mathsf{J}^\lambda||\right) &= \mathsf{M}_D(q)\left(\mathsf{S}^{\mathrm{Hilb}}\Big|_{t_D=0}\mathsf{J}^\lambda/||\mathsf{J}^\lambda||\right) - \left(\mathsf{S}^{\mathrm{Hilb}}\Big|_{t_D=0}\right)\mathsf{M}_D(0)\mathsf{J}^\lambda/||\mathsf{J}^\lambda|| \\ &= \mathsf{M}_D(q)\left(\mathsf{S}^{\mathrm{Hilb}}\Big|_{t_D=0}\mathsf{J}^\lambda/||\mathsf{J}^\lambda||\right) + c(\lambda;t_1,t_2)\left(\mathsf{S}^{\mathrm{Hilb}}\Big|_{t_D=0}\mathsf{J}^\lambda/||\mathsf{J}^\lambda||\right). \end{aligned}$$

[8] We use t_D to denote the coordinate of D.

Since $\mathsf{S}^{\text{Hilb}}\Big|_{t_D=0,q=0} = \text{Id}$, we have $\left(\mathsf{S}^{\text{Hilb}}\Big|_{t_D=0}\mathsf{J}^\lambda/||\mathsf{J}^\lambda||\right)\Big|_{q=0} = \mathsf{J}^\lambda/||\mathsf{J}^\lambda||$. Comparing the result with the proof of Proposition 7, we conclude

$$\mathsf{S}^{\text{Hilb}}\Big|_{t_D=0}\mathsf{J}^\lambda/||\mathsf{J}^\lambda|| = \Phi^\lambda,$$

as $\mathcal{F}$-valued power series.

Let $\mathsf{A} : \mathcal{F} \to \mathcal{F}$ be defined by $\mathsf{A}(e_\lambda) = \mathsf{J}^\lambda/||\mathsf{J}^\lambda||$. The above discussion yields the following result.

Proposition 8. *As power series in $1/z$, we have* $\mathsf{S}^{\textit{Hilb}}\Big|_{t_D=0}\mathsf{A} = \mathsf{S}L_0^{-1}$.

By definition, S^{Hilb} is a formal power series in q. By Proposition 8, S^{Hilb} is analytic in q.

By the divisor equation for primary Gromov-Witten invariants, we have

$$q\frac{\partial}{\partial q}(D\star_{t_D}) - \frac{\partial}{\partial t_D}(D\star_{t_D}) = 0\,.$$

A direct calculation then shows that the two differential operators

$$z\frac{\partial}{\partial t_D} - (D\star_{t_D}) \quad \text{and} \quad zq\frac{\partial}{\partial q} - z\frac{\partial}{\partial t_D} - (-)(D\cdot)$$

commute. Therefore, equation (2.2) and Proposition 8 uniquely determine $\mathsf{S}^{\text{Hilb}}(q, t_D)$.

2.2. **Symmetric products.** We introduce another copy of the Fock space $\mathcal{F}$ which we denote by $\widetilde{\mathcal{F}}$. An additive isomorphism

$$\widetilde{\mathcal{F}} \otimes_{\mathbb{C}} \mathbb{C}[t_1, t_2] \simeq \bigoplus_{n\geq 0} H^*_{\mathsf{T}}(I\text{Sym}^n(\mathbb{C}^2), \mathbb{C})\,,$$

is given by identifying $|\mu\rangle \in \widetilde{\mathcal{F}}$ with the fundamental class $[I_\mu]$ of the component of the inertia orbifold $I\text{Sym}^n(\mathbb{C}^2)$ indexed by μ. The orbifold Poincaré pairing $(-,-)^{\mathsf{Sym}}$ induces via this identification a pairing on $\widetilde{\mathcal{F}}$,

$$\widetilde{\eta}(\mu, \nu) = \frac{1}{(t_1t_2)^{\ell(\mu)}}\frac{\delta_{\mu\nu}}{\mathfrak{z}(\mu)}.$$

Following [25, Equation (1.6)], we define

$$|\widetilde{\mu}\rangle = (-\sqrt{-1})^{\ell(\mu)-|\mu|}|\mu\rangle \in \widetilde{\mathcal{F}}.$$

We will use the following linear isomorphism

$$\mathsf{C} : \mathcal{F} \to \widetilde{\mathcal{F}}\,, \qquad |\mu\rangle \mapsto |\widetilde{\mu}\rangle\,, \tag{2.5}$$

which is compatible with the pairings η and $\widetilde{\eta}$.

We recall the definition of the ramified Gromov-Witten invariants of $\text{Sym}^n(\mathbb{C}^2)$ following [25, Section 3.2]. Consider the moduli space $\overline{\mathcal{M}}_{g,r+b}(\text{Sym}^n(\mathbb{C}^2))$ of stable maps to $\text{Sym}^n(\mathbb{C}^2)$ and let

$$\overline{\mathcal{M}}_{g,r,b}(\text{Sym}^n(\mathbb{C}^2)) = \left[\left(ev_{r+1}^{-1}(I_{(2)}) \cap ... \cap ev_{r+b}^{-1}(I_{(2)})\right)/\Sigma_b\right]$$

where the symmetric group Σ_b acts by permuting the last b marked points. Define ramified descendent Gromov-Witten invariants by

$$\left\langle \prod_{i=1}^{r} I_{\mu^i}\psi^{k_i} \right\rangle_{g,b}^{\mathrm{Sym}^n(\mathbb{C}^2)} = \int_{[\overline{\mathcal{M}}_{g,r,b}(\mathrm{Sym}^n(\mathbb{C}^2))]^{vir}} \prod_{i=1}^{r} ev_i^*([I_{\mu^i}])\psi^{k_i}\,.$$

Let $\mathsf{S}^{\mathsf{Sym}}(u,\tilde{t})$ be the generating function of genus 0 ramified descendent Gromov-Witten invariants of $\mathrm{Sym}^n(\mathbb{C}^2)$,

$$\tilde{\eta}(a,\mathsf{S}^{\mathsf{Sym}}(u,\tilde{t})b) = \tilde{\eta}(a,b) + \sum_{k\geq 0} z^{-1-k} \sum_{m,d} \frac{u^d}{m!}\langle a, \underbrace{\tilde{t}I_{(2)},\ldots,\tilde{t}I_{(2)}}_{m}, b\psi_{m+2}^k\rangle_{0,d}^{\mathrm{Sym}^n(\mathbb{C}^2)}. \tag{2.6}$$

By definition, $\mathsf{S}^{\mathsf{Sym}}$ is a formal power series in $1/z$ whose coefficients are in $\mathrm{End}(\widetilde{\mathcal{F}})[\tilde{t}][[u]]$, written in the basis $\{|\widetilde{\mu}\rangle\}$. $\mathsf{S}^{\mathsf{Sym}}$ satisfies the following two differential equations:

$$z\frac{\partial}{\partial\tilde{t}}\mathsf{S}^{\mathsf{Sym}}(u,\tilde{t}) = (I_{(2)}\star_{\tilde{t}})\mathsf{S}^{\mathsf{Sym}}(u,\tilde{t})\,, \tag{2.7}$$

$$\frac{\partial}{\partial u}\mathsf{S}^{\mathsf{Sym}}(u,\tilde{t}) = \frac{\partial}{\partial\tilde{t}}\mathsf{S}^{\mathsf{Sym}}(u,\tilde{t})\,. \tag{2.8}$$

Here $(I_{(2)}\star_{\tilde{t}}) = (I_{(2)}\star_{\tilde{t}I_{(2)}})$ is the operator of quantum multiplication by the divisor $I_{(2)}$ at the point $\tilde{t}I_{(2)}$,

$$\tilde{\eta}((I_{(2)}\star_{\tilde{t}})a,b) = \sum_{m,d}\frac{u^d}{m!}\langle I_{(2)}, a, \underbrace{\tilde{t}I_{(2)},\ldots,\tilde{t}I_{(2)}}_{m}, b\rangle_{0,d}^{\mathrm{Sym}^n(\mathbb{C}^2)}\,.$$

Equation (2.7) follows from the genus 0 topological recursion relations for orbifold Gromov-Witten invariants, see [26]. Equation (2.8) follows from divisor equations for *ramified* orbifold Gromov-Witten invariants, see [5].

We first compare the operators $(D\star_{t_D D})$ and $(I_{(2)}\star_{\tilde{t}I_{(2)}})$. For simplicity, write (2) for the partition $(2,1^{n-2})$. By [25, Theorem 4], we have

$$\begin{aligned}\langle D, \underbrace{D,\ldots,D}_{k}, \lambda,\mu\rangle^{\mathrm{Hilb}} &= (-1)^{k+1}\langle (2), \underbrace{(2),\ldots,(2)}_{k}, \lambda,\mu\rangle^{\mathrm{Hilb}}\\ &= (-1)^{k+1}\langle (\tilde{2}), \underbrace{(\tilde{2}),\ldots,(\tilde{2})}_{k}, \tilde{\lambda},\tilde{\mu}\rangle^{\mathrm{Sym}}\\ &= \langle -(\tilde{2}), \underbrace{-(\tilde{2}),\ldots,-(\tilde{2})}_{k}, \tilde{\lambda},\tilde{\mu}\rangle^{\mathrm{Sym}},\end{aligned}$$

where $(\tilde{-})$ is defined in [25, Equation (1.6)]. Therefore, under the identification $|\mu\rangle \mapsto |\tilde{\mu}\rangle$, we have

$$D\star_{t_D D} = -(\tilde{2})\star_{t_D(-(\tilde{2}))}\,. \tag{2.9}$$

Now,

$$(\tilde{2}) = (-i)^{n-1-n}I_{(2)} = (-i)^{-1}I_{(2)} = iI_{(2)}\,.$$

Hence we have, after $-q = e^{iu}$,

$$D\star_{t_D D} = (-i)I_{(2)}\star_{\tilde{t}I_{(2)}}, \quad \tilde{t} = (-i)t_D\,. \tag{2.10}$$

Consider now $\mathsf{S}^{\text{Sym}}\Big|_{\tilde{t}=0}$. By (2.7) and (2.8), we have

$$z\frac{\partial}{\partial u}\mathsf{S}^{\text{Sym}}(u,\tilde{t}) = (I_{(2)}\star_{\tilde{t}})\mathsf{S}^{\text{Sym}}(u,\tilde{t})\,.$$

Setting $\tilde{t}=0$ and using (2.4) and (2.10), we find

$$z\frac{\partial}{\partial u}\left(\mathsf{S}^{\text{Sym}}\Big|_{\tilde{t}=0}\right) = i\mathsf{M}_D(-e^{iu})\left(\mathsf{S}^{\text{Sym}}\Big|_{\tilde{t}=0}\right)\,.$$

Since $\frac{\partial}{\partial u} = iq\frac{\partial}{\partial q}$, we find that, after $-q=e^{iu}$,

$$zq\frac{\partial}{\partial q}\left(\mathsf{S}^{\text{Sym}}\Big|_{\tilde{t}=0}\right) = \mathsf{M}_D(q)\left(\mathsf{S}^{\text{Sym}}\Big|_{\tilde{t}=0}\right)\,. \tag{2.11}$$

Recall $\mathsf{S} = \Theta\mathsf{Y}_z L^{-1}L_0$ also satisfied the same equation. We may then compare $\Theta\mathsf{Y}_z L^{-1}L_0$ and $\left(\mathsf{S}^{\text{Sym}}\Big|_{\tilde{t}=0}\right)$ by comparing them at $u=0$ which corresponds to $q=-1$. Set

$$B = \mathsf{S}\Big|_{q=-1} = \Theta\mathsf{Y}_z L^{-1}L_0\Big|_{q=-1}\,.$$

Since $\mathsf{S}^{\text{Sym}}\Big|_{\tilde{t}=0,u=0} = \text{Id}$, we have, after $-q=e^{iu}$,

$$\mathsf{S}^{\text{Sym}}\Big|_{\tilde{t}=0} = \mathsf{C}\mathsf{S}B^{-1}\mathsf{C}^{-1}\,. \tag{2.12}$$

By Proposition 8, we have

$$\mathsf{C}\mathsf{S}B^{-1}\mathsf{C}^{-1} = \mathsf{C}\mathsf{S}^{\text{Hilb}}\Big|_{t_D=0}\mathsf{A}L_0B^{-1}\mathsf{C}^{-1}\,. \tag{2.13}$$

Since $\mathsf{A}L_0\mathsf{A}^{-1} = q^{D/z}$,

$$\mathsf{A}L_0B^{-1} = \mathsf{A}L_0\mathsf{A}^{-1}\mathsf{A}B^{-1} = q^{D/z}\mathsf{A}B^{-1}.$$

Define $\mathsf{K} = B\mathsf{A}^{-1}$. We can then rewrite (2.13) as

$$\mathsf{S}^{\text{Sym}}\Big|_{\tilde{t}=0} = \mathsf{C}\mathsf{S}^{\text{Hilb}}\Big|_{t_D=0}q^{D/z}\mathsf{K}^{-1}\mathsf{C}^{-1}\,. \tag{2.14}$$

By the divisor equation for orbifold Gromov-Witten invariants in [5] (see also [25, Section 3.2]), we have

$$\frac{\partial}{\partial u}(I_{(2)}\star_{\tilde{t}}) - \frac{\partial}{\partial \tilde{t}}(I_{(2)}\star_{\tilde{t}}) = 0\,.$$

A direct calculation then shows that the two differential operators

$$z\frac{\partial}{\partial \tilde{t}} - (I_{(2)}\star_{\tilde{t}}) \quad \text{and} \quad \frac{\partial}{\partial u} - \frac{\partial}{\partial \tilde{t}}$$

commute. Therefore $\mathsf{S}^{\text{Sym}}(u,\tilde{t})$ is uniquely determined by equation (2.7) and $\mathsf{S}^{\text{Sym}}\Big|_{\tilde{t}=0}$. By (2.10), we have

$$z\frac{\partial}{\partial t_D} - (D\star_{t_D}) = i\left(z\frac{\partial}{\partial \tilde{t}} - (I_{(2)}\star_{\tilde{t}})\right)\,,$$

after $-q=e^{iu}$. Then equation (2.14) implies the following result.

Theorem 9. *After* $-q = e^{iu}$ *and* $\tilde{t} = (-i)t_D$*, we have*

$$\mathsf{S}^{\mathrm{Sym}}(u, \tilde{t}) = \mathsf{C}\mathsf{S}^{\mathrm{Hilb}}(q, t_D) q^{D/z} \mathsf{K}^{-1} \mathsf{C}^{-1}.$$

2.3. **Proof of Theorem 1.** By the definition of B and Proposition 7, K is an $\mathrm{End}(\mathcal{F})$-valued power series in $1/z$ of the form

$$\mathsf{K} = \mathrm{Id} + O(1/z)\,.$$

By Theorem 9 and the fact that $\mathsf{S}^{\mathrm{Hilb}}$ and $\mathsf{S}^{\mathrm{Sym}}$ are symplectic, it follows that K is also symplectic.

Next, we explicitly evaluate K. By the definition of B and [23, Theorem 4], we have

$$\begin{aligned} B &= \left(\Theta \mathsf{Y}_z L^{-1} L_0\right)\Big|_{q=-1} \\ &= \frac{1}{(2\pi\sqrt{-1})^{|\cdot|}} \Theta \mathbf{\Gamma}_z \mathsf{H}_z \left(\mathsf{G}_{\mathrm{DT}z}^{-1} L_0\right)\Big|_{q=-1} L^{-1}\,. \end{aligned} \tag{2.15}$$

Here $|\cdot| = \sum_{k>0} \alpha_{-k}\alpha_k$ is the energy operator. G_{DT} is the diagonal matrix in the basis $\{e_\lambda\}$ with eigenvalues

$$q^{-c(\lambda;t_1,t_2)} \prod_{\text{w: tangent weights at } \lambda} \frac{1}{\Gamma(\mathsf{w}+1)}\,,$$

see [23, Section 3.1.2]. The operator $\mathbf{\Gamma}$ is given by

$$\mathbf{\Gamma}|\mu\rangle = \frac{(2\pi\sqrt{-1})^{\ell(\mu)}}{\prod_i \mu_i} \mathsf{G}_{\mathrm{GW}}(t_1, t_2)|\mu\rangle\,,$$

see [23, Section 3.3], where

$$\mathsf{G}_{\mathrm{GW}}(t_1, t_2)|\mu\rangle = \prod_i g(\mu_i, t_1) g(\mu_i, t_2)|\mu\rangle\,,$$

and

$$g(\mu_i, t_1) g(\mu_i, t_2) = \frac{\mu_i^{\mu_i t_1} \mu_i^{\mu_i t_2}}{\Gamma(\mu_i t_1)\Gamma(\mu_i t_2)}\,,$$

see [23, Section 3.1.2]. Define

$$\mathbf{\Gamma}_z = \mathbf{\Gamma}\left(\frac{t_1}{z}, \frac{t_2}{z}\right).$$

Since

$$\mathsf{K} = B\mathsf{A}^{-1} = \frac{1}{(2\pi\sqrt{-1})^{|\cdot|}} \Theta \mathbf{\Gamma}_z \mathsf{H}_z \left(\mathsf{G}_{\mathrm{DT}z}^{-1} L_0\right)\Big|_{q=-1} L^{-1} \mathsf{A}^{-1},$$

and $||\mathsf{J}^\lambda|| = \prod_{\text{w: tangent weights at } \lambda} \mathsf{w}^{1/2}$, we see that K is the operator given by

$$\mathsf{K}(\mathsf{J}^\lambda) = \frac{z^{|\lambda|}}{(2\pi\sqrt{-1})^{|\lambda|}} \prod_{\text{w: tangent weights at } \lambda} \Gamma(\mathsf{w}/z + 1) \Theta \mathbf{\Gamma}_z \mathsf{H}_z^\lambda\,. \tag{2.16}$$

The proof Theorem 1 is complete. ∎

3. Descendent correspondence

3.1. **Variables.** We compare the descendent Gromov-Witten theories of $\mathsf{Hilb}^n(\mathbb{C}^2)$ and $\mathrm{Sym}^n(\mathbb{C}^2)$. The following identifications will be used throughout:

$$-\,q = e^{iu}\,, \quad \tilde{t} = (-i)t_D\,. \tag{3.1}$$

3.2. **Genus** 0**.** Following [11], consider the Givental spaces

$$\mathcal{H}^{\mathrm{Hilb}} = H^*_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2)) \otimes_{\mathbb{C}[t_1,t_2]} \mathbb{C}(t_1,t_2)[[q]]((z^{-1}))\,,$$
$$\mathcal{H}^{\mathrm{Sym}} = H^*_{\mathsf{T}}(\mathrm{Sym}^n(\mathbb{C}^2)) \otimes_{\mathbb{C}[t_1,t_2]} \mathbb{C}(t_1,t_2)[[u]]((z^{-1}))\,,$$

equipped with the symplectic forms

$$(f,g)^{\mathcal{H}^{\mathrm{Hilb}}} = \mathrm{Res}_{z=0}(f(-z),g(z))^{\mathrm{Hilb}}\,, \quad f,g\in\mathcal{H}^{\mathrm{Hilb}}\,,$$
$$(f,g)^{\mathcal{H}^{\mathrm{Sym}}} = \mathrm{Res}_{z=0}(f(-z),g(z))^{\mathrm{Sym}}\,, \quad f,g\in\mathcal{H}^{\mathrm{Sym}}\,.$$

The choice of bases

$$\{|\mu\rangle \big| \mu\in \mathrm{Part}(n)\} \subset H^*_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2))\,, \qquad \{|\widetilde{\mu}\rangle \big| \mu\in \mathrm{Part}(n)\} \subset H^*_{\mathsf{T}}(\mathrm{Sym}^n(\mathbb{C}^2))\,,$$

yields Darboux coordinate systems $\{p^\mu_a, q^\nu_b\}$, $\{\widetilde{p}^\mu_a, \widetilde{q}^\nu_b\}$. General points of $\mathcal{H}^{\mathrm{Hilb}}, \mathcal{H}^{\mathrm{Sym}}$ can be written in the form

$$\underbrace{\sum_{a\geq 0}\sum_{\mu} p^\mu_a |\mu\rangle \frac{(t_1t_2)^{\ell(\mu)}\mathfrak{z}(\mu)}{(-1)^{|\mu|-\ell(\mu)}}(-z)^{-a-1}}_{\mathbf{p}} + \underbrace{\sum_{b\geq 0}\sum_{\nu} q^\nu_b|\nu\rangle z^b}_{\mathbf{q}} \in \mathcal{H}^{\mathrm{Hilb}}\,,$$

$$\underbrace{\sum_{a\geq 0}\sum_{\mu} \widetilde{p}^\mu_a |\widetilde{\mu}\rangle \frac{(t_1t_2)^{\ell(\mu)}\mathfrak{z}(\mu)}{1}(-z)^{-a-1}}_{\widetilde{\mathbf{p}}} + \underbrace{\sum_{b\geq 0}\sum_{\nu} \widetilde{q}^\nu_b|\widetilde{\nu}\rangle z^b}_{\widetilde{\mathbf{q}}} \in \mathcal{H}^{\mathrm{Sym}}\,.$$

Define the Lagrangian cones associated to the generating functions of genus 0 descendent and ancestor Gromov-Witten invariants as follows:

$$\mathcal{L}^{\mathrm{Hilb}} = \{(\mathbf{p}\,,\mathbf{q})\big|\mathbf{p} = d_{\mathbf{q}}\mathcal{F}^{\mathrm{Hilb}}_0\} \subset \mathcal{H}^{\mathrm{Hilb}}\,, \quad \mathcal{L}^{\mathrm{Hilb}}_{an,t_D} = \{(\mathbf{p},\mathbf{q})\big|\mathbf{p} = d_{\mathbf{q}}\mathcal{F}^{\mathrm{Hilb}}_{an,t_D,0}\} \subset \mathcal{H}^{\mathrm{Hilb}}\,,$$
$$\mathcal{L}^{\mathrm{Sym}} = \{(\widetilde{\mathbf{p}},\widetilde{\mathbf{q}})\big|\widetilde{\mathbf{p}} = d_{\widetilde{\mathbf{q}}}\mathcal{F}^{\mathrm{Sym}}_0\} \subset \mathcal{H}^{\mathrm{Sym}}\,, \quad \mathcal{L}^{\mathrm{Sym}}_{an,\tilde{t}} = \{(\widetilde{\mathbf{p}},\widetilde{\mathbf{q}})\big|\widetilde{\mathbf{p}} = d_{\widetilde{\mathbf{q}}}\mathcal{F}^{\mathrm{Sym}}_{a,\tilde{t},0}\} \subset \mathcal{H}^{\mathrm{Sym}}\,,$$

where

$$\mathcal{F}^{\mathrm{Hilb}}_0(\mathbf{t}) = \sum_{d,k\geq 0}\frac{q^d}{k!}\langle\underbrace{\mathbf{t}(\psi),...,\mathbf{t}(\psi)}_{k}\rangle^{\mathrm{Hilb}}_{0,d}\,, \quad \mathcal{F}^{\mathrm{Hilb}}_{an,t_D,0}(\mathbf{t}) = \sum_{d,k,l\geq 0}\frac{q^d}{k!l!}\langle\underbrace{\mathbf{t}(\bar{\psi}),...,\mathbf{t}(\bar{\psi})}_{k},\underbrace{t_DD,...,t_DD}_{l}\rangle^{\mathrm{Hilb}}_{0,d}\,,$$

$$\mathcal{F}^{\mathrm{Sym}}_0(\widetilde{\mathbf{t}}) = \sum_{b,k\geq 0}\frac{u^b}{k!}\langle\underbrace{\widetilde{\mathbf{t}}(\psi),...,\widetilde{\mathbf{t}}(\psi)}_{k}\rangle^{\mathrm{Sym}}_{0,b}\,, \quad \mathcal{F}^{\mathrm{Sym}}_{an,\tilde{t},0}(\widetilde{\mathbf{t}}) = \sum_{b,k,l\geq 0}\frac{u^b}{k!l!}\langle\underbrace{\widetilde{\mathbf{t}}(\bar{\psi}),...,\widetilde{\mathbf{t}}(\bar{\psi})}_{k},\underbrace{tI_{(2)},...,tI_{(2)}}_{l}\rangle^{\mathrm{Sym}}_{0,b}\,.$$

Here, $\mathbf{q} = \mathbf{t} - 1z$ and $\widetilde{\mathbf{q}} = \widetilde{\mathbf{t}} - 1z$ are dilaton shifts.

By the descendent/ancestor relations [8], we have

$$\mathcal{L}^{\mathrm{Hilb}} = \mathsf{S}^{\mathrm{Hilb}}(q,t_D)^{-1}\mathcal{L}^{\mathrm{Hilb}}_{an,t_D}\,, \quad \mathcal{L}^{\mathrm{Sym}} = \mathsf{S}^{\mathrm{Sym}}(u,\tilde{t})^{-1}\mathcal{L}^{\mathrm{Sym}}_{an,\tilde{t}}\,.$$

By the genus 0 crepant resolution correspondence proven[9] in [5], we have

$$\mathsf{C}\mathcal{L}^{\text{Hilb}}_{an,t_D} = \mathcal{L}^{\text{Sym}}_{an,\tilde{t}}\,.$$

Theorem 10. *We have* $\mathcal{L}^{Sym} = \mathsf{C}\mathsf{K}q^{-D/z}\mathcal{L}^{Hilb}$.

Proof. Using Theorem 9, we calculate

$$\begin{aligned}\mathcal{L}^{\text{Sym}} =& \mathsf{S}^{\text{Sym}}(u,\tilde{t})^{-1}\mathcal{L}^{\text{Sym}}_{an,\tilde{t}}\\ =& \mathsf{S}^{\text{Sym}}(u,\tilde{t})^{-1}\mathsf{C}\mathcal{L}^{\text{Hilb}}_{an,t_D}\\ =& \mathsf{C}\mathsf{K}q^{-D/z}\mathsf{S}^{\text{Hilb}}(q,t_D)^{-1}\mathcal{L}^{\text{Hilb}}_{an,t_D}\\ =& \mathsf{C}\mathsf{K}q^{-D/z}\mathcal{L}^{\text{Hilb}}\,.\end{aligned}$$

□

3.3. **Higher genus.** Consider the total descendent potentials,

$$\mathcal{D}^{\text{Hilb}} = \exp\left(\sum_{g\geq 0}\hbar^{g-1}\mathcal{F}^{\text{Hilb}}_g\right)\,,\quad \mathcal{F}^{\text{Hilb}}_g(\mathbf{t}) = \sum_{d,k\geq 0}\frac{q^d}{k!}\langle\underbrace{\mathbf{t}(\psi),...,\mathbf{t}(\psi)}_{k}\rangle^{\text{Hilb}}_{g,d}\,,$$

$$\mathcal{D}^{\text{Sym}} = \exp\left(\sum_{g\geq 0}\hbar^{g-1}\mathcal{F}^{\text{Sym}}_g\right)\,,\quad \mathcal{F}^{\text{Sym}}_g(\widetilde{\mathbf{t}}) = \sum_{b,k\geq 0}\frac{u^b}{k!}\langle\underbrace{\widetilde{\mathbf{t}}(\psi),...,\widetilde{\mathbf{t}}(\psi)}_{k}\rangle^{\text{Sym}}_{g,b}\,,$$

and the total ancestor potentials[10],

$$\mathcal{A}^{\text{Hilb}}_{an,t_D} = \exp\left(\sum_{g\geq 0}\hbar^{g-1}\mathcal{F}^{\text{Hilb}}_{an,t_D,g}\right)\,,\quad \mathcal{F}^{\text{Hilb}}_{an,t_D,g}(\mathbf{t}) = \sum_{d,k,l\geq 0}\frac{q^d}{k!l!}\langle\underbrace{\mathbf{t}(\bar{\psi}),...,\mathbf{t}(\bar{\psi})}_{k},\underbrace{t_D D,...,t_D D}_{l}\rangle^{\text{Hilb}}_{g,d}\,,$$

$$\mathcal{A}^{\text{Sym}}_{an,\tilde{t}} = \exp\left(\sum_{g\geq 0}\hbar^{g-1}\mathcal{F}^{\text{Sym}}_{an,\tilde{t},g}\right)\,,\quad \mathcal{F}^{\text{Sym}}_{an,\tilde{t},g}(\widetilde{\mathbf{t}}) = \sum_{b,k,l\geq 0}\frac{u^b}{k!l!}\langle\underbrace{\widetilde{\mathbf{t}}(\bar{\psi}),...,\widetilde{\mathbf{t}}(\bar{\psi})}_{k},\underbrace{tI_{(2)},...,tI_{(2)}}_{l}\rangle^{\text{Sym}}_{g,b}\,.$$

Givental's quantization formalism [11] produces differential operators by quantizing quadratic Hamiltonians associated to linear symplectic transforms by the following rules:

$$\widehat{q^\mu_a q^\nu_b} = \frac{q^\mu_a q^\nu_b}{\hbar}, \widehat{q^\mu_a p^\nu_b} = q^\mu_a\frac{\partial}{\partial q^\nu_b}, \widehat{p^\mu_a p^\nu_b} = \hbar\frac{\partial}{\partial q^\mu_a}\frac{\partial}{\partial q^\nu_b}\,,$$

$$\widehat{\widetilde{q}^\mu_a \widetilde{q}^\nu_b} = \frac{\widetilde{q}^\mu_a \widetilde{q}^\nu_b}{\hbar}, \widehat{\widetilde{q}^\mu_a \widetilde{p}^\nu_b} = \widetilde{q}^\mu_a\frac{\partial}{\partial \widetilde{q}^\nu_b}, \widehat{\widetilde{p}^\mu_a \widetilde{p}^\nu_b} = \hbar\frac{\partial}{\partial \widetilde{q}^\mu_a}\frac{\partial}{\partial \widetilde{q}^\nu_b}\,.$$

By the descendent/ancestor relations [8], we have

$$\mathcal{D}^{\text{Hilb}} = e^{F^{\text{Hilb}}_1(t_D)}\widehat{\mathsf{S}^{\text{Hilb}}(q,t_D)}^{-1}\mathcal{A}^{\text{Hilb}}_{an,t_D}\,,$$

$$\mathcal{D}^{\text{Sym}} = e^{F^{\text{Sym}}_1(\tilde{t})}\widehat{\mathsf{S}^{\text{Sym}}(u,\tilde{t})}^{-1}\mathcal{A}^{\text{Sym}}_{an,\tilde{t}}\,,$$

[9]In particular, the results of [5] implies that $\mathcal{L}^{\text{Hilb}}_{an,t_D}$ is analytic in q.

[10]The results of [25] imply that $\mathcal{A}^{\text{Hilb}}_{an,t_D}$ depends analytically in q.

where F_1^{Hilb} and F_1^{Sym} are generating functions of genus 1 primary invariants with insertions D and $I_{(2)}$ respectively. F_1^{Sym} and F_1^{Hilb} can be easily matched using [25, Theorem 4].

Theorem 11. *We have* $e^{-F_1^{Sym}(\tilde{t})}\mathcal{D}^{Sym} = \widehat{\mathsf{C}}\widehat{\mathsf{K}q^{-D/z}}\left(e^{-F_1^{Hilb}(t_D)}\mathcal{D}^{Hilb}\right)$.

Proof. By [25, Theorem 4], we have $\widehat{\mathsf{C}}\mathcal{A}^{\mathrm{Hilb}}_{an,t_D} = \mathcal{A}^{\mathrm{Sym}}_{an,\tilde{t}}$. Using Theorem 9, we calculate

$$\widehat{\mathsf{S}^{\mathrm{Sym}}(u,\tilde{t})}^{-1}\mathcal{A}^{\mathrm{Sym}}_{an,\tilde{t}} = \widehat{\mathsf{C}}\widehat{\mathsf{K}q^{-D/z}}\widehat{\mathsf{S}^{\mathrm{Hilb}}(q,t_D)}^{-1}\mathcal{A}^{\mathrm{Hilb}}_{an,t_D}\,.$$

Therefore, we conclude

$$\begin{aligned} e^{-F_1^{\mathrm{Sym}}(\tilde{t})}\mathcal{D}^{\mathrm{Sym}} &= \widehat{\mathsf{S}^{\mathrm{Sym}}(u,\tilde{t})}^{-1}\mathcal{A}^{\mathrm{Sym}}_{an,\tilde{t}} \\ &= \widehat{\mathsf{C}}\widehat{\mathsf{K}q^{-D/z}}\widehat{\mathsf{S}^{\mathrm{Hilb}}(q,t_D)}^{-1}\mathcal{A}^{\mathrm{Hilb}}_{an,t_D} \\ &= \widehat{\mathsf{C}}\widehat{\mathsf{K}q^{-D/z}}\left(e^{-F_1^{\mathrm{Hilb}}(t_D)}\mathcal{D}^{\mathrm{Hilb}}\right)\,. \end{aligned}$$

□

4. Fourier-Mukai transformation

4.1. Proof of Theorem 4. We first localize the top row of the diagram of Theorem 4:

$$\begin{array}{ccc} K_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2))_{\mathrm{loc}} & \xrightarrow{\mathbb{FM}} & K_{\mathsf{T}}(\mathsf{Sym}^n(\mathbb{C}^2))_{\mathrm{loc}} \\ \Psi^{\mathrm{Hilb}}\big\downarrow & & \big\downarrow\Psi^{\mathrm{Sym}} \\ \widetilde{\mathcal{H}}^{\mathrm{Hilb}} & \xrightarrow{\;\mathsf{CK}\big|_{z\mapsto -z}\;} & \widetilde{\mathcal{H}}^{\mathrm{Sym}}\,. \end{array}$$

Here, loc denotes tensoring by $\mathrm{Frac}(R(\mathsf{T}))$, the field of fractions of the representation ring $R(\mathsf{T})$ of the torus T. The maps Ψ^{Hilb} and Ψ^{Sym} are still well-defined since the T-equivariant Chern character of a representation is invertible. The commutation of the above diagram immediately implies the commutation of the diagram of Theorem 4.

Let $k_\lambda \in K_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2))$ be the skyscraper sheaf supported on the fixed point indexed by λ. The set $\{k_\lambda \big| \lambda \in \mathrm{Part}(n)\}$ is a basis of $K_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2))_{\mathrm{loc}}$ as a $\mathrm{Frac}(R(\mathsf{T}))$-vector space. The commutation of the localized diagram is then a consequence of the following equality: for all $\lambda \in \mathrm{Part}(n)$,

$$\mathsf{CK}\big|_{z\mapsto -z} \circ \Psi^{\mathrm{Hilb}}(k_\lambda) = \Psi^{\mathrm{Sym}} \circ \mathbb{FM}(k_\lambda)\,. \tag{4.1}$$

To prove (4.1), we will match the two sides by explicit calculation.

4.2. Iritani's Gamma class. For a vector bundle $\mathcal{V}$ on a Deligne-Mumford stack $\mathcal{X}$,

$$\mathcal{V} \to \mathcal{X}\,,$$

Iritani has defined a characteristic class called the *Gamma class*. Let

$$I\mathcal{X} = \coprod_i \mathcal{X}_i$$

be the decomposition of the inertia stack $I\mathcal{X}$ into connected components. By pulling back $\mathcal{V}$ to $I\mathcal{X}$ and restricting to $\mathcal{X}_i$, we obtain a vector bundle $\mathcal{V}\big|_{\mathcal{X}_i}$ on $\mathcal{X}_i$. The stabilizer element g_i of $\mathcal{X}$

associated to the component $\mathcal{X}_i$ acts on $\mathcal{V}_{\mathcal{X}_i}$. The bundle $\mathcal{V}\big|_{\mathcal{X}_i}$ decomposes under g_i into a direct sum of eigenbundles

$$\mathcal{V}\big|_{\mathcal{X}_i} = \oplus_{0\leq f<1}\mathcal{V}_{i,f}\,,$$

where g_i acts on $\mathcal{V}_{i,f}$ by multiplication by $\exp(2\pi\sqrt{-1}f)$. The orbifold Chern character of $\mathcal{V}$ is defined to be

$$\widetilde{\mathrm{ch}}(\mathcal{V}) = \bigoplus_i \sum_{0\leq f<1} \exp(2\pi\sqrt{-1}f)\,\mathrm{ch}(\mathcal{V}_{i,f}) \in H^*(I\mathcal{X})\,, \tag{4.2}$$

where $\mathrm{ch}(-)$ is the usual Chern character.

For each i and f, let $\delta_{i,f,j}$, for $1\leq j\leq \mathrm{rank}\,\mathcal{V}_{i,f}$, be the Chern roots of $\mathcal{V}_{i,f}$. Iritani's Gamma class[11] is defined to be

$$\Gamma(\mathcal{V}) = \bigoplus_i \prod_{0\leq f<1} \prod_{j=1}^{\mathrm{rank}\,\mathcal{V}_{i,f}} \Gamma(1-f+\delta_{i,f,j})\,. \tag{4.3}$$

As usual, $\Gamma_{\mathcal{X}} = \Gamma(T\mathcal{X})$.

If the vector bundle $\mathcal{V}$ is equivariant with respect to a T-action, the Chern character and Chern roots above should be replaced by their equivariant counterparts to define a T-equivariant Gamma class.

If $\mathcal{X}$ is a scheme, then the Gamma class simplifies considerably since there are no stabilizers. Directly from the definition, the restriction of Γ_{Hilb} to the fixed point indexed by λ is

$$\Gamma_{\mathsf{Hilb}}\Big|_\lambda = \prod_{\mathsf{w}:\text{ tangent weights at }\lambda} \Gamma(\mathsf{w}+1)\,.$$

Recall that the inertia stack $I\mathrm{Sym}^n(\mathbb{C}^2)$ is a disjoint union indexed by conjugacy classes of S_n. For a partition μ of n, the component $I_\mu\subset I\mathrm{Sym}^n(\mathbb{C}^2)$ indexed by the conjugacy class of cycle type μ is the stack quotient

$$[\mathbb{C}^{2n}_\sigma/C(\sigma)]\,,$$

where $\sigma\in S_n$ has cycle type μ, $\mathbb{C}^{2n}_\sigma\subset\mathbb{C}^{2n}$ is the σ-invariant part, and $C(\sigma)\subset S_n$ is the centralizer of σ.

Lemma 12. *The restriction of* Γ_{Sym} *to the component* I_μ *is given by*

$$\Gamma_{\mathsf{Sym}}\Big|_\mu = (t_1t_2)^{\ell(\mu)}(2\pi)^{n-\ell(\mu)}\left(\prod_i \mu_i\right)\left(\prod_i \mu_i^{-\mu_i t_1}\mu_i^{-\mu_i t_2}\right)\left(\prod_i \Gamma(\mu_i t_1)\Gamma(\mu_i t_2)\right).$$

Proof. Using the description of eigenspaces of $T_{\mathsf{Sym}^n(\mathbb{C}^2)}$ on the component of $I\mathrm{Sym}^n(\mathbb{C}^2)$ indexed by μ (see [25, Section 6.2]), we find that

$$\Gamma_{\mathsf{Sym}}\Big|_\mu = \prod_i\prod_{l=0}^{\mu_i-1}\Gamma\left(1-\frac{l}{\mu_i}+t_1\right)\Gamma\left(1-\frac{l}{\mu_i}+t_2\right).$$

[11]The substitution of cohomology classes into Gamma function makes sense because the Gamma function $\Gamma(1+x)$ has a power series expansion at $x=0$.

Using the formula

$$\prod_{k=0}^{m-1}\Gamma\left(z+\frac{k}{m}\right)=(2\pi)^{\frac{m-1}{2}}m^{\frac{1}{2}-mz}\Gamma(mz)\,,$$

we find

$$\prod_{l=0}^{\mu_i-1}\Gamma\left(1-\frac{l}{\mu_i}+t_1\right)=t_1(2\pi)^{\frac{\mu_i-1}{2}}\mu_i^{\frac{1}{2}-\mu_i t_1}\Gamma(\mu_i t_1)\,,$$

and similarly for the other factor. Therefore,

$$\Gamma_{\mathsf{Sym}}\Big|_{\mu}=(t_1t_2)^{\ell(\mu)}(2\pi)^{n-\ell(\mu)}\left(\prod_i \mu_i\right)\left(\prod_i \mu_i^{-\mu_i t_1}\mu_i^{-\mu_i t_2}\right)\left(\prod_i \Gamma(\mu_i t_1)\Gamma(\mu_i t_2)\right)\,,$$

which is the desired formula. □

4.3. **Calculation of $\mathsf{CK}\circ\Psi^{\mathbf{Hilb}}$.** Since k_λ is supported at the T-fixed point of $\mathsf{Hilb}^n(\mathbb{C}^2)$ indexed by λ, the T-equivariant Chern character $\mathrm{ch}(k_\lambda)$ is also supported there. Using the Koszul resolution (or Grothendieck-Riemann-Roch), we calculate

$$\mathrm{ch}(k_\lambda)=\mathsf{J}^\lambda\prod_{\mathsf{w}:\text{ tangent weights at }\lambda}\frac{1-e^{-\mathsf{w}}}{\mathsf{w}}\,. \tag{4.4}$$

We have used the fact that the class of the T-fixed point of $\mathsf{Hilb}^n(\mathbb{C}^2)$ indexed by λ corresponds to the factor

$$\frac{\mathsf{J}^\lambda}{\prod_{\mathsf{w}}\mathsf{w}}\,.$$

By the definition of $\deg_0^{\mathrm{Hilb}}$, we have

$$(2\pi\sqrt{-1})^{\frac{\deg_0^{\mathrm{Hilb}}}{2}}\mathrm{ch}(k_\lambda)=\frac{(2\pi\sqrt{-1})^{\frac{\deg_0^{\mathrm{Hilb}}}{2}}\mathsf{J}^\lambda}{\prod_{\mathsf{w}}2\pi\sqrt{-1}\mathsf{w}}\prod_{\mathsf{w}:\text{ tangent weights at }\lambda}(1-e^{-2\pi\sqrt{-1}\mathsf{w}})\,.$$

Write $\mathsf{J}^\lambda=\sum_\epsilon \mathsf{J}^\lambda_\epsilon(t_1,t_2)|\epsilon\rangle$. Since $\mathsf{J}^\lambda_\epsilon$ is $(t_1t_2)^{\ell(\epsilon)}$ times a homogeneous polynomial in t_1,t_2 of degree $n-\ell(\epsilon)$, we have[12]

$$\begin{aligned}(2\pi\sqrt{-1})^{\frac{\deg_0^{\mathrm{Hilb}}}{2}}\mathsf{J}^\lambda&=\sum_\epsilon(2\pi\sqrt{-1})^{\frac{\deg_0^{\mathrm{Hilb}}}{2}}\mathsf{J}^\lambda_\epsilon(t_1,t_2)|\epsilon\rangle\\&=\sum_\epsilon \mathsf{J}^\lambda_\epsilon(2\pi\sqrt{-1}t_1,2\pi\sqrt{-1}t_2)(2\pi\sqrt{-1})^{n-\ell(\epsilon)}|\epsilon\rangle\\&=\sum_\epsilon \mathsf{J}^\lambda_\epsilon(t_1,t_2)(2\pi\sqrt{-1})^{n+\ell(\epsilon)}(2\pi\sqrt{-1})^{n-\ell(\epsilon)}|\epsilon\rangle\\&=(2\pi\sqrt{-1})^{2n}\sum_\epsilon \mathsf{J}^\lambda_\epsilon(t_1,t_2)|\epsilon\rangle\\&=(2\pi\sqrt{-1})^{2n}\mathsf{J}^\lambda.\end{aligned}$$

[12]The calculation also follows from the fact that J^λ is the class a T-fixed point (of real degree $4n$).

After putting the above formulas together, we obtain

$$\Gamma_{\text{Hilb}} \cup (2\pi\sqrt{-1})^{\frac{\deg_0^{\text{Hilb}}}{2}} \text{ch}(k_\lambda) = \frac{(2\pi\sqrt{-1})^{2n} \mathsf{J}^\lambda}{\prod_{\mathsf{w}} 2\pi\sqrt{-1}\mathsf{w}} \prod_{\text{w: tangent weights at } \lambda} \Gamma(\mathsf{w}+1)(1-e^{-2\pi\sqrt{-1}\mathsf{w}})\,.$$

Recall the following identity for the Gamma function:

$$\Gamma(1+t)\Gamma(1-t) = \frac{2\pi\sqrt{-1}t}{e^{\pi\sqrt{-1}t} - e^{-\pi\sqrt{-1}t}}\,. \tag{4.5}$$

We have

$$\begin{aligned}\Gamma(\mathsf{w}+1)(1-e^{-2\pi\sqrt{-1}\mathsf{w}}) =& \Gamma(\mathsf{w}+1)(e^{\pi\sqrt{-1}\mathsf{w}} - e^{-\pi\sqrt{-1}\mathsf{w}})(e^{-\pi\sqrt{-1}\mathsf{w}}) \\ =& \frac{2\pi\sqrt{-1}\mathsf{w}}{\Gamma(1-\mathsf{w})}(e^{-\pi\sqrt{-1}\mathsf{w}})\,.\end{aligned}$$

Hence

$$\Gamma_{\text{Hilb}} \cup (2\pi\sqrt{-1})^{\frac{\deg_0^{\text{Hilb}}}{2}} \text{ch}(k_\lambda) = ((2\pi\sqrt{-1})^{2n} \mathsf{J}^\lambda) \prod_{\text{w: tangent weights at } \lambda} \frac{1}{\Gamma(1-\mathsf{w})} e^{-\pi\sqrt{-1}\mathsf{w}}\,.$$

Since the operator $z^{\rho^{\text{Hilb}}}$ is the operator of multiplication by $z^{c_1^{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2))}$, we have

$$\begin{aligned} & z^{\rho^{\text{Hilb}}} \left(\Gamma_{\text{Hilb}} \cup (2\pi\sqrt{-1})^{\frac{\deg_0^{\text{Hilb}}}{2}} \text{ch}(k_\lambda) \right) \\ &= z^{n(t_1+t_2)} ((2\pi\sqrt{-1})^{2n} \mathsf{J}^\lambda) \prod_{\text{w: tangent weights at } \lambda} \frac{1}{\Gamma(1-\mathsf{w})} e^{-\pi\sqrt{-1}\mathsf{w}} \\ &= z^{n(t_1+t_2)} e^{-\pi\sqrt{-1}n(t_1+t_2)} ((2\pi\sqrt{-1})^{2n} \mathsf{J}^\lambda) \prod_{\text{w: tangent weights at } \lambda} \frac{1}{\Gamma(1-\mathsf{w})}\,, \end{aligned}$$

where we use

$$c_1^{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2))\Big|_\lambda = \sum_{\text{w: tangent weights at } \lambda} \mathsf{w} = n(t_1+t_2)\,.$$

By the definition of μ^{Hilb}, we have

$$z^{-\mu^{\text{Hilb}}}(\phi) = z^n z^{-\deg_0^{\text{Hilb}}/2}(\phi) = z^n (\frac{\phi}{z^{k/2}})$$

for $\phi \in H^k_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2), \mathbb{C})$, we have

$$\begin{aligned} & z^{-\mu^{\text{Hilb}}} z^{\rho^{\text{Hilb}}} \left(\Gamma_{\text{Hilb}} \cup (2\pi\sqrt{-1})^{\frac{\deg_0^{\text{Hilb}}}{2}} \text{ch}(k_\lambda) \right) \\ &= z^n z^{n(t_1+t_2)/z} e^{-\pi\sqrt{-1}n(t_1+t_2)/z} \left(\frac{2\pi\sqrt{-1}}{z} \right)^{2n} \mathsf{J}^\lambda \prod_{\text{w: tangent weights at } \lambda} \frac{1}{\Gamma(1-\mathsf{w}/z)}\,. \end{aligned}$$

Here, the operator $z^{-\deg_0^{\mathrm{Hilb}}/2}$ acts on $z^{n(t_1+t_2)}$ as follows:

$$\begin{aligned} z^{-\deg_0^{\mathrm{Hilb}}/2}\left(z^{n(t_1+t_2)}\right) =& z^{-\deg_0^{\mathrm{Hilb}}/2}\left(e^{n(t_1+t_2)\log z}\right) \\ =& z^{-\deg_0^{\mathrm{Hilb}}/2}\left(\sum_{k\geq 0}\frac{(n(t_1+t_2)\log z)^k}{k!}\right) \\ =& \sum_{k\geq 0}\frac{(n\log z)^k z^{-\deg_0^{\mathrm{Hilb}}/2}((t_1+t_2)^k)}{k!} \\ =& \sum_{k\geq 0}\frac{(n\log z)^k((t_1+t_2)^k/z^k)}{k!} \\ =& \sum_{k\geq 0}\frac{(n\log z((t_1+t_2)/z))^k}{k!} \\ =& z^{n(t_1+t_2)/z}. \end{aligned}$$

The actions of $z^{-\deg_0^{\mathrm{Hilb}}/2}$ on $e^{-\pi\sqrt{-1}n(t_1+t_2)}$ and $\Gamma(1+\mathsf{w})$ are similarly determined.

By Equation (2.16), we have

$$\mathsf{K}\big|_{z\mapsto -z}(\mathsf{J}^\lambda) = \frac{(-z)^{|\lambda|}}{(2\pi\sqrt{-1})^{|\lambda|}}\left(\prod_{\mathsf{w}\text{: tangent weights at }\lambda}\Gamma(-\mathsf{w}/z+1)\right)\Theta'\boldsymbol{\Gamma}_{-z}\mathsf{H}^\lambda_{-z}\,,$$

where we define $\Theta'|\mu\rangle = (-z)^{\ell(\mu)}|\mu\rangle$. Hence,

$$\begin{aligned} &\mathsf{K}\big|_{z\mapsto -z}\left(z^{-\mu^{\mathrm{Hilb}}}z^{\rho^{\mathrm{Hilb}}}\left(\Gamma_{\mathrm{Hilb}}\cup(2\pi\sqrt{-1})^{\frac{\deg_0^{\mathrm{Hilb}}}{2}}\mathrm{ch}(k_\lambda)\right)\right) \\ =& z^n z^{n(t_1+t_2)/z}e^{-\pi\sqrt{-1}n(t_1+t_2)/z}\left(\frac{2\pi\sqrt{-1}}{z}\right)^{2n}\mathsf{K}\big|_{z\mapsto -z}(\mathsf{J}^\lambda)\prod_{\mathsf{w}\text{: tangent weights at }\lambda}\frac{1}{\Gamma(1-\mathsf{w}/z)} \\ =& z^n z^{n(t_1+t_2)/z}e^{-\pi\sqrt{-1}n(t_1+t_2)/z}\left(\frac{2\pi\sqrt{-1}}{z}\right)^{2n}\frac{(-z)^{|\lambda|}}{(2\pi\sqrt{-1})^{|\lambda|}}\Theta'\boldsymbol{\Gamma}_{-z}\mathsf{H}^\lambda_{-z}\prod_{\mathsf{w}\text{: tangent weights at }\lambda}\frac{\Gamma(-\mathsf{w}/z+1)}{\Gamma(1-\mathsf{w}/z)} \\ =& (-1)^n z^n z^{n(t_1+t_2)/z}e^{-\pi\sqrt{-1}n(t_1+t_2)/z}\left(\frac{2\pi\sqrt{-1}}{z}\right)^{n}\Theta'\boldsymbol{\Gamma}_{-z}\mathsf{H}^\lambda_{-z}\,. \end{aligned}$$

By the definition of $\boldsymbol{\Gamma}_{-z}$, we have

$$\boldsymbol{\Gamma}_{-z}|\mu\rangle = \frac{(2\pi\sqrt{-1})^{\ell(\mu)}}{\prod_i\mu_i}\prod_i\frac{\mu_i^{-\mu_i t_1/z}\mu_i^{-\mu_i t_2/z}}{\Gamma(-\mu_i t_1/z)\Gamma(-\mu_i t_2/z)}|\mu\rangle\,.$$

Also, $\mathsf{C}|\mu\rangle = |\widetilde{\mu}\rangle$, we thus obtain

$$\mathsf{CK}\big|_{z\mapsto -z}\left(z^{-\mu^{\mathrm{Hilb}}}z^{\rho^{\mathrm{Hilb}}}\left(\Gamma_{\mathrm{Hilb}}\cup(2\pi\sqrt{-1})^{\frac{\deg_0^{\mathrm{Hilb}}}{2}}\mathrm{ch}(k_\lambda)\right)\right) = \Delta^{\mathrm{Hilb}}(\mathsf{H}^\lambda_{-z})\,, \tag{4.6}$$

where $\Delta^{\text{Hilb}} : \mathcal{F} \to \widetilde{\mathcal{F}}$ is the operator defined as follows:

$$\begin{aligned}
&\Delta^{\text{Hilb}}|\mu\rangle \\
&=(-1)^n z^n z^{n(t_1+t_2)/z} e^{-\pi\sqrt{-1}n(t_1+t_2)/z} \left(\frac{2\pi\sqrt{-1}}{z}\right)^n (-z)^{\ell(\mu)} \frac{(2\pi\sqrt{-1})^{\ell(\mu)}}{\prod_i \mu_i} \prod_i \frac{\mu_i^{-\mu_i t_1/z} \mu_i^{-\mu_i t_2/z}}{\Gamma(-\mu_i t_1/z)\Gamma(-\mu_i t_2/z)} |\widetilde{\mu}\rangle \\
&=(-1)^{n+\ell(\mu)} z^{n(t_1+t_2)/z} e^{-\pi\sqrt{-1}n(t_1+t_2)/z} (2\pi\sqrt{-1})^{n+\ell(\mu)} z^{\ell(\mu)} \frac{1}{\prod_i \mu_i} \prod_i \frac{\mu_i^{-\mu_i t_1/z} \mu_i^{-\mu_i t_2/z}}{\Gamma(-\mu_i t_1/z)\Gamma(-\mu_i t_2/z)} |\widetilde{\mu}\rangle \,.
\end{aligned} \tag{4.7}$$

4.4. **Haiman's result.** The homomorphism $\mathbb{FM}$ has been calculated by Haiman [12, 13]. Denote by F the operator of taking Frobenius series of bigraded S_n-modules, as defined in [12, Definition 3.2.3]. Note that T-equivariant sheaves on

$$\text{Sym}^n(\mathbb{C}^2) = [(\mathbb{C}^2)^n/S_n]$$

are $\mathsf{T} \times S_n$-equivariant sheaves on $\mathbb{C}^2$, and hence can be identified with bigraded S_n-equivariant $\mathbb{C}[\mathbf{x}, \mathbf{y}]$-modules[13]. Therefore, the composition

$$\Phi = F \circ \mathbb{FM}$$

makes sense and takes values in a certain algebra of symmetric functions, see [12, Proposition 5.4.6]. For the analysis of the diagram of Theorem 4, we will need the following result of Haiman.

Theorem 13 ([12], Equation (95)). *Let $k_\lambda \in K_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2))$ be the skyscraper sheaf supported on the T-fixed point indexed by λ. Then*

$$\Phi(k_\lambda) = \widetilde{H}_\lambda(z; q, t)\,.$$

The Macdonald polynomial $\widetilde{H}_\lambda(z; q, t)$ is a symmetric function in an infinite set of variables

$$z = \{z_1, z_2, z_3, ...\}$$

and depends on two parameters q, t. As explained in [25, Section 9.1], $\widetilde{H}_\lambda(z; q, t)$ of [12] is the same as H^λ after the following identification: the parameters (q, t) and (t_1, t_2) are related by

$$(q, t) = (e^{2\pi\sqrt{-1}t_1}, e^{2\pi\sqrt{-1}t_2})\,.$$

Symmetric functions in z are viewed as elements of $\widetilde{\mathcal{F}}$ via the following convention. For a partition μ, the power-sum symmetric function

$$p_\mu = \prod_k \Big(\sum_{i\geq 1} z_i^{\mu_k}\Big)$$

is identified with $\mathfrak{z}(\mu)|\mu\rangle$.

To make use of Haiman's result, we must compare the operator F taking Frobenius series with the orbifold Chern character $\widetilde{\text{ch}}$. Let V^λ be the irreducible S_n-representation indexed by $\lambda \in \text{Part}(n)$. We construct the bigraded S_n-equivariant $\mathbb{C}[\mathbf{x}, \mathbf{y}]$-module $V^\lambda \otimes \mathbb{C}[\mathbf{x}, \mathbf{y}]$, which is equivalent to a T-equivariant sheaf $\mathcal{V}^\lambda$ on $\text{Sym}^n(\mathbb{C}^2)$. Define the operator $\delta : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}}$ by

$$\delta|\mu\rangle = \prod_i (1 - q^{\mu_i})(1 - t^{\mu_i})|\mu\rangle\,.$$

[13]Here, $\mathbf{x} = \{x_1, ..., x_n\}$ and $\mathbf{y} = \{y_1, ..., y_n\}$.

By [12, Section 5.4.3], we have

$$F_{V^\lambda\otimes\mathbb{C}[\mathbf{x},\mathbf{y}]} = s_\lambda\Big[\frac{Z}{(1-q)(1-t)}\Big],$$

where s_λ is the Schur function. Here Z denotes the collection of variables $z_1, z_2, \ldots$ that the functions are symmetric with respect to, according to the convention of [12]. Using the definition of plethystic substitution $Z \mapsto Z/(1-q)(1-t)$, see [12, Section 3.3], we obtain

$$\delta(F_{V^\lambda\otimes\mathbb{C}[\mathbf{x},\mathbf{y}]}) = s_\lambda.$$

On the other hand, by the definition of orbifold Chern character[14] recalled in Equation (4.2), we have

$$\widetilde{\mathrm{ch}}(\mathcal{V}^\lambda) = s_\lambda\,.$$

Since $K_\mathsf{T}(\mathrm{Sym}^n(\mathbb{C}^2))$ is freely spanned as a $R(T)$-module by $V^\lambda\otimes\mathbb{C}[\mathbf{x},\mathbf{y}]$, we find

$$\delta\circ F = \widetilde{\mathrm{ch}}\,,$$

after identifying[15] $q=e^{-t_1}, t=e^{-t_2}$. Therefore,

$$\begin{aligned}\widetilde{\mathrm{ch}}(\mathbb{FM}(k_\lambda)) =&\delta(F(\mathbb{FM}(k_\lambda)))\\ =&\delta(\Phi(k_\lambda))\\ =&\delta(\widetilde{H}_\lambda)\,, \quad q=e^{-t_1}\,,\ \ t=e^{-t_2}.\end{aligned}$$

4.5. **Calculation of $\Psi^{\mathrm{Sym}}\circ\mathbb{FM}$.** We have

$$(2\pi\sqrt{-1})^{\frac{\deg_0^{\mathrm{Sym}}}{2}}\widetilde{\mathrm{ch}}(\mathbb{FM}(k_\lambda)) = \delta(\widetilde{H}_\lambda)\,,\quad q=e^{-2\pi\sqrt{-1}t_1}\,,\ \ t=e^{-2\pi\sqrt{-1}t_2}\,.$$

We have used the definition of $\deg_0^{\mathrm{Sym}}$ and the fact that $|\mu\rangle\in\widetilde{\mathcal{F}}$ as a class in $H^*_\mathsf{T}(I\mathrm{Sym}^n(\mathbb{C}^2))$ has degree 0.

By Lemma 12, we have

$$\Gamma_{\mathrm{Sym}}\cup(2\pi\sqrt{-1})^{\frac{\deg_0^{\mathrm{Sym}}}{2}}\widetilde{\mathrm{ch}}(\mathbb{FM}(k_\lambda)) = \delta_2(\widetilde{H}_\lambda)\,,\quad q=e^{-2\pi\sqrt{-1}t_1}\,,\ \ t=e^{-2\pi\sqrt{-1}t_2},$$

where $\delta_2:\widetilde{\mathcal{F}}\to\widetilde{\mathcal{F}}$ is defined by

$$\begin{aligned}\delta_2|\mu\rangle = (t_1t_2)^{\ell(\mu)}(2\pi)^{n-\ell(\mu)}\left(\prod_i\mu_i\right)\left(\prod_i\mu_i^{-\mu_it_1}\mu_i^{-\mu_it_2}\right)\\ \times\left(\prod_i\Gamma(\mu_it_1)\Gamma(\mu_it_2)\right)\left(\prod_i(1-e^{-2\pi\sqrt{-1}\mu_it_1})(1-e^{-2\pi\sqrt{-1}\mu_it_2})\right)|\mu\rangle\,.\end{aligned}$$

Since $c_1^\mathsf{T}(\mathrm{Sym}^n(\mathbb{C}^2))\Big|_\mu = n(t_1+t_2)$, we have

$$z^{\rho^{\mathrm{Sym}}}\left(\Gamma_{\mathrm{Sym}}\cup(2\pi\sqrt{-1})^{\frac{\deg_0^{\mathrm{Sym}}}{2}}\widetilde{\mathrm{ch}}(\mathbb{FM}(k_\lambda))\right) = z^{n(t_1+t_2)}\delta_2(\widetilde{H}_\lambda)\,,\quad q=e^{-2\pi\sqrt{-1}t_1}\,,\ \ t=e^{-2\pi\sqrt{-1}t_2}.$$

[14] The natural basis of $H^*_\mathsf{T}(I\mathrm{Sym}^n(\mathbb{C}^2))$ is identified with $\{|\mu\rangle\big|\mu\in\mathrm{Part}(n)\}\subset\widetilde{\mathcal{F}}$.

[15] The choice of $\mathsf{T}=(\mathbb{C}^*)^2$-action on $\mathbb{C}^2$ in [12, Section 5.1.1] is dual to ours.

Next, we write

$$z^{-\mu^{\mathsf{Sym}}} z^{\rho^{\mathsf{Sym}}} \left(\Gamma_{\mathsf{Sym}} \cup (2\pi\sqrt{-1})^{\frac{\deg_0^{\mathsf{Sym}}}{2}} \widetilde{\mathrm{ch}}(\mathbb{FM}(k_\lambda)) \right) = \delta_3(\mathsf{H}^\lambda_{-z}) \,,$$

where $\delta_3 : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}}$ is defined by

$$\begin{aligned} \delta_3|\mu\rangle = {} & z^n z^{n(t_1+t_2)/z} (t_1t_2/z^2)^{\ell(\mu)} (2\pi)^{n-\ell(\mu)} \left(\prod_i \mu_i \right) \left(\prod_i \mu_i^{-\mu_i t_1/z} \mu_i^{-\mu_i t_2/z} \right) \\ & \times \left(\prod_i \Gamma(\mu_i t_1/z)\Gamma(\mu_i t_2/z) \right) \left(\prod_i (1-e^{-2\pi\sqrt{-1}\mu_i t_1/z})(1-e^{-2\pi\sqrt{-1}\mu_i t_2/z}) \right) z^{-(n-\ell(\mu))} |\mu\rangle \,. \end{aligned}$$

We have used the definition of μ^{Sym} and the fact that $|\mu\rangle \in \widetilde{\mathcal{F}}$ as a class in $H^*_{\mathsf{T}}(I\mathrm{Sym}^n(\mathbb{C}^2))$ has age-shifted degree $2(n-\ell(\mu))$. We have also used

$$z^{\deg_{\mathrm{CR}}/2} \left(\widetilde{H}_\lambda \big|_{q=e^{-2\pi\sqrt{-1}t_1},\ t=e^{-2\pi\sqrt{-1}t_2}} \right) = \widetilde{H}_\lambda \big|_{q=e^{-2\pi\sqrt{-1}t_1/z},\ t=e^{-2\pi\sqrt{-1}t_2/z}} \,,$$

which is equal to H^λ_{-z}.

By (4.5), we have

$$\begin{aligned} \Gamma(t)\Gamma(-t) = & \frac{\Gamma(1+t)}{t} \frac{\Gamma(1-t)}{-t} \\ = & \frac{1}{-t} \frac{2\pi\sqrt{-1}}{e^{\pi\sqrt{-1}t} - e^{-\pi\sqrt{-1}t}} \\ = & \frac{2\pi\sqrt{-1}}{-t} \frac{1}{(1-e^{-2\pi\sqrt{-1}t})e^{\pi\sqrt{-1}t}} \,. \end{aligned}$$

Hence

$$\Gamma(t)(1-e^{-2\pi\sqrt{-1}t}) = (-1)e^{-\pi\sqrt{-1}t} 2\pi\sqrt{-1} \frac{1}{t} \frac{1}{\Gamma(-t)} \,.$$

We then obtain

$$\begin{aligned} & \left(\prod_i \Gamma(\mu_i t_1/z)\Gamma(\mu_i t_2/z) \right) \left(\prod_i (1-e^{-2\pi\sqrt{-1}\mu_i t_1/z})(1-e^{-2\pi\sqrt{-1}\mu_i t_2/z}) \right) \\ = & (-1)^{2\ell(\mu)} e^{-\pi\sqrt{-1}n(t_1+t_2)/z} (2\pi\sqrt{-1})^{2\ell(\mu)} \left(\prod_i \frac{z}{\mu_i t_1} \frac{z}{\mu_i t_2} \right) \left(\prod_i \frac{1}{\Gamma(-\mu_i t_1/z)\Gamma(-\mu_i t_2/z)} \right) \\ = & (-1)^{2\ell(\mu)} e^{-\pi\sqrt{-1}n(t_1+t_2)/z} (2\pi\sqrt{-1})^{2\ell(\mu)} \left(\frac{z^2}{t_1t_2} \right)^{\ell(\mu)} \left(\prod_i \frac{1}{\mu_i} \right)^2 \left(\prod_i \frac{1}{\Gamma(-\mu_i t_1/z)\Gamma(-\mu_i t_2/z)} \right) . \end{aligned}$$

Therefore, we can write $\delta_3|\mu\rangle$ as

$$\begin{aligned}
&z^n z^{n(t_1+t_2)/z}(t_1t_2/z^2)^{\ell(\mu)}(2\pi)^{n-\ell(\mu)}\left(\prod_i \mu_i\right)\left(\prod_i \mu_i^{-\mu_i t_1/z}\mu_i^{-\mu_i t_2/z}\right)\\
&\times(-1)^{2\ell(\mu)}e^{-\pi\sqrt{-1}n(t_1+t_2)/z}(2\pi\sqrt{-1})^{2\ell(\mu)}\left(\frac{z^2}{t_1t_2}\right)^{\ell(\mu)}\left(\prod_i\frac{1}{\mu_i}\right)^2\\
&\times\left(\prod_i\frac{1}{\Gamma(-\mu_it_1/z)\Gamma(-\mu_it_2/z)}\right)z^{-(n-\ell(\mu))}|\mu\rangle\\
=\ &z^{\ell(\mu)}z^{n(t_1+t_2)/z}e^{-\pi\sqrt{-1}n(t_1+t_2)/z}\frac{1}{\prod_i\mu_i}\prod_i\frac{\mu_i^{-\mu_it_1/z}\mu_i^{-\mu_it_2/z}}{\Gamma(-\mu_it_1/z)\Gamma(-\mu_it_2/z)}\\
&\times(2\pi)^{n-\ell(\mu)}(2\pi\sqrt{-1})^{2\ell(\mu)}(-1)^{2\ell(\mu)}|\mu\rangle\,.
\end{aligned}$$

4.6. **Proof of Theorem 4.** The last step of the proof is the matching

$$\delta_3|\mu\rangle=\Delta^{\text{Hilb}}|\mu\rangle\,. \tag{4.8}$$

By comparing the expression above for $\delta_3|\mu\rangle$ with Equation (4.7), we see the matching (4.8) follows from the following equality in $\widetilde{\mathcal{F}}$:

$$(-1)^{n+\ell(\mu)}(2\pi\sqrt{-1})^{n+\ell(\mu)}|\widetilde{\mu}\rangle=(2\pi)^{n-\ell(\mu)}(2\pi\sqrt{-1})^{2\ell(\mu)}(-1)^{2\ell(\mu)}|\mu\rangle\,. \tag{4.9}$$

We verify (4.9) as follows. By definition, $|\widetilde{\mu}\rangle=(-\sqrt{-1})^{\ell(\mu)-n}|\mu\rangle$. Thus,

$$(-1)^{n+\ell(\mu)}(2\pi\sqrt{-1})^{n+\ell(\mu)}|\widetilde{\mu}\rangle=(-1)^{n+\ell(\mu)}(2\pi\sqrt{-1})^{n+\ell(\mu)}(-\sqrt{-1})^{\ell(\mu)-n}|\mu\rangle\,.$$

We calculate

$$\begin{aligned}
(-1)^{n+\ell(\mu)}(2\pi\sqrt{-1})^{n+\ell(\mu)}(-\sqrt{-1})^{\ell(\mu)-n}&=(2\pi)^{n+\ell(\mu)}(-1)^{2\ell(\mu)}\sqrt{-1}^{2\ell(\mu)}\,,\\
(2\pi)^{n-\ell(\mu)}(2\pi\sqrt{-1})^{2\ell(\mu)}(-1)^{2\ell(\mu)}&=(2\pi)^{n+\ell(\mu)}(-1)^{2\ell(\mu)}\sqrt{-1}^{2\ell(\mu)}.
\end{aligned}$$

This proves (4.9), hence (4.8).

In summary, our calculations establish the equation

$$\begin{aligned}
&z^{-\mu^{\text{Sym}}}z^{\rho^{\text{Sym}}}\left(\Gamma_{\text{Sym}}\cup(2\pi\sqrt{-1})^{\frac{\deg_0^{\text{Sym}}}{2}}\widetilde{\text{ch}}(\mathbb{FM}(k_\lambda))\right)\\
&=\mathsf{CK}\big|_{z\mapsto -z}\left(z^{-\mu^{\text{Hilb}}}z^{\rho^{\text{Hilb}}}\left(\Gamma_{\text{Hilb}}\cup(2\pi\sqrt{-1})^{\frac{\deg_0^{\text{Hilb}}}{2}}\text{ch}(k_\lambda)\right)\right)\,,
\end{aligned}$$

which completes the proof of Theorem 4 . □

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