CARRIER GRAPHS FOR REPRESENTATIONS OF THE RANK TWO FREE GROUP INTO ISOMETRIES OF HYPERBOLIC THREE SPACE

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ABSTRACT. Carrier graphs were first introduced for closed hyperbolic 3-manifolds by White. In this paper, we first generalize this definition to carrier graphs for representations of a rank two free group into the isometry group of hyperbolic three space. Then we prove the existence and the finiteness of minimal carrier graphs for those representations which are discrete, faithful and geometrically finite, and more generally, those that satisfy certain finiteness conditions first introduced by Bowditch.

CONTENTS

1. INTRODUCTION

Let M be a hyperbolic 3-manifold of rank n. A connected finite graph Γ is called an *n-graph* if all vertices are trivalent and the rank of the graph is n. A *carrier n-graph* for M is an n-graph together with a map $f : \Gamma \to M$ such that $f_* : \pi_1(\Gamma) \to \pi_1(M)$ is an epimorphism. This definition was first introduced by White in [2] for a closed connected hyperbolic 3-manifold. A carrier n-graph is said to be *minimal* if the image of f has the minimal length among all carrier n-graphs for M. By studying the minimal carrier graph, White showed that the injectivity radius of a closed hyperbolic 3-manifold M is bounded above by a constant only depending on its rank. In [1], Siler studied the geometric properties of the carrier graph for a general hyperbolic 3-manifold. However the existence and uniqueness of the minimal carrier graph of a hyperbolic 3-manifold in general is still unclear. In this paper, we consider this problem for geometrically finite M whose fundamental group is isomorphic to $\mathbb{F}_2$ the rank two free group.

Any such M is isometric to a quotient of $\mathbb{H}^3$ by the action of the image of a representation $\rho : \mathbb{F}_2 \to \mathrm{PSL}(2, \mathbb{C})$ which is discrete and faithful. By the result of Culler in [7], these representations can be lifted to representations in $\mathrm{SL}(2, \mathbb{C})$ up to $\pm\mathrm{Id}$. Hence any such M can be considered as a point in the $\mathrm{SL}(2, \mathbb{C})$-character variety $\chi(\mathrm{SL}(2, \mathbb{C}))$ which consists of all representations of $\mathbb{F}_2$ to $\mathrm{SL}(2, \mathbb{C})$ up to conjugacy. There is an open subset in $\chi(\mathrm{SL}(2, \mathbb{C}))$ which is invariant under the action of $Out(\mathbb{F}_2)$ satisfying certain conditions first introduced by Bowditch. Moreover, the $Out(\mathbb{F}_2)$-action on it is properly discontinuous. We extend our study to the representations corresponding to the points in this open subset.

The character variety $\chi = \chi(\mathrm{SL}(2, \mathbb{C}))$ has been studied by many people. The theorem of Vogt and Fricke implies that it can be identified with $\mathbb{C}^3$, where the coordinates are given by taking the traces of the image of a fixed superbasis of $\mathbb{F}_2$. By considering the natural embedding of $\mathrm{SL}(2, \mathbb{R})$ into $\mathrm{SL}(2, \mathbb{C})$, this space also contains Fricke spaces of orientable hyperbolic surfaces with Euler characteristic -1. In [9], Goldman described those Fricke spaces as subspaces of χ using the above coordinate system. Of particular interest is the Fricke space of a once-puncture torus, where in [12], McShane proved a remarkable identity. The coordinates for this space satisfies the Markoff equation. By using this fact, in [3], Bowditch gave an alternative proof of this identity, and then in [4], generalized this identity to the type preserving representations satisfying certain conditions which he called *Q-conditions*. In [16], by dropping the type-preserving restriction, Tan, Wong and Zhang proved a variation of McShane identity for general representations satisfying the *Bowditch Q-conditions*, or *BQ-conditions* for short. A representation satisfying BQ-conditions will be called a *BQ-representation*.

In this paper, we will consider the subspace of conjugacy classes of all BQ-representations and we call it the *Bowditch space*. The Bowditch space contains the Schottky representations, that is, the discrete faithful convex cocompact representations arising from the holonomy representations of convex cocompact hyperbolic three manifolds with fundamental group $\mathbb{F}_2$. However, the whole subspace is strictly larger than that, see [16] or [15]. For example, it contains representations of $\mathbb{F}_2$ into $\mathrm{SL}(2, \mathbb{R})$ arising from hyperbolic structures on the torus with one cone singularity. When the cone angle is irrational, such a representation is not discrete.

We define the carrier 2-graph for a generic irreducible representation ρ which is a generalization of the one defined by White in [2]. Roughly speaking, a carrier 2-graph for ρ is an equivalence class of a pair (Γ, Ψ), where Γ is a 2-graph marked by a basis of $\mathbb{F}_2$, Ψ is a ρ-equivariant homeomorphism from the universal cover $\widetilde{\Gamma}$ to $\mathbb{H}^3$ and the equivalence relation is defined by considering the $\mathrm{Inn}(\mathbb{F}_2)$-action on the pair. In particular, Ψ induces an automorphism h_Ψ of $\mathbb{F}_2$. By pulling back the intrinsic metric on $\Psi(\widetilde{\Gamma})$ induced by the hyperbolic metric, we obtain a metric on Γ, and the *length* of a carrier 2-graph is defined to be the sum of its edge lengths with respect to this metric. A carrier 2-graph is *minimal* if it has the shortest length among all carrier 2-graphs.

To study the existence of the minimal carrier 2-graph, we decompose the space of all carrier 2-graph into a disjoint union of subsets by considering h_Ψ. In each subset, a carrier 2-graph with the shortest length, if it exists, is called a *critical carrier 2-graph*. Then our result can be stated as follows:

Theorem 1.1. *If $\rho : \mathbb{F}_2 \to \mathrm{SL}(2, \mathbb{C})$ satisfies the BQ-conditions, then critical carrier 2-graphs exist and there are finitely many of them.*

Since a minimal carrier 2-graph is also a critical carrier 2-graph, as a corollary, we show that

Corollary 1.2. *If a representation ρ satisfies the BQ-conditions, then it admits finitely many minimal carrier 2-graphs.*

The main ingredients in the proof is the convexity lemma in hyperbolic geometry and a detailed analysis of the Steiner tree for a triple of pairwise disjoint geodesics in $\mathbb{H}^3$. Informally, the Steiner tree is a graph connecting triple of pairwise disjoint geodesics with minimal length among all such graphs. This is a generalization of the Steiner tree for points in Euclidean plane.

The outline of the paper is as follows. In section 2, we recall some necessary background on 3-dimensional hyperbolic geometry.

In section 3, we recall background on rank two free groups and the definition of the Bowditch Q-conditions.

In section 4, we will recall the definition of the tree of superbases for $\mathbb{F}_2$ and discuss three different edge orientations on it induced by an irreducible representation of $\mathbb{F}_2$ into $\mathrm{SL}(2, \mathbb{C})$. In particular, we show that if the representation satisfies the Bowditch Q-conditions, then for any edge orientation introduced in this section, there is a compact attracting subtree.

In section 5, we prove the existence and uniqueness of the Fermat point of a hyperbolic triangle, which is a result similar to the Euclidean case.

In section 6, we generalize the definition of the Fermat point result for a triple of geodesics in $\mathbb{H}^3$, from which we define the Steiner tree and prove its existence. We also give a characterization of its combinatorial type.

In section 7, we define the carrier graph for a representation.

In section 8, we give the proof of the main theorem and its corollary.

In the last section, we will discuss the result for BQ-representations which preserve a hyperbolic plane in $\mathbb{H}^3$ as examples.

2. Three dimensional hyperbolic geometry

2.1. **Upper-half space model.** The upper half space model of 3-dimensional hyperbolic space is defined to be the following set:

$$\mathbb{H}^3 = \{(a, b, c) \in \mathbb{R}^3 \mid c > 0\},$$

equipped with the hyperbolic metric:

$$(\mathrm{d}s)^2 = \frac{(\mathrm{d}a)^2 + (\mathrm{d}b)^2 + (\mathrm{d}c)^2}{c^2}.$$

The boundary $\partial \mathbb{H}^3$ of $\mathbb{H}^3$ is the one point compactification of the plane:

$$\{(a, b, c) \in \mathbb{R}^3 \mid c = 0\}.$$

It can be identified with the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$ with $u = a + ib$.

In the upper half space model, the geodesics are either vertical lines or half circles orthogonal, at its both ends, to the plane defined by $c = 0$. Each geodesic is uniquely determined by its end points on $\partial \mathbb{H}^3$, hence there is a natural 1-1 correspondence between (oriented) geodesics and (ordered) pairs of distinct points on $\partial \mathbb{H}^3$. Given two distinct points u and u' in $\hat{\mathbb{C}}$, we denote by $[u, u']$ the non-oriented geodesic determined by them and by (u, u') (resp. (u', u)) its two oriented versions oriented from u to u' (resp. from u' to u).

Given two distinct geodesics, their positions relative to each other have three possible types:

- intersecting in $\mathbb{H}^3$,
- parallel, i.e. sharing one end point in $\partial \mathbb{H}^3$,
- disjoint (or ultra-parallel), i.e. disjoint in $\mathbb{H}^3 \cup \partial \mathbb{H}^3$.

The distance between two disjoint geodesics is positive and can be realized by their intersection points with their common perpendicular geodesic. The distance between two parallel geodesics is 0 which is not realizable. The distance between two intersecting geodesics is 0 realized by their intersection point.

2.2. **Orientation preserving isometries of** $\mathbb{H}^3$**.** The orientation preserving isometry group of $\mathbb{H}^3$ can be identified with $\mathrm{PSL}(2,\mathbb{C})$. Its elements can be classified by their fixed points in $\mathbb{H}^3 \cup \partial\mathbb{H}^3$. A non-identity element in $\mathrm{PSL}(2,\mathbb{C})$ is called

- *loxodromic* if it preserves a geodesic and acts as translation on it;
- *parabolic* if it has a unique fixed point contained in $\partial\mathbb{H}^3$;
- *elliptic* if it fixes a geodesic pointwise in $\mathbb{H}^3$.

In particular, we call an element in $\mathrm{PSL}(2,\mathbb{C})$ an *involution* if it is elliptic of order 2. The geodesic fixed by a loxodromic element or an elliptic element is called its *axis*.

Definition 2.1. The *real translation distance* of $\phi \in \mathrm{PSL}(2,\mathbb{C})$ is defined to be the infimum:

$$a(\phi) := \inf_{p \in \mathbb{H}^3} \mathrm{d}_{\mathbb{H}^3}(p, \phi(p)),$$

where $\mathrm{d}_{\mathbb{H}^3}$ is the distance in $\mathbb{H}^3$ with respect to the hyperbolic metric.

If ϕ is a loxodromic element, then $a(\phi)$ is positive and realized by any point on the axis of ϕ; if ϕ is elliptic, then $a(\phi)$ is 0 and realized by any point on the axis of ϕ; if ϕ is parabolic, then $a(\phi)$ equals 0 and is not realizable.

A well known fact is that any element $\phi \in \mathrm{PSL}(2,\mathbb{C})$ can be written as a composition of two involutions of $\mathbb{H}^3$. Consider the axes of these two involutions. For any ϕ different from identity, they are

- disjoint, if ϕ is loxodromic, and the distance between them equals to $a(\phi)/2$;
- parallel, if ϕ is parabolic;
- intersecting with each other, if ϕ is elliptic.

This decomposition is not unique. Now consider two elements ϕ and ψ in $\mathrm{PSL}(2,\mathbb{C})$. We have the following well-known theorem:

Theorem 2.2 (Coxeter extension)**.** *Two elements ϕ and ψ do not have any common fixed point on $\partial\mathbb{H}^3$, if and only if there is a unique triple of involutions $(\iota_1, \iota_2, \iota_3)$ such that*

$$\begin{aligned} \phi &= \iota_3\iota_2, \\ \psi &= \iota_2\iota_1, \\ (\phi\psi)^{-1} &= \iota_1\iota_3. \end{aligned} \qquad (1)$$

Remark 2.3. In the next section, we will see that the condition that ϕ and ψ do not have common fixed points in $\partial\mathbb{H}^3$ is equivalent to the condition that the subgroup of $\mathrm{PSL}(2,\mathbb{C})$ generated by ϕ and ψ is irreducible.

Assume that ϕ and ψ do not have common fixed points on $\partial\mathbb{H}^3$. We consider $(\iota_1, \iota_2, \iota_3)$ the unique triple of involutions associated to $(\phi, \psi, (\phi\psi)^{-1})$. Let γ_1, γ_2 and γ_3 be the the axes of ι_1, ι_2 and ι_3 respectively. If moreover ϕ, ψ and $(\phi\psi)^{-1}$ are all loxodromic elements, then the γ_i's are disjoint in $\mathbb{H}^3 \cup \partial\mathbb{H}^3$. Therefore we have a unique right angled hexagon which is bounded by γ_i's and their pairwise common perpendicular geodesics. It is possible that this right angled hexagon is degenerate, since the sides along γ_i's may have zero length.

2.3. **Lifting** $\mathrm{PSL}(2,\mathbb{C})$ **to** $\mathrm{SL}(2,\mathbb{C})$**.** In order to talk about the trace, we would like to consider $\mathrm{SL}(2,\mathbb{C})$ instead of $\mathrm{PSL}(2,\mathbb{C})$. A reference for this part can be found in Section V and Section VI in [8].

Any $\phi \in \mathrm{PSL}(2,\mathbb{C})$ has two lifts ξ and $-\xi$ in $\mathrm{SL}(2,\mathbb{C})$. The actions of $\pm\xi$ on $\mathbb{H}^3$ are the same as that of ϕ. In particular, the subgroup $\pm\mathrm{Id}$ acts trivially on $\mathbb{H}^3$. Therefore, the action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{H}^3$ is not free. If ϕ is a loxodromic (resp. parabolic, elliptic) element, then its lifts are also called loxodromic (resp. parabolic, elliptic) elements in $\mathrm{SL}(2,\mathbb{C})$. Elements of different types in $\mathrm{SL}(2,\mathbb{C})$ can be distinguished by considering their traces: an element $\xi \in \mathrm{SL}(2,\mathbb{C})$ different from $\pm\mathrm{Id}$ is

- loxodromic if and only if its trace is not contained in the segment $[-2, 2]$;
- parabolic if and only if its trace is 2 or -2;
- elliptic if and only if its trace is contained in $(-2, 2)$.

As with $\mathrm{PSL}(2,\mathbb{C})$-elements, the geodesic fixed by a loxodromic element or an elliptic element is called its axis. The translation distance $a(\xi)$ of $\xi \in \mathrm{SL}(2,\mathbb{C})$ can be defined in the same way by considering their action on $\mathbb{H}^3$.

From the discussion in the previous part, any isometry ϕ can be written as a composition $\iota_2\iota_1$ of two involutions. Hence an alternative way to find the lift of ϕ is by considering the lifts of ι_2 and ι_1.

More precisely, let ι be an involution whose axis is $[u, u']$, and consider its two associated oriented geodesics (u, u') and (u', u). The two lifts of ι associated to (u, u') and (u', u) respectively are:

$$\frac{i}{u - u'}\begin{pmatrix} u + u' & -2uu' \\ 2 & -u - u' \end{pmatrix} \quad \text{and} \quad \frac{i}{u' - u}\begin{pmatrix} u + u' & -2uu' \\ 2 & -u - u' \end{pmatrix}.$$

If $u = \infty$, then they become

$$\begin{pmatrix} i & -2u'i \\ 0 & -i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -i & 2u'i \\ 0 & i \end{pmatrix}.$$

We denote (u, u') and (u', u) by γ and $\overline{\gamma}$, and the corresponding lifts by r and $\overline{r}$ respectively. We call them the π-*rotations*. Moreover we can verify that:

$$r^2 = \overline{r}^2 = -\mathrm{Id},$$
$$r\overline{r} = \overline{r}r = \mathrm{Id},$$

which then implies that

$$r^{-1} = -r = \overline{r}.$$

Each involution ι_i has two lifts r_i and $\overline{r}_i$. Then there are four different combinations satisfying the following relations:

$$\xi = r_2r_1 = \overline{r}_2\overline{r}_1,$$
$$-\xi = \overline{r}_2r_1 = r_2\overline{r}_1,$$

where ξ and $-\xi$ are the two lifts of ϕ.

Therefore, the $\mathrm{SL}(2,\mathbb{C})$ version of Theorem 2.2 can be stated as follows:

Proposition 2.2'. Let ξ and η be two elements of $\mathrm{SL}(2,\mathbb{C})$. Then ξ and η do not have common fixed points on $\partial\mathbb{H}^3$, if and only if there exists exactly two triples of π-rotations: (r_1, r_2, r_3) and its inverse $(\overline{r}_1, \overline{r}_2, \overline{r}_3)$, such that

$$\begin{aligned} \xi &= r_3r_2 = \overline{r}_3\overline{r}_2, \\ \eta &= r_2r_1 = \overline{r}_2\overline{r}_1, \\ (\xi\eta)^{-1} &= -r_1r_3 = -\overline{r}_1\overline{r}_3. \end{aligned}$$

A *double-cross* $(\gamma; \gamma_1, \gamma_2)$ consists of a triple of distinct oriented geodesics, such that γ is the common perpendicular geodesic of γ_1 and γ_2. We denote the end points of γ, γ_1 and γ_2, by (u, u'), (v, v') and (w, w') respectively. We denote by p_1 and p_2 the intersection points of γ_1 and γ_2 with γ respectively. Let P_1 (resp. P_2) denote the hyperbolic plane containing γ and γ_1 (resp. γ_2). We denote by P_1^+ (resp. P_2^+) the half plane in P_1 (resp. P_2) bounded by γ containing $p_1 v'$ (resp. $p_2 w'$).

The *angle* of the double-cross $(\gamma; \gamma_1, \gamma_2)$ is defined to be the angle from P_1^+ to P_2^+ with respect to the orientation of γ, taking values in $[0, 2\pi)$. To each double-cross, we can associate to it a complex number $l(\gamma; \gamma_1, \gamma_2)$, such that the real part is the signed distance from p_1 to p_2, and the imaginary part is the angle of the double-cross.

Given four distinct points u_1, u_2, u_3 and u_4 in Riemann sphere, we can define their cross-ratio:

$$\mathrm{Cr}(u_1, u_2; u_3, u_4) := \frac{(u_1 - u_3)(u_2 - u_4)}{(u_1 - u_4)(u_2 - u_3)}.$$

Then we have the following relation:

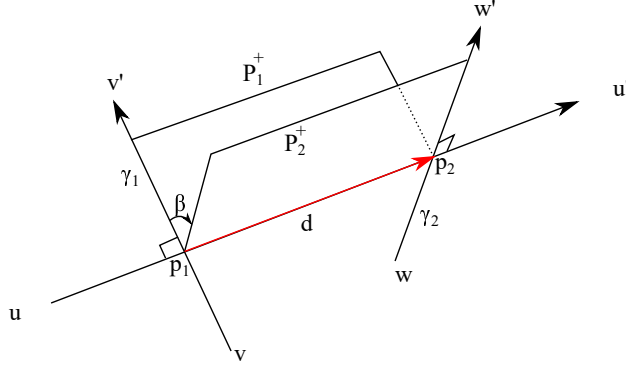$$\mathrm{Cr}(v', w'; u', u) = \exp(l(\gamma; \gamma_1, \gamma_2))$$

FIGURE 1. $l(\gamma; \gamma_1, \gamma_2) = d + i\beta$

Let $\xi \in \mathrm{SL}(2, \mathbb{C})$ be a non parabolic element different from the identity. To a decomposition $\xi = r_2 r_1$, we can associate a double-cross $(\gamma; \gamma_1, \gamma_2)$ where $\gamma = \gamma(\xi)$ is the oriented axis of ξ, and γ_i is the oriented axis of r_i. We can verify the following relation:

$$\mathrm{tr}(\xi) = \mathrm{tr}(r_2 r_1) = -2 \cosh l(\xi). \tag{2}$$

We call twice of $l(\gamma(\xi); \gamma_1, \gamma_2)$ the *complex translation distance* of ξ. Notice that this quantity is independent of choice of the decomposition of ξ, therefore we will simply denote it by $l(\xi)$.

Remark 2.4. If ξ is parabolic with $\xi = r_2 r_1$, then their axes γ_1 and γ_2 share a endpoint u in the boundary of $\mathbb{H}^3$ which is the fixed point of the action of ξ. We may make the following convention: if two oriented geodesic γ_1 and γ_2 point to or away from u at the same time, then the complex translation distance of ξ is 0; otherwise the complex translation distance of ξ is $2\pi i$. The the above formula (2) can be extended to parabolic elements.

3. FREE GROUP OF TWO GENERATORS AND ITS REPRESENTATIONS INTO $\mathrm{SL}(2, \mathbb{C})$

3.1. **Superbases and the corresponding tree.** Let us consider the rank two free group $\mathbb{F}_2$. In this part, we recall superbases for $\mathbb{F}_2$ which can be viewed as vertices of a natural trivalent tree upon which the outer-automorphism group of $\mathbb{F}_2$ acts. The notion of superbases was first introduced by Conway [6] for $\mathbb{Z}^2$. Superbases and the corresponding trivalent tree play an important role in the work of Bowditch [4] and its generalization by Tan-Wong-Zhang [16], as well as in ours. For more details about these objects, the reader may refer to [9], [5] and [10].

A *basis* of $\mathbb{F}_2$ is an ordered pair $(X, Y) \in \mathbb{F}_2 \times \mathbb{F}_2$, such that X and Y generate $\mathbb{F}_2$ freely. An element of $\mathbb{F}_2$ is said to be *primitive* if it can be extended to a basis of $\mathbb{F}_2$. A *basic triple* of $\mathbb{F}_2$ is an ordered triple

$$(X, Y, Z) \in \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2,$$

such that (X, Y) is a basis of $\mathbb{F}_2$ and $XYZ = id$ where id is the identity element of $\mathbb{F}_2$. The *automorphism group* $\mathrm{Aut}(\mathbb{F}_2)$ of $\mathbb{F}_2$ acts transitively and freely on the set of bases, hence on the set of ordered triples, as well.

An element in $\mathbb{F}_2$ induces an automorphism of $\mathbb{F}_2$ obtained by taking conjugation of $\mathbb{F}_2$ by this element. Such an automorphism is called an *inner-automorphism*. The inner-automorphisms of $\mathbb{F}_2$ form a normal subgroup of $\mathrm{Aut}(\mathbb{F}_2)$, called the *inner-automorphism group* and denoted by $\mathrm{Inn}(\mathbb{F}_2)$. The quotient group

$$\mathrm{Aut}(\mathbb{F}_2)/\mathrm{Inn}(\mathbb{F}_2),$$

is called the *outer-automorphism group* of $\mathbb{F}_2$ and denoted by $\mathrm{Out}(\mathbb{F}_2)$.

Given a basis (X, Y), we can associate to it an *elliptic involution* $\mathfrak{e}_{(X,Y)}$ which is an automorphism of $\mathbb{F}_2$ such that:

$$\mathfrak{e}_{(X,Y)}(X) = X^{-1}$$
$$\mathfrak{e}_{(X,Y)}(Y) = Y^{-1}$$

Elliptic involutions associated to different bases differ by an inner automorphism. We denote by $\mathrm{Inn}^{\mathfrak{e}}(\mathbb{F}_2)$ the subgroup of $\mathrm{Aut}(\mathbb{F}_2)$ generated by elliptic involutions and $\mathrm{Inn}(\mathbb{F}_2)$. It contains $\mathrm{Inn}(\mathbb{F}_2)$ as an index two subgroup. The quotient group

$$\mathrm{Inn}^{\mathfrak{e}}(\mathbb{F}_2)/\mathrm{Inn}(\mathbb{F}_2),$$

is the center of $\mathrm{Out}(\mathbb{F}_2)$.

Definition 3.1. An *ordered superbasis* of $\mathbb{F}_2$ is an $\mathrm{Inn}^{\mathfrak{e}}(\mathbb{F}_2)$-orbit of basic triples.

The $\mathrm{Aut}(\mathbb{F}_2)$-action on basic triples induces an $\mathrm{Out}(\mathbb{F}_2)$-action on ordered superbases. This action is transitive.

The symmetric group $\mathfrak{S}_3$ over 3 symbols acts on the space of basic triples by the action generated by:

$$(X, Y, Z) \overset{(12)}{\longmapsto} (Y^{-1}, X^{-1}, Z^{-1});$$
$$(X, Y, Z) \overset{(23)}{\longmapsto} (X^{-1}, Z^{-1}, Y^{-1}).$$

This $\mathfrak{S}_3$-action on basic triples commutes with the $\mathrm{Inn}^{\mathfrak{e}}(\mathbb{F}_2)$-action, therefore it induces an $\mathfrak{S}_3$-action on ordered superbases.

Definition 3.2. An *unordered superbasis* (or a *superbasis* for short) of $\mathbb{F}_2$ is an $\mathfrak{S}_3$-orbit of ordered superbases. Two distinct unordered superbases are said to be *neighbors* if they have representatives sharing two elements up to taking inverse.

Example 3.3. The superbasis corresponding to $(X, Y, (XY)^{-1})$ and that corresponding to (X, Y^{-1}, YX^{-1}) are neighbors.

Remark 3.4. To simplify the notation, in the reminder of the paper, we will use (X, Y, Z) to denote both an ordered triple and its corresponding superbasis. The meaning will be clear by the context.

A graph can be constructed from the superbases of $\mathbb{F}_2$. The vertices are the superbases and two vertices are connected by an edge if the corresponding superbases are neighbors. This graph is a trivalent tree. We call it the ***tree of superbases*** and denote it by Σ. For our convenience, we also consider the metric on Σ induced by setting each edge with length 1.

Remark 3.5. From the discussion, we can see that the group $\mathrm{Out}(\mathbb{F}_2)$ is isomorphic to
$$\mathbb{Z}_2 \times ((\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2) \rtimes \mathfrak{S}_3),$$
where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ in the first factor corresponds to the elliptic involution.

3.2. $\mathrm{SL}(2, \mathbb{C})$-**character variety of** $\mathbb{F}_2$**.** Recall that the $\mathrm{SL}(2, \mathbb{C})$-character variety χ of $\mathbb{F}_2$ is the space of conjugacy classes of homomorphisms from $\mathbb{F}_2$ to $\mathrm{SL}(2, \mathbb{C})$. A classical result of Fricke and Vogt about χ is stated as follows:

Theorem 3.6 (Fricke [18], Vogt [17])**.** *Let* $f : \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}) \to \mathbb{C}$ *be a regular function which is invariant under the diagonal action of* $\mathrm{SL}(2, \mathbb{C})$ *by conjugation. There exists a polynomial function* $F(x, y, z) \in \mathbb{C}[x, y, z]$ *such that: for any pair* $(\xi, \eta) \in \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$*, we have*
$$f(\xi, \eta) = F(\mathrm{tr}(\xi), \mathrm{tr}(\eta), \mathrm{tr}(\xi\eta)). \tag{3}$$
Furthermore, for all $(x, y, z) \in \mathbb{C}^3$*, there exists* $(\xi, \eta) \in \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$ *such that*
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathrm{tr}(\xi) \\ \mathrm{tr}(\eta) \\ \mathrm{tr}(\xi\eta) \end{bmatrix}.$$
Conversely, if $x^2 + y^2 + z^2 - xyz \neq 4$ *and* $(\xi, \eta), (\xi', \eta') \in \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$ *satisfy,*
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathrm{tr}(\xi) \\ \mathrm{tr}(\eta) \\ \mathrm{tr}(\xi\eta) \end{bmatrix} = \begin{bmatrix} \mathrm{tr}(\xi') \\ \mathrm{tr}(\eta') \\ \mathrm{tr}(\xi'\eta') \end{bmatrix},$$
Then, there exists an element $g \in \mathrm{SL}(2, \mathbb{C})$ *such that* $(\xi, \eta) = (g\xi'g^{-1}, g\eta'g^{-1})$*.*

Fixing a superbasis (X, Y, Z), we can define a map
$$\Phi : \mathrm{Hom}(\mathbb{F}_2, \mathrm{SL}(2, \mathbb{C})) \to \mathbb{C}^3,$$
sending ρ to $(x, y, z) = (\mathrm{tr}\,\rho(X), \mathrm{tr}\,\rho(Y), \mathrm{tr}\,\rho(Z))$.

Definition 3.7. A representation $\rho : \mathbb{F} \to \mathrm{SL}(2, \mathbb{C})$ is said to be *reducible* if its image preserves a non-trivial proper subspace when acting on $\mathbb{C}^2$. A representation is said to be *irreducible* if it is not reducible.

It is well-known that a representation ρ is reducible if and only if it is $\mathrm{SL}(2, \mathbb{C})$-conjugate to a representation by upper triangular matrices in $\mathrm{SL}(2, \mathbb{C})$. By using Fricke's trace identity, we have:
$$\mathrm{tr}([\xi, \eta]) = \mathrm{tr}^2(\xi) + \mathrm{tr}^2(\eta) + \mathrm{tr}^2(\xi\eta) - \mathrm{tr}(\xi)\mathrm{tr}(\eta)\mathrm{tr}(\xi\eta) - 2. \tag{4}$$

Moreover, Nielson [14] showed that all commutators of pairs of free generators of $\mathbb{F}_2$ are conjugate to each other up to taking inverse. Therefore, we conclude that a representation is reducible if and only if its trace of commutator of a pair of generators equals 2. By the above theorem, two irreducible representations ρ_1 and ρ_2 are conjugate to each other if and only if $\Phi(\rho_1) = \Phi(\rho_2)$.

Remark 3.8. Using this coordinate system, we can see that the elliptic involution acts trivially on the χ. The $\mathfrak{S}_3$ acts as permutations of the three coordinates. Hence to see the $\mathrm{Out}(\mathbb{F}_2)$-action on χ, it is enough to study the $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$-action. This explains why we are interested in the tree of superbasis.

3.3. **Bowditch's Q-conditions.** In this section, we recall the work of Bowditch [4] and its generalization by Tan-Wong-Zhang [16].

Let us consider Σ the trivalent tree of superbases associated to $\mathbb{F}_2$. Let $V(\Sigma)$ and $E(\Sigma)$ denote the vertex set and the edge set of Σ respectively. It admits a proper embedding in the Poincaré disk $\mathbb{D}$ as the dual graph of Farey tessellation $\mathcal{F}$ of $\mathbb{D}$.

More precisely, the conformal map from $\mathbb{D}$ to the upper half plane model of hyperbolic plane induces an identification of the boundary S^1 of $\mathbb{D}$ with $\mathbb{R} \cup \infty$. A point on S^1 is called *rational* if it is mapped to p/q under this identification, where p and q are coprime integers. As a convention 0 is considered as $0/1$ and ∞ is considered as $1/0$. Two rational points p/q and s/t are said to be *Farey neighbors* if $|pt - qs| = 1$. The *Farey tessellation* $\mathcal{F}$ of $\mathbb{D}$ is given by connecting every pair of Farey neighbors by geodesics. It is an ideal triangulation of $\mathbb{D}$. Therefore the dual graph is an infinite trivalent tree.

Given any element W of $\mathbb{F}_2$, we denote by $[W]$ its $\mathrm{Inn}^{\mathfrak{e}}(\mathbb{F}_2)$-orbit. We denote by Ω the set of all $[W]$ with W primitive. There is an 1-1 correspondence between rational points on S^1 and Ω described in the following way. Let (X_0, Y_0) be a basis of $\mathbb{F}_2$. Then any primitive element can be written as a word of X_0 and Y_0 with integer powers. Consider the abelianzation of $\mathbb{F}_2$ to $\mathbb{Z}^2$ sending X_0 to $(1, 0)$ and Y_0 to $(0, 1)$. Any primitive element becomes (q, p) with p and q coprime. A theorem of Nielsen [14] tells us that two primitive elements of $\mathbb{F}_2$ are conjugate if and only if they have the same image in $\mathbb{Z}^2$. Therefore the rational number p/q determines an $\mathrm{Inn}^{\mathfrak{e}}(\mathbb{F}_2)$-orbit of a primitive element in $\mathbb{F}_2$. This induces an identification between rational points on S^1 and Ω.

We say that two orbits $[X]$ and $[Y]$ are *Farey neighbors* if their corresponding rational points are Farey neighbors. It is easy to check the following two facts:

(1) $[X]$ and $[Y]$ are Farey neighbors if and only if we can find $X' \in [X]$ and $Y' \in [Y]$ such that (X', Y') is a basis of $\mathbb{F}_2$;
(2) $[X]$, $[Y]$ and $[Z]$ are vertices of an ideal triangle in the complement of $\mathcal{F}$ if and only if we can find three elements, one in each orbit, which form an ordered triple.

Hence, an ideal triangle in the complement of $\mathcal{F}$ corresponds to a unique superbasis of $\mathbb{F}_2$. As a result, we find a map identifying the dual tree of $\mathcal{F}$ with the tree of superbases Σ. To simplify the notation, the dual tree of $\mathcal{F}$ will also be denoted by Σ.

Remark 3.9. The embedding of Σ in $\mathbb{D}$ depends on the choice of (X_0, Y_0). We will fix one embedding in the reminder of this paper.

Consider the complement of Σ in $\mathbb{D}$. Each connected component is bounded by a bi-infinite geodesic on Σ asymptotic to a same rational point on S^1 along both directions. This gives a 1-1 correspondence between connected components of $\mathbb{D} \setminus \Sigma$ and rational points on S^1, which moreover induces a labeling of connected components by $\mathrm{Inn}^{\mathfrak{e}}(\mathbb{F}_2)$-orbits of primitive elements of $\mathbb{F}_2$. Then each edge $e \in E(\Sigma)$ can be labeled by $(X, Y; Z, Z')$ where $[X]$ and $[Y]$ are the two connected components adjacent to e and $[Z]$ and $[Z']$ are the two connected components only meeting e at vertices.
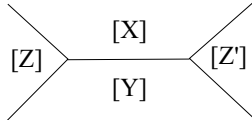

FIGURE 2. Labelling an edge

A representation ρ induces a function defined on Ω sending $[X]$ to $x = \mathrm{tr}\rho(X)$ which we call the *trace function* f_ρ for ρ. The f_ρ-values of three regions meeting at any vertex satisfy the following relation:

$$x^2 + y^2 + z^2 = xyz + \mu, \tag{5}$$

where $\mu = 2 + \mathrm{tr}\rho([X, Y])$ for a basis (X, Y).

Given four elements corresponding to an edge, the traces of their images under ρ satisfy the following relation:

$$z' + z = xy, \tag{6}$$
$$z'z = x^2 + y^2 - \mu \tag{7}$$

This allows us to compute the traces of all primitive elements inductively from the trace of one superbasis.

A representation is said to be *type preserving* if $\mu = 0$. Let ρ be a type preserving representation. In [4], Bowditch studied the growth rate of f_ρ when computing the traces of primitive elements using the above inductive method. In particular, he was interested in comparing f_ρ with the Fibonacci function defined as follows.

Let $e = (X, Y; Z, Z')$ be a fixed edge of the tree Σ. We set $F_e([X]) = F_e([Y]) = 1$ and $F_e([Z]) = F_e([Z']) = 2$. Then for a vertex $v_n = (X_n, Y_n, Z_n)$ at distance n to e, we set $F_e([Z_n]) = F_e([X_n]) + F_e([Y_n])$ if the distance from Z_n to e is greater than the other two distances. In this way, we define a function $F_e : \Omega \to \mathbb{N}$ which is called the *Fibonacci function* with respect to the edge e.
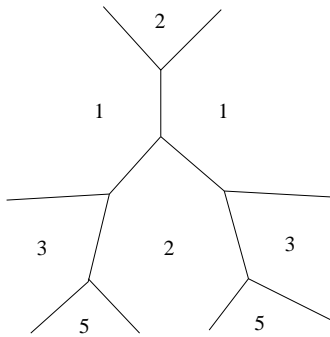

FIGURE 3. Fibonacci function

Let $f : \Omega \to [0, +\infty)$ be any function.

Definition 3.10. The function f is said to have an *upper Fibonacci bound* if there exists a constant $K > 0$ such that $f([X]) \leq KF_e([X])$ for all $[X] \in \Omega$;

The function f is said to have a *lower Fibonacci bound* if there exists a constant $k > 0$ such that $f([X]) \geq kF_e([X])$ for all but finitely many $[X] \in \Omega$;

The function f is said to have a *Fibonacci growth* if it has both lower and upper Fibonacci bounds.

Remark 3.11. The Fibonacci function F_e with respect to e has Fibonacci growth with respect to the Fibonacci function $F_{e'}$ with respect to any other edge e'. Hence, the above definition is independent of the choice of e.

Let f_ρ be the trace function for a type-preserving representations ρ. In [3] Bowditch proved:

Theorem 3.12. *The function f_ρ satisfies:*

(1) $f_\rho^{-1}([-2, 2]) = \emptyset$;
(2) $|f_\rho|^{-1}([0, 2])$ *is finite.*

if and only if the function $\log^+ |f_\rho|$ has Fibonacci growth, where $|\cdot|$ stands for taking the modulus of a complex number and $\log^+ := \max\{0, \log\}$.

In [16], Tan, Wong and Zhang generalized Bowditch's work by dropping the type preserving restriction. In particular, they proved that the above theorem is true for any representation ρ with $\mu \neq 4$.

Definition 3.13. An irreducible representation ρ satisfies *BQ-conditions* if its trace function f_ρ satisfies the following two conditions:

(1) $f_\rho^{-1}([-2, 2]) = \emptyset$;
(2) $|f_\rho|^{-1}([0, 2])$ is finite.

4. Edge orientations on Σ and corresponding attracting subtrees

4.1. **Basic Definitions.** Let us consider the tree of superbasis Σ.

Definition 4.1. We say that Σ is equipped with an *edge orientation* if each of its edges is equipped with an orientation.

One way to give an edge orientation on Σ can be described as follows. Denote by $\widetilde{\Omega}$ the set:

$$\widetilde{\Omega} := \{([W], v) \in \Omega \times V(\Sigma) \mid v \text{ is a vertex on the boundary of } [W]\}.$$

Let f be a function on $\widetilde{\Omega}$ with values in $\mathbb{R}_{\geq 0}$. It induces an edge orientation on Σ in the following way: for the edge $e = (X, \overline{Y}; Z, Z')$ with vertices $v = (X, Y, Z)$ and $v' = (X, Y, Z')$,

- if $|f([Z], v)| > |f([Z'], v')|$, e is oriented from v to v';
- if $|f([Z], v)| < |f([Z'], v')|$, e is oriented from v' to v;
- if $|f([Z], v)| = |f([Z'], v')|$, we choose one of the two orientations arbitrarily.

Remark 4.2. A function on Ω induces a function on $\widetilde{\Omega}$ by pre-composing the projection from $\widetilde{\Omega}$ to Ω. In this case, we call the latter the *lift* of the former.

Remark 4.3. The edge orientation induced by f is not unique if the equality holds at some edges. Below we will see that in our case the equality holds only at finitely many edges and the induced orientation is unique up to finitely many edges.

Definition 4.4. A subgraph Σ_0 of Σ is called an *attracting subtree* of Σ for the given edge orientation if it satisfies the following two conditions:

- it is connected;
- Every edge e not contained in Σ_0 is oriented towards Σ_0.

Remark 4.5. Since Σ is simply connected, for any edge orientation, the minimal attracting subtree, if it exists, is unique.

4.2. **Three edge orientations induced by an irreducible representation.** In this subsection, we show that an irreducible representation ρ induces three functions on $\widetilde{\Omega}$ with values in $\mathbb{R}_{\geq 0}$ which in turn induces three edge orientations on Σ. The first two are the lifts of functions on Ω, while the third one is not. In the following, we discuss them case by case.

(I) *The modulus of traces function.* To define the first function, we consider the trace function f_ρ induced by ρ and lift it to a function defined on $\widetilde{\Omega}$ which is denoted by $\widetilde{f}_\rho$. By post-composing by the modulus function on $\mathbb{C}$, we obtain the *function of modulus of traces* $|\widetilde{f}_\rho|$ on $\widetilde{\Omega}$. The corresponding edge orientation was first studied by Bowditch [4] for the case where ρ is type preserving. By generalizing a result of Bowditch, Tan-Wong-Zhang proved the following result:

Proposition 4.6 (Tan-Wong-Zhang [16]). *If an irreducible representation ρ satisfies the BQ-conditions, then there exists a compact attracting subtree in Σ with respect to the edge orientation induced by $|\widetilde{f}_\rho|$.*

(II) *The real translation distance function.* Consider the real translation distance function a_ρ defined as follows:

$$a_\rho : \Omega \quad \to \quad \mathbb{R}_{\geq 0}$$
$$[W] \quad \mapsto \quad a(\rho(W))$$

where $a(\rho(W))$ is the real translation distance of $\rho(W)$ in $\mathbb{H}^3$ defined in the previous section. Its lift $\widetilde{a}_\rho$ is the *real translation distance function* on $\widetilde{\Omega}$.

By our discussion in the previous section, the trace $\mathrm{tr}\,(\xi)$ of a loxodromic element $\xi \in \mathrm{SL}(2,\mathbb{C})$ is

$$-2\cosh\frac{a(\xi) + i\alpha(\xi)}{2},$$

where $a(\xi) + i\alpha(\xi)$ is the complex translation distance of ξ. Notice that for a pair of positive number a and α, the difference

$$\left| 2\cosh\frac{a + i\alpha}{2} \right| - \exp\frac{a}{2}$$

is $O(\exp(-\frac{a}{2}))$ as a goes to infinity. Therefore Proposition 4.6 above implies a similar result for $\widetilde{a}_\rho$:

Proposition 4.7. *If an irreducible representation ρ satisfies the BQ-conditions, then there exists a compact attracting subtree in Σ with respect to the edge orientation induced by $\widetilde{a}_\rho$.*

Proof. By Theorem 3.3 of [16] and the fact that Fibonacci growth implies that the BQ-conditions are satisfied, a representation ρ satisfies the BQ-conditions if and only if $\log^+ |f_\rho|$ has Fibonacci growth. Let v be a vertex of Σ. The Fibonacci growth of $\log^+ |f_\rho|$ implies that the difference

$$||f_\rho|(W) - \exp \frac{a_\rho(W)}{2}|$$

for W with distance N to v is $O(\exp(-kN))$ as N goes to infinity, where k is the constant appearing in the lower Fibonacci growth inequality for $\log^+ |f_\rho|$. Another consequence of the Fibonacci growth result is that for any compact subset in $\mathbb{R}_{\geq 0}$, its $\log^+ |f_\rho|$-pre-image is finite. Combining these two facts, we can conclude that the edge orientation induced by $|\widetilde{f}_\rho|$ and that induced by $\widetilde{a}_\rho$ coincide on all but finitely many edges. Then this proposition follows from Proposition 4.6. $\square$

(III) *Angle function* The third function is called the *angle function* denoted by A_ρ. For the study in this part, we assume moreover that the ρ-image of all primitive elements are loxodromic which is the case for BQ-representations.

Let $([X], v) \in \widetilde{\Omega}$. Consider a representative (X, Y, Z) of the superbasis corresponding to v. Since all primitive elements are sent to loxodromic elements by ρ, the axes of $\rho(X)$, $\rho(Y)$ and $\rho(Z)$ exist, and we can orient them, so that their translation directions are the positive directions respectively. We denote the three oriented axes by δ_X, δ_Y and δ_Z respectively. We denote by γ_X, γ_Y and γ_Z the axes of π-rotations $r(X)$, $r(Y)$ and $r(Z)$ respectively, where $r(X)$, $r(Y)$ and $r(Z)$ satisfy:

$$\begin{aligned}
\rho(X) &= r(Y)r(Z), \\
\rho(Y) &= r(Z)r(X), \\
\rho(Z) &= -r(X)r(Y).
\end{aligned}$$

Then A_ρ at $([X], v)$ is defined to be the complementary angle of the angle between the axes of $\rho(Y)$ and $\rho(Z)$. Notice that the value of A_ρ depends on $[X]$, as well as v which is different from the previous two cases. We will show:

Proposition 4.8. *An irreducible representation ρ satisfies the BQ-conditions, then there exists a compact attracting subtree in Σ with respect to the edge orientation induced by A_ρ.*

Combining the above three propositions, we have the following corollary:

Corollary 4.9. *If an irreducible representation ρ satisfies BQ-condition, then there exists a compact subtree of Σ which is attracting with respect to all three edge orientations.*

4.3. **Proof of Proposition 4.8.**

4.3.1. *Cosine rule for right-angled hexagons.* In Section VI of [8], Fenchel gave an interpretation of the cosine rule and sine rule for a right-angled hexagon using what he called line matrices. They are called π-rotations in our work. We recall the cosine rule following his idea. As a convention, if two sides of H are non-degenerate, we assume that they are non-collinear. We also assume that two adjacent sides of H are not both degenerate.

Let us first assume that H is non-degenerate. Let s_1, s_2, s_3, s_4, s_5 and s_6 denote its sides in a cyclic order. All indices in this part will be considered up to mod 6. Let γ_n denote the geodesic containing s_n. We choose the orientation for each γ_n which is compatible with this cyclic order of s_n's. Let r_n denote the π-rotation with respect to γ_n. Since γ_{n-1} and γ_{n+1} are disjoint, the product $r_{n+1}r_{n-1}$ is a loxodromic element in $\mathrm{SL}(2,\mathbb{C})$ whose orientated axis is γ_n. Let l_n denote half of its complex translation distance. Then we have:

$$\cosh l_n = -\frac{1}{2}\mathrm{tr}\,(r_{n+1}r_{n-1}),$$

$$\sinh l_n = \frac{1}{2}i\,\mathrm{tr}\,(r_{n+1}r_n r_{n-1}).$$

The cosine rule for H is written as follows:

$$\cosh l_n = \cosh l_{n-2}\cosh l_{n+2} + \sinh l_{n-2}\sinh l_{n+2}\cosh l_{n+3}. \tag{8}$$

Now let us consider the degenerate case. There are three cases:

- there is one degenerate side;
- there are two degenerate sides which are non-adjacent;
- there are three alternate sides of H which are degenerate, so that H becomes a triangle.

If s_n is degenerate, then γ_n is the common perpendicular geodesic of γ_{n-1} and γ_{n+1}. The orientation on γ_n chosen so that the angle counted from the positive direction of γ_{n-1} to the positive direction of γ_{n+1} is positive with value in $(0,\pi)$. Then the identity (8) holds in the degenerate cases by replacing "cosh" and "sinh" by "cos" and "sin" respectively for the degenerate sides.

Next we would like to show that the identity (8) still holds when we choose arbitrary combination of orientations of γ_i's. To see this, we may rewrite (8) using π-rotations as follows:

$$-\frac{1}{2}\mathrm{tr}\,(r_{n+1}r_{n-1}) - \frac{1}{4}\mathrm{tr}\,(r_{n-1}r_{n-3})\mathrm{tr}\,(r_{n-3}r_{n+1})$$
$$= \frac{1}{8}\mathrm{tr}\,(r_{n-1}r_{n-2}r_{n-3})\mathrm{tr}\,(r_{n-3}r_{n+2}r_{n+1})\mathrm{tr}\,(r_{n-2}r_{n+2})$$

When we change the orientation of γ_n, the corresponding π-rotation is changed from r_n to $-r_n$. Since the π-rotation along each side appears either once in each term or twice in some terms, we can check that (8) still holds with arbitrary sides orientations.

4.3.2. *right-angled hexagon for an ordered triple.* Let ρ be a BQ-representation. Let (X,Y,Z) be an ordered triple. Let $(r(X),r(Y),r(Z))$ be a triple of π-rotations satisfying:

$$\rho(X) = r(Y)r(Z),$$
$$\rho(Y) = r(Z)r(X),$$
$$\rho(Z) = -r(X)r(Y).$$

Let δ_X, δ_Y, δ_Z, γ_X, γ_Y and γ_Z denote the oriented axes of $\rho(X)$, $\rho(Y)$, $\rho(Z)$, $r(X)$, $r(Y)$ and $r(Z)$ respectively. The orientations on δ_X, δ_Y and δ_Z are chosen to be the same as the translation directions of actions of $\rho(X)$, $\rho(Y)$ and $\rho(Z)$ respectively. The six geodesics bound a right-angled hexagon denoted by $H(X,Y,Z)$.
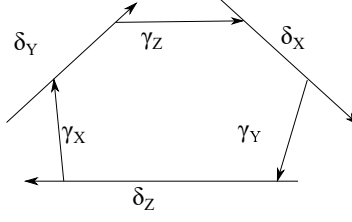
FIGURE 4. Right-angled hexagon $H(X, Y, Z)$

We denote by $r_\delta(X)$, $r_\delta(Y)$ and $r_\delta(Z)$ the three π-rotations along the oriented axes δ_X, δ_Y and δ_Z. We consider the complex transport distances:

$$
\begin{aligned}
2l(\delta_X) &= a(\delta_X) + i\alpha(\delta_X), \\
2l(\delta_Y) &= a(\delta_Y) + i\alpha(\delta_Y), \\
2l(\delta_Z) &= a(\delta_Z) + i\alpha(\delta_Z), \\
2l(\gamma_X) &= a(\gamma_X) + i\alpha(\gamma_X), \\
2l(\gamma_Y) &= a(\gamma_Y) + i\alpha(\gamma_Y), \\
2l(\gamma_Z) &= a(\gamma_Z) + i\alpha(\gamma_Z),
\end{aligned}
$$

of $r(Z)r(X)$, $r(Z)r(X)$, $r(X)r(Y)$, $r_\delta(Y)r_\delta(Z)$, $r_\delta(Z)r_\delta(X)$ and $r_\delta(X)r_\delta(Y)$, respectively. By Formula (2) above, we have:

$$
\begin{aligned}
2\cosh l(\delta_X) &= -\mathrm{tr}\,(r(Y)r(Z)), \\
2\cosh l(\delta_Y) &= -\mathrm{tr}\,(r(Z)r(X)), \\
2\cosh l(\delta_Z) &= -\mathrm{tr}\,(r(X)r(Y)), \\
2\cosh l(\gamma_X) &= -\mathrm{tr}\,(r_\delta(Y)r_\delta(Z)), \\
2\cosh l(\gamma_Y) &= -\mathrm{tr}\,(r_\delta(Z)r_\delta(X)), \\
2\cosh l(\gamma_Y) &= -\mathrm{tr}\,(r_\delta(X)r_\delta(Y)).
\end{aligned}
$$

Using the cosine rule for $H(X, Y, Z)$, we have the following relations:

$$
\cosh l(\gamma_X) = \frac{\cosh l(\delta_X) - \cosh l(\delta_Y) \cosh l(\delta_Z)}{\sinh l(\delta_Y) \sinh l(\delta_Z)}, \tag{9}
$$

$$
\cosh l(\gamma_Y) = \frac{\cosh l(\delta_Y) - \cosh l(\delta_Z) \cosh l(\delta_X)}{\sinh l(\delta_Z) \sinh l(\delta_X)}, \tag{10}
$$

$$
\cosh l(\gamma_Z) = \frac{\cosh l(\delta_Z) - \cosh l(\delta_X) \cosh l(\delta_Y)}{\sinh l(\delta_X) \sinh l(\delta_Y)}. \tag{11}
$$

Let $(x, y, z) := (\mathrm{tr}\,\rho(X), \mathrm{tr}\,\rho(Y), \mathrm{tr}\,\rho(Z))$. Then we have:

$$
\begin{aligned}
x &= -2\cosh l(\delta_X), \\
y &= -2\cosh l(\delta_Y), \\
z &= 2\cosh l(\delta_Z).
\end{aligned}
$$

We recall that they satisfy Relation (5):

$$
x^2 + y^2 + z^2 = xyz + \mu,
$$

where μ only depends on ρ. We can rewrite it as:

$$
\frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy} = 1 + \frac{\mu}{xyz}
$$

since ρ satisfies the BQ-conditions and so $x, y, z \neq 0$. Without loss of generality, we assume that $|z| \geq |y| \geq |x|$. Recall that the function $\log^+ |f_\rho|$ has Fibonacci growth. We consider the Fibonacci growth inequality for $\log^+ |f_\rho|$ with respect to a initial edge e on Σ. Let k and K be the two constant in the inequality for the lower bound and upper bound respectively. If (X, Y, Z) is a vertex on Σ with distance N to e and N is large enough, then we have:

$$|z| \geq |y| \geq \exp(k(N-1)).$$

This implies:

Lemma 4.10. *The sum*

$$\frac{1}{x}\left(\frac{z}{y} + \frac{y}{z}\right)$$

converges to 1 uniformly with error $O(\exp(-kN))$ as N goes to infinity.

This in turn implies:

Lemma 4.11. *The quantity*

$$\frac{-1}{2\sinh l(\delta_X)}\left(\frac{z}{y} - \frac{y}{z}\right)$$

converges to 1 uniformly with error $O(\exp(-kN))$ as N goes to infinity.

Proof.

$$
\begin{aligned}
&\frac{1}{4\sinh^2 l(\delta_X)}\left(\frac{z}{y} - \frac{y}{z}\right)^2 - 1 \\
={} &\frac{1}{4\sinh^2 l(\delta_X)}\left(\left(\frac{z}{y} - \frac{y}{z}\right)^2 - 4\sinh^2 l(\delta_X)\right) \\
={} &\frac{x^2}{4\sinh^2 l(\delta_X)}\left(\frac{1}{x^2}\left(\frac{z}{y} - \frac{y}{z}\right)^2 - \frac{4\sinh^2 l(\delta_X)}{x^2}\right) \\
={} &\coth^2 l(\delta_X)\left(\frac{1}{x^2}\left(\frac{y}{z} + \frac{z}{y}\right)^2 - 1\right).
\end{aligned}
$$

Notice that the zeros for the functions "cosh" and "sinh" are $(\pi/2 + n\pi)i$, $(\pi + 2n\pi)i$ and $2n\pi i$. If ρ is a BQ-representation, then the values of $\coth l(\delta_X)$ are bounded away from 0 and ∞ uniformly for all primitive elements X. By Lemma 4.10, the above quantity converges to 0 uniformly with error $O(\exp(-kN))$ as N goes to infinity.

Notice that the hyperbolic cosine function is injective on the subset of $\mathbb{C}$ consisting of complex numbers whose real part is positive and imaginary part is in $[0, 2\pi)]$. By Lemma 4.10 and the facts that $|z| \geq |y|$ and $x = -2\cosh l(\delta_X)$, we obtain the lemma. □

To finish the proof of 4.8, we prove the following proposition:

Proposition 4.12. *The triple $(l(\gamma(X)), l(\gamma(Y)), l(\gamma(Z)))$ converges to $(i\pi, i\pi, 0)$ uniformly with error $O(\exp(-kN))$ as N goes to ∞.*

Proof. It is enough to show the uniform convergence of

$$(\cosh l(\gamma_X), \cosh l(\gamma_Y), \cosh l(\gamma_Z))$$

to $(-1, -1, 1)$ with the same error control as N goes to ∞.

(I) For $\cosh l(\gamma_X)$, we consider Formula (9):

$$
\begin{aligned}
&\cosh l(\gamma_X) \\
=& \frac{\cosh l(\delta_X) - \cosh l(\delta_Y) \cosh l(\delta_Z)}{\sinh l(\delta_Y) \sinh l(\delta_Z)} \\
=& \frac{\cosh l(\delta_X)}{\sinh l(\delta_Y) \sinh l(\delta_Z)} - \frac{\cosh l(\delta_Y) \cosh l(\delta_Z)}{\sinh l(\delta_Y) \sinh l(\delta_Z)}.
\end{aligned}
$$

The proof of Proposition 4.7 shows that the real translation distance grows uniformly to infinity as N goes to infinity. When N is big, by our assumption, we have $\sinh l(\delta_Z)$ and $\cosh l(\delta_Z)$ (resp. $\sinh l(\delta_Y)$ and $\cosh l(\delta_Y)$) are close to $\exp(l(\delta_Z))/2$ (resp. $\exp(l(\delta_Z))/2$) with difference $O(\exp(-kN))$ as N goes to infinity. Hence, we conclude that $\cosh l(\gamma_X)$ converges to -1 uniformly with error $O(\exp(-kN))$ as N goes to infinity

(II) For $\cosh l(\gamma_Y)$, we us formula (10):

$$
\begin{aligned}
&\cosh l(\gamma_Y) \\
=& \frac{\cosh l(\delta_Y) - \cosh l(\delta_Z) \cosh l(\delta_X)}{\sinh l(\delta_Z) \sinh l(\delta_X)} \\
=& \left(\frac{\cosh l(\delta_Y)}{\cosh l(\delta_Z) \cosh l(\delta_X)} - 1 \right) \frac{\cosh l(\delta_Z) \cosh l(\delta_X)}{\sinh l(\delta_Z) \sinh l(\delta_X)} \\
=& \left(\frac{2y}{xz} - 1 \right) \frac{\cosh l(\delta_Z) \cosh l(\delta_X)}{\sinh l(\delta_Z) \sinh l(\delta_X)} \\
=& \left(\left(\frac{z}{xy} + \frac{y}{xz} \right) - 1 + \left(\frac{y}{xz} - \frac{z}{xy} \right) \right) \frac{\cosh l(\delta_Z) \cosh l(\delta_X)}{\sinh l(\delta_Z) \sinh l(\delta_X)}.
\end{aligned}
$$

By Lemma 4.10 and Lemma 4.11, we can conclude that $\cosh l(\gamma_Y)$ converges to -1 uniformly with error $O(\exp(-kN))$ as N goes to infinity.

(III) The proof for the convergence of $\cosh l(\gamma_Z)$ is similar to that for $\cosh l(\gamma_Y)$:

$$
\begin{aligned}
&\cosh l(\gamma_Z) \\
=& \frac{\cosh l(\delta_Z) - \cosh l(\delta_X) \cosh l(\delta_Y)}{\sinh l(\delta_X) \sinh l(\delta_Y)} \\
=& \left(\frac{\cosh l(\delta_Z)}{\cosh l(\delta_X) \cosh l(\delta_Y)} - 1 \right) \frac{\cosh l(\delta_X) \cosh l(\delta_Y)}{\sinh l(\delta_X) \sinh l(\delta_Y)} \\
=& \left(\frac{2z}{xy} - 1 \right) \frac{\cosh l(\delta_X) \cosh l(\delta_Y)}{\sinh l(\delta_X) \sinh l(\delta_Y)} \\
=& \left(\left(\frac{z}{xy} + \frac{y}{xz} \right) - 1 + \left(\frac{z}{xy} - \frac{y}{xz} \right) \right) \frac{\cosh l(\delta_Z) \cosh l(\delta_X)}{\sinh l(\delta_Z) \sinh l(\delta_X)}.
\end{aligned}
$$

By Lemma 4.10 and Lemma 4.11, we can conclude that $\cosh l(\gamma_Z)$ converges to 1 uniformly with error $O(\exp(-kN))$ as N goes to infinity. $\square$

Remark 4.13. Using the above proposition, in [11], Lee and Xu prove the equivalence between the BQ-conditions and the primitive stability for $\mathrm{PSL}(2, \mathbb{C})$-representations of $\mathbb{F}_2$. The latter was first defined and studied by Minsky in [13] where he studied the action of $\mathrm{Out}(\mathbb{F}_n)$ on the $\mathrm{PSL}(2, \mathbb{C})$-character variety of $\mathbb{F}_n$.

5. The Fermat point of a hyperbolic triangle

In Euclidean geometry, we may find different special points associated to a triangle satisfying different properties. We are interested in the one which is called the Fermat point. More precisely, consider a triangle in the Euclidean plane. Then its Fermat point is the one which realizes the minimum of the sum of distances to its three vertices, among all points in the plane. In this section, we introduce its hyperbolic counterpart.

5.1. **Convexity lemma in hyperbolic geometry.** Before going further, let us first recall the convexity property of the distance function in hyperbolic geometry which plays an important role in most of the proofs in the reminder of the paper. For any two points p and q in $\mathbb{H}^n$, we denote by pq the geodesic segment connecting them and by $|pq|$ the hyperbolic length of this segment.

Lemma (Convexity Lemma). *Let p_1, p_2, p_3 and p_4 be four points in n-dimensional hyperbolic space $\mathbb{H}^n$. Let p and p' be two points on p_1p_2 and p_3p_4 respectively, such that $|p_1p|/|p_1p_2| = |p_4p'|/|p_3p_4| = t$ for some $t \in (0, 1)$. Then,*

$$|pp'| \leq (1 - t)|p_1p_4| + t|p_2p_3|. \tag{12}$$

Moreover equality is realized if and only if all four points lie on the same geodesic and q is between p and p', where q is on p_1p_3 such that $|p_1q|/|p_1p_3| = t$.

This is a standard result in the differential geometry of spaces with non-positive curvature. To make the paper self-contained, we give a proof of this lemma in the appendix. As a convention, by the convexity lemma we will always refer to this lemma.

5.2. **Compact hyperbolic triangles.** A *compact triangle* in $\mathbb{H}^2$ is a triangle whose sides are geodesic segments with finite lengths. We will not consider the degenerate case, where the three vertices are collinear. All indices in this section are considered up to mod 3. The following definitions will be useful in the subsequent discussions:

Definition 5.1. A triangle in $\mathbb{H}^2$ is called $2\pi/3$-*acute* if all three internal angles are strictly smaller than $2\pi/3$; otherwise, it will be called $2\pi/3$-*obtuse*.

Remark 5.2. In particular, by our definition, a triangle with an internal angle equal to $2\pi/3$ is $2\pi/3$-obtuse.

Let Δ be a compact triangle in the hyperbolic plane $\mathbb{H}^2$ with vertices v_1, v_2 and v_3. Its complement in $\mathbb{H}^2$ has two connected components, each of which is an open set. We consider the bounded one and call it the Δ-*domain*.

Definition 5.3. A point p in $\mathbb{H}^2$ distinct from v_1, v_2 and v_3 is said to be a *balanced* point of Δ if it has the following property:

$$\angle v_1pv_2 = \angle v_2pv_3 = \angle v_3pv_1 = \frac{2\pi}{3}. \tag{13}$$

We denote by $\widetilde{p}$ a balanced point.

Proposition 5.4. *The triangle Δ admits a balanced point if and only if it is $2\pi/3$-acute. Moreover if it exists, it is unique and contained in the Δ-domain.*

Proof. We notice that if a point p is contained in the complement of the Δ-domain, then we have $\angle v_{j-1}pv_j + \angle v_j pv_{j+1} = \angle v_{j-1}pv_{j+1}$ for some j. Therefore to prove this proposition, it is enough to consider the points contained in the Δ-domain.

To prove the "only if" part, we assume that Δ is $2\pi/3$-obtuse and $\angle v_2v_1v_3 \geq 2\pi/3$. Let p be a point in the Δ-domain. Let q denote the intersection point between the geodesic containing v_2p and the side v_1v_3.
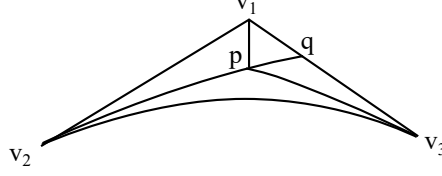

FIGURE 5. $2\pi/3$-obtuse

We then have the following relation:

$$\angle v_2pv_3 > \angle v_2qv_3 > \angle v_2v_1v_3 \geq \frac{2\pi}{3}. \tag{14}$$

Therefore p cannot be balanced, hence the " only if " part.

Now we assume that Δ is $2\pi/3$-acute. We may assume that $\angle v_2v_3v_1 < \pi/3$. Consider the side v_2v_3. For any angle $\alpha \in (0,\pi)$. We say that a path in $\mathbb{H}^2$ is α-equiangular with respect to v_2v_3 if $\angle v_2pv_3$ is constant (say $= \alpha$) when p moves along this path. One may check that for each angle α, there are two equiangular paths, symmetric with respect to v_2v_3. Moreover each one of them is an open path with endpoints at v_2 and v_3, and bounds a convex subset together with v_2v_3. Since we have:

$$\angle v_2v_3v_1 \quad < \quad \pi/3,$$
$$\angle v_1v_2v_3 \quad < \quad 2\pi/3,$$

one of the $2\pi/3$-equiangular paths of v_2v_3 intersects v_1v_3. For same reason, one of the $2\pi/3$-equiangular paths with respect to v_1v_3 intersects v_2v_3. Therefore these two paths intersect in the Δ-domain. The intersection point is a balanced point for Δ.
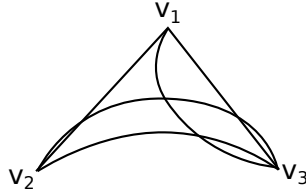

FIGURE 6. $2\pi/3$-acute

Let $\widetilde{p}$ be a balanced point of Δ. Notice that the segments $\widetilde{p}v_1$, $\widetilde{p}v_2$ and $\widetilde{p}v_3$ separate Δ into three $2\pi/3$-obtuse triangles. Using the same argument as in the " only if " part, we can show that $\widetilde{p}$ is the only balanced point. $\square$

5.3. **The Fermat Point of a compact triangle.** Consider Δ the compact triangle in the hyperbolic plane $\mathbb{H}^2$ with vertices v_1, v_2 and v_3. We define a function $L : \mathbb{H}^2 \to \mathbb{R}_{>0}$ by sending each point p to $L(p) = |v_1 p| + |v_2 p| + |v_3 p|$.

Definition 5.5. The *Fermat point* of Δ is a point in $\mathbb{H}^2$ realizing the minimum of the function L.

Theorem 5.6. *The Fermat point of the triangle Δ exists and is unique. Moreover,*

(1) *if Δ is $2\pi/3$-acute, then its balanced point is its Fermat point;*
(2) *if Δ is $2\pi/3$-obtuse, then its Fermat point is the vertex with the biggest internal angle.*

We first prove the following lemma:

Lemma 5.7. *If p lies outside the closure of Δ-domain, we can always find a point q on Δ, such that $L(p) > L(q)$.*

Proof. Assume that p lies outside Δ. There are two possible cases:

(1) there exists j such that pv_j intersects $v_{j-1}v_{j+1}$;
(2) there exists j such that the vertex v_j lies in the $\Delta pv_{j-1}v_{j+1}$-domain.

Without loss of generality, in the following discussion, we assume that $j = 3$ in both cases. In the first case, we have
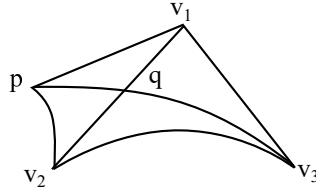
$$|pp_3| > |qp_3|.$$

FIGURE 7. First case

At the same time, we have the triangular inequality for the triangle Δpv_1v_2:

$$|pv_1| + |pv_2| > |v_1v_2| = |qv_1| + |qv_2|.$$

Hence, we have $L(p) > L(q)$ in the first case.

In the second case, we have:

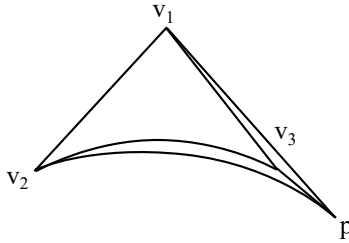$$\angle pv_3v_1 + \angle pv_3v_2 + \angle v_1v_3v_1 = 2\pi.$$

FIGURE 8. Second case

Therefore, either $\angle pv_3v_1$ or $\angle pv_3v_2$ is bigger than $\pi/2$. We assume that it is $\angle pv_3v_1$. Then we have:

$$|v_1v_3| < |pv_1|.$$

On the other hand, by the triangle inequality for Δpv_3v_2, we have:

$$|v_3v_2| < |pv_3| + |pv_2|.$$

Therefore, we have $L(p) > L(v_3)$ in the second case. $\hfill\square$

Proof of Theorem 5.6. By the above lemma, to show the existence and uniqueness of the minimum point for the function L, it is enough to consider the points contained in the closure of the Δ-domain. We prove the theorem by showing that in each case, the point mentioned in the statement of the theorem has strictly smaller L-value than any other point in the closure of the Δ-domain.

Case I: Δ is $2\pi/3$-obtuse. Without loss of generality, let v_1 be the vertex of Δ with interior angle bigger than or equal to $2\pi/3$. Let p be a point in the closure of Δ-domain different from v_1. Then the theorem is equivalent to saying that for any point p in the Δ-domain different from v_1 satisfies:

$$L(p) > L(v_1).$$

Let us consider the angles $\angle v_2v_1p$ and $\angle v_3v_1p$ and denote them by α_p and β_p respectively for simplification of notation. By assumption, we have

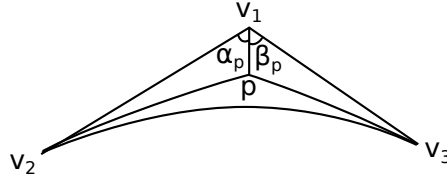$$\alpha_p + \beta_p = \angle v_2v_1v_3 \geq \frac{2\pi}{3}.$$

FIGURE 9. α_p and β_p

We first discuss the case where one of them is bigger or equal to $\pi/2$. Without loss of generality, we assume that $\alpha_p \geq \pi/2$. Then it is easy to check:

$$\begin{aligned} |pv_2| &\geq |v_1v_2|; \\ |pv_1| + |pv_3| &\geq |v_1v_3|. \end{aligned}$$

Moreover the two equalities can be realized at the same time if and only if $p = v_1$. Therefore, if $p \neq v_1$, we have

$$L(p) > L(v_1).$$

Secondly, we observe that if $|pv_1| \geq |v_1v_2|$ or $|pv_1| \geq |v_1v_3|$, since $|v_2v_3|$ is strictly bigger than $|v_1v_2|$ and $|v_1v_3|$, we have again

$$L(p) > L(v_1).$$

Now we discuss the remaining case. We may assume that

$$\alpha_p < \pi/2;$$
$$\beta_p < \pi/2;$$
$$|pv_1| < |v_1v_2|;$$
$$|pv_1| < |v_1v_3|.$$

By the cosine rule for Δv_1pv_2 and Δv_1pv_3, we have the following two identities:

$$\cosh|pv_2| \quad = \quad \cosh|v_1v_2|\cosh|pv_1| - \sinh|v_1v_2|\sinh|pv_1|\cos\alpha_p; \qquad (15)$$
$$\cosh|pv_3| \quad = \quad \cosh|v_1v_3|\cosh|pv_1| - \sinh|v_1v_3|\sinh|pv_1|\cos\beta_p. \qquad (16)$$

Our goal is to show that there is $t \in (0,1)$ such that

$$\cosh|pv_2| \quad \geq \quad \cosh((1-t)|pv_1|)\cosh(|v_1v_2| - t|pv_1|); \qquad (17)$$
$$\cosh|pv_3| \quad \geq \quad \cosh(t|pv_1|)\cosh(|v_1v_3| - (1-t)|pv_1|). \qquad (18)$$

To simplify the notation, we set

$$a' = |v_1v_2|;$$
$$b' = |v_1v_3|;$$
$$a = |pv_2|;$$
$$b = |pv_3|;$$
$$c = |pv_1|.$$

We rewrite the right hand side of (15) as follows:

$$\cosh a$$
$$= \quad \cosh a'\cosh(tc + (1-t)c) - \sinh a'\sinh(tc + (1-t)c)\cos\alpha_p$$
$$= \quad \cosh a'\Big(\cosh(tc)\cosh((1-t)c) + \sinh(tc)\sinh((1-t)c)\Big) -$$
$$\quad - \sinh a'\Big(\sinh(tc)\cosh((1-t)c) + \cosh(tc)\sinh((1-t)c)\Big)\cos\alpha_p$$
$$= \quad \cosh((1-t)c)\cosh(a' - tc) + (1 - \cos\alpha_p)\sinh(tc)\sinh(a' + (1-t)c) -$$
$$\quad - \cos\alpha_p\sinh((1-t)c)\sinh(a' - tc).$$

Similarly, the right hand side of (16) can be rewritten as:

$$\cosh b$$
$$= \quad \cosh(tc)\cosh(b' - (1-t)c) + \cos\beta_p\sinh((1-t)c)\sinh(b' + tc)$$
$$\quad -(1 - \cos\beta_p)\sinh(tc)\sinh(b' - (1-t)c).$$

To show (17) and (18) hold, it is enough to show that there exists $t \in (0,1)$ such that:

$$0 < \frac{\cos\alpha_p}{1 - \cos\alpha_p} \leq \frac{\sinh(tc)}{\sinh((1-t)c)} \leq \frac{1 - \cos\beta_p}{\cos\beta_p} < \infty. \qquad (19)$$

If we remove the middle term, the remaining relations hold. To see this, we notice that

$$\frac{\pi}{6} < \alpha_p < \frac{\pi}{2},$$
$$\frac{\pi}{6} < \beta_p < \frac{\pi}{2},$$

hence, the above relation is equivalent to

$$\cos \alpha_p + \cos \beta_p \leq 1,$$

which moreover can be written as

$$2 \cos \frac{\alpha_p + \beta_p}{2} \cos \frac{\alpha_p - \beta_p}{2} \leq 1.$$

The above inequality holds, since we have:

$$\frac{2\pi}{3} \leq \alpha_p + \beta_p \leq \pi.$$

On the other hand, the function

$$\frac{\sinh(tc)}{\sinh((1-t)c)}$$

has image $(0, \infty)$ for $t \in (0, 1)$ and is monotonically increasing as t grows. Hence we have the existence of t for (19) to hold. Using the inequalities (17) and (18) and the monotonicity of the hyperbolic cosine function, we obtain moreover that for some $t \in (0, 1)$:

$$\begin{aligned} a &\geq a' - tc; \\ b &\geq b' - (1-t)c, \end{aligned}$$

where the two equalities hold if and only if when $p = v_1$. In that case, we have $c = 0$. Hence we have

$$L(p) = a + b + c > a' + b' = L(p_1),$$

for any p in the closure of Δ-domain different from v_1.

Case II: Δ is $2\pi/3$-acute. We first prove the following lemma:

Lemma 5.8. *Let q be a point in $\mathbb{H}^2$. Let r_1, r_2 and r_3 be three distinct rays in $\mathbb{H}^2$ starting at q such that the angle between any two of them is $2\pi/3$. Let q_1, q_2 and q_3 be three points on r_1, r_2 and r_3 respectively. Then,*

$$|qq_1| + |qq_2| + |qq_3| < |q_1q_2| + |q_1q_3|.$$

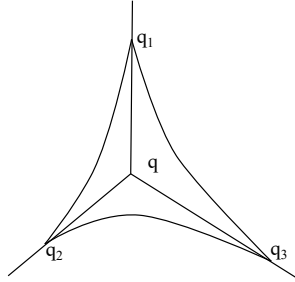*Proof of Lemma 5.8.* The proof uses hyperbolic trigonometry as follows.

FIGURE 10

The cosine rule for Δqq_1q_2 and Δqq_1q_3 gives the following two identities:

$$\cosh |q_1q_2| = \cosh |qq_1| \cosh |qq_2| - \sinh |qq_1| \sinh |qq_2| \cos \frac{2\pi}{3};$$

$$\cosh |q_1q_3| = \cosh |qq_1| \cosh |qq_3| - \sinh |qq_1| \sinh |qq_3| \cos \frac{2\pi}{3}.$$

Equivalently, we have:

$$\cosh|q_1q_2| = \cosh|qq_1|\cosh|qq_2| + \frac{1}{2}\sinh|qq_1|\sinh|qq_2|;$$

$$\cosh|q_1q_3| = \cosh|qq_1|\cosh|qq_3| + \frac{1}{2}\sinh|qq_1|\sinh|qq_3|.$$

Therefore we have:

$$\cosh|q_1q_2| = \cosh^2(\frac{1}{2}|qq_1|)\cosh|qq_2| + \sinh^2(\frac{1}{2}|qq_1|)\cosh|qq_2|$$

$$+ \sinh(\frac{1}{2}|qq_1|)\sinh|qq_2|\cosh(\frac{1}{2}|qq_1|);$$

$$\cosh|q_1q_3| = \cosh^2(\frac{1}{2}|qq_1|)\cosh|qq_3| + \sinh^2(\frac{1}{2}|qq_1|)\cosh|qq_3|$$

$$+ \sinh(\frac{1}{2}|qq_1|)\sinh|qq_3|\cosh(\frac{1}{2}|qq_1|).$$

This implies:

$$\cosh|q_1q_2| = \cosh(\frac{1}{2}|qq_1|)\cosh(\frac{1}{2}|qq_1| + |qq_2|) + \sinh^2(\frac{1}{2}|qq_1|)\cosh|qq_2|;$$

$$\cosh|q_1q_3| = \cosh(\frac{1}{2}|qq_1|)\cosh(\frac{1}{2}|qq_1| + |qq_3|) + \sinh^2(\frac{1}{2}|qq_1|)\cosh|qq_3|.$$

Since the hyperbolic cosine function is positve and strictly monotonically increasing on $\mathbb{R}_{>0}$, we have the following two inequalities:

$$|q_1q_2| > \frac{1}{2}|qq_1| + |qq_2|;$$

$$|q_1q_3| > \frac{1}{2}|qq_1| + |qq_3|.$$

Taking the sum of the above two inequalities complete the proof of the lemma.

$\square$

Now consider Δ and its balanced point $\widetilde{p}$. The three segments $v_1\widetilde{p}$, $v_2\widetilde{p}$ and $v_3\widetilde{p}$ separate the closure of the Δ-domain into three closed subsets bounded by triangles $\Delta v_1v_2\widetilde{p}$, $\Delta v_2v_3\widetilde{p}$ and $\Delta v_1v_3\widetilde{p}$ respectively. It is sufficient to discuss the case where p is contained in one of the three closed subsets. Without loss of generality, we assume p is in the part bounded by $\Delta v_1v_2\widetilde{p}$. Moreover, we may assume that v_3p intersects $v_1\widetilde{p}$ at q, see figure below.

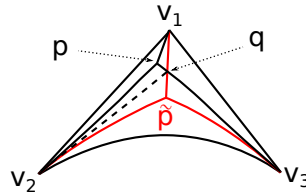FIGURE 11. $2\pi/3$-acute

Since the triangle $\Delta v_1v_2\widetilde{p}$ is $2\pi/3$-obtuse, by the discussion for Case I above, we have

$$|v_1q| + |v_2q| \le |v_1p| + |v_2p| + |pq|,$$

where equality holds if and only if $p = q$. On the other hand, by Lemma 5.8, we have

$$|\widetilde{p}v_2| + |\widetilde{p}v_3| + |\widetilde{p}q| + |qv_1| \leq |qv_2| + |qv_3| + |qv_1|,$$

where equality holds if and only if $q = \widetilde{p}$. This implies that $L(\widetilde{p}) \leq L(p)$ and equality holds if and only if $p = \widetilde{p}$. Hence, $\widetilde{p}$ is the Fermat point of Δ. $\square$

Remark 5.9. Since the closed set bounded by Δ is compact and the function L is continuous, another way to show the existence of the Fermat point is by using the extreme value theorem. Then we have uniqueness by the convexity lemma. Furthermore, by Lemma 5.8, for each case, only the point mentioned in the theorem can be a local minimum for L from which we obtain the theorem.

6. The Fermat Point and Steiner tree of a triple of geodesics in $\mathbb{H}^3$

We would like to generalize the definition and discussion above to the Fermat point of a triple of geodesics in $\mathbb{H}^3$. More precisely, given three geodesics in $\mathbb{H}^3$, the Fermat point is defined to be the point realizing the minimum of the sum of the distances to the three geodesics among all points in $\mathbb{H}^3$. With the help of the Fermat point, we can find a graph connecting the three given geodesics with shortest length as possible, which will be called the Fermat tree.

We will also be interested in the question whether the length of the double by the involution of $\mathbb{H}^3$ determined by one of the three geodesics is still shortest. Hence in the end of this section, we will introduce another modified length for a graph connecting a triple of geodesics which will be called the Steiner length. The graph with the shortest Steiner length will be called the Steiner tree for the given triple of geodesics. The result of this section will be used in the proof of the main theorem.

6.1. **Triple of geodesics of $\mathbb{H}^3$ in general position.**

In the previous section, we considered compact triangles in $\mathbb{H}^2$. To avoid the degenerate cases, we adopt the convention that the three vertices are distinct and do not lie on the same geodesic. Similarly, we would also like to avoid certain degenerate cases when studying triples of geodesics. For this purpose, we say that three geodesics in $\mathbb{H}^3$ are in *general position* if

(1) they are pairwise disjoint;
(2) each pair does not share any end point on the boundary of $\mathbb{H}^3$;
(3) they are not all orthogonal to some geodesic in $\mathbb{H}^3$.

Such a triple will be called a *generic triple*.

We can embed $\mathbb{H}^2$ into $\mathbb{H}^3$ isometrically. Then we can consider the three vertices of a triangle in $\mathbb{H}^2$ as the intersection points between $\mathbb{H}^2$ and three geodesics in $\mathbb{H}^3$ orthogonal to it respectively. Hence the triangle case discussed in the previous section can be considered as a special case of this part.

Convention: In the reminder of the paper, all triples of geodesics that we consider will be generic, and we will omit the word "generic". The indices in the discussion will be considered up to mod 3. For consistency with the triangle case, we will use the same notation.

Let $(\gamma_1, \gamma_2, \gamma_3)$ be a triple of geodesics in $\mathbb{H}^3$. Let p be a point in $\mathbb{H}^3$. We denote by p_1, p_2 and p_3 its orthogonal projections to γ_1, γ_2 and γ_3 respectively.

Definition 6.1. The point p is said to be *planar* with respect to $(\gamma_1, \gamma_2, \gamma_3)$ if one of the following conditions is satisfied:

(1) p is not contained in any of the three geodesics, and it is contained in the hyperbolic plane determined by its projections p_1, p_2 and p_3;

(2) p is on γ_j for some j, such that $\angle p_{j-1}pu = \angle p_{j+1}pu'$, where $\gamma_j = [u, u']$.

Remark 6.2. If p satisfies the first condition in the above definition, the plane containing it and its projections are unique. This is because that the plane contains the triangle determined by its three projections. If this triangle is degenerate, there will be some geodesic orthogonal to the three geodesics, contradicting to the fact that the three geodesics are in general position.

Proposition 6.3. *There is a unique planar point of type* (2) *for each geodesic* γ_j *in the triple.*

Proof. Let $\gamma_j = [u, u']$. For a point $p \in \gamma_j$, the angle $\angle p_{j+1}pu'$ is strictly increasing from 0 to π when p moves from u to u', while at the same time, the angle $\angle p_{j-1}pu$ is strictly decreasing from π to 0. Hence there is a unique point p where we have $\angle p_{j+1}pu' = \angle p_{j-1}pu$. $\qquad\square$

We denote the unique planar point on γ_j by v_j. We denote by v_{jj-1} and v_{jj+1} its orthogonal projections to γ_{j-1} and γ_{j+1} respectively. We borrow the terminology from the triangle case and give the following definition:

Definition 6.4. The angle $\angle v_{jj-1}v_jv_{jj+1}$ is called the *internal angle* for γ_j.

Similar to the triangle case, we define

Definition 6.5. A planar point p in $\mathbb{H}^3$ is said to be a *balanced point* for $(\gamma_1, \gamma_2, \gamma_3)$ if p is not contained in any of the three geodesics and it has the following property:

$$\angle p_1 p p_2 = \angle p_2 p p_3 = \angle p_1 p p_3 = \frac{2\pi}{3}. \tag{20}$$

As before, we use $\widetilde{p}$ to denote a balanced point for a triple of geodesics.

Definition 6.6. A triple $(\gamma_1, \gamma_2, \gamma_3)$ is said to be $2\pi/3$-*acute* if all internal angles are strictly smaller than $2\pi/3$. If there is one internal angle greater or equal to $2\pi/3$, it is called $2\pi/3$-*obtuse*.

To a triple $(\gamma_1, \gamma_2, \gamma_3)$, we can associate a unique right angled hexagon H by considering the common perpendicular geodesics for the pairs of geodesics in the triple. Since we assume that the triple is in general position, this right-angled hexagon is compact as a subset of $\mathbb{H}^3$.

Definition 6.7. The right angled hexagon H associated to a triple $(\gamma_1, \gamma_2, \gamma_3)$ is said to be $2\pi/3$-*acute* (resp. $2\pi/3$-*obtuse*) if the triple is $2\pi/3$-acute (resp. $2\pi/3$-obtuse).

Remark 6.8. Notice that it is possible that the right angled hexagon is degenerate, because some of its sides on γ_1, γ_2 and γ_3 may have 0 length. In particular, if all these three sides have 0 length, we get to the triangle case. Then the definitions of being $2\pi/3$-acute and being $2\pi/3$-obtuse coincide with those that we defined in the previous section for a triangle.

6.2. **Fermat point for a triple of geodesics.** For any point $p \in \mathbb{H}^3$, we denote by $d_1(p)$, $d_2(p)$ and $d_3(p)$ the distances from p to γ_1, γ_2 and γ_3 respectively. We define the function $L : \mathbb{H}^3 \to \mathbb{R}$ by

$$L(p) = d_1(p) + d_2(p) + d_3(p). \tag{21}$$

By the convexity lemma, we can see that the function L for the triangle case is a restriction of the one defined by (21) to the plane containing the triangle. Similar to the triangle case, we give the following definition:

Definition 6.9. The *Fermat point* for $(\gamma_1, \gamma_2, \gamma_3)$ is the point in $\mathbb{H}^3$ which realizes the minimum of the function L. The graph obtained by connecting the Fermat point to its orthogonal projections to γ_1, γ_2 and γ_3 is called the *Fermat tree* for $(\gamma_1, \gamma_2, \gamma_3)$.

Similar to the triangle case, we have

Theorem 6.10. *The Fermat point for $(\gamma_1, \gamma_2, \gamma_3)$ exists and is unique.*

Proof. We use the extreme value theorem to prove this theorem.

For a positive number $R \in \mathbb{R}_{\geq 0}$, we denote by $N_j(R)$ the R neighborhood of γ_j. Since γ_j's are disjoint in $\overline{\mathbb{H}^3}$, the set

$$N(R) := N_1(R) \cap N_2(R) \cap N_3(R) \tag{22}$$

is compact for any R. Moreover, for $R < R'$, we have $N(R) \subset N(R')$.

Let R_H be the sum of the lengths of all sides of the right angled hexagon H associated to $(\gamma_1, \gamma_2, \gamma_3)$. Then $N(R_H)$ is non-empty, since the points on $H \cap \gamma_1$ are contained in $N(R_H)$. Let q be a point on $H \cap \gamma_1$, then we have $L(q) < R_H$. Therefore for any $p \in \mathbb{H}^3 \setminus N(R_H)$, we always have $L(p) > R_H > L(q)$.

Meanwhile, since L is continuous on $\mathbb{H}^3$ and $N(R_H)$ is compact, we can use the extreme value theorem and get a point in $N(R_H)$ realizing the minimum of L on $N(R_H)$. By the discussion above, this point also realizes the minimum of L on $\mathbb{H}^3$. By definition, it is the Fermat point for $(\gamma_1, \gamma_2, \gamma_3)$.

The uniqueness of the Fermat point follows from the convexity lemma.

Assume that there are two distinct points p and p' both realizing the minimum of L. Let p_m denote the midpoint of the geodesic segment pp'. Then by the convexity lemma, for any geodesic γ_j in the triple, we have:

$$d_j(p_m) \leq \frac{1}{2} d_j(p) + \frac{1}{2} d_j(p'),$$

where equality is attained when the geodesic passing through p_1 and p_2 is either γ_j or orthogonal to γ_j. Therefore, we have

$$L(p_m) \leq \frac{1}{2} L(p) + \frac{1}{2} L(p'),$$

where equality holds if and only if for any $j = 1, 2, 3$, the geodesic γ_j is either the same as the geodesic passing through p and p', or orthogonal to it. Recall that as a convention, we only consider the triple $(\gamma_1, \gamma_2, \gamma_3)$ in general position. Hence, the inequality is strict, which contradicts the fact that $L(p_1) = L(p_2)$ is the minimal value of L on $\mathbb{H}^3$. $\square$

By considering the property of the Fermat point of a triangle, an immediate observation is:

Observation 6.11. If $\widetilde{p}$ is the Fermat point of a triple $(\gamma_1, \gamma_2, \gamma_3)$, then it is planar and it is the Fermat point of the triangle $\Delta\widetilde{p}_1\widetilde{p}_2\widetilde{p}_3$. Therefore, if the Fermat point is not contained in any γ_j, it must be balanced. If it is on γ_j for some j, then the angle $\angle\widetilde{p}_{j-1}\widetilde{p}\widetilde{p}_{j+1}$ is bigger than or equal to $2\pi/3$.

The least obvious part of this observation is that when the Fermat point $\widetilde{p}$ is on γ_j for some j, it must be planar. Without loss of generality, we may assume that $j = 1$. The fact that $\widetilde{p}$ must be planar is a consequence of the following lemma:

Lemma 6.12. *The unique planar point v_1 on γ_1 realizes the minimum of the sum of the distances to γ_2 and γ_3 among all points on γ_1.*

Proof. Let $\gamma_1 = [u, u']$. The existence of a point realizing the minimum of the sum of distances to γ_2 and γ_3 is clear, since the distance from $p \in \gamma_1$ to γ_2 and that from p to γ_3 approaches infinity when p goes to u or u'.

Consider a point $p \in \gamma_1$ and its projections p_2 and p_3 to γ_2 and γ_3 respectively. We consider the one parameter subgroup of $\mathrm{PSL}(2, \mathbb{C})$ consisting of all elliptic isometries with axis γ_1. There is an isometry in this subgroup such that the image p_3' of p_3 under this isometry is contained in the plane determined by γ_1 and p_2, moreover p_3' and p_2 are separated by γ_1. Notice that $|pp_3| = |pp_3'|$.
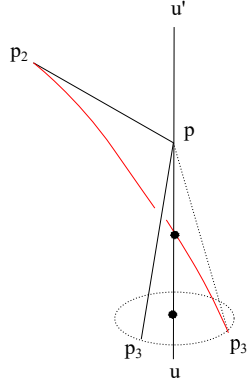

FIGURE 12. The planar point realizes the minimum.

If a point $p \in \gamma_1$ realizes the minimum of the sum of distance to γ_2 and γ_3 among all points in γ_1, then p must realize the minimum of sum of distances to p_2 and p_3 among all points in γ_1, hence it realizes the minimum of the sum of the distance between p_2 and p_3'. Hence p is contained in the geodesic segment p_2p_3' and we have

$$\angle p_2pu' = \angle p_3'pu = \angle p_3pu.$$

Since v_1 is the only point with this property, we have $p = v_1$. $\square$

Our next result is to show that the reciprocal of the observation is also true:

Theorem 6.13. *Let $(\gamma_1, \gamma_2, \gamma_3)$ be a triple of geodesics in general position. Then,*
   (1) if the triple admits a balanced point, then the balanced point is the Fermat point;
   (2) if the triple is $2\pi/3$-obtuse, then the Fermat point of this triple is the vertex of the unique internal angle bigger or equal to $2\pi/3$.

We first give two immediate corollaries that we can deduce from the above theorem:

Corollary 6.14. *A triple admits a balanced point if and only if it is $2\pi/3$-acute, and the balanced point, if it exists, is unique.*

Corollary 6.15. *If $(\gamma_1, \gamma_2, \gamma_3)$ is $2\pi/3$-obtuse, then there is a unique internal angle which is bigger or equal to $2\pi/3$. Hence, for any triple, the sum of interior angles is bounded from above by $7\pi/3$.*

The remainder of this subsection is occupied with the proof of the above theorem. We first give the proof for Fact (1) in the theorem:

Proof of Fact (1) in Theorem 6.13. Let $\widetilde{p}$ be a balanced point for $(\gamma_1, \gamma_2, \gamma_3)$. Let $\widetilde{p}_j$ denote the orthogonal projection of $\widetilde{p}$ to γ_j for $j = 1, 2, 3$. Let P denote the plane containing the triangle $\Delta\widetilde{p}_1\widetilde{p}_2\widetilde{p}_3$. Let P_j denote the plane orthogonal to $\widetilde{p}\widetilde{p}_j$ for $j = 1, 2, 3$. Therefore, the geodesic γ_j will be contained in P_j. We define the function $L' : \mathbb{H}^3 \to \mathbb{R}_{\geq 0}$ sending each point to the sum of its distances to the three planes P_1, P_2 and P_3.

Let p be any point in $\mathbb{H}^3$. By definition, we have $L(p) \geq L'(p)$. Let p' denote the image of p under the reflection of $\mathbb{H}^3$ with respect to P. Then we have $L'(p) = L'(p')$. Let p_m denote the mid point of pp'. Then p_m is in P. Moreover, we have $L'(p_m) \leq L'(p)$, where the equality holds if and only if p is in P. In that case, the three points p, p' and p_m coincide.
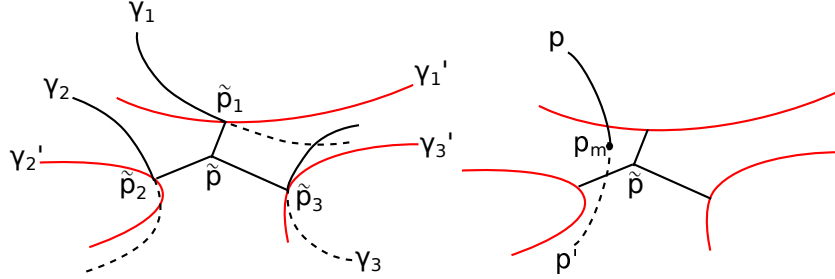

FIGURE 13. The three red geodesics and $\widetilde{p}$ are coplane.

Since $L'(\widetilde{p}) = L(\widetilde{p})$, to show that $L(\widetilde{p}) < L(p)$ for any $p \in \mathbb{H}^3$ different from $\widetilde{p}$, it is enough to prove the following fact:

$$L'(\widetilde{p}) < L'(p_m). \tag{23}$$

Let γ'_1, γ'_2 and γ'_3 be the intersection geodesics of P with P_1, P_2 and P_3 respectively. Notice that they may have intersections points with each other. By their construction, the point $\widetilde{p}$ is not contained in any of these three geodesics. By the convexity lemma, $L'(p_m)$ equals to the sum of distances from p_m to γ'_1, γ'_2 and γ'_3 respectively.

By the above discussion, to prove Fact (1), it is enough to prove that a balanced point $\widetilde{p}$ realizes the minimum of the sum of distances to γ'_1, γ'_2 and γ'_3 among all points in P.

The complement of the three geodesics γ'_1, γ'_2 and γ'_3 in P has several connected open components. We consider the one containing the balanced point $\widetilde{p}$ and denote

it by D. Depending on the number of intersection points, the shape of D has one of the following forms:
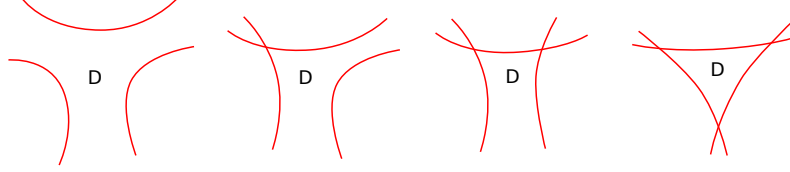

FIGURE 14. Possible shapes of D

By using a similar argument to the proof of Lemma 5.7, we can show that for any point $p \in P \setminus \overline{D}$, there is a point $q \in \overline{D}$, such that $L'(q) < L'(p)$. Therefore, from now on, we only need to consider points in $\overline{D}$.

We consider the geodesic segments $\widetilde{p}\widetilde{p}_1$, $\widetilde{p}\widetilde{p}_2$ and $\widetilde{p}\widetilde{p}_3$. Their union separates $\overline{D}$ into three parts. Below we will prove that $L'(p) \geq L(\widetilde{p})$ for any p contained in the part whose boundary contains $\widetilde{p}\widetilde{p}_2$ and $\widetilde{p}\widetilde{p}_3$ where equality holds if and only if $p = \widetilde{p}$. The proof for points in other two parts will be the same.

Without loss of generality, we assume that pp_1 intersects $\widetilde{p}\widetilde{p}_3$ at q. Let q_2 denote the orthogonal projection of q to γ'_2.
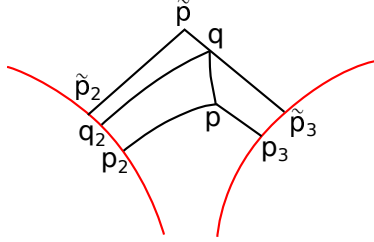

FIGURE 15

Since $\angle \widetilde{p}_2\widetilde{p}\widetilde{p}_3 = 2\pi/3$, we have $\angle \widetilde{p}_3qq_2 \geq 2\pi/3$ where equality holds if and only if $q = \widetilde{p}$. We first prove the following inequality:

$$|q\widetilde{p}_3| + |qq_2| \leq |pp_2| + |pp_3| + |pq|, \tag{24}$$

where equality holds if and only if $p = q$.

If $p \neq q$, the geodesic segment pq separates $\angle \widetilde{p}_3qq_2$ into two angles $\angle \widetilde{p}_3qp$ and $\angle pqq_2$. We denote them by α and β respectively. Since $q \in \widetilde{p}\widetilde{p}_3$, we have $\alpha < \pi/3$. If $\beta \geq \pi/2$, then we have

$$\begin{aligned} |qq_2| &< |pp_2|, \\ |q\widetilde{p}_3| &\leq |pp_3| + |pq|. \end{aligned}$$

Hence, the inequality in (24) holds in this case.

Now we assume that $\beta < \pi/2$. By using the cosine rule for a quadrilateral with two right angles, we have the following relations:

$$\begin{aligned} \sinh |pp_3| &= \sinh |q\widetilde{p}_3| \cosh |pq| - \cosh |q\widetilde{p}_3| \sinh |pq| \cos \alpha, \\ \sinh |pp_2| &= \sinh |qq_2| \cosh |pq| - \cosh |qq_2| \sinh |pq| \cos \beta. \end{aligned}$$

We use the same technique as in the triangle case. We use a, b, a', b' and c to denote $|q\widetilde{p_3}|$, $|qq_2|$, $|pp_3|$, $|pp_2|$ and $|pq|$ respectively. Let $t \in [0,1]$. We rewrite the above relations as follows:

$$\sinh a' = \sinh a \cosh(tc + (1-t)c) - \cosh a \sinh(tc + (1-t)c)\cos\alpha,$$
$$\sinh b' = \sinh b \cosh(tc + (1-t)c) - \cosh b \sinh(tc + (1-t)c)\cos\beta.$$

Then we have:

$$\sinh a' = \sinh(a - tc)\cosh((1-t)c) + \sinh(tc)\cosh(a + (1-t)c)(1 - \cos\alpha) -$$
$$- \sinh((1-t)c)\cosh(a - tc)\cos\alpha,$$
$$\sinh b' = \sinh(b - (1-t)c)\cosh(tc) + \sinh((1-t)c)\cosh(b + tc)(1 - \cos\beta) -$$
$$- \sinh(tc)\cosh(b - (1-t)c)\cos\beta.$$

Our aim is to find $t \in [0,1]$, such that the two inequalities

$$\sinh a' \geq \sinh(a - tc)\cosh((1-t)c),$$
$$\sinh b' \geq \sinh(b - (1-t)c)\cosh(tc),$$

hold. This would imply the inequality in (24)

First, we notice that if $a + b \leq c$, then we have the inequality in (24) directly.

Let us assume that $a + b > c$. If $a < c$ or $b < c$, we let $t = 1$ or $t = 0$ respectively and get the above two inequalities. Therefore we also have the inequality in (24) in this case.

Assume that $a > c$ and $b > c$. It is enough to find $t \in (0,1)$ such that the following inequalities hold:

$$0 < \frac{\cos\alpha}{1 - \cos\alpha} \leq \frac{\sinh(tc)}{\sinh((1-t)c)} \leq \frac{1 - \cos\beta}{\cos\beta} < \infty.$$

The same argument as in the proof of the $2\pi/3$-obtuse case in Theorem 5.6 implies the existence of such t. Therefore, the inequality in (24) holds in this case. As a consequence, we have

$$L'(p) \geq L'(q),$$

where the equality holds if and only if $p = q$.

The remaining part of the proof is to compare $L'(q)$ with $L(\widetilde{p})$. The idea is similar to the proof of Lemma 5.8. Let q_1 be the orthogonal projection of q to γ'_1. Assume that $q \neq \widetilde{p}$. By the cosine rule for a quadrilateral with two right angles, we have the following relations:

$$\sinh|qq_2| = \sinh|\widetilde{p}\widetilde{p_2}|\cosh|\widetilde{p}q| + \frac{1}{2}\cosh|\widetilde{p}\widetilde{p_2}|\sinh|\widetilde{p}q|,$$
$$\sinh|qq_1| = \sinh|\widetilde{p}\widetilde{p_1}|\cosh|\widetilde{p}q| + \frac{1}{2}\cosh|\widetilde{p}\widetilde{p_1}|\sinh|\widetilde{p}q|.$$

They imply the following two equalities:

$$\sinh|qq_2| = \sinh(|\widetilde{p}\widetilde{p_2}| + \frac{|\widetilde{p}q|}{2})\cosh\frac{|\widetilde{p}q|}{2} + \sinh|\widetilde{p}\widetilde{p_2}|\sinh^2\frac{|\widetilde{p}q|}{2},$$
$$\sinh|qq_1| = \sinh(|\widetilde{p}\widetilde{p_1}| + \frac{|\widetilde{p}q|}{2})\cosh\frac{|\widetilde{p}q|}{2} + \sinh|\widetilde{p}\widetilde{p_1}|\sinh^2\frac{|\widetilde{p}q|}{2},$$

from which we conclude:

$$|qq_2| \;>\; |\widetilde{p}\widetilde{p_2}| + \frac{|\widetilde{p}q|}{2},$$

$$|qq_1| \;>\; |\widetilde{p}\widetilde{p_1}| + \frac{|\widetilde{p}q|}{2}.$$

Hence, we have $L'(q) > L(\widetilde{p})$ which moreover implies that $L(p) > L(\widetilde{p})$. Therefore the balanced point is the Fermat point for the triple $(\gamma_1, \gamma_2, \gamma_3)$. $\square$

To prove Fact (2) in the theorem, we use similar ideas. The technical part is how to use the inequality (24) that we proved above.

Proof of Fact (2) in Theorem 6.13. Without loss of generality, we assume that the internal angle at $v_1 \in \gamma_1$ is greater or equal to $2\pi/3$, where v_1 is the unique planar point on γ_1. Let v_{12} and v_{13} denote its orthogonal projection to γ_2 and γ_3.

If the internal angle at v_1 is π, then we are done. In that case, the three points v_{12}, v_1 and v_{13} are collinear and the geodesic segment $v_{12}v_{13}$ is orthogonal to both γ_2 and γ_3. By the convexity lemma, the segment $v_{12}v_{13}$ realizes the distance between γ_2 and γ_3. Given any point $p \in \mathbb{H}^3$, when we connect it to its orthogonal projections to γ_1, γ_2 and γ_3 with geodesic segments, we always get a connected graph containing a subgraph connecting γ_2 to γ_3. Therefore the value $L(p)$ is always greater or equal to $L(v_1)$, and the equality is realized only when $p = v_1$. Hence v_1 is the Fermat point.

From now on, we assume that the internal angle at v_1 is in $[2\pi/3, \pi)$. The first step is similar to the previous case. We would like to simplify the problem to the planar case. We consider the unique plane P containing v_1, v_{12} and v_{13}. Since v_1 is planar, there is a unique geodesic $\gamma(v_1)$ orthogonal to γ_1 with respect to which the geodesic containing v_1v_{12} and that containing v_1v_{13} are symmetric. It is contained in P and separates the angle $\angle v_{12}v_1v_{13}$ equally.

We denote by P_1 the plane orthogonal to $\gamma(v_1)$ at v_1, by P_2 (resp. P_3) the plane orthogonal to v_1v_{12} (resp. v_1v_{13}) at v_2 (resp. v_3). We denote by γ'_1, γ'_2 and γ'_3 the intersection of P with P_1, P_2 and P_3 respectively.
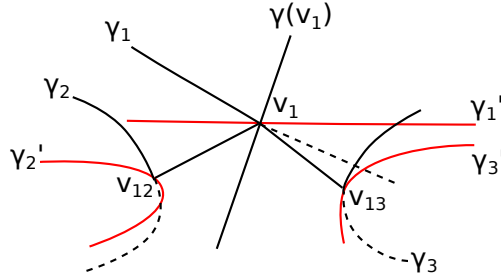

FIGURE 16

Using a similar argument as in the previous case, to see v_1 is the Fermat point, it is enough to show that v_1 realizes the minimum of the sum of distances to γ'_1, γ'_2 and γ'_3 among all points in P. In particular, it is enough to consider the connected component of the complement of $\gamma'_1 \cup \gamma'_2 \cup \gamma'_3$ in P containing v_1v_{12} and v_1v_{13}. We denote the closure of this connected component by D. As in the previous case, we

use L' to denote the function sending each point p to the sum of the distance from p to γ_1', γ_2' and γ_3'.

Given a point $p \in D$, we denote by p_1, p_2 and p_3 its orthogonal projections to γ_1', γ_2' and γ_3'. We consider the path of points p such that $\angle p_{j-1}pp_{j+1} = 2\pi/3$ and denote the path by C_j. Since $\angle v_{12}v_1v_{13} \geq 2\pi/3$, the intersection between C_1 and D has two connected components. One intersects γ_1' and γ_2', the other intersects γ_1' and γ_3'. We denote them by C_{12} and C_{13} respectively. The paths C_2 and C_3 are contained in D, such that C_2 intersects γ_1' and γ_3', while C_3 intersects γ_1' and γ_2'. The subset D_0 of D bounded by C_3, C_{12}, γ_1' and γ_2' is convex, as is the subset D_0' bounded by C_2, C_{13}, γ_1' and γ_3'. Moreover, the sets D_0 and D_0' are disjoint, and the segment pp_1 for each point $p \in D$ intersects at most one of them. In particular, if $\angle v_{12}v_1v_{13} = 2\pi/3$, the paths C_{12}, C_{13}, C_2 and C_3 meet at v_1.
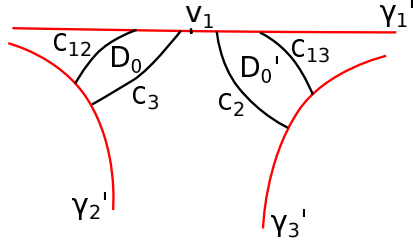

Figure 17

We now claim that given a point p in D, we can find a zig-zag path ending at γ_1' starting from p such that the L'-value is strictly decreasing along this path. More precisely, if pp_j intersects with D_0, we denote by q_1 the point when pp_j leaves D_0. By inequality (24), we have $L'(q_1) \leq L'(p)$, where the equality holds if and only if $p = q_1$. Since q is on the boundary of D_0, we can repeat the above process for q_1 and find another point q_2 on the boundary of D_0 such that $L'(q_2) < L(q_1)$. We keep repeating. If $\angle v_{12}v_1v_{13} > 2\pi/3$, then the zig-zag path will stop on a point $q \in \gamma_1'$ within finite steps. Otherwise, we will have a zigzag path with infinite number of segments converging to v_1.
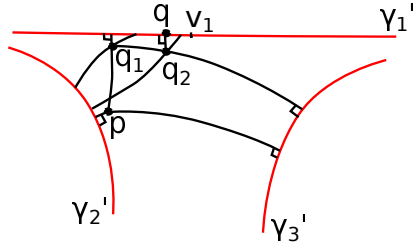

Figure 18

As a conclusion, for any point $p \in D$ with pp_j intersecting D_0 for some j, there is a point q on γ_1' whose L'-value is strictly smaller than p. The same argument works for p with pp_j intersecting D_0' for some j. Notice that for any point p in D, there is always some j such that pp_j intersecting D_0 or D_0'. Hence, we have the claim at the beginning of this paragraph.

Since v_1 realizes the minimum of the L'-value among all points on γ_1', Fact (2) holds.

$\square$

6.3. **The Steiner tree for triples of geodesics in $\mathbb{H}^3$.** Given a triple $(\gamma_1, \gamma_2, \gamma_3)$, we consider the corresponding triple of involutions (r_1, r_2, r_3). In this part, we assume that G is a finite graph connecting the three geodesics in the triple, whose edges are all geodesics segments. Hence the intersection between an edge of G and γ_j for any j is either a point or the entire edge. For our convenience, if the intersection is a point, we add this point to the set of vertices of G. If edges in G intersect, the intersection points are also added in the vertex set.

We are interested in the length of the graph D which arises from doubling G using r_j for some j, and then taking the quotient by the action of the group $\langle r_1 r_2, r_2 r_3 \rangle$. Note that D is a graph of rank at least two. The length of D will be the length of a carrier graph of the associated representation. Moreover, generically the length of D is twice of the Steiner length of G which is defined as follows:

Definition 6.16. The *Steiner length* $\mathsf{L}(G)$ of a graph G is defined by:

$$\mathsf{L}(G) = L(G_1) + \frac{1}{2}L(G_2), \tag{25}$$

where G_2 is the intersection $G \cap (\gamma_1 \cup \gamma_2 \cup \gamma_3)$ and G_1 is the complement of G_2 in G.

Remark 6.17. It is possible that G has two distinct edges e and e' collinear, adjacent to a same vertex v lying in γ_j for some j, while e and e' are both orthogonal to γ_j. In that case, the intersection $G \cap r_j(G)$ contains $(e \cup e') \cap r_j(e \cup e')$ which is not contained in γ_j. In this case, the double of the Steiner length of G is strictly bigger than the usual length of D, since the Steiner length for $(e \cup e') \cap r_j(e \cup e')$ equals to the usual length and is counted twice when doubling.

However, this is not an issue for our main purpose, since we are interested in the length of the double of the graph. We can always replace G by a different graph G' without changing the double graph D, such that the above situation does not happen to G'.

More precisely, since G is connected, without loss of generality, we may assume that e can be connected to γ_{j-1} by a subgraph F_1, without passing e'. Now there are two possible cases. The first case is that e can also be connected to γ_{j+1} without passing through e' by a subgraph F_2. In this case, we can cut out $F_1 \cup F_2$ from G and denote by F' the connected component containing e'. Then we denote by G' the union $(G \setminus F') \cup r_j(F')$. Notice that G' is connected and finite. It connects γ_j's. Moreover, its double is still D, but the double of its Steiner length now equals the usual length of D.

The second case is that there is no such a subgraph F_2 described above. Then by cutting G at v, we have two connected component. We denote by F_1' the one containing F_1. We consider G'' the union $F_1' \cup r_j(G \setminus F_1')$. Notice that G'' is connected, finite. It connects γ_{j-1}, γ_j and $r_j(\gamma_{j+1})$. Notice that its double is also D and the double of its Steiner length equals to the usual length of D.

Consider the Fermat point for $(\gamma_1, \gamma_2, \gamma_3)$. Recall that the Fermat graph is obtained by connecting the Fermat point to its orthogonal projections to the geodesics in the triple. In particular, the L-value equals to the usual length of the Fermat

graph. By the property of the Fermat point, we see that this graph has the smallest usual length among all graphs connecting γ_i's. Hence it is a good candidate for our purpose. However, its Steiner length is not always the shortest. Below is an example where this is the case.

Example 6.18. We consider the case where γ_1, γ_2 and γ_3 are lying in the same plane and disjoint from each other in $\mathbb{H}^3 \cup \partial\mathbb{H}^3$. We further assume that none of them separates the other two into two different half planes. Assume that the Fermat point p of $(\gamma_1, \gamma_2, \gamma_3)$ is on γ_1. Let p_2 and p_3 be its orthogonal projections to γ_2 and γ_3 respectively. By our discussion in the previous section, the angle $\angle p_2 p p_3$ is greater than or equal to $2\pi/3$. Suppose that it is greater. Let G_p denote the Fermat tree of $(\gamma_1, \gamma_2, \gamma_3)$.

We construct a new graph G' from G in the following way. Choose points q_1 and q_2 on γ_1 such that the angles $\angle p_2 q_2 q_3$ and $\angle p_3 q_3 q_2$ are both $2\pi/3$. It is easy to see that p is contained in the segment $q_2 q_3$. The new graph G' is the union of the geodesic segments $p_2 q_2$, $q_2 q_3$ and $q_3 p_3$.
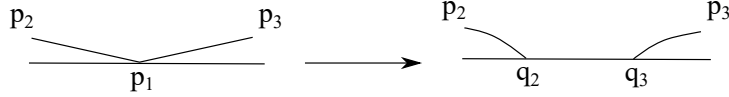

FIGURE 19. Breaking the Fermat point

By Lemma 5.8, we know that:

$$|q_2 p_2| + \frac{1}{2}|q_2 p| < |p p_2|.$$
$$|q_3 p_3| + \frac{1}{2}|q_3 p| < |p p_3|.$$

Therefore, the new graph has shorter Steiner length. Note that this particular graph may not have the shortest Steiner length since p_i is not necessarily the projection of q_i to γ_i for $i = 2, 3$.

Definition 6.19. The *Steiner tree* of $(\gamma_1, \gamma_2, \gamma_3)$ is a connected graph G connecting them with the shortest Steiner length $\mathsf{L}(G)$.

Our next result is about the existence of the Steiner tree for a triple of geodesics and its characterization.

Proposition 6.20. *For any triple $(\gamma_1, \gamma_2, \gamma_3)$, the Steiner tree exists. Moreover it is one of the following two types:*

   (a) *It is G_p;*
   (b) *It is a piece-wise geodesic with three segments where the middle one is contained in one of γ_i's and the other two segments intersects γ_{i-1} and γ_{i+1} orthogonally. Moreover, the angle between two adjacent segments is $2\pi/3$.*

Proof. In this proof, by length we will always mean the Steiner length. Recall that we only consider finite connected graphs. All graphs that we will consider always connect the three geodesics γ_j's.

Since removing one edge to break a loop always reduces the length of the graph, it is enough to only consider simply connected graphs. In the rest of the proof, we assume that this is always the case.

We first prove the following lemma:

Lemma 6.21. *Given any finite simply connected graph G, its length is always bigger or equal to that of a graph with one of the following three types:*

(1) *One valence 3 vertex and three valence 1 vertices;*
(2) *One valence 2 vertex and two valence 1 vertices;*
(3) *Two valence 2 vertices and two valence 1 vertices.*

Proof of Lemma 6.21. Consider the path P_{12} (resp. P_{13}) with shortest Steiner length in G connecting γ_1 and γ_2 (resp. γ_1 and γ_3). Notice that P_{12} (resp. P_{13}) is a piecewise geodesic and meet each one of γ_1 and γ_2 (resp. γ_1 and γ_3) once. Let p_2 and p_3 denote the intersections $P_{12} \cap \gamma_1$ and $P_{13} \cap \gamma_1$ respectively.

Notice that the choices for P_{12} and P_{13} may not be unique. We choose and fix one choice. In the following, we discuss case by case.

Case 1: P_{12} intersects γ_3. We consider the orientation of P_{12} from γ_1 to γ_2 and denote by p_1 the point where P_{12} meets γ_3 for the first time and by p_2 the point where P_{12} leaves γ_3 for the last time. Let q_1 and q_2 be the projection of p_1 and p_2 to γ_1 and γ_2 respectively. Then the graph G' given by the union of segments $q_1 p_1$, $p_1 p_2$ and $p_2 q_2$ is of type (2) or (3) with shorter length than G.

Case 2: P_{13} intersects γ_2. The argument is the same as in Case 1.

Case 3: both $P_{12} \cap \gamma_3$ and $P_{13} \cap \gamma_2$ are empty and $P_{12} \cap P_{13} \neq \emptyset$. We consider the point on P_{12} where it starts from γ_2 and meets P_{13} for the first time. Then by connecting this point to its projections to γ_j's respectively, we obtain a graph of type (1) with length shorter than G.

Case 4: both $P_{12} \cap \gamma_3$ and $P_{13} \cap \gamma_2$ are empty and $P_{12} \cap P_{13} = \emptyset$. We denote by Q_{23} the shortest path in G connecting P_{12} and P_{13}. Hence, the path Q_{23} does not contain any edges in P_{12} or in P_{13}, and only meet each one of them once. Denote by o_2 and o_3 the vertices $P_{12} \cap Q_{23}$ and $P_{13} \cap Q_{23}$ respectively. All vertices in the subgraph Q_{23} except o_2 and o_3 have valence 2.

Case 4.1: Q_{23} intersects γ_1, γ_2 and γ_3. Since Q_{23} is a piecewise geodesic, it contains a subgraph which starts from γ_i, intersects γ_j for some times, then ends at γ_k, where $\{i, j, k\} = \{1, 2, 3\}$. Using the same argument as for case 1, we can find a path of type (2) or (3) with shorter length than G.

Case 4.2: Q_{23} is disjoint from γ_2, but intersects γ_1 or γ_3. Then we consider Q'_{23} the union of Q_{23} and the subgraph of P_{12} between γ_2 and o_2. By our construction, the graph Q'_{23} is a simple piecewise geodesic connecting γ_1, γ_2 and γ_3, hence it contains a subgraph starting from γ_2, intersecting γ_j form some times, then ending at γ_k with $\{j, k\} = \{1, 3\}$. Using the same argument as for case 1, this we can find a path of type (2) or (3) with shorter length than G.

Case 4.3: Q_{23} is disjoint from γ_3, but intersects γ_1 or γ_2. We connect Q_{23} to γ_3 using the subgraph of P_{13} between o_3 and γ_3, then the proof is the same as in Case 4.2.

Case 4.4: Q_{23} is disjoint from γ_1, but intersects γ_2 or γ_3. We connect Q_{23} to γ_1 using the subgraph of P_{12} between o_2 and γ_1, then the proof is the same as in Case 4.2.

Case 4.5: Q_{23} is disjoint from γ_1 and γ_2, but intersects γ_3. We connect P_{12} to γ_3 using the subgraph of Q_{23} between o_2 and its first intersection point with γ_3

counting from o_2. Then we consider the graph given by connecting o_2 to its three projections to γ_1, γ_2 and γ_3. It is of type (1) with length shorter than that of G.

Case 4.6: Q_{23} is disjoint from γ_1 and γ_3, but intersects γ_2. We connect P_{13} to γ_2 using the subgraph of Q_{23} between o_3 and its first intersection point with γ_2 counting from o_3, then the proof is the same as in Case 4.5.

Case 4.7: Q_{23} is disjoint from γ_2 and γ_3, but intersects γ_1. We connect Q_{23} to γ_2 and γ_3 using parts of P_{12} and P_{13}, such that the resulting graph meets each of γ_2 and γ_3 once, and has only valence 2 vertices except those two on γ_2 and γ_3. Then the proof is the same as in Case 1. $\square$

Returning to the proof of the proposition, if G is of type (1) or type (2), we can moreover assume that G has no edge contained in any of the γ_j's. Otherwise, either the edge on γ_j is removable, or there are two of γ_j's intersect each other, which contradict to the fact that γ_j's are in general position. Therefore, its Steiner length is greater than or equal to that of the Fermat tree.

If G is of type (3), we may moreover assume that the middle edge is on γ_j for some j. Otherwise, its length is the usual length, hence greater than that of the Fermat tree. Without loss of generality, we may assume that the graph is the union of p_2q_2, q_2q_3 and q_3p_3, where q_2q_3 is contained in γ_1, and p_2 and p_3 are in γ_2 and γ_3 respectively. We may further assume that p_2 (resp. p_3) is the orthogonal projection of q_2 (resp. q_3), otherwise, we can shorten the Steiner length by replacing p_2 (resp. p_3) by the orthogonal projection. Then G is determined by the positions of q_2 and q_3 and the Steiner length is a function of q_2 and q_3. By the extreme value theorem, there is a graph G which realizes the minimal Steiner length among all type (3) graphs with middle edge on γ_1. Moreover it is either of type (2) or of type (3). When it is of type (2), its length is greater than that of the Fermat tree. When it is of type (3), by Lemma 5.8 and the proof of Case I in Theorem 5.6, the angle between each pair of adjacent edges must be $2\pi/3$. $\square$

Remark 6.22. Unlike the uniqueness result of the Fermat point (or Fermat graph), it is possible to have up to two Steiner trees for a triple $(\gamma_1, \gamma_2, \gamma_3)$. An example would be a triple of coplanar geodesics. Consider a triple $(\gamma_1, \gamma_2, \gamma_3)$ contained in the same plane, such that no one separates other two. We assume that the distances between pairs of them are $(a, a, 2a)$ with $a > 0$. By studying the planar points for this triple of geodesics, we can see that there exists a balanced point for this triple of geodesics. By our previous discussion, this balanced point is the Fermat point p, and the Fermat tree G is given by connecting the p to p_j's which are its projections to γ_j's respectively.

Assume that the distance between γ_1 and γ_2 is $2a$. Consider the involution r_3 with respect to γ_3. We denote by γ_1' and γ_2' the images of γ_1 and γ_2 under r_3 respectively. We denote by D the double of G under r_3. An observation is that the subgroup generated by the elliptic element η of $\mathbb{H}^3$ fixing p_3, preserving the plane with angle $\pi/2$ preserves the set $\{\gamma_1, \gamma_2, \gamma_1', \gamma_2'\}$. Hence the image $\eta(D)$ is still a graph connecting γ_1, γ_2, γ_1' and γ_2'. Moreover the middle edge of $\eta(D)$ lying on γ_3. Therefore, the graph $\eta(D)$ is the double of a graph G' connecting $(\gamma_1, \gamma_2, \gamma_3)$ with one edge on γ_3.

By our assumption, we can see that $L(G) = L(G')$. Both of them are Steiner tree for $(\gamma_1, \gamma_2, \gamma_3)$, while G is of type (a) and G' is of type (b)
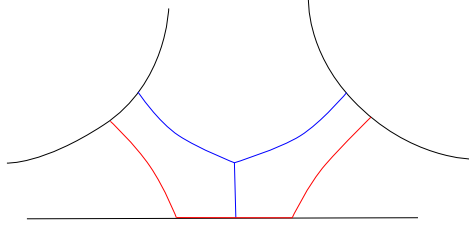
FIGURE 20. Two Steiner trees

6.4. **The fourth edge orientation on Σ.** Recall that Σ is the tree of superbases. In the earlier section, we have discussed three edge orientation on Σ. In this part, we would like to associated to Σ the fourth edge orientation using the Steiner tree introduced above.

More precisely, let ρ be an irreducible representation of $\mathbb{F}_2$ into $\mathrm{SL}(2, \mathbb{C})$ sending all primitive elements to loxodromic elements. Given an ordered triple (X, Y, Z), by proposition 2.3, there is a triple of π-rotation (r_X, r_Y, r_Z) such that:

$$
\begin{aligned}
\rho(X) &= r_Y r_Z, \\
\rho(Y) &= r_Z r_X, \\
\rho(Z) &= -r_X r_Y.
\end{aligned}
$$

We denote by γ_X, γ_Y and γ_Z the axes of r_X, r_Y and r_Z respectively. Since $\rho(X)$, $\rho(Y)$ and $\rho(Z)$ are all loxodromic elements by assumption, the triple $(\gamma_X, \gamma_Y, \gamma_Z)$ is generic.

Although the triple of geodesics depends on the choice of representative of a superbasis, the relative position among the three geodesics in the triple does not. Given two representatives of a superbasis, the two associated triples of geodesics are different by a diagonal action of an isometry of $\mathbb{H}^3$. In this way, we associate to a superbasis a triple of geodesics up to isometry. By our work in the earlier part of this section, it admits a Steiner tree.

Recall that $\widetilde{\Omega}$ consists of pairs $([W], v)$ where $[W]$ is the $\mathrm{Inn}^{\mathfrak{e}}$-orbit of a primitive element W and v is a vertex on the boundary of connected component in $\mathbb{D} \setminus \Sigma$ identified with $[W]$. We can define a function $L_\rho : \widetilde{\Omega} \to \mathbb{R}_{>0}$ sending $([W], v)$ to the Steiner length of the Steiner tree associated to v. This function L_ρ then induces an edge orientation on Σ, such that the orientation goes from vertex with bigger L_ρ-value to the one with smaller L_ρ-value.

Proposition 6.23. *If ρ satisfies BQ-conditions, then there is a compact attracting subtree for the edge orientation induced by L_ρ.*

Proof. Consider the right angled hexagon $H(X, Y, Z)$ associated to $(\gamma_X, \gamma_Y, \gamma_Z)$. Without loss of generality, we may assume that with respect to the fixed basis (X_0, Y_0), the element Z has bigger word length than X and Y. Let e denote the edge of Σ adjacent to the components $[X_0]$ and $[Y_0]$.

In the proof of proposition 4.8, we saw that if the vertex (X, Y, Z) is at distance N to the edge e with N large enough, the complex lengths of the three sides of $H(X, Y, Z)$ on γ_X, γ_Y and γ_Z will be uniformly close to $i\pi$, $i\pi$ and 0 respectively. It does not depend on the choice of the geodesic on Σ.

The internal angle on γ_Z is bounded from below by the transport angle along γ_Z, and the planar point on γ_Z is contained in the side of $H(X, Y, Z)$ on γ_Z. Therefore,

there exists a positive integer N, such that if the distance from (X, Y, Z) to the starting vertex is bigger than N, the associated right angled hexagon is $2\pi/3$-obtuse. Moreover, the associated Steiner tree is of type (2), meaning that it has one valence 2 vertex on γ_Z and two valence 1 vertices on γ_X and γ_Y respectively. The Steiner length is more or less the sum of the lengths of two sides of $H(X, Y, Z)$ on the axes of $\rho(X)$ and $\rho(Y)$. The difference is bounded from above by the sum of the lengths of the sides of $H(X, Y, Z)$ on γ_X, γ_Y and γ_Z.

Consider going one step farther than (X, Y, Z) and denote the vertex by (X', Y, Z). Then the right angled hexagon $H(X', Y, Z)$ is also $2\pi/3$-obtuse and its Steiner tree $G(X', Y, Z)$ is of type (2). Up to isometry, the triple of geodesics for (X', Y, Z) is $(\gamma_X, \gamma_Y, r_X(\gamma_Z))$. The valence 2 vertex $v_{X'}$ of $G(X', Y, Z)$ is on γ_X. The internal angle on γ_X change from $2\pi/3$-acute to $2\pi/3$-obtuse. Hence the Steiner length of $G(X, Y, Z)$ is strictly smaller than that of $G(X', Y, Z)$. Therefore, there is a compact attracting subtree of Σ with respect to the edge orientation induces by L_ρ.
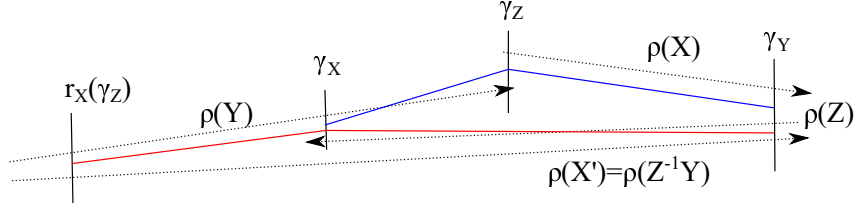

FIGURE 21. More one step further

$\square$

7. THE CARRIER 2-GRAPHS FOR AN IRREDUCIBLE REPRESENTATION

Recall that the definition of a carrier n-graph Γ for a hyperbolic 3-manifold M involves an immersion of Γ into M. Let ρ be a lift of the representation of $\pi_1(M)$ into $\mathrm{SL}(2, \mathbb{C})$ corresponding to the hyperbolic structure on M. By taking the lift to the universal cover, we obtain an immersion of the universal cover $\widetilde{\Gamma}$ of Γ into $\mathbb{H}^3$ which is ρ-equivariant. This immersion of $\widetilde{\Gamma}$ is unique up to the action of $\rho(\pi_1(M))$. Following this idea, we generalize the definition of a carrier graph to that for an irreducible representation. Recall that we are interested in irreducible representations $\rho : \mathbb{F}_2 \to \mathrm{SL}(2, \mathbb{C})$ and we will only consider 2-graphs.

Remark 7.1. One may define carrier graphs using any graph with no restriction on valence, however by lemma 5.8, the carrier graph with minimal length, if exists, is always a trivalent graph. As such, we only will consider trivalent 2-graphs.

7.1. **Definition.** Let Γ be a trivalent 2-graph. Then there are only two possible combinatorial types of Γ which we call the Buckle and the Dumbbell, see figure below.
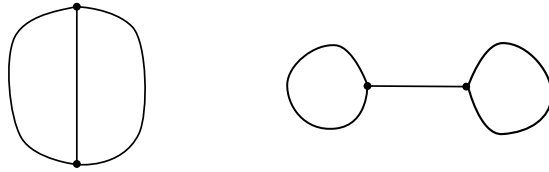
FIGURE 22. A Buckle and a Dumbbell

We can give Γ a marking by assigning a free basis (X, Y) to two distinct simple loops on Γ respectively. We call it a marked 2-graph. For a marked 2-graph $(\Gamma, (X, Y))$, a fundamental domain D in $\widetilde{\Gamma}$ can be chosen to be the one with four boundary points p_1, p_1', p_2 and p_2', such that $X(p_1) = p_1'$ and $Y(p_2) = p_2'$. As a normalization, when Γ is equipped with a metric l, the points p_1 and p_2 will be chosen so that their projections on Γ are the two mid-points of two edges of Γ where they lie on respectively. We will only consider this fundamental domain D for each marked 2-graph. Depending on the marked 2-graph, the fundamental domain D has 2 possible combinatorial types given in the figure below:
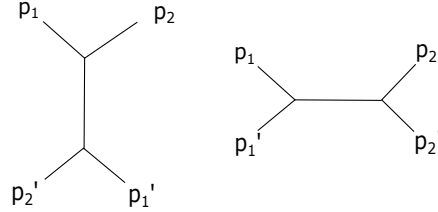

FIGURE 23. Possible types of D

Definition 7.2. Two marked 2-graphs $(\Gamma_1, (X_1, Y_1))$ and $(\Gamma_2, (X_2, Y_2))$ are said to be *equivalent*, if the following holds:

(1) There exists a homeomorphism from Γ_1 to Γ_2;
(2) This homeomorphism induces an automorphism of $\mathbb{F}_2$ sending (X_1, Y_1) to (X_2, Y_2).

If we consider the space of all marked 2-graphs and take its quotient by the above equivalence relation, we obtain 2 classes corresponding to the above two types of fundamental domain respectively. To avoid talking about class and taking representative, we will choose and fix one representative in each class with the same marking (X_0, Y_0). These two graphs will be the topological model for the later discussion. As a convention, in the reminder of this paper, by a 2-graph, we mean one of the two topological models and simply denote it by Γ.

Remark 7.3. A marking on a graph induces an isomorphism between the fundamental group of the graph and $\mathbb{F}_2$. By choosing and fixing a marking (X_0, Y_0), we choose and fix the isomorphism.

Now let ρ be an irreducible representation of $\mathbb{F}_2$ into $\mathrm{SL}(2, \mathbb{C})$.

Definition 7.4. A local homeomorphism $\Psi : \widetilde{\Gamma} \to \mathbb{H}^3$ is said to be ρ-*equivariant* if there exists a free basis (X, Y) of $\mathbb{F}_2$ such that the following commutative diagram holds:

$$
\begin{array}{ccc}
\widetilde{\Gamma} & \xrightarrow{\;\;W\;\;} & \widetilde{\Gamma} \\
\Big\downarrow{\scriptstyle \Psi} & & \Big\downarrow{\scriptstyle \Psi} \\
\mathbb{H}^3 & \xrightarrow{\;\rho(h_\Psi(W))\;} & \mathbb{H}^3
\end{array}
$$

where h_Ψ is the automorphism of $\mathbb{F}_2$ sending X_0 and Y_0 to X and Y respectively.

Definition 7.5. A *marked carrier* 2-*graph* for ρ is a pair (Γ, Ψ), where

- Γ is a 2-graph;

- $\Psi : \widetilde{\Gamma} \to \mathbb{H}^3$ is a ρ-equivariant local homeomorphism.

The pullback of the intrinsic metric on $\Psi(\widetilde{\Gamma})$ by Ψ induces a metric on Γ. We denote by $l(\Gamma, \Psi)$ this metric on Γ.

Definition 7.6. The *length* of Γ with respect to $l(\Gamma, \Psi)$ is defined to be the sum of the $l(\Gamma, \Psi)$-length of all the edges. We denote it by $L(\Gamma, \Psi)$.

Definition 7.7. Two marked carrier 2-graphs (Γ_1, Ψ_1) and (Γ_2, Ψ_2) are said to be *equivalent* if

(1) Γ_1 and Γ_2 are the same 2-graph;
(2) there exists an element W of $\mathbb{F}_2$, such that $Im(\Psi_1) = \rho(W)(Im(\Psi_2))$;
(3) the two automorphisms h_{Ψ_2} and h_{Ψ_1} satisfies:

$$h_{\Psi_2} = h' h_{\Psi_1},$$

where $h' \in \mathrm{Inn}^{\mathfrak{e}}\mathbb{F}_2$.

An equivalence class of marked carrier graph is called a *carrier 2-graph* for ρ. We denote it by $(\Gamma, [\Psi])$. Its *length* $L(\Gamma, [\Psi])$ is defined to be the length $L(\Gamma, \Psi)$ for a representative (Γ, Ψ) in the equivalence class.

Remark 7.8. This definition is similar to the one for the equivalence class of marked hyperbolic structure on 3-manifold. More precisely, the conditions (1) says that the two 2-graphs homeomorphic to each other, the condition (2) implies that the two 2-graphs equipped with induced metrics are isometric and the condition (3) says that the marking on the two 2-graphs are the same. For 3-manifold, the condition (2) is equivalent to the two pullback metric are isometric. This is not the case for carrier graph. Instead, the condition (2) is strictly stronger than that two pullback metrics are isometric, since we consider the intrinsic metrics.

7.2. **Decomposition of the space of carrier 2-graphs.** Since we are interested in looking for the carrier 2-graph with shortest length, it is reasonable to only consider the geodesic carrier 2-graphs, meaning that those carrier 2-graphs $(\Gamma, [\Psi])$, such that all edges of $\widetilde{\Gamma}$ are sent to geodesics segments by Ψ. By lemma 5.8, we can moreover assume that the edges of $\widetilde{\Gamma}$ are sent to geodesic segments with strictly positive length.

Let $\mathcal{CG}_\rho$ denote the space of all geodesic carrier 2-graphs associated to ρ. By our definition, the carrier graphs can be classified using h_Ψ. Let (X, Y) be a basis of $\mathbb{F}_2$. We denote by $\mathcal{CG}_\rho([X], [Y])$ the subspace consisting of equivalence classes of all marked carrier 2-graphs with $h_\Psi(X_0) \in [X]$ and $h_\Psi(Y_0) \in [Y]$. Therefore we can write $\mathcal{CG}_\rho$ into a disjoint union of $\mathcal{CG}_\rho([X], [Y])$'s.

Definition 7.9. A carrier 2-graph $(\Gamma, [\Psi]) \in \mathcal{CG}_\rho$ is said to be *critical*, if it realizes the minimum of the length function restricted on $\mathcal{CG}_\rho([X], [Y])$ for some (X, Y). It is called *minimal* if it has the minimal length among all carrier graphs.

Now let us recall the main result:

Theorem 1.1. If $\rho : \mathbb{F}_2 \to \mathrm{SL}(2, \mathbb{C})$ satisfies the BQ-conditions, then it admits finitely many critical carrier 2-graphs.

As a corollary, we have

Corollary 1.2. If $\rho : \mathbb{F}_2 \to \mathrm{SL}(2, \mathbb{C})$ satisfies the BQ-conditions, then it admits finitely many minimal carrier 2-graphs.

8. PROOF OF THE MAIN RESULT

Let ρ be a BQ-representation. Let (X, Y) be a basis of $\mathbb{F}_2$. We first consider the carrier 2-graphs in $\mathcal{CG}_\rho([X], [Y])$. To simplify the notation, we will also use D, p_1, p_1', p_2, p_2', v_1 and v_2 to denote their Ψ-images. We consider the two ordered triple (X, Y, Z) and (X, Y^{-1}, Z').

Consider the triple SL$(2, \mathbb{C})$-elements $(\rho(X), \rho(Y), \rho(Z))$. By Proposition 2.3, we can find a triple of π-rotations (r_X, r_Y, r_Z) with axes $(\gamma_X, \gamma_Y, \gamma_Z)$ in general position, such that:

$$\begin{aligned}
\rho(X) &= r_Y r_Z, \\
\rho(Y) &= r_Z r_X, \\
\rho(Z) &= -r_X r_Y, \\
\rho(Z') &= (r_Z r_X r_Z) r_Y.
\end{aligned}$$

We first prove the following lemma:

Lemma 8.1. *If $r_Z(D) \neq D$, then there is a new carrier graph (Γ', Ψ') with its length smaller than that of (Γ, Ψ).*

Proof. Assume that $r_Z(D) \neq D$. We first consider the case where Γ is a buckle. We consider the geodesic segments $p_1 r_Z(p_1')$, $p_1' r_Z(p_1)$, $p_2 r_Z(p_2')$, $p_2' r_Z(p_2)$, $v_1 r_Z(v_2)$ and $v_2 r_Z(v_1)$. Below is a picture for the planar case.
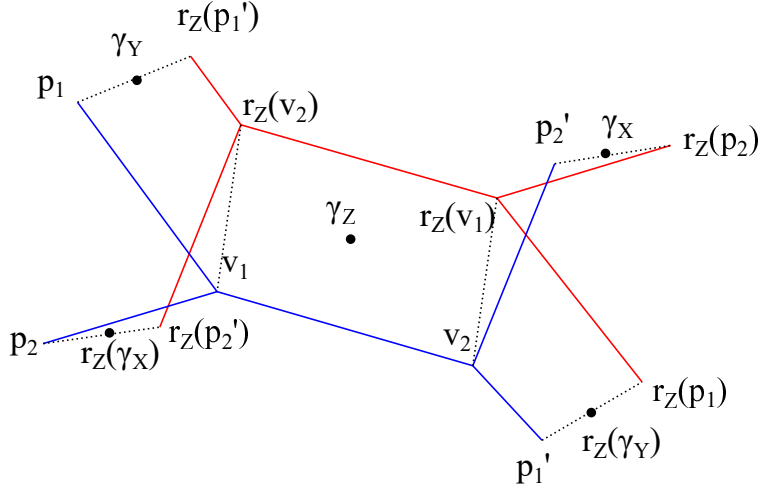

FIGURE 24

Then we can find a new graph D' homotopic to D relative to these 6 geodesic segments, such that its vertices are the mid points of the six geodesic segments respectively. The hyperbolic convexity lemma implies that the length of D' is smaller than that of D. Therefore the new carrier graph (Γ', Ψ') induced by D' has smaller length than (Γ, Ψ).

If Γ is a dumbbell, then we consider the geodesic segments $v_1 r_Z(v_1)$ and $v_2 r_Z(v_2)$ instead of $v_1 r_Z(v_2)$ and $v_2 r_Z(v_1)$. The rest of the proof is the same as the first case. $\qquad\square$

Now assume that $r_Z(D) = D$. Then the graph D can be viewed as branched 2-cover of a graph G in $\mathbb{H}^3$ connecting $(\gamma_X, \gamma_Y, \gamma_Z)$ or $(r_z(\gamma_X), \gamma_Y, \gamma_Z)$ with covering map induced by r_Z, such that the branching part is either $v_1 v_2$ or the midpoint of $v_1 v_2$.

We consider $G(X, Y, Z)$ (resp. $G(X, Y^{-1}, Z')$) the Steiner tree for $(\gamma_X, \gamma_Y, \gamma_Z)$ (resp. $(r_z(\gamma_X), \gamma_Y, \gamma_Z)$) and denote by $D(X, Y, Z)$ (resp. $D(X, Y^{-1}, Z')$) its double. By the definition of the Steiner tree, we can see that if (Γ, Ψ) is a marked carrier 2-graph with $(h_\Psi(X_0), h_\Psi(Y_0)) = (X, Y)$ or (X, Y^{-1}), then its length is strictly longer than that of $D(X, Y, Z)$ or that of $D(X, Y^{-1}, Z')$. Therefore, up to isometry, the carrier graph realizing the infimum of the length among all carrier 2-graph in $\mathcal{CG}_\rho([X], [Y])$ is induced by the shorter one of $D(X, Y, Z)$ and $D(X, Y^{-1}, Z')$. Therefore to look for the minimal graph, it is enough to consider those induced by the Steiner trees associated to superbases.

Given an edge orientation on the tree of superbasis, we call a vertex is a sink if all three adjacent edges are oriented towards it. By the proposition 6.23, we know that there exists a compact attracting subtree Σ_0 with respect to the edge orientation induced by L_ρ. This moreover implies that there are finitely many sinks.

Lemma 8.2. *If a vertex is a sink, then the corresponding Steiner tree induces a critical carrier graph.*

Proof. It is enough to show that if the Steiner tree does not induces a critical carrier graph, the Steiner tree of a neighbor vertex has strictly shorter length.

Consider the superbasis (X, Y, Z). If the corresponding carrier graph (Γ, Ψ) is not critical, it has a valence 4 vertex. Then the Steiner tree for $(\gamma_X, \gamma_Y, \gamma_Z)$ consists of one vertex and two edges. Assume that the vertex is on the geodesic γ_Z. We denote by v this vertex and v_X and v_Y its projections to γ_X and γ_Y respectively. By our previous discussion, the angle between vv_X and vv_Y is bigger or equal to $2\pi/3$, and the angle between each edge and the γ_Z is bigger or equals to $\pi/3$. This implies that the angle between vv_X and $vr_Z(v_Y)$ is smaller than $2\pi/3$. Hence it is not the Steiner tree for the triple $(\gamma_X, r_Z(\gamma_Y), \gamma_Z)$. Therefore the new triple of geodesics admits a shorter Steiner tree, which contradicts to the facts that (X, Y, Z) is a sink. $\square$

This shows the existence of critical carrier graphs. The finiteness is given by the fact that the Steiner graph associated to a vertex outside the compact attracting subtree consists of one vertex and two edges. Therefore we prove the main theorem.

The corollary follows from the fact that a minimal carrier graph is also a critical carrier graph.

9. EXAMPLE

Consider $\mathbb{H}^2$ as a totally geodesic submanifold in $\mathbb{H}^3$. In this section we will consider the BQ-representations which preserve $\mathbb{H}^2$ in $\mathbb{H}^3$. They all come from the hyperbolic structure on the surfaces with Euler characteristic -1. Based on the topological type of the corresponding surface, these representations can be classified into the following 4 types:

(1) once-punctured torus;
(2) three-punctured sphere;
(3) once-punctured Möbius band;
(4) once-punctured Klein bottle.

For more details, one may read [9] and [10].

A minimal carrier 2-graph for these BQ-representations is induced by a superbasis (X, Y, Z) of which the triple of geodesics $(\gamma_X, \gamma_Y, \gamma_Z)$ are of special configuration, and the number of minimal carrier 2-graph is at most 2. We will give the description of these configuration case by case.

We recall three types of a Steiner tree for a triple of geodesics:

(1) One valence 3 vertex and three valence 1 vertices;
(2) One valence 2 vertex and two valence 1 vertices;
(3) Two valence 2 vertices and two valence 1 vertices.

9.1. **Once-punctured torus.** Let ρ be an irreducible representation. Given a superbasis (X, Y, Z), we assume that the three geodesics γ_X , γ_Y and γ_Z associated to $(\rho(X), \rho(Y), \rho(Z))$ are all orthogonal to $\mathbb{H}^2$. Such a representation is a BQ representation if and only if it corresponds to a hyperbolic structure on once-punctured torus. If the boundary holonomy is hyperbolic, the hyperbolic structure is complete and of infinite volume; if the boundary holonomy is parabolic, the hyperbolic structure is complete and of finite volume; if the boundary holonomy is elliptic, then the hyperbolic structure is not complete and the boundary becomes a conic singularity. In the first two cases, the representation is discrete and faithful, while in the last case, it is not discrete and possibly not faithful, either.

For each superbasis, the corresponding right angled hexagon is a triangle and the corresponding Steiner tree is either of type (1) or of type (2). Therefore the Steiner tree is the same as the Fermat tree in this case. We start with the triangle $\Delta v_1 v_2 v_3$ for one superbasis. To get the triangle for one neighbor superbasis, we consider the involution of $\mathbb{H}^2$ with respect to v_j for some j, and consider its action on a different vertex v_{j+1} to get a new point. The new triangle is determined by v_j, $r_j(v_{j+1})$ and v_{j-1}. It is clear that the two triangle have same lengths on two sides adjacent to v_j and the internal angles at v_j are complementary to each other.
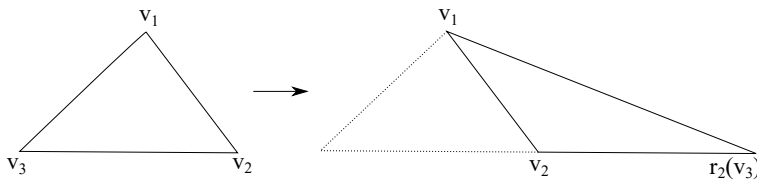

FIGURE 25. Changing superbases

In this way, we can obtain one triangle for each superbases inductively. We can verify the following two facts:

a. there are finitely many superbases of which the associated triangles are $2\pi/3$-acute;
b. if there is a basis (X, Y) of $\mathbb{F}_2$, such that the axes of $\rho(X)$ and $\rho(Y)$ are orthogonal, then there are only two superbases (X, Y, Z) and (X, Y^{-1}, Z') of which the associated triangles are non obtuse in the usual sense, and are right angled triangles; otherwise there is a unique superbasis of which the associated triangle is non obtuse and is acute.

The Fermat trees for those $2\pi/3$-acute triangles induce critical carrier 2-graphs, while the Fermat tree for the acute triangle or the right-angled triangle induces the minimal carrier 2-graph.

9.2. **three-holed sphere.** Let ρ be an irreducible representation. Given a super-basis (X, Y, Z), we assume that the three geodesics γ_X, γ_Y and γ_Z associated to $(\rho(X), \rho(Y), \rho(Z))$ are contained in $\mathbb{H}^2$ and disjoint from one another. Such a representation is a BQ representation if and only if it corresponds to a hyperbolic structure on thrice-punctured sphere. Since the three peripheral elements are all primitive elements, they must be sent to hyperbolic elements in $\mathrm{SL}(2, \mathbb{R})$ by ρ. Therefore, the hyperbolic structure on the three punctured sphere is complete and with infinite volume. The representation is always discrete and faithful.

The triple of geodesics $(\gamma_X, \gamma_Y, \gamma_Z)$ has one of the following two configurations:

a. there exists one of the three geodesics separating the other two into two different half planes;
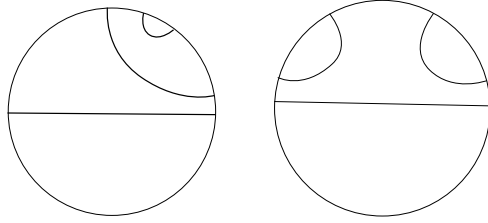b. no one separate the other two into two half planes.

FIGURE 26. Possible configurations

There is a unique superbasis of which the associated triple of geodesics are of the second configuration. It admits

- either a unique Steiner tree of type (1) or type (3);
- or two Steiner tree, such that one is of type (1), while the other is of type (3).

For the former case, the carrier 2-graph induced by the Steiner tree is the unique minimal one, and for the latter case, the two carrier 2-graph induced by the two Steiner trees are both minimal (see Remark 6.22).

9.3. **Once-punctured Möbius band.** The once-punctured Möbius band and the once-punctured Kleinian bottle are both non-oriented surfaces. Their corresponding representations have been studied by Goldman-McShane-Stantchev-Tan in [10].

Consider a BQ-representation ρ corresponding to a hyperbolic structure on a once-punctured Möbius band. Given a superbasis (X, Y, Z), the three geodesics γ_X, γ_Y and γ_Z associated to $(\rho(X), \rho(Y), \rho(Z))$ have one orthogonal to $\mathbb{H}^2$ and two others contained in $\mathbb{H}^2$. Without loss of generality, we may assume that γ_X is the one orthogonal to $\mathbb{H}^2$ and we denote by v_X its intersection point with $\mathbb{H}^2$. Since the triple of geodesics are in general position, there are two possible configurations:

a. one of γ_Y and γ_Z separates the other two into two half planes in $\mathbb{H}^2$;
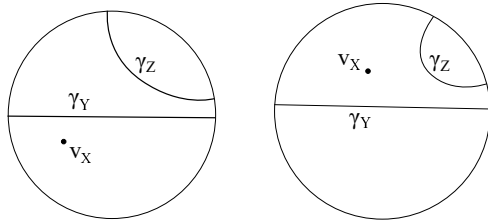b. none of γ_Y and γ_Z separate the other two.

FIGURE 27. Possible configurations

Let r_X denote the involution of with respect to γ_X. For a triple of second configuration, by applying r_X on γ_Y or γ_Z, the new triple is almost always of the first configuration. The exceptions are two triples associated to two superbases which are neighbors to each other. They are the special superbases among all superbases for the representation ρ. The two triples of geodesics associated to these two superbases are both of the second configuration and their internal angles at v_X are complementary to each other.

If the two internal angles at v_X are both right angles, then the Steiner tree associated to these two triples induce the only two minimal carrier 2-graphs for ρ; if one is bigger than the other, then the Steiner tree for the triple with smaller internal angle at v_X induces the unique minimal carrier 2-graph of ρ.
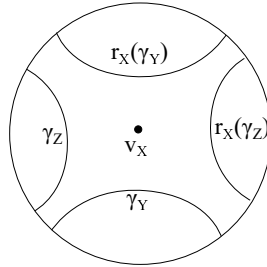

FIGURE 28. Realizing minimal carrier graph

9.4. **Once-punctured Klein bottle.** Consider a BQ-representation ρ corresponding to a hyperbolic structure on a once-punctured Klein bottle. Given a superbasis (X, Y, Z), the triple of geodesics $(\gamma_X, \gamma_Y, \gamma_Z)$ associated to $(\rho(X), \rho(Y), \rho(Z))$ has two orthogonal to $\mathbb{H}^2$, and the last one is contained in $\mathbb{H}^2$. Without loss of generality, we may assume that γ_X is the one contained in $\mathbb{H}^2$ and we denote by v_Y and v_Z the intersection points between γ_Y and γ_Z with $\mathbb{H}^2$ respectively. Since the triple of geodesics are in general position, there are two possible configurations:

    a. the geodesic γ_X separates v_Y from v_Z in $\mathbb{H}^2$;

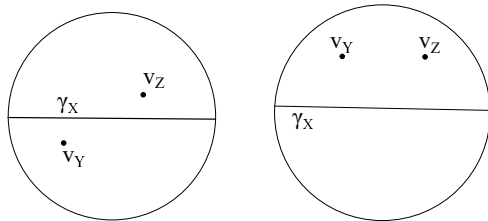   b. the two point v_Y and v_Z are on the same side of γ_X in $\mathbb{H}^2$.

FIGURE 29. Possible configurations

The geodesic in $\mathbb{H}^2$ passing v_Y and v_Z is the axis of $\rho(X)$. Denote it by δ_X. The triple induces the minimal carrier 2-graph is the one such that

- the geodesic δ_X is disjoint from γ_X;
- the distance between δ_X and γ_X is closest among all disjoint pairs $(\delta_{X'}, \gamma_{X'})$'s;
- either the two point v_Y and v_Z are separated by the common perpendicular geodesic γ between δ_X and γ_X, or one of them is on γ.

If v_Y and v_Z are separated by γ, then there is a unique minimal carrier 2-graph; if one of the two points is on γ and assume that it is v_Y, then there are two minimal carrier 2-graph induced by $(\gamma_X, \gamma_Y, \gamma_Z)$ and $(\gamma_X, \gamma_Y, r_Y(\gamma_Z))$.
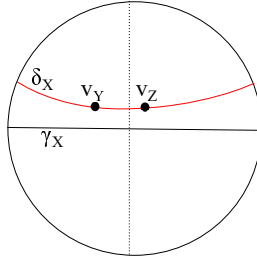

FIGURE 30. Realizing minimal carrier graph

Appendix A. Proof of the convexity lemma

Lemma (Hyperbolic Convexity Lemma). *Let a, b, c and d be four points in n-dimensional hyperbolic space $\mathbb{H}^n$. Let e and f be two points on ab and cd respectively, such that $|ae|/|ab| = |df|/|cd| = t$ for some $t \in (0,1)$. Then,*

$$|ef| \leq (1-t)|ad| + t|bc|. \tag{26}$$

Moreover the equality is realized if and only if all four points lie on the same geodesic and m is between e and f.

Proof. Let us first consider the triangle Δabc with vertices a, b and c. There is always a 2 dimensional hyperbolic plane containing this triangle. Let m be a point on ac such that $|am|/|ac| = t$, where $0 < t < 1$. Then $|em|/|bc| \leq t$. the proof is as follows.

We will use the Poincaré disk model $\mathbb{D}$ of $\mathbb{H}^2$. By translations and rotations of $\mathbb{D}$, we can assume that the vertex a is at the origin and b is point on the positive side of the real axis. Recall that the hyperbolic metric is given by the following formula under polar coordinates:

$$\mathrm{d}\,s(r, \theta) = \frac{2\sqrt{(\mathrm{d}\,r)^2 + r^2(\mathrm{d}\,\theta)^2}}{(1 - r^2)}. \tag{27}$$

Then the point on the geodesic segment bc has the coordinates $(r_1(\theta), \theta)$ with $\theta \in [0, \alpha]$ where $\alpha = \angle bac$. Let us assume first that $0 < \alpha < \pi$. Then we consider the path γ connecting points e and m which is parametrized by $(t \cdot r_1(\theta), \theta)$.
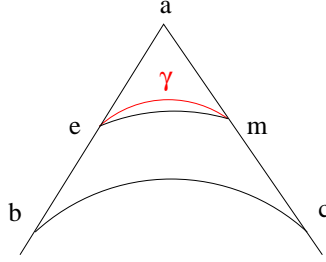
FIGURE 31

The length of γ can be given by the following two integrals:

$$
\begin{aligned}
|\gamma| &= \int_0^\alpha \frac{2t\sqrt{(\frac{\mathrm{d}\,r_1}{\mathrm{d}\,\theta})^2 + r_1^2}\,\mathrm{d}\,\theta}{(1 - t^2 r_1^2)} \\
&= t\int_0^\alpha \frac{2\sqrt{(\frac{\mathrm{d}\,r_1}{\mathrm{d}\,\theta})^2 + r_1^2}\,\mathrm{d}\,\theta}{(1 - t^2 r_1^2)} \\
&< t\int_0^\alpha \frac{2\sqrt{(\frac{\mathrm{d}\,r_1}{\mathrm{d}\,\theta})^2 + r_1^2}\,\mathrm{d}\,\theta}{(1 - r_1^2)} \\
&= t|bc|.
\end{aligned}
$$

Since em is the geodesic connecting e and m, its length is strictly shorter than that of γ. This shows that $|em|/|bc| < t$ when $0 < \alpha < \pi$.

When $\alpha = 0$ or π, then we have the equality. Combining the two cases, we show that $|em|/|bc| \leq t$. By applying the same discussion to the triangle Δacd, we can show that $|mf|/|ad| \leq (1 - t)$.
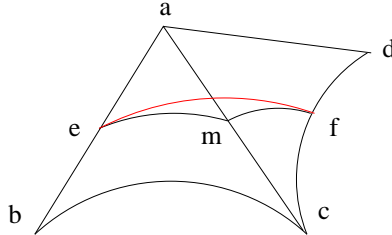

FIGURE 32

By the triangular inequality, we can conclude the following inequality:

$$|ef| \leq |em| + |fm| \leq t|bc| + (1 - t)|ad| \tag{28}$$

The equality is realized if and only if

$$
\begin{aligned}
|em| &= t|bc| \\
|fm| &= (1 - t)|ad| \\
|ef| &= |em| + |fm|
\end{aligned}
$$

The first equality is realized if and only if a, b and c are collinear. The second equality is realized if and only if a, c and d are collinear. The third one is realized

if and only if e, m and f are on the same geodesic and m is between e and f. This complete the proof of this lemma. $\square$

Remark A.1. To be more precise, the equality $|ef| = t|bc| + (1-t)|ad|$ is realized if and only if a, b, c and d are collinear and in one of the following orders: $\overrightarrow{abdc}$, $\overrightarrow{adbc}$, $\overrightarrow{abcd}$, $\overrightarrow{badc}$, $\overrightarrow{bacd}$, $\overrightarrow{bcad}$, $\overrightarrow{dacb}$, $\overrightarrow{dcab}$, $\overrightarrow{cdab}$, $\overrightarrow{dcba}$, $\overrightarrow{cbda}$ and $\overrightarrow{cdba}$.

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