Ground state solutions for Bessel fractional equations with irregular nonlinearities

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July 20, 2018

We consider the semilinear fractional equation $(I-\Delta)^s u = a(x)|u|^{p-2}u$ in $\mathbb{R}^N$, where $N \geq 3$, $0 < s < 1$, $2 < p < 2N/(N-2s)$ and a is a bounded weight function. Without assuming that a has an asymptotic profile at infinity, we prove the existence of a ground state solution.

Keywords and phrases: Bessel fractional operator, fractional Laplacian
2010 Mathematics Subject Classification: 35KJ60, 35Q55, 35S05

Dedicated to Francesca

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1 Introduction

To pursue further the study that we began in [19, 20], we consider in this paper the equation

$$(I-\Delta)^s u = a(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $a \in L^\infty(\mathbb{R}^N)$, $N > 2$, $0 < s < 1$ and $2 < p < 2_s^\star = 2N/(N-2s)$.
When $s = 1$, (1.1) formally reduces to the semilinear elliptic equation

$$-\Delta u + u = a(x)|u|^{p-2}u,$$

which has been widely studied over the years. This equation can be seen as a particular case of the stationary Nonlinear Schrödinger Equation

$$-\Delta u + V(x)u = a(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

When both V and a are constants, we refer to the seminal papers [7, 8] and to the references therein. Since the *non-compact* group of translations acts on $\mathbb{R}^N$, when V and a are general functions the analysis becomes subtler, and solutions exist according to some properties of these potentials. For instance, when both V and a are radially symmetric, (1.2) is invariant under rotations, and it becomes legitimate to look for radially symmetric solutions: see [12].
Without any *a priori* symmetry assumption, the lack of compactness in (1.2) must be overcome with a careful analysis, and the behavior of V and a at infinity plays a crucial rôle. The first attempt to solve (1.2) in the case $\lim_{|x|\to+\infty} V(x) = +\infty$ and a is a constant appeared in [16]. With similar techniques, it is possible to solve (1.2) under the assumption $\limsup_{|x|\to+\infty} a(x) \leq 0$. So many papers dealing with (1.2) (or with even more general equations) appeared in the literature afterwards that we refrain from any attempt to give a complete overview.

If $0 < s < 1$, our equation becomes *non-local*, since the fractional power $(I-\Delta)^s$ of the positive operator $I-\Delta$ in $L^2(\mathbb{R}^N)$ is no longer a differential operator. It is strictly related to the more popular *fractional laplacian* $(-\Delta)^s$, but it behaves worse under scaling. We offer a very quick review of this operator.

For $s > 0$ we introduce the *Bessel function space*

$$L^{s,2}(\mathbb{R}^N) = \left\{ f \in L^2(\mathbb{R}^N) \mid f = G_s \star g \text{ for some } g \in L^2(\mathbb{R}^N) \right\},$$

where the Bessel convolution kernel is defined by

$$G_s(x) = \frac{1}{(4\pi)^{s/2}\Gamma(s/2)} \int_0^\infty \exp\left(-\frac{\pi}{t}|x|^2\right) \exp\left(-\frac{t}{4\pi}\right) t^{\frac{s-N}{2}-1}\, dt.$$

The Bessel space is endowed with the norm $\|f\| = \|g\|_2$ if $f = G_s \star g$. The operator $(I-\Delta)^{-s}u = G_{2s} \star u$ is usually called Bessel operator of order s.
In Fourier variables the same operator reads

$$G_s = \mathcal{F}^{-1} \circ \left(\left(1+|\xi|^2\right)^{-s/2} \circ \mathcal{F} \right),$$

so that

$$\|f\| = \left\| (I-\Delta)^{s/2} f \right\|_2.$$

For more detailed information, see [2, 22] and the references therein.

In the paper [13] the pointwise formula

$$(I-\Delta)^s u(x) = c_{N,s}\, \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|)\, dy + u(x)$$

was derived for functions $u \in C_c^2(\mathbb{R}^N)$. Here $c_{N,s}$ is a positive constant depending only on N and s, P.V. denotes the principal value of the singular integral, and K_ν is the modified Bessel function of the second kind with order ν (see [13, Remark 7.3] for more details). However a closed formula for K_ν is not known.

We summarize the main properties of Bessel spaces. For the proofs we refer to [14, Theorem 3.1], [22, Chapter V, Section 3].

Theorem 1.1. *1. $L^{s,2}(\mathbb{R}^N) = W^{s,2}(\mathbb{R}^N) = H^s(\mathbb{R}^N)$, where the sign of equality must be understood in the sense of an isomorphism.*

2. If $s \geq 0$ and $2 \leq q \leq 2_s^* = 2N/(N-2s)$, then $L^{s,2}(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$; if $2 \leq q < 2_s^*$ then the embedding is locally compact.

3. Assume that $0 \leq s \leq 2$ and $s > N/2$. If $s - N/2 > 1$ and $0 < \mu \leq s - N/2 - 1$, then $L^{s,2}(\mathbb{R}^N)$ is continuously embedded into $C^{1,\mu}(\mathbb{R}^N)$. If $s - N/2 < 1$ and $0 < \mu \leq s - N/2$, then $L^{s,2}(\mathbb{R}^N)$ is continuously embedded into $C^{0,\mu}(\mathbb{R}^N)$.

Remark 1.2. According to Theorem 1.1, the Bessel space $L^{s,2}(\mathbb{R}^N)$ is topologically undistinguishable from the Sobolev fractional space $H^s(\mathbb{R}^N)$. Since our equation involves the Bessel norm, we will not exploit this characterization.

Going back to (1.1), it must be said that in the case $s \in (0,1)$ less is known than in the *local* case $s = 1$. Equation (1.1) arises from the more general Schrödinger-Klein-Gordon equation

$$\mathrm{i}\frac{\partial \psi}{\partial t} = (I-\Delta)^s \psi - \psi - f(x,\psi)$$

describing the the behaviour of bosons, spin-0 particles in relativistic fields. We refer to [15, 19–21] for very recent results about the existence of variational solutions. When $s = 1/2$, the operator $(I-\Delta)^{1/2} = \sqrt{I-\Delta}$ is also called *pseudorelativistic* or *semirelativistic*, and it is very important in the study of several physical phenomena. The interested reader can refer to [10, 11] and to the references therein for more information.

Remark 1.3. The identity operator I is often replaced by a multiple $m^2 I$, for some real number $m \neq 0$. The operator reads then $(-\Delta + m^2)^s$, but for our purposes this generality does not give any advantage.

A common feature in the current literature is that the existence of solutions to (1.1) is related to the behavior of the potential function a at infinity. This is a very useful tool for applying concentration-compactness methods or for working in weighted Lebesgue spaces. In the present paper, following [1], we investigate (1.1) under much weaker assumptions on a, see Section 2. The first existence results for semilinear elliptic equations with *irregular* potentials appeared, as far as we know, in [9].

2 The variational setting

We introduce some tools that will be used systematically in the rest of the paper.

Definition 2.1.
- For any $y \in \mathbb{R}^N$, we define the translation operator τ_y acting on a (suitably regular) function f as $\tau_y f \colon x \mapsto f(x-y)$.
- In a normed space X, we denote by $B(x,r)$ the ball centered at $x \in X$ with radius $r > 0$, and by $\overline{B}(x,r)$ its closure. The boundary of $B(0,1)$ will be denoted by $S(X)$.
- For any $a \in L^\infty(\mathbb{R}^N)$, we define

$$\mathscr{P} = \overline{B}(0, |a|_\infty) \subset L^\infty(\mathbb{R}^N).$$

  Looking at $L^\infty(\mathbb{R}^N)$ as the dual space of $L^1(\mathbb{R}^N)$, the set $\mathscr{P}$ will be endowed with the weak* topology. It is well-known that $\mathscr{P}$ becomes a compact metrizable space, see [17, Theorem 3.15 and Theorem 3.16].
- For any $a \in L^\infty(\mathbb{R}^N)$, we define the subset $\mathscr{A} = \left\{ \tau_y a \mid y \in \mathbb{R}^N \right\}$ of $\mathscr{P}$, endowed with the relative topology. Finally, we introduce $\mathscr{B} = \overline{\mathscr{A}} \setminus \mathscr{A}$.
- For any $a \in L^\infty(\mathbb{R}^N)$, we define

$$\bar{a} = \sup \left\{ \operatorname{ess\,sup} u \mid u \in \mathscr{B} \right\}. \tag{2.1}$$

  If $\mathscr{B} = \emptyset$, we agree that $\bar{a} = -\infty$.

The following is the main assumption of the present paper.

(A) The function $a \in L^\infty(\mathbb{R}^N)$ is such that $a^+ = \max\{a, 0\}$ is not identically zero, and either (i) $\bar{a} \leq 0$ or (ii) $\bar{a} \leq a$.

Weak solutions to (1.1) are critical points of the functional $I_a \colon L^{s,2}(\mathbb{R}^N) \to \mathbb{R}^N$ defined by

$$I_a(u) = \frac{1}{2} \|u\|^2_{L^{s,2}} - \frac{1}{p} \int_{\mathbb{R}^N} a|u|^p.$$

Definition 2.2. A solution $u \in L^{s,2}(\mathbb{R}^N)$ is called a ground-state solution to (1.1) if I_a attains at u the infimum over the set of all solutions to (1.1), namely

$$I_a(u) = \min \left\{ I_a(v) \mid v \in L^{s,2}(\mathbb{R}^N) \text{ solves } (1.1) \right\}.$$

We now state the main result of our paper.

Theorem 2.3. *Equation (1.1) has (at least) a positive ground state provided that $2 < p < 2^\star_s$ and $a \in L^\infty(\mathbb{R}^N)$ satisfies **(A)**.*

3 The construction of a Nehari manifold

We introduce the Nehari set of I_a as

$$\mathscr{N}_a = \left\{ u \in L^{s,2}(\mathbb{R}^N) \mid u \neq 0,\ DI_a(u)[u] = 0 \right\}.$$

Definition 3.1. $c_a = \inf_{u \in \mathscr{N}_a} I_a(u)$. We agree that $c_a = +\infty$ if $\mathscr{N}_a = \emptyset$.

To proceed further, we need a "dual" characterization of the essential supremum.

Lemma 3.2. *Let $a \in L^\infty(\mathbb{R}^N)$. There results*

$$\operatorname{ess\,sup} a = \sup \left\{ \int_{\mathbb{R}^N} a\varphi \mid \varphi \in L^1(\mathbb{R}^N),\ \varphi \geq 0,\ \int_{\mathbb{R}^N} \varphi = 1 \right\}. \tag{3.1}$$

Proof. Whenever $\varphi \in L^1(\mathbb{R}^N)$, $\varphi \geq 0$, $\int_{\mathbb{R}^N} \varphi = 1$, we compute

$$\int_{\mathbb{R}^N} a\varphi \leq \operatorname{ess\,sup} a \int_{\mathbb{R}^N} \varphi = \operatorname{ess\,sup} a.$$

Hence

$$\operatorname{ess\,sup} a \geq \sup \left\{ \int_{\mathbb{R}^N} a\varphi \mid \varphi \in L^1(\mathbb{R}^N),\ \varphi \geq 0,\ \int_{\mathbb{R}^N} \varphi = 1 \right\}. \tag{3.2}$$

On the other hand, if we set

$$\sup \left\{ \int_{\mathbb{R}^N} a\varphi \mid \varphi \in L^1(\mathbb{R}^N),\ \varphi \geq 0,\ \int_{\mathbb{R}^N} \varphi = 1 \right\} = b$$

and we assume that $\operatorname{ess\,sup} a > b$, then for some $\delta > 0$ we can say that the set $\Omega = \left\{ x \in \mathbb{R}^N \mid a(x) \geq b + \delta \right\}$ has positive measure. Let us define $\varphi = \chi_\Omega / \mathcal{L}^N(\Omega)$, so that

$$\int_{\mathbb{R}^N} a\varphi = \frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega a \geq b + \delta,$$

contrary to (3.2). This completes the proof. ☐

Recall from assumption **(A)** that $a^+ \neq 0$ as an element of $L^\infty(\mathbb{R}^N)$. Therefore Lemma 3.2 yields a function $\varphi \in S(L^1(\mathbb{R}^N))$ such that $\varphi \geq 0$ and $\int_{\mathbb{R}^N} a\varphi > 0$. By a standard mollification argument, we can assume without loss of generality that $\varphi \in C_c^\infty(\mathbb{R}^N)$.

Since $L^{s,2}(\mathbb{R}^N)$ is continuously embedded into $L^p(\mathbb{R}^N)$ for every $2 < p < 2_s^\star$, we can set

$$S_p = \sup \left\{ \frac{|u|_p}{\|u\|_{L^{s,2}}} \mid u \in L^{s,2}(\mathbb{R}^N),\ u \neq 0 \right\} \in (0, +\infty).$$

We write

$$\mathscr{B}_a^+ = \left\{ u \in L^{s,2}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} a|u|^p > 0 \right\}$$

and

$$\mathscr{S}_a^+ = \mathscr{B}_a^+ \cap S(L^{s,2}(\mathbb{R}^N)).$$

Lemma 3.3. *The set $\mathscr{B}_a^+$ is non-empty and open in $L^{s,2}(\mathbb{R}^N)$.*

Proof. We already know that $\varphi \in \mathscr{B}_a^+$. Furthermore, the map $u \mapsto \int_{\mathbb{R}^N} a|u|^p$ is continuous from $L^{s,2}(\mathbb{R}^N)$ to $\mathbb{R}$, since $a \in L^\infty(\mathbb{R}^N)$ and $2 < p < 2^\star$. This immediately implies that $\mathscr{B}_a^+$ is an open subset of $L^{s,2}(\mathbb{R}^N)$. ☐

Lemma 3.4. *There exists a homeomorphism $\mathscr{S}_a^+ \to \mathscr{N}_a$ whose inverse map is $u \mapsto u/\|u\|_{L^{s,2}}$.*

Proof. For any $u \in L^{s,2}(\mathbb{R}^N) \setminus \{0\}$ we consider the *fibering map*

$$h(t) = I_a(tu), \qquad (t \geq 0).$$

It follows easily that h has a positive critical point if, and only if, $u \in \mathscr{B}_a^+$. It is a Calculus exercise to check that, in this case, the critical point of h is the unique non-degenerate global maximum $\bar{t}(u) > 0$ of h. By direct computation, $tu \in \mathscr{N}_a$ if, and only if, $t = \bar{t}(u)$. Explicitly,

$$\bar{t}(u) = \frac{\|u\|_{L^{s,2}}^2}{\int_{\mathbb{R}^N} a|u|^p}.$$

This shows that the map $u \mapsto \bar{t}(u)$ is continuous from $\mathscr{B}_a^+$ to $(0, +\infty)$. The rest of the proof follows easily. ☐

Lemma 3.5. *The set $\mathscr{N}_a$ is closed in $L^{s,2}(\mathbb{R}^N)$.*

Proof. If $u \in \mathscr{N}_a$, then

$$\|u\|_{L^{s,2}}^2 = \int_{\mathbb{R}^N} a|u|^p \leq \int_{\mathbb{R}^N} a^+|u|^p \leq S_p|a^+|_\infty \|u\|_{L^{s,2}}^p.$$

It follows that

$$\inf_{u \in \mathscr{N}_a} \|u\|_{L^{s,2}} \geq \frac{1}{S_p|a^+|_\infty^{1/(p-2)}}. \tag{3.3}$$

As a consequence, 0 is not a cluster point of $\mathscr{N}_a$, which turns out to be closed. ☐

It is now standard to invoke the Implicit Function Theorem to prove that $\mathscr{N}_a$ is a C^2-submanifold of $L^{s,2}(\mathbb{R}^N)$ and that (3.3) implies

$$\inf_{u \in \mathscr{N}_a} I_a(u) \geq \left(\frac{1}{2} - \frac{1}{p}\right) \frac{1}{S_p^2|a^+|_\infty^{2/(p-2)}}.$$

More importantly, $\mathscr{N}_a$ is a *natural constraint* for I_a, i.e. every critical point of the restriction $\bar{I}_a$ of I_a to $\mathscr{N}_a$ is a nontrivial critical point of I_a. The following result was proved in [15, Proposition 3.2], and allows us to consider only positive ground states.

Proposition 3.6. *Any weak solution to (1.1) is strictly positive.*

Proposition 3.7. *Let $\bar{I}_a$ be the restriction of the functional I_a to the manifold $\mathscr{N}_a$. Every Palais-Smale sequence at level c for $\bar{I}_a$ is also a Palais-Smale sequence at level c for I_a.*

Proof. Assume that $\{u_n\}_n \subset \mathscr{N}_a$ is a Palais-Smale sequence at level c for $\bar{I}_a$, namely

$$\lim_{n \to +\infty} \bar{I}_a(u_n) = c$$

and

$$\lim_{n\to+\infty} D\bar{I}_a(u_n) = 0$$

in the norm topology. It suffices to show that the sequence $\{\nabla I_a(u_n)\}_n$ converges to zero in $L^{s,2}(\mathbb{R}^N)$. Let us abbreviate $\psi(u) = DI_a(u)[u]$, so that $\mathscr{N}_a = \psi^{-1}(\{0\}) \setminus \{0\}$. From the fact that $u_n \in \mathscr{N}_a$, we deduce that $I_a(u_n) = (1/2 - 1/p)\|u_n\|^2_{L^{s,2}}$, and hence the sequence $\{u_n\}_n$ is bounded. This implies that

$$\sup_n \frac{\|\nabla\psi(u_n)\|_{L^{s,2}}}{\|u_n\|_{L^{s,2}}} < +\infty. \tag{3.4}$$

Explicitly, we have that, for every $n \in \mathbb{N}$,

$$\langle \nabla\psi(u_n) \mid u_n \rangle = (2-p)\|u_n\|^2_{L^{s,2}} < 0 \tag{3.5}$$

and

$$\nabla\bar{I}_a(u_n) = \nabla I_a(u_n) - \frac{\langle \nabla I_a(u_n) \mid \nabla\psi(u_n)\rangle}{\|\nabla\psi(u_n)\|^2_{L^{s,2}}} \nabla\psi(u_n). \tag{3.6}$$

Observe that $\nabla I_a(u_n) \perp u_n$ because $u_n \in \mathscr{N}_a$. If we consider the quantity

$$\|\nabla\psi(u_n)\|^2_{L^{s,2}} - \left(\frac{\langle \nabla I_a(u_n) \mid \nabla\psi(u_n)\rangle}{\|\nabla I_a(u_n)\|^2_{L^{s,2}}} \right)^2,$$

we immediately see that it equals the square of the norm of the projection of the vector $\nabla\psi(u_n)$ onto the subspace of $L^{s,2}(\mathbb{R}^N)$ orthogonal to the unit vector $\nabla I_a(u_n)/\|\nabla I_a(u_n)\|$. Since this subspace contains in particular the vector $u_n/\|u_n\|_{L^{s,2}}$, it follows from the Pythagorean Theorem that

$$\|\nabla\psi(u_n)\|^2_{L^{s,2}} - \left(\frac{\langle \nabla I_a(u_n) \mid \nabla\psi(u_n)\rangle}{\|\nabla I_a(u_n)\|^2_{L^{s,2}}} \right)^2 \geq \left(\frac{\langle \nabla\psi(u_n) \mid u_n\rangle}{\|u_n\|_{L^{s,2}}} \right)^2. \tag{3.7}$$

This yields, recalling (3.6), (3.5) and (3.4),

$$\begin{aligned}
\left\|\nabla\bar{I}_a(u_n)\right\|_{L^{s,2}} \|\nabla I_a(u_n)\|_{L^{s,2}} &\geq \langle \nabla\bar{I}_a(u_n) \mid \nabla I_a(u_n)\rangle \\
&= \frac{\|\nabla I_a(u_n)\|^2_{L^{s,2}}}{\|\nabla\psi(u_n)\|^2_{L^{s,2}}} \left(\|\nabla\psi(u_n)\|^2_{L^{s,2}} - \left(\frac{\langle \nabla I_a(u_n) \mid \nabla\psi(u_n)\rangle}{\|\nabla I_a(u_n)\|^2_{L^{s,2}}} \right)^2 \right)^2 \\
&\geq \frac{\|\nabla I_a(u_n)\|^2_{L^{s,2}}}{\|\nabla\psi(u_n)\|^2_{L^{s,2}}} \left(\frac{\langle \nabla\psi(u_n) \mid u_n\rangle}{\|u_n\|_{L^{s,2}}} \right)^2 \\
&= \frac{\|\nabla I_a(u_n)\|^2_{L^{s,2}}}{\|\nabla\psi(u_n)\|^2_{L^{s,2}}} (2-p)^2 \|u_n\|^2_{L^{s,2}} \\
&\geq C\|\nabla I_a(u_n)\|^2_{L^{s,2}}.
\end{aligned}$$

This argument proves that $\lim_{n\to+\infty} \|\nabla I_a(u_n)\|_{L^{s,2}} = 0$, and we conclude. □

4 Splitting and vanishing sequences

The analysis of Palais-Smale sequences can be harder than in the more familiar case of a potential function a that has a precise asymptotic behavior at infinity. For this reason, we recall a language taken from [1].

Definition 4.1. A map $F\colon X \to Y$ between two Banach spaces splits in the BL sense[1] if for any sequence $\{u_n\}_n \subset X$ such that $u_n \rightharpoonup u$ in X there results

$$F(u_n - u) = F(u_n) - F(u) + o(1)$$

in the norm topology of Y.

Lemma 4.2. *Suppose that $\{u_n\}_n \subset L^{s,2}(\mathbb{R}^N)$ and $\{y_n\}_n \subset \mathbb{R}^N$ are such that $\tau_{-y_n} u_n \rightharpoonup u_0$ in $L^{s,2}(\mathbb{R}^N)$. Then*

$$I_{\tau_{-y_n} a}(\tau_{-y_n} u_n) - I_{\tau_{-y_n} a}(\tau_{-y_n} u_n - u_0) - I_{\tau_{-y_n} a}(u_0) = o(1)$$

and

$$DI_{\tau_{-y_n} a}(\tau_{-y_n} u_n) - DI_{\tau_{-y_n} a}(\tau_{-y_n} u_n - u_0) - DI_{\tau_{-y_n} a}(u_0) = o(1).$$

Proof. Since the maps $F(u) = p^{-1}|u|^p$ and $F'(u) = |u|^{p-2}u$ both split from $L^{s,2}(\mathbb{R}^N)$ into $L^1(\mathbb{R}^N)$, see [19, Lemma 4.4], we can write

$$\begin{aligned}\int_{\mathbb{R}^N} &|(\tau_{-y_n} a)\left(F(\tau_{-y_n} u_n) - F(\tau_{-y_n} u_n - u_0) - F(u_0)\right)| \\ &\leq |a|_\infty \int_{\mathbb{R}^N} |F(\tau_{-y_n} u_n) - F(\tau_{-y_n} u_n - u_0) - F(u_0)| = o(1)\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N} &\left|(\tau_{-y_n} a)\left(F'(\tau_{-y_n} u_n) - F'(\tau_{-y_n} u_n - u_0) - F'(u_0)\right)\right|^{p/(p-1)} \\ &\leq |a|_\infty^{p/(p-1)} \int_{\mathbb{R}^N} \left|F'(\tau_{-y_n} u_n) - F'(\tau_{-y_n} u_n - u_0) - F'(u_0)\right|^{p/(p-1)}.\end{aligned}$$

Recalling that the squared norm splits in the BL sense, the proof is complete. ☐

Definition 4.3. A sequence $\{u_n\}_n \subset L^{s,2}(\mathbb{R}^N)$ vanishes if $\tau_{x_n} u_n \rightharpoonup 0$ in $L^{s,2}(\mathbb{R}^N)$ for any sequence $\{x_n\}_n$ of points in $\mathbb{R}^N$.

Remark 4.4. Any vanishing sequence is necessarily bounded in $L^{s,2}(\mathbb{R}^N)$, and by the Rellich-Kondratchev theorem (see [6, Corollary 7.2]) $\tau_{x_n} u_n \to 0$ strongly in $L^2_{\text{loc}}(\mathbb{R}^N)$ for every sequence $\{x_n\}_n \subset \mathbb{R}^N$. This yields that, for every $R > 0$,

$$\lim_{n \to +\infty} \sup \left\{ \int_{B(x,R)} |u_n|^2 \mid x \in \mathbb{R}^N \right\} = 0.$$

By the fractional version of Lions' vanishing lemma [18, Proposition II.4], we deduce that $u_n \to 0$ strongly in $L^q(\mathbb{R}^N)$ for every $2 < q < 2^\star_s$.

[1] BL stands for Brezis and Lieb.

Definition 4.5. If $\{u_n\}_n$ is a sequence from $L^{s,2}(\mathbb{R}^N)$, we say that $\{DI_a(u_n)\}_n$ *-vanishes if $DI_{\tau_{x_n}a}(u_n) \rightharpoonup^\star 0$ in the weak* topology for every sequence $\{x_n\}_n \subset \mathbb{R}^N$.

Remark 4.6. It follows from the definition of the gradient and from the definition of the weak* topology that $\{DI_a(u_n)\}_n$ *-vanishes if, and only if, $\{\nabla I_a(u_n)\}_n$ vanishes in $L^{s,2}(\mathbb{R}^N)$ in the sense of Definition 4.3.

Lemma 4.7. *Suppose that $\{u_n\}_n \subset L^{s,2}(\mathbb{R}^N)$, $\{y_n\}_n \subset \mathbb{R}^N$ and $a^* \in L^\infty(\mathbb{R}^N)$ are such that $\{DI_a(u_n)\}_n$ *-vanishes, $\tau_{-y_n}u_n \rightharpoonup u_0$ weakly in $L^{s,2}(\mathbb{R}^N)$ and $\tau_{-y_n}a \rightharpoonup^* a^*$ weakly*. If $v_n = u_n - \tau_{y_n}u_0$, then*

$$\lim_{n\to+\infty} (I_a(u_n) - I_a(v_n)) = I_{a^*}(u_0) \tag{4.1}$$

$$\lim_{n\to+\infty} \left(\|u_n\|^2_{L^{s,2}} - \|v_n\|^2_{L^{s,2}}\right) = \|u_0\|^2_{L^{s,2}} \tag{4.2}$$

$$DI_{a^*}(u_0) = 0. \tag{4.3}$$

*Furthermore, also $\{DI_a(v_n)\}_n$ *-vanishes.*

Proof. From the assumption that $\tau_{-y_n}a \rightharpoonup^* a^*$ we deduce that $I_{a^*}(u_0) = I_{\tau_{-y_n}}(u_0) + o(1)$. Combining with Lemma 4.2 we get (4.1). Equation (4.2) follows from the splitting properties of the squared norm. We prove now (4.3).
Fix any $v \in L^{s,2}(\mathbb{R}^N)$. We have that $\lim_{n\to+\infty} F'(\tau_{-y_n}u_n)v = F'(u_0)v$ in $L^1(\mathbb{R}^N)$ due to the fact that $\tau_{-y_n}u_n \to u_0$ strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ (see again [6]). Therefore

$$\begin{aligned} DI_{a^*}(u_0)[v] &= \langle u_0 \mid v\rangle - \int_{\mathbb{R}^N} \tau_{-y_n}a\, F'(u_0)v + o(1) \\ &= \langle \tau_{-y_n}u_n \mid v\rangle - \int_{\mathbb{R}^N} \tau_{-y_n}a\, F'(\tau_{-y_n}u_n)v + o(1) \\ &= DI_{\tau_{-y_n}a}(\tau_{-y_n}u_n)[v] + o(1) = o(1), \end{aligned}$$

where we have used the assumption that $\{DI_a(u_n)\}_n$ *-vanishes. This completes the proof of (4.3).
To conclude the proof, we suppose that $\{x_n\}_n$ is a sequence of points from $\mathbb{R}^N$ and that $v \in L^{s,2}(\mathbb{R}^N)$. We distinguish two cases.

(i) Up to a subsequence, $\lim_{n\to+\infty} |x_n + y_n| = +\infty$. This implies that $\tau_{-x_n-y_n}v \rightharpoonup 0$ weakly in $L^{s,2}(\mathbb{R}^N)$, and thus $F'(u_0)\tau_{-x_n-y_n}v \to 0$ strongly in $L^1(\mathbb{R}^N)$. This yields

$$DI_{\tau_{-y_n}a}(u_0)[\tau_{-x_n-y_n}v] = o(1). \tag{4.4}$$

Equation (4.4), Lemma 4.2 and the fact that $\{DI_a(v_n)\}_n$ *-vanishes, we obtain

$$\begin{aligned} DI_{\tau_{x_n}a}(\tau_{x_n}v_n)[v] &= DI_{\tau_{-y_n}a}(\tau_{-y_n}v_n)[\tau_{-x_n-y_n}v] \\ &= DI_{\tau_{-y_n}a}(\tau_{-y_n}u_n)[\tau_{-x_n-y_n}v] - DI_{\tau_{-y_n}a}(u_0)[\tau_{-x_n-y_n}v] + o(1) \\ &= DI_{\tau_{-y_n}a}(\tau_{-y_n}u_n)[\tau_{-x_n-y_n}v] + o(1) \\ &= DI_{\tau_{x_n}a}(\tau_{x_n}u_n)[v] + o(1) \\ &= o(1). \end{aligned}$$

Since the limit is independent of the subsequence, this shows that $\{DI_a(v_n)\}_n$ *-vanishes in this case.

(ii) Up to a subsequence, $\lim_{n\to+\infty}(x_n+y_n) = -\xi \in \mathbb{R}^N$. In this case,

$$\begin{aligned} DI_{\tau_{x_n}a}(\tau_{x_n}v_n)[v] &= DI_{\tau_{-y_n}a}(\tau_{-y_n}v_n)[\tau_\xi v] + o(1) \\ &= DI_{\tau_{-y_n}a}(\tau_{-y_n}u_n)[\tau_\xi] - DI_{\tau_{-y_n}a}(u_0)[\tau_\xi v] + o(1) \\ &= -DI_{\tau_{-y_n}a}(u_0)[\tau_\xi v] + o(1) \\ &= -DI_{a^*}(u_0)[\tau_\xi v] + o(1) \\ &= o(1), \end{aligned}$$

and we conclude as before.

□

Proposition 4.8. *Let $\{u_n\}_n$ be a Palais-Smale sequence for I_a at level $c \in \mathbb{R}$. One of the following alternatives must hold:*

(a) $\lim_{n\to+\infty} u_n = 0$ *strongly in* $L^{s,2}(\mathbb{R}^N)$*;*

(b) after passing to a subsequence, there exist a positive integer k, k sequences $\{y_n^i\}_n \subset \mathbb{R}^N$, k functions $a^i \in L^\infty(\mathbb{R}^N)$, and k functions $u^i \in L^{s,2}(\mathbb{R}^N) \setminus \{0\}$ for $i = 1, \dots, k$ such that $DI_{a^i}(u^i) = 0$ for every $i = 1, \dots, k$ and such that the following hold true:

$$\lim_{n\to+\infty} \left\| u_n - \sum_{i=1}^k \tau_{y_n^i} u^i \right\|_{L^p} = 0, \tag{4.5}$$

$$c \geq \sum_{i=1}^k I_{a^i}(u^i), \tag{4.6}$$

$$\lim_{n\to+\infty} \tau_{-y_n^i} a = a^i \quad \textit{in the weak* topology}, \tag{4.7}$$

and

$$\lim_{n\to+\infty} \left| y_n^i - y_n^j \right| = +\infty \quad \textit{if } i \neq j. \tag{4.8}$$

Proof. It follows from the assumptions that the sequence $\{u_n\}_n$ is bounded in $L^{s,2}(\mathbb{R}^N)$ and $\{DI_a(u_n)\}_n$ *-vanishes. We distinguish two cases.
If $\{u_n\}_n$ vanishes, then by Remark 4.4 $\{u_n\}_n$ converges strongly to zero in $L^p(\mathbb{R}^N)$. Recalling that $DI_a(u_n)[u_n] = o(1)$, we conclude that $\{u_n\}_n$ converges to zero strongly in $L^{s,2}(\mathbb{R}^N)$.
If, on the contrary, $\{u_n\}_n$ does not vanish, then there exist a function $u^1 \in L^{s,2}(\mathbb{R}^N)$ and a sequence $\{y_n^1\}_n \subset \mathbb{R}^N$ such that, after passing to a subsequence, and writing $u_n^1 = u_n$, we have $\tau_{-y_n^1} u_n^1 \rightharpoonup u^1$ weakly. Recalling that $\mathscr{P}$ is compact, we may also assume that $\{\tau_{-y_n^1} a\}_n$ weakly* converges to $a^1 \in L^\infty(\mathbb{R}^N)$. We then define $u_n^2 = u_n^1 - \tau_{y_n^1} u^1$, so that $\tau_{-y_n^1} u_n^2 \rightharpoonup 0$ weakly.
Lemma 4.7 ensures that

$$\begin{gathered} \lim_{n\to+\infty} I_a(u_n^1) - I_a(u_n^2) = I_{a^1}(u^1), \\ \lim_{n\to+\infty} \left\| u_n^1 \right\|_{L^{s,2}}^2 - \left\| u_n^2 \right\|_{L^{s,2}}^2 = 0, \\ DI_{a^1}(u^1) = 0 \end{gathered}$$

and $\{DI_a(u_n^2)\}_n$ *-vanishes. If $\{u_n^2\}_n$ vanishes, then it converges to zero in $L^p(\mathbb{R}^N)$ and thus also $\{u_n^1 - \tau_{y_n^1} u^1\}_n$ converges to zero in $L^p(\mathbb{R}^N)$. Otherwise there exist $a^2 \in L^\infty(\mathbb{R}^N)$, $u^2 \in L^{s,2}(\mathbb{R}^N) \setminus \{0\}$ and a sequence $\{y_n^2\}_n \subset \mathbb{R}^N$ such that, up to a subsequence, $\lim_{n\to+\infty} \tau_{-y_n^2} a = a^2$ weakly* and $\lim_{n\to+\infty} \tau_{-y_n^2} u_n^2 = u^2$ weakly. Necessarily, $\lim_{n\to+\infty} |y_n^1 - y_n^2| = 0$, since $\lim_{n\to+\infty} \tau_{-y_n^1} u_n^2 = 0$ weakly.

Iterating this construction, we obtain sequences $\{y_n^1\}_n \subset \mathbb{R}^N$, functions $a^i \in L^\infty(\mathbb{R}^N)$ and functions $u^i \in L^{s,2}(\mathbb{R}^N) \setminus \{0\}$ for $i = 1, 2, 3, \ldots$ Since each u^i is a non-trivial critical point of I_{a^i}, we have that $(a^i)^+ \neq 0$. On the other hand, $|(a^i)^+|_\infty \leq |a|_\infty$. Hence $u^i \in \mathcal{N}_{a^i}$ for every i and by (3.3) there exists a constant $C > 0$, independent of i, such that $\|u^i\|_{L^{s,2}} \geq C$. For every j we also have

$$0 \leq \|u_n^{j+1}\|_{L^{s,2}}^2 = \|u_n\|_{L^{s,2}}^2 - \sum_{i=1}^{j} \|u^i\|_{L^{s,2}}^2 + o(1),$$

which implies that the iteration must stop after finitely many steps. Therefore there exists a positive integer k such that $\{u_n^{k+1}\}_n$ vanishes, $\{u_n^{k+1}\}_n$ converges to zero strongly in $L^p(\mathbb{R}^N)$ and (4.5) holds true. Similarly,

$$-\int_{\mathbb{R}^N} a \left|u_n^{k+1}\right|^p \leq I_a(u_n^{k+1}) = I_a(u_n) - \sum_{i=1}^{k} I_{a^i}(u^i) + o(1),$$

and also (4.6) follows from $c = \lim_{n\to+\infty} I_a(u_n)$. The proof is complete. □

5 Existence of a ground state

The proof of the following comparison lemma is probably known, but we reproduce here for the reader's convenience.

Lemma 5.1. *Suppose that a_1, $a_2 \in L^\infty(\mathbb{R}^N)$. If $a_1 \geq a_2$, then $c_{a_1} \leq c_{a_2}$. If, in addition, $a_1 \neq a_2$ and I_{a_2} possesses a ground state, then $c_{a_1} < c_{a_2}$.*

Proof. Without loss of generality, we assume that $a_2^+ = \max\{a_2, 0\}$ is not identically equal to zero, otherwise there is nothing to prove. If $u \in \mathcal{N}_{a_2}$, then

$$\int_{\mathbb{R}^N} a_1 |u|^p \geq \int_{\mathbb{R}^N} a_2 |u|^p > 0.$$

We can therefore define

$$t = \left(\frac{\int_{\mathbb{R}^N} a_2 |u|^p}{\int_{\mathbb{R}^N} a_1 |u|^p} \right)^{1/(p-2)} \leq 1. \tag{5.1}$$

Then we have

$$DI_{a_1}(tu)[tu] = t^2 \left(\|u\|_{L^{s,2}}^2 - t^{p-2} \int_{\mathbb{R}^N} a_1 |u|^p \right) = t^2 DI_{a_2}(u)[u] = 0,$$

and hence $tu \in \mathcal{N}_{a_1}$. Since

$$I_{a_2}(u) = \frac{1}{2}\|u\|_{L^{s,2}}^2 - \frac{1}{p}\int_{\mathbb{R}^N} a_2 |u|^p = \left(\frac{1}{2} - \frac{1}{p}\right) \|u\|_{L^{s,2}}^2 \geq \left(\frac{1}{2} - \frac{1}{p}\right) \|tu\|_{L^{s,2}}^2 = J_{a_1}(u) \geq c_{a_1},$$

we conclude that $c_{a_2} = \inf_{u \in \mathcal{N}_{a_2}} I_{a_2}(u) \geq c_{a_1}$. Furthermore, if $a_1 \neq a_2$ (as elements of $L^\infty(\mathbb{R}^N)$) and u is a ground state of I_{a_2}, then $|u| > 0$. In (5.1) we then have $t < 1$, and it follows that $c_{a_2} = I_{a_2}(u) > I_{a_1}(tu) \geq c_{a_1}$. □

Recall the definition (2.1) of $\bar{a}$. We have

Proposition 5.2. *There results*

$$c_a < c_{\bar{a}}.$$

Proof. We first consider (i) of assumption **(A)**. Since $\bar{a} \leq 0$, we have $c_{\bar{a}} = \infty$. But $c_a \in \mathbb{R}$ because $a^+ \neq 0$, and there is nothing more to prove. We can assume that $\bar{a} > 0$ in the rest of the proof. If (ii) of assumption **(A)** holds, recalling that $\bar{a} > -\infty$ entails $\mathscr{B} \neq \emptyset$ we can conclude that $a \neq \bar{a}$. Now Lemma 5.1 implies that $c_a < c_{\bar{a}}$, since $I_{\bar{a}}$ has a ground state by the arguments of [3, Theorem 1.1]. $\square$

We are now ready to prove our main existence result.

Proof of Theorem 2.3. We have $\mathscr{N}_a \neq \emptyset$ and $c_a < \infty$ because $a^+ \neq 0$. From (3.3) we get $c_a > 0$. An application of Ekeland's Principle yields in a standard way a mimnimizing sequence $\{u_n\}_n \subset \mathscr{N}_a$ for the functional $\bar{I}_a$ defined as the restriction of I_a to $\mathscr{N}_a$. This sequence is also a (PS)-sequence for $\bar{I}_a$ at the level c_a. By Proposition 3.7 $\{u_n\}_n$ is a (PS)-sequence for I_a at the level c_a. The strong convergence of $\{u_n\}_n$ to zero is easily ruled out, since $I_a(u_n) \to c_a > 0$. Proposition 4.8 yields then a number $k \in \mathbb{N}$, functions $a^i \in \overline{\mathscr{A}}$ and non-trivial critical points u^i of I_{a^i} such that

$$c_a \geq \sum_{i=1}^{k} I_{a^i}(u^i).$$

From the knowledge that each u^i is a non-trivial critical point of I_{a^i} we deduce $(a^i)^+ \neq 0$ for every $i = 1, \ldots, k$. Again by (3.3) we get $I_{a^i}(u^i) > 0$ for every $i = 1, \ldots, k$.
Suppose that for *some* index i there results $a^i \in \mathscr{B}$. Then $a^i \leq \bar{a}$, and Lemma 5.1 together with Proposition 5.2 yield $I_{a^i}(u^i) \geq c_{a^i} \geq c_{\bar{a}} > c_a$. This is a contrdiction. Therefore each a^i is a translation of a, and $I_{a^i}(u^i) \geq c_a$ for every $i = 1, \ldots, k$. This forces $k = 1$, and a translation of u^1 is a ground state of I_a. $\square$

6 An example

Assumption **(A)** can be rephrased in a more familiar way for continuous bounded potentials.

Proposition 6.1. *For any $a \in L^\infty(\mathbb{R}^N)$, define*

$$\hat{a} = \lim_{R \to +\infty} \operatorname*{ess\,sup}_{x \in \mathbb{R}^N \setminus B(0,R)} a(x).$$

*If **(A)** holds true with $\bar{a}$ replaced by $\hat{a}$, then **(A)** holds true with $\bar{a}$.*

Proof. If $\mathscr{B} = \emptyset$, then $\bar{a} = -\infty$ and **(A)** holds true. We may assume that $\mathscr{B} \neq \emptyset$, so that a cannot be constant. Let us prove that

$$\bar{a} \leq \hat{a}. \tag{6.1}$$

Pick $b \in \mathscr{B}$. There is a sequence $\{x_n\}_n \subset \mathbb{R}^N$ such that $\tau_{x_n} a \rightharpoonup^\star b$. Translations are continuous in the weak* topology of $L^\infty(\mathbb{R}^N)$, since they are continuous in $L^1(\mathbb{R}^N)$. For the sake of contradiction, suppose that $\{x_n\}_n$ contains a bounded subsequence. Up to a further subsequence,

there must exist a point $\xi \in \mathbb{R}^N$ such that $x_n \to \xi$ and $\tau_{x_n} a \rightharpoonup^\star \tau_\xi a$. Since $\mathscr{P}$ is metrizable, $\tau_\xi a = b \notin \mathscr{A}$, a contradiction. Therefore $\lim_{n\to+\infty} |x_n| = +\infty$.
Let $\varepsilon > 0$ be given, and apply Lemma 3.2: there exists $\varphi \in L^1(\mathbb{R}^N)$ with $\varphi \geq 0$ and $\|\varphi\|_{L^1} = 1$ such that

$$\int_{\mathbb{R}^N} b\varphi \geq \operatorname{ess\,sup} b - \frac{\varepsilon}{2}.$$

Choose $\tilde{\psi} \in C_c^\infty(\mathbb{R}^N)$ such that $\tilde{\psi} \geq 0$ and

$$\left\| \varphi - \tilde{\psi} \right\|_{L^1} \leq \frac{\varepsilon}{4\|b\|_{L^\infty}}.$$

Now $\psi = \tilde{\psi}/\|\tilde{\psi}\|_{L^1} \in C_c^\infty(\mathbb{R}^N)$ satisfies

$$\|\varphi - \psi\|_{L^1} \leq \frac{\varepsilon}{2\|b\|_{L^\infty}},$$

$\psi \geq 0$ and $\|\psi\|_{L^1} = 1$. This implies

$$\int_{\mathbb{R}^N} b\psi = \int_{\mathbb{R}^N} b\varphi - \int_{\mathbb{R}^N} b(\varphi - \psi) \geq \int_{\mathbb{R}^N} b\varphi - \|b\|_{L^\infty} \|\psi - \varphi\|_{L^1} \geq \operatorname{ess\,sup} b - \varepsilon.$$

Suppose that $\operatorname{supp} \psi \subset B(0,R)$: then

$$\begin{aligned} \operatorname{ess\,sup} b - \varepsilon \leq \int_{\mathbb{R}^N} b\psi &= \lim_{n\to+\infty} \int_{\mathbb{R}^N} (\tau_{x_n} a)\psi \\ &\leq \lim_{n\to+\infty} \operatorname*{ess\,sup}_{x \in B(-x_n,R)} a(x) \int_{\mathbb{R}^N} \psi \leq \lim_{n\to+\infty} \operatorname*{ess\,sup}_{x\in\mathbb{R}^N \setminus B(0,|x_n|-R)} a(x) = \hat{a}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\operatorname{ess\,sup} b \leq \hat{a}$. If (i) of assumption **(A)** holds, then (6.1) yields $\bar{a} \leq \hat{a} \leq 0$. If (ii) holds, then (6.1) yields $\bar{a} \leq \hat{a} \leq a$, and the proof is complete. ☐

An immediate consequence of Theorem 2.3 is then the following.

Corollary 6.2. *If a is a bounded continuous function such that either $\limsup_{|x|\to+\infty} a(x) \leq 0$ or $\limsup_{|x|\to+\infty} a(x) \leq a$, then equation (1.1) has (at least) a positive ground state as soon as $2 < p < 2_s^\star$.*

Acknowledgement

The author is member of the *Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni* (GNAMPA) of the *Istituto Nazionale di Alta Matematica* (INdAM). The manuscript was realized within the auspices of the INdAM – GNAMPA Projects *Problemi non lineari alle derivate parziali* (Prot_U-UFMBAZ-2018-000384).

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