MOMENTS OF AN EXPONENTIAL SUM RELATED TO THE DIVISOR FUNCTION

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ABSTRACT. We use the circle method to obtain tight bounds on the L^p norm of an exponential sum involving the divisor function for $p > 2$.

1. Introduction

Let $X \geq 1$ be sufficiently large. For a function $f : \mathbb{N} \to \mathbb{C}$, let

$$M_f(\alpha) := \sum_{n \leq X} f(n)e(n\alpha)$$

where as usual, $e(\alpha) := e^{2\pi i\alpha}$. Information on the structure of $f(n)$ can be obtained by studying the size of L^p-integrals of $M_f(\alpha)$, and bounds on them are often useful in applications of the circle method. In particular, often, when bounding the contribution from "minor arcs" in an application of the circle method, one is led to bounding the L^∞ norm of an exponential sum in the minor arcs times $\int_0^1 |G(\alpha)|^p d\alpha$ for some $G(\alpha)$, which is often of the form $M_f(\alpha)$ for various choices of f.

For example, a proof of the minor arc bounds in Vinogradov's theorem (that all sufficiently large odd integers are the sum of 3 primes) involves bounding $M_\Lambda(\alpha)$ by $O_A(X \log^{-A} X)$ for a particular choice of minor arcs, which along with the fact that by Parseval's identity and the prime number theorem $\int_0^1 |M_\Lambda(\alpha)|^2 d\alpha \ll X \log X$ implies that $|M_\Lambda(\alpha)|^3$ is bounded above by $X^2 \log^{-A+1} X$ on average on the minor arcs. Here, Λ is the von Mangoldt function

Write

$$I_f(p) := \int_0^1 |M_f(\alpha)|^p d\alpha. \tag{1.1}$$

In the case $p = 1, f = \tau$, it was shown in [GP] that

$$\sqrt{X} \ll I_\tau(1) \ll \sqrt{X} \log X, \tag{1.2}$$

where

$$\tau(n) := \sum_{d|n} 1$$

and the methods used to prove the lower bound in that paper should extend to allow one to show that

$$I_\tau(p) \gg X^{p/2}$$

for $p<1$. For sequences other than $\tau(n)$, similar results have been established in the case $p=1$. For example, with μ the Möbius function, we have that $X^{1/6} \ll I_\mu(1) \ll X^{1/2}$ where the upper bound follows from Parseval's identity, and the lower bound follows from Theorem 3 in [BR]. Estimates for $I_f(1)$ in the case f is an indicator function for the primes have been obtained by Vaughan [Va1] and Goldston [Go], and in the case f is the indicator function for integers not divisible by the rth power of any prime by Balog and Ruzsa [BR]. Later, a result of Keil [Ke] finds with f the indicator function for the r-free numbers the exact order of magnitude of all moments but $1+\frac{1}{r}$ in which case the exact order of magnitude is found within a factor of $\log X$.

In this paper, we shall focus on the case $f=\tau$, the divisor function. Note that we have that by Parseval's identity

$$I_\tau(2) = \sum_{n\le X} \tau(n)^2 \sim \frac{1}{\pi^2} X(\log X)^3. \tag{1.3}$$

We shall obtain tight estimates on $I_\tau(p)$ for $p>2$. In particular, we prove the following result.

Theorem 1.1. *We have that for $p>2$*

$$X^{p-1}(\log X)^p \ll \int_0^1 |M_\tau(\alpha)|^p d\alpha \ll X^{p-1}(\log X)^p. \tag{1.4}$$

where the implied constants depend only on p.

1.1. **Notation.** Throughout this paper, all implied constants will be assumed to depend only on p unless otherwise specified. In addition, we write $\|\alpha\|$ to denote $\inf_{n\in\mathbb{Z}} |\alpha-n|$. We write $f \asymp g$ to denote that $g \ll f \ll g$ where the two implied constants need not be the same. In addition, any statement with ε holds for all $\varepsilon>0$ and implied constants depend on ε too if it appears.

2. Preliminaries and setup

Note that we have that since $\tau(n) = \sum_{d|n} 1 = \sum_{uv=n} 1$

$$\begin{aligned} M_\tau(\alpha) &= \sum_{n\le X} \tau(n)e(n\alpha) = \sum_{uv\le X} e(\alpha uv) \\ &= 2\sum_{\substack{uv\le X\\ u<v}} e(\alpha uv) + \sum_{\substack{uv\le X\\ u=v}} e(\alpha uv) = 2T(\alpha) + E(\alpha). \end{aligned} \tag{2.1}$$

Also, let

$$v(\beta) := \sum_{n\le X} e(n\beta).$$

We record the following well-known bound on $v(\beta)$ which we will use later.

Lemma 2.1. *We have that for $\beta \notin \mathbb{Z}$, $|v(\beta)| \ll \min(X, \|\beta\|^{-1})$.*

Proof. Note that we have that if $\beta \in \mathbb{Z}$, we are done since then $v(\beta) = \lfloor X \rfloor$. Otherwise, we have that by summing the geometric series,
(2.2)
$$|v(\beta)| = \left|\frac{e((\lfloor X \rfloor + 1)\beta) - e(\beta)}{e(\beta) - 1}\right| = \frac{|1 - e(\lfloor X \rfloor \beta)|}{|1 - e(\beta)|} = \frac{\sin(\pi \lfloor X \rfloor \alpha)}{\pi \alpha} \ll \frac{1}{|\sin 2\pi\alpha|} \ll \frac{1}{\|\alpha\|}$$
□

In addition, we shall also use the following result on moments of $v(\beta)$.

Lemma 2.2. *For $p > 2$, we have that*
$$\int_0^1 |v(\beta)|^p \asymp X^{p-1}.$$

Proof. Note that by the third equality in (2.2), we have that
(2.3)
$$\int_0^1 |v(\beta)|^p d\beta \geq \int_{[(2X)^{-1},(4X)^{-1}]} \sin(\pi \lfloor X \rfloor \beta)(\sin \pi\beta)^{-p} d\beta \gg \int_{[(2X)^{-1},(4X)^{-1}]} X^{-p} d\beta \gg X^{p-1}.$$
In addition, note that for positive integers s, by considering the underlying Diophantine system, we have that
$$\int_0^1 |v(\beta)|^{2s} d\beta = |\{1 \leq x_1, \ldots, x_s, y_1, \ldots, y_s \leq X : x_1 + \cdots + x_s = y_1 + \cdots + y_s\}|$$
$$= \sum_{n \leq X} \binom{n-1}{s-1}^2 \sim C_s X^{2s-1}$$
for some $C_s > 0$, so the upper bound, and therefore the desired result, follows from Hölder's inequality. □

We will use the circle method to prove the main result. In particular, we shall show that the main contribution to the integral $I_\tau(p)$ comes from α close to rationals with small denominator by estimating I_τ around rationals with small denominator and bounding it everywhere else. To that end, let
$$\mathfrak{M}(q, a) = \{\alpha \in [0, 1] : |q\alpha - a| \leq PX^{-1}\}$$
with $P = X^\nu$ for $\nu > 0$ sufficiently small, and let
$$\mathfrak{M} = \bigcup_{q \leq P} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathfrak{M}(q, a), \mathfrak{m} = [0, 1] \setminus \mathfrak{M}.$$
For any measurable $\mathfrak{B} \subseteq [0, 1)$, let
$$I_f(p; \mathfrak{B}) := \int_{\mathfrak{B}} |M_f(\alpha)| d\alpha.$$
We shall prove Theorem 1.1 by using the fact that $I_\tau(p) = I_\tau(p; \mathfrak{M}) + I_\tau(p; \mathfrak{m})$, showing that $I_\tau(p; \mathfrak{m}) = o(X^{p-1}(\log X)^p)$ and showing that $I_\tau(p; \mathfrak{M}) \asymp X^{p-1}(\log X)^p$.

3. The minor arcs

Our bound on the minor arcs will depend on the following result, which is nontrivial for $X^\varepsilon \ll q \ll X^{1-\varepsilon}$.

Proposition 3.1. *If $|q\alpha - a| \le q^{-1}$ for some $(a,q)=1$, $q \ge 1$, then*

$$M_\tau(\alpha) \ll X\log(2Xq)(q^{-1} + X^{-1/2} + qX^{-1}). \tag{3.1}$$

Proof. We have that by (2.1) and the trivial bound $|E(\alpha)| \le X^{1/2}$

$$M_\tau(\alpha) = 2T(\alpha) + O(X^{1/2})$$

so it suffices to show that $T(\alpha) \ll X\log(2Xq)(q^{-1} + X^{-1/2} + qX^{-1})$, since we can absorb the $O(X^{1/2})$ into the bound since

$$X\log(2Xq)(q^{-1} + X^{-1/2} + qX^{-1}) \gg X^{1/2}\log X.$$

To this end, note that by the triangle inequality and Lemma 2.1

$$|T(\alpha)| \le \sum_{u \le X^{-1}} \left| \sum_{u < v \le X/u} e(\alpha uv) \right| \ll \sum_{u \le X^{1/2}} \min(X/u, \|\alpha u\|^{-1}).$$

The desired result then follows from Lemma 2.2 in [Va]. □

From this, we obtain the following result.

Lemma 3.2. *We have that*

$$I_\tau(p;\mathfrak{m}) \ll X^{p-1-\nu/2}(\log X)^4. \tag{3.2}$$

Proof. Note that we have that

$$\int_{\mathfrak{m}} |M_\tau(\alpha)|^p d\alpha \le \left(\sup_{\alpha\in\mathfrak{m}} |M_\tau(\alpha)|\right)^{p-2} \int_{\mathfrak{m}} |M_\tau(\alpha)|^2 d\alpha \ll X(\log X)^3 \left(\sup_{\alpha\in\mathfrak{m}} |M_\tau(\alpha)|\right)^{p-2}.$$

Suppose that $\alpha \in \mathfrak{m}$. Then, by Dirichlet's theorem, we have that there exist a, q such that $(a,q) = 1, q \le P^{-1}X, |q\alpha - a| \le PX^{-1}$, so it follows that $q > P$, since otherwise, α would be in $\mathfrak{M}(q,a)$. Then, by Proposition 3.1, we have that $|M_\tau(\alpha)| \ll X^{1-\nu/2}\log X$, and the desired result follows. □

Now, we proceed to estimate the major arcs. To that end, we first record the following estimate.

Proposition 3.3. *For $(a,q)=1$, $q \ge 1$, we have*

$$\sum_{n \le X} \tau(n) e\left(\frac{an}{q}\right) = \frac{X}{q}\left(\log\frac{X}{q^2} + 2\gamma - 1\right) + O((X^{1/2} + q)\log 2q).$$

Proof. This is shown in the proof of Lemma 2.5 in [PV]. We shall reproduce its proof below. Note that we have that by (2.1)

$$\sum_{n\le X}\tau(n)e\left(\frac{an}{q}\right)=\sum_{u\le X^{1/2}}\left(\sum_{v\le X/u}2-\sum_{v\le X^{1/2}}1\right)e(auv/q).$$

For $q\nmid u$, we have that the inner sums are $\ll \|au/q\|^{-1}$. The contribution from the remaining terms is then

$$\frac{X}{q}\left(\log\frac{X}{q^2}+2\gamma-1\right)+O(X^{1/2})$$

from which the desired result follows. □

Now, it follows then from this and partial summation that for $\alpha\in\mathfrak{M}(q,a)$, we have

$$M_\tau(\alpha)=\frac{1}{q}\left(\log\frac{X}{q^2}+2\gamma-1\right)v(\alpha-a/q)+O(X^{1/2+\nu}\log X). \tag{3.3}$$

Therefore, we have that (by using the binomial theorem for $p\in\mathbb{Z}^+$, and then using Hölder's inequality to bound the remaining error terms)

$$|M_\tau(\alpha)|^p=q^{-p}(\log X-2\log q+2\gamma-1)^p|v(\alpha-a/q)|^p+O(X^{p-1/2+\nu}(\log X)^p)$$

so it follows that

$$\int_{\mathfrak{M}}|M_\tau(\alpha)|^p d\alpha= \tag{3.4}$$

$$\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{-PX^{-1}}^{PX^{-1}}q^{-p}(\log X-2\log q+2\gamma-1)^p|v(\alpha-a/q)|^p d\beta+O(X^{p-3/2+4\nu}(\log X)^p)$$

$$=\mathfrak{S}(X,P)\int_{-PX^{-1}}^{PX^{-1}}|v(\beta)|^p d\beta+O(X^{p-3/2+4\nu}(\log X)^p)$$

where

$$\mathfrak{S}(X,P):=\sum_{q\le P}\varphi(q)q^{-p}(\log X-2\log q+2\gamma-1)^p.$$

It is easy to show that by partial summation, we have

$$\mathfrak{S}(X,P)\asymp(\log X)^p. \tag{3.5}$$

Also, note that by Lemmas 2.1 and 2.2, we have that

$$\int_{-PX^{-1}}^{PX^{-1}}|v(\beta)|^p d\beta\gg X^{p-1}-\int_{[PX^{-1},1-PX^{-1}]}X^{-p}d\beta\gg X^{p-1}.$$

Theorem 1.1 then folows since this implies that $I_\tau(p;\mathfrak{M})\asymp X^{p-1}(\log X)^p$.

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