A finite difference approximation of a two dimensional time fractional advection-dispersion problem

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Abstract

The main purpose of this paper is the construction and analysis of an implicit finite difference scheme for the numerical solution of a two dimensional time-fractional advection-dispersion equation with variable coefficients. The dispersion term is in nondivergence form and the fractional derivative is taken in the sense of Caputo. Equations of this sort are potentially useful as models of contaminant transport in groundwater. Provided some mild assumptions are satisfied, proofs of consistency, stability and convergence are obtained. Furthermore, we offer a general but simple framework for the matrices required in computations and everything is tested by a well selected set of numerical experiments.

keywords: Caputo fractional derivative, two dimensional time fractional advection-dispersion problem, finite difference approximation, stability, convergence.

1 Introduction

Fractional derivatives are associated with memory and hereditary properties of materials and processes. They are known since the seventeenth century but only recently have become an important subject of applied mathematics. They might be applied on time and/or space variables and are suitable for a variety of topics, for instance, the behavior of viscoelastic materials ([2]) and the anomalous diffusion of a contaminant in porous media ([4]). For these and other uses of fractional derivatives the reader is invited to consult [9, 8, 5].

There are a variety of fractional derivatives, i.e. Caputo, Riemann-Liouville, Grünwald-Letnikov and many others. Moreover, the fractional derivatives can be single-term or multi-term, according to the number of differentiation orders which can be real or complex numbers. Our interest is on the modeling of transport phenomena in porous media through a two dimensional time-fractional advection-dispersion equation with variable coefficients in which the diffusion term is given in nondivergence form and the differentiation order is a real number between 0 and 1.

Many authors have proposed numerical solutions for time-fractional differential equations. For instance, [13] introduces a two dimensional single term time-fractional diffusion equation with variable coefficients and diffusion term in nondivergence form. The present work owes several ideas to this paper. Some authors solve one dimensional time-fractional differential equations with diffusion term in nondivergence form. We mention [11], in which the interest is

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in the direct problem and [7] which solves a one dimensional time-fractional diffusion equation as a tool in the process of solving an inverse problem.

Among the authors who face two dimensional time-fractional differential equations, we mention [1], in which the problem is similar to ours but the coefficients are constant and [3], which deals with a two dimensional inverse source problem and introduces a particular case of our numerical scheme for the necessary solution of the direct problem.

In this paper, for the implicit approximation of the Caputo fractional derivative we implement the known scheme very well described in [6]. This scheme appears elsewhere, for instance, in [13]. For the advection and dispersion terms we implement standard central finite difference schemes.

The rest of the paper is divided in three sections. Section 2 defines the equation and the numerical methods. The next section contains the consistency, stability and convergence statements along with their proofs. The numerical experiments and final remarks are presented in Section 4 .

2 The problem and the numerical method

The prediction of the environmental consequences of groundwater contamination is an important goal for researchers. Our interest is to help in this prediction through a numerical approximation of a mathematical model based on a partial differential equation known as an advection-dispersion equation. Our equation has variable coefficients and a time fractional derivative rather than the classical time derivative. Other features of our model are: It considers the contaminant transport through a two dimensional porous medium with variable advection and dispersion function coefficients given by two components each. Moreover, the diffusion terms are in nondivergence form. For this matter we follow references [13, 10, 12]. All of them show that nondivergence diffusion terms are worth and with Caputo time-fractional derivatives provide useful models of anomalous diffusion.

2.1 The initial-boundary value problem

We consider the two-dimensional initial-boundary value problem

$$\begin{aligned} u_t^{(\alpha)}(x,y,t) + a(x,y,t)u_x(x,y,t) + b(x,y,t)u_y(x,y,t) \\ = c(x,y,t)u_{xx}(x,y,t) + d(x,y,t)u_{yy}(x,y,t) + f(x,y,t) \end{aligned} \tag{1}$$

with initial condition

$$u(x,y,0) = \psi(x,y), \qquad (x,y) \in \Omega := (x_L, x_R) \times (y_L, y_R) \subset \mathbb{R}^2, \tag{2}$$

and Dirichlet boundary condition

$$u(x,y,t) = 0, \qquad (x,y) \in \partial\Omega \times (0,T], \tag{3}$$

where:

1. $u\,(x,y,t)$ is the contaminant concentration.
2. c is the longitudinal dispersion variable coefficient.
3. d is the transversal dispersion variable coefficient.
4. a and b are the longitudinal and transversal advection coefficients respectively. They are basically the seepage or average pore water velocity and if one of the directions is predominant, only one of the advection function coefficients is nonzero.

5. f is a known source or sink term.

6. $u_t^{(\alpha)}$ is the Caputo fractional derivative of order α given by

$$u_t^{(\alpha)}(x,y,t) := \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t \dfrac{u_t(x,y,\xi)}{(t-\xi)^\alpha}\, d\xi, & 0<\alpha<1, \\ u_t(x,y,t), & \alpha = 1, \end{cases} \tag{4}$$

7. The variable coefficient functions a, b, c, d satisfy the following uniform bounds: There are two positive constants A and D so that

$$\begin{array}{lcl} 0 \le a(x,y,t) \le A, & \text{and} & 0 < D \le c(x,y,t), \\ 0 \le b(x,y,t) \le A, & & 0 < D \le d(x,y,t). \end{array} \tag{5}$$

The next subsection deals with the proposed finite difference approximation.

2.2 The numerical scheme

Let the mesh points $x_i = x_L + i\Delta x$, $0 \le i \le N_x$, $y_j = y_L + j\Delta y$, $0 \le j \le N_y$ and $t_k = k\Delta t$, $0 \le k \le N_t$, where $\Delta x = (x_R - x_L)/N_x$ and $\Delta y = (y_R - y_L)/N_y$ are the spatial grid sizes in the x- and y-direction, respectively, and $\Delta t = T/N_t$ is the time step size. The values of the functions u, a, b, c, d and f at the grid points are denoted by $u_{i,j}^k = u(x_i, y_j, t_k)$, $a_{i,j}^k = a(x_i, y_j, t_k)$, $b_{i,j}^k = b(x_i, y_j, t_k)$, $c_{i,j}^k = c(x_i, y_j, t_k)$, $d_{i,j}^k = d(x_i, y_j, t_k)$ and $f_{i,j}^k = f(x_i, y_j, t_k)$, respectively. The initial condition is set as $u_{i,j}^0 = \psi_{i,j} = \psi(x_i, y_j)$. The Dirichlet boundary condition at $x = x_L$ is set as $u_{0,j}^k = 0$ and similarly on the other three sides of the boundary.

The Caputo fractional derivative at time t_{k+1} is approximated by

$$u_t^{(\alpha)}(x_i, y_j, t_{k+1}) = \sigma_{\alpha,\Delta t} \sum_{s=0}^{k} \omega_s^{(\alpha)} \left(u_{i,j}^{k-s+1} - u_{i,j}^{k-s} \right) + O\left((\Delta t)^{2-\alpha} \right), \tag{6}$$

for $k = 0, \ldots, N_t - 1$, where $\sigma_{\alpha,\Delta t} = \dfrac{1}{(\Delta t)^\alpha \Gamma(2-\alpha)}$ and $\omega_s^{(\alpha)} = (s+1)^{1-\alpha} - s^{1-\alpha}$ for $s = 0, \ldots, N_t$, as described in [6].

Derivatives with respect to space coordinates are approximated by central difference formulae. Let $v_{i,j}^k$ be the numerical approximation to $u_{i,j}^k$. The discrete version of (1) is the implicit finite difference scheme (**IFDS**) given by

$$\begin{aligned} \sigma_{\alpha,\Delta t} \sum_{s=0}^{k} \omega_s^{(\alpha)} \left(v_{i,j}^{k-s+1} - v_{i,j}^{k-s} \right) + a_{i,j}^{k+1} \frac{v_{i+1,j}^{k+1} - v_{i-1,j}^{k+1}}{2\Delta x} + b_{i,j}^{k+1} \frac{v_{i,j+1}^{k+1} - v_{i,j-1}^{k+1}}{2\Delta y} \\ = c_{i,j}^{k+1} \frac{v_{i+1,j}^{k+1} - 2v_{i,j}^{k+1} + v_{i-1,j}^{k+1}}{(\Delta x)^2} + d_{i,j}^{k+1} \frac{v_{i,j+1}^{k+1} - 2v_{i,j}^{k+1} + v_{i,j-1}^{k+1}}{(\Delta y)^2} + f_{i,j}^{k+1}, \end{aligned} \tag{7}$$

for $i = 1, \ldots, N_x - 1$, $j = 1, \ldots, N_y - 1$ and $k = 0, \ldots, N_t - 1$.

2.3 Consistency

In order to prove consistency of scheme (**IFDS**), it is convenient to denote (1) by

$$S(u) = S\left(\partial_t, \partial_x, \partial_y, \partial_{xx}, \partial_{yy}\right) u = f(x,y,t),$$

where

$$S(u) = u_t^{(\alpha)}(x,y,t) + a(x,y,t)u_x(x,y,t) + b(x,y,t)u_y(x,y,t) \\ -c(x,y,t)u_{xx}(x,y,t) - d(x,y,t)u_{yy}(x,y,t)$$

Likewise, we establish the following alternative notation for scheme (**IFDS**)

$$S_\Delta(v) = S_{\Delta t,\Delta x,\Delta y}(v_{i,j}^{k+1}),$$

where

$$S_\Delta(v) = \sigma_{\alpha,\Delta t}\sum_{s=0}^{k}\omega_s^{(\alpha)}\left(v_{i,j}^{k-s+1} - v_{i,j}^{k-s}\right) + a_{i,j}^{k+1}\frac{v_{i+1,j}^{k+1} - v_{i-1,j}^{k+1}}{2\Delta x} + b_{i,j}^{k+1}\frac{v_{i,j+1}^{k+1} - v_{i,j-1}^{k+1}}{2\Delta y} \\ - c_{i,j}^{k+1}\frac{v_{i+1,j}^{k+1} - 2v_{i,j}^{k+1} + v_{i-1,j}^{k+1}}{(\Delta x)^2} - d_{i,j}^{k+1}\frac{v_{i,j+1}^{k+1} - 2v_{i,j}^{k+1} + v_{i,j-1}^{k+1}}{(\Delta y)^2}.$$

for $i = 1,\ldots,N_x-1$, $j = 1,\ldots,N_y-1$ and $k = 0,\ldots,N_t-1$. It is known that if u is a smooth function, then at interior points of its domain the following equalities hold:

$$\begin{aligned} u_t^{(\alpha)}(x_i,y_j,t_k) - \sigma_{\alpha,\Delta t}\sum_{s=0}^{k}\omega_s^{(\alpha)}\left(u_{i,j}^{k-s+1} - u_{i,j}^{k-s}\right) &= O\left((\Delta t)^{2-\alpha}\right) \\ u_x(x_i,y_j,t_k) - \frac{u_{i+1,j}^{k+1} - u_{i-1,j}^{k+1}}{2\Delta x} &= O\left((\Delta x)^2\right) \\ u_y(x_i,y_j,t_k) - \frac{u_{i,j+1}^{k+1} - u_{i,j-1}^{k+1}}{2\Delta y} &= O\left((\Delta y)^2\right) \\ u_{xx}(x_i,y_j,t_k) - \frac{u_{i+1,j}^{k+1} - 2u_{i,j}^{k+1} + u_{i-1,j}^{k+1}}{(\Delta x)^2} &= O\left((\Delta x)^2\right) \\ u_{yy}(x_i,y_j,t_k) - \frac{u_{i,j+1}^{k+1} - 2u_{i,j}^{k+1} + u_{i,j-1}^{k+1}}{(\Delta y)^2} &= O\left((\Delta y)^2\right) \end{aligned}$$

Let us denote the addition of all right hand sides above by

$$O(\Delta) = O\left((\Delta t)^{2-\alpha}, (\Delta x)^2, (\Delta y)^2\right). \tag{8}$$

Thus,

$$S(u) - S_\Delta(u) = O\left((\Delta t)^{2-\alpha}, (\Delta x)^2, (\Delta y)^2\right) \tag{9}$$

and we have proved

Lemma 2.1. *The finite difference scheme (**IFDS**) is consistent with the partial differential equation* (1).

Other way to write (9) is

$$f_{i,j}^{k+1} - S_\Delta(u) = O\left((\Delta t)^{2-\alpha}, (\Delta x)^2, (\Delta y)^2\right). \tag{10}$$

By setting

$$\mu_1 = \frac{(\Delta t)^\alpha}{2\Delta x}, \quad \mu_2 = \frac{(\Delta t)^\alpha}{2\Delta y}, \quad \mu_3 = \frac{(\Delta t)^\alpha}{(\Delta x)^2}, \quad \mu_4 = \frac{(\Delta t)^\alpha}{(\Delta y)^2}, \quad \tau = \frac{1}{\sigma_{\alpha,\Delta t}} = (\Delta t)^\alpha\Gamma(2-\alpha), \tag{11}$$

and

$$
\begin{aligned}
p_{i,j}^k &= \Gamma(2-\alpha)\left[\mu_3 c_{i,j}^k - \mu_1 a_{i,j}^k\right], \quad q_{i,j}^k = \Gamma(2-\alpha)\left[\mu_3 c_{i,j}^k + \mu_1 a_{i,j}^k\right], \\
r_{i,j}^k &= \Gamma(2-\alpha)\left[\mu_4 d_{i,j}^k - \mu_2 b_{i,j}^k\right], \quad h_{i,j}^k = \Gamma(2-\alpha)\left[\mu_4 d_{i,j}^k + \mu_2 b_{i,j}^k\right], \\
e_{i,j}^k &= 1 + p_{i,j}^k + q_{i,j}^k + r_{i,j}^k + h_{i,j}^k = 1 + 2\,\Gamma(2-\alpha)\left[\mu_3 c_{i,j}^k + \mu_4 d_{i,j}^k\right],
\end{aligned}
\tag{12}
$$

we split scheme (**IFDS**) in two stages:

1. For $k=0$, it is

$$
\begin{aligned}
-\left(p_{i,j}^1 v_{i+1,j}^1 + q_{i,j}^1 v_{i-1,j}^1\right) + e_{i,j}^1 v_{i,j}^1 - \left(r_{i,j}^1 v_{i,j+1}^1 + h_{i,j}^1 v_{i,j-1}^1\right) \\
= v_{i,j}^0 + \tau f_{i,j}^1
\end{aligned}
\tag{13}
$$

   for $i = 1, \ldots, N_x - 1$ and $j = 1, \ldots, N_y - 1$

2. For $k = 1, \ldots, N_t - 1$, the scheme is

$$
\begin{aligned}
-\left(p_{i,j}^{k+1} v_{i+1,j}^{k+1} + q_{i,j}^{k+1} v_{i-1,j}^{k+1}\right) + e_{i,j}^{k+1} v_{i,j}^{k+1} - \left(r_{i,j}^{k+1} v_{i,j+1}^{k+1} + h_{i,j}^{k+1} v_{i,j-1}^{k+1}\right) \\
= v_{i,j}^k - \sum_{s=1}^{k} \omega_s^{(\alpha)} \left(v_{i,j}^{k-s+1} - v_{i,j}^{k-s}\right) + \tau f_{i,j}^{k+1}
\end{aligned}
\tag{14}
$$

   where $i = 1, \ldots, N_x - 1$ and $j = 1, \ldots, N_y - 1$.

The next lemma provides the main features of the quadrature weights $\omega_s^{(\alpha)}$.

Lemma 2.2. *The quadrature weights $\omega_s^{(\alpha)}$ are positive and $\omega_s^{(\alpha)} > \omega_{s+1}^{(\alpha)}$ for all $s = 0, 1, \ldots$*

The nonnegativity of all variable coefficients of scheme (13)-(14) is a desirable feature. Definitions of q and h in (12) establish that they are nonnegative functions. For the other coefficients, nonnegativity is achieved provided a mild assumption on the grid sizes is imposed. The details are in the following lemma.

Lemma 2.3. *If the variable coefficients a, b, c and d satisfy the bounds* (5) *and* $\max\{\Delta x, \Delta y\} \leq 2D/A$, *then $p_{i,j}^k \geq 0$ and $r_{i,j}^k \geq 0$ for each $i = 1, \ldots, N_x - 1$, $j = 1, \ldots, N_y - 1$ and $k = 1, \ldots, N_t - 1$.*

Proof. The proof consists on the following straightforward computations:

$$
p_{i,j}^k = \Gamma\left(2-\alpha\right) \frac{(\Delta t)^\alpha}{(\Delta x)^2} \left[c_{i,j}^k - \frac{\Delta x}{2} a_{i,j}^k\right] \geq \frac{\tau}{(\Delta x)^2} \left[D - \frac{\Delta x}{2} A\right] \geq 0
$$

and

$$
r_{i,j}^k = \frac{\tau}{(\Delta y)^2} \left[d_{i,j}^k - \frac{\Delta y}{2} b_{i,j}^k\right] \geq \frac{\tau}{(\Delta y)^2} \left[D - \frac{\Delta y}{2} A\right] \geq 0. \qquad \square
$$

2.4 The linear system

Let

$$
v^k = \begin{bmatrix} v_{*,1}^k & v_{*,2}^k & \cdots & v_{*,N_y-1}^k \end{bmatrix}^T \tag{15}
$$

where $v_{*,j}^k = \begin{bmatrix} v_{1,j}^k & v_{2,j}^k & \cdots & v_{N_x-1,j}^k \end{bmatrix}^T, \quad j = 1, \ldots, N_y - 1.$

In the particular case $N_x = N_y = N$, the $(N-1)^2$ equations (13)-(14) may be written in matrix form

$$
A^{(k+1)} v^{k+1} = y^k \tag{16}
$$

for each $0 \le k < N_t$, where $A^{(k)}$ is the $(N-1)^2 \times (N-1)^2$ matrix of coefficients resulting from the system of difference equations at the gridpoints at level $t = t_k$, $v^k = \begin{bmatrix} v^k_{*,1} & v^k_{*,2} & \cdots & v^k_{*,N-1} \end{bmatrix}^T$ with $v^k_{*,j} = \begin{bmatrix} v^k_{1,j} & v^k_{2,j} & \cdots & v^k_{N-1,j} \end{bmatrix}^T$, and $y^k = \begin{bmatrix} y^k_{*,1} & y^k_{*,2} & \cdots & y^k_{*,N-1} \end{bmatrix}^T$ with

$$y^k_{*,j} = \begin{cases} \psi_{*,j} + \tau f^1_{*,j}, & k = 0, \\ \gamma v^1_{*,j} + \gamma \psi_{*,j} + \tau f^2_{*,j}, & k = 1, \\ \gamma v^k_{*,j} + \sum_{s=1}^{k-1} \left(\omega^{(\alpha)}_s - \omega^{(\alpha)}_{s+1} \right) v^{k-s}_{*,j} + \omega^{(\alpha)}_k \psi_{*,j} + \tau f^{k+1}_{*,j}, & 1 < k < N_t, \end{cases}$$

where $\psi_{*,j} = [\psi_{1,j} \ \ \psi_{2,j} \ \ \cdots \ \ \psi_{N-1,j}]^T$, $f^k_{*,j} = \begin{bmatrix} f^k_{1,j} & f^k_{2,j} & \cdots & f^k_{N-1,j} \end{bmatrix}^T$ and $\gamma = \left(2 - 2^{1-\alpha}\right)$.

Eq. (16) requires, at each time step, to solve a linear system where the right-hand side y^k utilizes all the history of the computed solution up to that time, and $A^{(k)}$ is a band matrix with a block structure. Each block is a $(N-1) \times (N-1)$ matrix and together they give $A^{(k)}$ the following form

$$A^{(k)} = \begin{bmatrix} T^k_1 & D^k_1 & 0 & \cdots & 0 \\ \widetilde{D}^k_1 & T^k_2 & D^k_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \widetilde{D}^k_{N-3} & T^k_{N-2} & D^k_{N-2} \\ 0 & \cdots & 0 & \widetilde{D}^k_{N-2} & T^k_{N-1} \end{bmatrix}. \tag{17}$$

In this expression each T^k_ℓ is a tridiagonal matrix given by

$$T^{(k)}_\ell = \begin{bmatrix} e^k_{1,\ell} & -p^k_{1,\ell} & 0 & \cdots & 0 \\ -q^k_{2,\ell} & e^k_{2,\ell} & -p^k_{2,\ell} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -q^k_{N-2,\ell} & e^k_{N-2,\ell} & -p^k_{N-2,\ell} \\ 0 & \cdots & 0 & -q^k_{N-1,\ell} & e^k_{N-1,\ell} \end{bmatrix},$$

while $D^k_\ell = [d^k_{i,j}]$ and $\widetilde{D}^k_\ell = [\widetilde{d}^k_{i.j}]$ are diagonal matrices defined by $d^k_{i,i} = -r^k_{i,\ell}$ and $\widetilde{d}^k_{i,i} = -h^k_{i,\ell}$, for $i = 1, \ldots, N-1$.

Remark 1. Note that for each $1 \le i \le (N-1)^2$, there exists exactly one $1 \le \ell_i \le N-1$ such that the resulting diagonal entry $A^{(k)}_{i,i}$ of (17) is determined by

$$A^{(k)}_{i,i} := e^k_{i,\ell_i} = 1 + p^k_{i,\ell_i} + q^k_{i,\ell_i} + r^k_{i,\ell_i} + h^k_{i,\ell_i}.$$

The off-diagonal entries $A^{(k)}_{i,j}$ with $i \ne j$, can be determined in the same way.

3 The approximation

In this section we prove the unconditional stability and the convergence of scheme (13)-(14). Both results are inspired by [13]. Let v^k be given by (15) for $k = 0, \dots, N_t$.

3.1 Stability

Theorem 3.1. *If the hypotheses of lemma 2.3 hold, scheme* (13)-(14) *for the homogeneous* $(f \equiv 0)$ *initial-boundary value problem* (1)-(2)-(3) *is unconditionally stable.*

Proof.

1. Scheme (13): Let $\|v^1\|_\infty = \left|v^1_{l,m}\right| = \max_{i,j} \left|v^1_{i,j}\right|$. We show that $\|v^1\|_\infty \le \|v^0\|_\infty$

$$
\begin{aligned}
\left|v^1_{l,m}\right| &= \left(1 + p^1_{l,m} + q^1_{l,m} + r^1_{l,m} + h^1_{l,m}\right)\left|v^1_{l,m}\right| \\
&- \left(p^1_{l,m}\left|v^1_{l,m}\right| + q^1_{l,m}\left|v^1_{l,m}\right|\right) - \left(r^1_{l,m}\left|v^1_{l,m}\right| + h^1_{l,m}\left|v^1_{l,m}\right|\right) \\
&\le \left(1 + p^1_{l,m} + q^1_{l,m} + r^1_{l,m} + h^1_{l,m}\right)\left|v^1_{l,m}\right| \\
&- \left(p^1_{l,m}\left|v^1_{l+1,m}\right| + q^1_{l,m}\left|v^1_{l-1,m}\right|\right) - \left(r^1_{l,m}\left|v^1_{l,m+1}\right| + h^1_{l,m}\left|v^1_{l,m-1}\right|\right) \\
&\le \left|\left(1 + p^1_{l,m} + q^1_{l,m} + r^1_{l,m} + h^1_{l,m}\right)v^1_{l,m}\right. \\
&\left. - \left(p^1_{l,m}v^1_{l+1,m} + q^1_{l,m}v^1_{l-1,m}\right) - \left(r^1_{l,m}v^1_{l,m+1} + h^1_{l,m}v^1_{l,m-1}\right)\right| \\
&= \left|v^0_{l,m}\right| \le \|v^0\|_\infty.
\end{aligned}
$$

2. Scheme (14): The proof is by induction over k. Suppose $\|v^n\|_\infty \le \|v^0\|_\infty$ for $n = 2, \dots, k$. We prove the inequality $\|v^{k+1}\|_\infty \le \|v^0\|_\infty$. From now on, we will write ω_s rather than $\omega_s^{(\alpha)}$. Notice that the right hand side in scheme (14) is

$$
\begin{aligned}
v^k_{i,j} - \sum_{s=1}^{k} \omega_s \left(v^{k-s+1}_{i,j} - v^{k-s}_{i,j}\right) &= \omega_0 v^k_{i,j} - \omega_1 v^k_{i,j} + \omega_1 v^{k-1}_{i,j} - \omega_2 v^{k-1}_{i,j} + \omega_2 v^{k-2}_{i,j} \\
&- \dots - \omega_k v^1_{i,j} + \omega_k v^0_{i,j} \\
&= (\omega_0 - \omega_1) v^k_{i,j} + (\omega_1 - \omega_2) v^{k-1}_{i,j} + (\omega_2 - \omega_3) v^{k-2}_{i,j} \\
&+ \dots + (\omega_{k-1} - \omega_k) v^1_{i,j} + \omega_k v^0_{i,j}.
\end{aligned}
$$

Now we look at the left hand side. Let $\|v^{k+1}\|_\infty = \left|v^{k+1}_{l,m}\right| = \max_{i,j}\left|v^{k+1}_{i,j}\right|$.

$$
\begin{aligned}
\left|v^{k+1}_{l,m}\right| &= \left(1 + p^{k+1}_{l,m} + q^{k+1}_{l,m} + r^{k+1}_{l,m} + h^{k+1}_{l,m}\right)\left|v^{k+1}_{l,m}\right| \\
&- \left(p^{k+1}_{l,m}\left|v^{k+1}_{l,m}\right| + q^{k+1}_{l,m}\left|v^{k+1}_{l,m}\right|\right) - \left(r^{k+1}_{l,m}\left|v^{k+1}_{l,m}\right| + h^{k+1}_{l,m}\left|v^{k+1}_{l,m}\right|\right) \\
&\le \left(1 + p^{k+1}_{l,m} + q^{k+1}_{l,m} + r^{k+1}_{l,m} + h^{k+1}_{l,m}\right)\left|v^{k+1}_{l,m}\right| \\
&- \left(p^{k+1}_{l,m}\left|v^{k+1}_{l+1,m}\right| + q^{k+1}_{l,m}\left|v^{k+1}_{l-1,m}\right|\right) - \left(r^{k+1}_{l,m}\left|v^{k+1}_{l,m+1}\right| + h^{k+1}_{l,m}\left|v^{k+1}_{l,m-1}\right|\right) \\
&\le \left|\left(1 + p^{k+1}_{l,m} + q^{k+1}_{l,m} + r^{k+1}_{l,m} + h^{k+1}_{l,m}\right)v^{k+1}_{l,m}\right. \\
&\left. - \left(p^{k+1}_{l,m}v^{k+1}_{l+1,m} + q^{k+1}_{l,m}v^{k+1}_{l-1,m}\right) - \left(r^{k+1}_{l,m}v^{k+1}_{l,m+1} + h^{k+1}_{l,m}v^{k+1}_{l,m-1}\right)\right| \\
&= \left|(\omega_0 - \omega_1) v^k_{l,m} + \dots + (\omega_{k-1} - \omega_k) v^1_{l,m} + \omega_k v^0_{l,m}\right| \\
&\le (\omega_0 - \omega_1)\left\|v^k\right\|_\infty + \dots + (\omega_{k-1} - \omega_k)\left\|v^1\right\|_\infty + \omega_k\left\|v^0\right\|_\infty \\
&\le (\omega_0 - \omega_1)\left\|v^0\right\|_\infty + \dots + (\omega_{k-1} - \omega_k)\left\|v^0\right\|_\infty + \omega_k\left\|v^0\right\|_\infty \\
&= \left\|v^0\right\|_\infty.
\end{aligned}
$$

□

This theorem allows us to prove an additional stability bound. Let v_{ij}^0 and $\tilde{v}_{ij}^0$ be the initial discrete values corresponding to two initial conditions ψ_{ij} and $\tilde{\psi}_{ij}$. We may think of two different measurements of the initial concentration. Furthermore, let v_{ij}^k and $\tilde{v}_{ij}^k$ be the corresponding discrete approximations obtained by the numerical schemes (13) and (14). Let $\varepsilon_{ij}^k = v_{ij}^k - \tilde{v}_{ij}^k$ and

$$E^k = \left[\varepsilon_{*,1}^k \;\; \varepsilon_{*,2}^k \;\; \cdots \;\; \varepsilon_{*,N_y-1}^k\right]^T \tag{18}$$

where $\varepsilon_{*,j}^k = \left[\varepsilon_{1,j}^k \;\; \varepsilon_{2,j}^k \;\; \cdots \;\; \varepsilon_{N_x-1,j}^k\right]^T, \quad j = 1, \ldots, N_y - 1.$

Corollary 3.2. *If the hypotheses of lemma 2.3 are satisfied, the numerical errors induced by initial-value conditions in scheme* (13)-(14) *for the inhomogeneous initial-boundary value problem* (1)-(2)-(3) *do not propagate. More precisely, they satisfy the bound*

$$\left\|E^k\right\|_\infty \leq \left\|E^0\right\|_\infty, \qquad k = 1, 2, \ldots.$$

3.2 Convergence

Stability and convergence proofs follow similar patterns. Let $\varepsilon_{ij}^k = u_{ij}^k - v_{ij}^k$ and

$$E^k = \left[\varepsilon_{*,1}^k \;\; \varepsilon_{*,2}^k \;\; \cdots \;\; \varepsilon_{*,N_y-1}^k\right]^T \tag{19}$$

where $\varepsilon_{*,j}^k = \left[\varepsilon_{1,j}^k \;\; \varepsilon_{2,j}^k \;\; \cdots \;\; \varepsilon_{N_x-1,j}^k\right]^T, \quad j = 1, \ldots, N_y - 1.$ The convergence of the scheme is given by the following theorem.

Theorem 3.3. *If the hypotheses of lemma 2.3 hold, then*

$$\left\|E^k\right\|_\infty \leq \|E^0\|_\infty + (\Delta t)^\alpha O(\Delta), \qquad k = 1, 2, \ldots \tag{20}$$

where $O(\Delta)$ *is defined by* (8).

Proof. The proof is by induction on k.

Case $k = 1$. We show that

$$\|E^1\|_\infty \leq \|E^0\|_\infty + (\Delta t)^\alpha O(\Delta).$$

Let $\|E^1\|_\infty = \left|\varepsilon_{lm}^1\right| = \max_{i,j}\left|\varepsilon_{i,j}^1\right|$. In this case the scheme under consideration is (13).

$$\begin{aligned}
\left|\varepsilon_{l,m}^1\right| &= \left(1 + p_{l,m}^1 + q_{l,m}^1 + r_{l,m}^1 + h_{l,m}^1\right)\left|\varepsilon_{l,m}^1\right| - \left(p_{l,m}^1\left|\varepsilon_{l,m}^1\right| + q_{l,m}^1\left|\varepsilon_{l,m}^1\right|\right) - \left(r_{l,m}^1\left|\varepsilon_{l,m}^1\right| + h_{l,m}^1\left|\varepsilon_{l,m}^1\right|\right)\\
&\leq \left(1 + p_{l,m}^1 + q_{l,m}^1 + r_{l,m}^1 + h_{l,m}^1\right)\left|\varepsilon_{l,m}^1\right| - \left(p_{l,m}^1\left|\varepsilon_{l+1,m}^1\right| + q_{l,m}^1\left|\varepsilon_{l-1,m}^1\right|\right)\\
&\quad - \left(r_{l,m}^1\left|\varepsilon_{l,m+1}^1\right| + h_{l,m}^1\left|\varepsilon_{l,m-1}^1\right|\right)\\
&\leq \left|\left(1 + p_{l,m}^1 + q_{l,m}^1 + r_{l,m}^1 + h_{l,m}^1\right)\varepsilon_{l,m}^1 - \left(p_{l,m}^1\varepsilon_{l+1,m}^1 + q_{l,m}^1\varepsilon_{l-1,m}^1\right)\right.\\
&\quad \left. - \left(r_{l,m}^1\varepsilon_{l,m+1}^1 + h_{l,m}^1\varepsilon_{l,m-1}^1\right)\right|\\
&= \left|\left(1 + p_{l,m}^1 + q_{l,m}^1 + r_{l,m}^1 + h_{l,m}^1\right)u_{l,m}^1 - \left(p_{l,m}^1 u_{l+1,m}^1 + q_{l,m}^1 u_{l-1,m}^1\right)\right.\\
&\quad - \left(r_{l,m}^1 u_{l,m+1}^1 + h_{l,m}^1 u_{l,m-1}^1\right) - \left(1 + p_{l,m}^1 + q_{l,m}^1 + r_{l,m}^1 + h_{l,m}^1\right)v_{l,m}^1\\
&\quad \left. + \left(p_{l,m}^1 v_{l+1,m}^1 + q_{l,m}^1 v_{l-1,m}^1\right) + \left(r_{l,m}^1 v_{l,m+1}^1 + h_{l,m}^1 v_{l,m-1}^1\right)\right|\\
&= \left|u_{l,m}^1 + \tau\left(S_\Delta\left(u_{l,m}^1\right) - \sigma_{\alpha,\Delta t}\left(u_{l,m}^1 - u_{l,m}^0\right)\right) - \left(1 + p_{l,m}^1 + q_{l,m}^1 + r_{l,m}^1 + h_{l,m}^1\right)v_{l,m}^1\right.\\
&\quad \left. + \left(p_{l,m}^1 v_{l+1,m}^1 + q_{l,m}^1 v_{l-1,m}^1\right) + \left(r_{l,m}^1 v_{l,m+1}^1 + h_{l,m}^1 v_{l,m-1}^1\right)\right|\\
&= \left|u_{l,m}^0 + \tau\left(S\left(u_{l,m}^1\right) + O(\Delta)\right) - \left(1 + p_{l,m}^1 + q_{l,m}^1 + r_{lm}^1 + h_{l,m}^1\right)v_{l,m}^1\right.\\
&\quad \left. + \left(p_{l,m}^1 v_{l+1,m}^1 + q_{l,m}^1 v_{l-1,m}^1\right) + \left(r_{l,m}^1 v_{l,m+1}^1 + h_{l,m}^1 v_{l,m-1}^1\right)\right|\\
&= \left|u_{l,m}^0 + \tau f_{l,m}^1 + \tau O(\Delta) - v_{l,m}^0 - \tau f_{lm}^1\right|\\
&= \left|\varepsilon_{l,m}^0 + \tau O(\Delta)\right|\\
&\leq \left\|E^0\right\|_\infty + (\Delta t)^\alpha O(\Delta).
\end{aligned}$$

Now suppose

$$\|E^s\|_\infty \leq \left\|E^0\right\|_\infty + (\Delta t)^\alpha O(\Delta)$$

holds for $s = 1, 2, \ldots, k$. We prove the result for $s = k+1$. Let $\|E^{k+1}\|_\infty = \left|\varepsilon_{lm}^{k+1}\right| = \max_{i,j}\left|\varepsilon_{ij}^{k+1}\right|$. By the same argument as before,

$$\begin{aligned}
\left|\varepsilon_{lm}^{k+1}\right| \leq & \Big|\left(1 + p_{l,m}^{k+1} + q_{l,m}^{k+1} + r_{l,m}^{k+1} + h_{l,m}^{k+1}\right) u_{l,m}^{k+1} - \left(p_{l,m}^{k+1} u_{l+1,m}^{k+1} + q_{l,m}^{k+1} u_{l-1,m}^{k+1}\right) \\
& - \left(r_{l,m}^{k+1} u_{l,m+1}^{k+1} + h_{l,m}^{k+1} u_{l,m-1}^{k+1}\right) \\
& - \left(1 + p_{l,m}^{k+1} + q_{l,m}^{k+1} + r_{l,m}^{k+1} + h_{l,m}^{k+1}\right) v_{l,m}^{k+1} + \left(p_{l,m}^{k+1} v_{l+1,m}^{k+1} + q_{l,m}^{k+1} v_{l-1,m}^{k+1}\right) \\
& + \left(r_{l,m}^{k+1} v_{l,m+1}^{k+1} + h_{l,m}^{k+1} v_{l,m-1}^{k+1}\right)\Big|.
\end{aligned}$$

In this case the scheme is (14) and as before, we write ω_s instead of $\omega_s^{(\alpha)}$. The last expression becomes

$$\begin{aligned}
& \left|u_{l,m}^{k+1} + \tau\left(S_\Delta\left(u_{l,m}^{k+1}\right) - \sigma_{\alpha,\Delta t}\sum_{s=0}^{k}\omega_s\left(u_{l,m}^{k-s+1} - u_{l,m}^{k-s}\right)\right)\right. \\
& \left. - v_{l,m}^{k} + \sum_{s=1}^{k}\omega_s\left(v_{l,m}^{k-s+1} - v_{l,m}^{k-s}\right) - \tau f_{l,m}^{k+1}\right| \\
& = \left|u_{l,m}^{k+1} + \tau S\left(u_{l,m}^{k+1}\right) + \tau O(\Delta) - \sum_{s=0}^{k}\omega_s\left(u_{l,m}^{k-s+1} - u_{l,m}^{k-s}\right)\right. \\
& \left. - v_{l,m}^{k} + \sum_{s=1}^{k}\omega_s\left(v_{l,m}^{k-s+1} - v_{l,m}^{k-s}\right) - \tau f_{l,m}^{k+1}\right| \\
& = \left|u_{l,m}^{k+1} + \tau\, f_{l,m}^{k+1} + \tau O(\Delta) - \sum_{s=0}^{k}\omega_s\left(u_{l,m}^{k-s+1} - u_{l,m}^{k-s}\right)\right. \\
& \left. - v_{l,m}^{k} + \sum_{s=1}^{k}\omega_s\left(v_{l,m}^{k-s+1} - v_{l,m}^{k-s}\right) - \tau f_{l,m}^{k+1}\right| \\
& = \left|\sum_{s=1}^{k}\left(\omega_{s-1} - \omega_s\right) u_{l,m}^{k-s+1} + \omega_k u_{l,m}^{0} + \tau\, O(\Delta)\; - v_{l,m}^{k} + \sum_{s=1}^{k}\omega_s\left(v_{l,m}^{k-s+1} - v_{l,m}^{k-s}\right)\right| \\
& = \left|\sum_{s=0}^{k-1}\left(\omega_s - \omega_{s+1}\right)\varepsilon_{l,m}^{k-s} + \omega_k \varepsilon_{l,m}^{0} + \tau\, O(\Delta)\right| \\
& \leq \sum_{s=0}^{k-1}\left(\omega_s - \omega_{s+1}\right)\left|\varepsilon_{l,m}^{k-s}\right| + \omega_k\left|\varepsilon_{l,m}^{0}\right| + \tau\, O(\Delta) \\
& \leq \sum_{s=0}^{k-1}\left(\omega_s - \omega_{s+1}\right)\left\|E^{k-s}\right\|_\infty + \omega_k\left\|E^0\right\|_\infty + \tau\, O(\Delta) \\
& \leq \sum_{s=0}^{k-1}\left(\omega_s - \omega_{s+1}\right)\left\|E^{0}\right\|_\infty + \omega_k\left\|E^0\right\|_\infty + \tau\, O(\Delta) \\
& = \left\|E^0\right\|_\infty + (\Delta t)^\alpha O(\Delta). \qquad \square
\end{aligned}$$

Δt	$\Delta x = \Delta y$	Max. error	Order
$\alpha = 0.1$			
1/16	1/4	1.440e-01	- - - -
1/32	1/8	4.070e-02	1.823
1/64	1/16	1.043e-02	1.964
1/128	1/32	2.607e-03	2.001
1/256	1/64	6.530e-04	1.997
$\alpha = 0.5$			
1/16	1/4	1.415e-01	- - - -
1/32	1/8	4.055e-02	1.803
1/64	1/16	1.045e-02	1.957
1/128	1/32	2.625e-03	1.993
1/256	1/64	6.627e-04	1.986
$\alpha = 0.9$			
1/16	1/4	1.588e-01	- - - -
1/32	1/8	4.434e-02	1.841
1/64	1/16	1.189e-02	1.899
1/128	1/32	3.365e-03	1.821
1/256	1/64	1.053e-03	1.676

Table 1: Absolute errors and order of convergence at $t = 1$ for Example 4.1.

4 Numerical experiments and final remarks

In order to demonstrate the reliability of our numerical method, three examples are presented. The absolute errors in the approximation v of u at time $t = t_k$ are measured by the maximum norm

$$\left\| v^k - u(t_k) \right\|_\infty := \max_{i,j} \left| v^k_{i,j} - u^k_{i,j} \right| .$$

Example 4.1. We consider the time fractional advection-dispersion equation

$$\begin{aligned} u_t^{(\alpha)}(x,y,t) + a(x,y,t)u_x(x,y,t) + b(x,y,t)u_y(x,y,t) \\ = c(x,y,t)u_{xx}(x,y,t) + d(x,y,t)u_{yy}(x,y,t) + f(x,y,t) \end{aligned}$$

on a finite square domain $\Omega = (0,1) \times (0,1)$ for $0 \le t \le 1$, with the initial condition

$$u(x,y,0) = \sin \pi x \sin \pi y, \qquad (x,y) \in \Omega,$$

and the boundary condition

$$u(x,y,t) = 0, \qquad (x,y) \in \partial\Omega \times (0,1].$$

The advection and dispersion coefficients are given by

$$a(x,y,t) = \frac{1}{\sin \pi y}, \qquad b(x,y,t) = \frac{1}{\sin \pi x},$$

Δt	$\Delta x = \Delta y$	Absolute error			Order of convergence		
$\alpha = 0.1$		ε =1e-1	ε =1e-3	ε =1e-5	ε =1e-1	ε =1e-3	ε =1e-5
1/16	1/4	1.103e-01	1.431e-01	1.451e-01	- - -	- - -	- - -
1/32	1/8	2.982e-02	5.154e-02	5.622e-02	1.887	1.473	1.368
1/64	1/16	7.703e-03	1.448e-02	1.711e-02	1.953	1.831	1.716
1/128	1/32	1.927e-03	3.386e-03	4.484e-03	1.999	2.097	1.932
$\alpha = 0.5$		ε =1e-1	ε =1e-3	ε =1e-5	ε =1e-1	ε =1e-3	ε =1e-5
1/16	1/4	1.108e-01	1.445e-01	1.468e-01	- - -	- - -	- - -
1/32	1/8	3.001e-02	5.079e-02	5.451e-02	1.884	1.508	1.430
1/64	1/16	7.817e-03	1.406e-02	1.617e-02	1.941	1.853	1.753
1/128	1/32	1.974e-03	3.169e-03	4.090e-03	1.986	2.150	1.983
$\alpha = 0.9$		ε =1e-1	ε =1e-3	ε =1e-5	ε =1e-1	ε =1e-3	ε =1e-5
1/16	1/4	1.228e-01	1.451e-01	1.470e-01	- -	- -	- -
1/32	1/8	3.377e-02	4.401e-02	4.523e-02	1.862	1.721	1.701
1/64	1/16	9.633e-03	1.219e-02	1.321e-02	1.810	1.852	1.776
1/128	1/32	2.899e-03	3.743e-03	3.981e-03	1.733	1.703	1.730

Table 2: Absolute errors and order of convergence at $t = 1$ for Example 4.2.

and

$$c(x, y, t) = \frac{x}{\pi^2\Gamma(3-\alpha)(t^2+1)}, \qquad d(x, y, t) = \frac{y}{\pi^2\Gamma(3-\alpha)(t^2+1)},$$

respectively, the source or sink function is

$$f(x, y, t) = \frac{1}{\Gamma(3-\alpha)}\left(2t^{2-\alpha} + x + y\right)\sin\pi x \sin\pi y + \pi(t^2+1)(\cos\pi x + \cos\pi y)$$

and the exact concentration is

$$u(x, y, t) = (t^2+1)\sin\pi x \sin\pi y.$$

Numerical experiments for fractional derivatives of orders $\alpha = 0.1, \alpha = 0.5$ and $\alpha = 0.9$ are listed in Table 1. The columns for absolute errors and order of convergence are the main features of this table. This is a somewhat extreme example due to the fact that the advection coefficients a and b do not safisfy the bounds (5)

Example 4.2. As a second example we consider the time fractional advection-dispersion equation

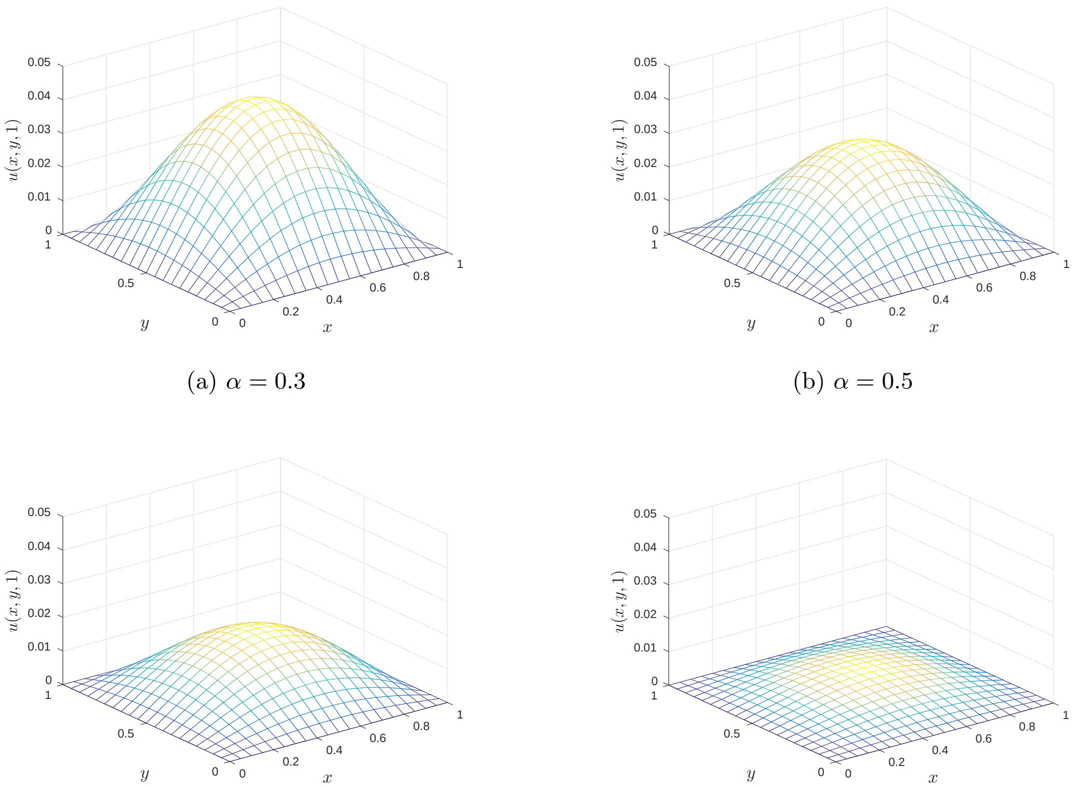
$$u_t^{(\alpha)}(x, y, t) + \frac{1}{1+x}u_y(x, y, t) + \frac{1}{1+y}u_y(x, y, t) = \varepsilon\Delta u(x, y, t) + f(x, y, t), \qquad \varepsilon > 0,$$

on $\Omega = (0, 1) \times (0, 1)$ for $0 \le t \le 1$, with initial condition

$$u(x, y, 0) = \sin\pi x \sin\pi y, \qquad (x, y) \in \Omega,$$

boundary condition

$$u(x, y, t) = 0, \qquad (x, y) \in \partial\Omega \times (0, 1],$$



(a) $\alpha = 0.3$ (b) $\alpha = 0.5$

(c) $\alpha = 0.7$ (d) $\alpha = 0.9$

Figure 1: Numerical solutions of Example 4.3 for $t = 1$

and source or sink term

$$f(x, y, t) = \left(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2\varepsilon\pi^2(t^2+1)\right)\sin\pi x \sin\pi y + \pi(t^2+1)\left(\frac{\cos\pi x \sin\pi y}{x+1} + \frac{\cos\pi \sin\pi y}{y+1}\right).$$

The exact solution is

$$u(x, y, t) = (t^2+1)\sin\pi x \sin\pi y.$$

This is a test for the behavior of the method in the presence of very small diffusion coefficients, an almost degenerate parabolic equation. Numerical results are provided in Table 2. As before, three different fractional derivative orders are taken into account and there are results for three different diffusion coefficients: $\varepsilon = 10^{-1}$, $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-5}$.

Example 4.3. Finally we solve the time fractional diffusion equation

$$u_t^{(\alpha)}(x, y, t) + u_x(x, y, t) + u_y(x, y, t) = \Delta u(x, y, t),$$

on the finite square domain $\Omega = (0, 1) \times (0, 1)$ for $0 \leq t \leq 1$, with the initial condition

$$u(x, y, 0) = \sin\pi x \sin\pi y, \qquad \text{for } (x, y) \in \Omega,$$

and the boundary condition

$$u(x, y, t) = 0, \qquad (x, y) \in \partial\Omega \times (0, 1].$$

Figure 1 illustrates the computed solutions for $t = 1$ for several values of α. No exact solutions are known for this problem but the pictures illustrate the continuous dependence of the solutions on the fractional differentiation order.

In summary, this paper introduces an implicit finite difference approximation for the solution of an initial boundary value problem for a two dimensional time fractional advection-dispersion equation with variable coefficients in which the fractional derivative is given in the sense of Caputo and the dispersion terms are in nondivergence form. Proofs of consistency, stability and convergence are included and so are illustrative numerical experiments. A useful feature of the paper is the computational framework based on matrices. Our scheme was successfully implemented for the solution of an inverse source problem in [3]. We certainly expect to develop other applications of this scheme in the near future.

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