A METHOD FOR PROVING RAMANUJAN SERIES FOR $1/\pi$

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Abstract. In a famous paper of 1914 Ramanujan gave a list of 17 extraordinary formulas for the number π. In this paper we explain a general method to prove them, based on an original idea of James Wan and in some own ideas.

1. Introduction

In his famous paper [15] of 1914 Ramanujan gave a list of 17 extraordinary formulas for the number π, which are of the following form

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{s}\right)_n \left(1-\frac{1}{s}\right)_n}{(1)_n^3} (a+bn)\, z^n = \frac{1}{\pi}, \qquad (c)_0 = 1, \quad (c)_n = \prod_{j=1}^{n} (c+j-1), \tag{1}$$

where $s \in \{2,3,4,6\}$, and z, b, a are algebraic numbers. Four of his formulas are

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3} \frac{42n+5}{64^n} = \frac{16}{\pi}, \tag{2}$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^3} (33n+4) \left(\frac{4}{125}\right)^n = \frac{15\sqrt{3}}{2\pi}, \tag{3}$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} \frac{26390n+1103}{99^{4n+2}} = \frac{\sqrt{2}}{4\pi}, \tag{4}$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} (63n+8) \left(\frac{-4}{5}\right)^{3n} = \frac{5\sqrt{15}}{\pi}, \tag{5}$$

corresponding to $s = 2,3,4,6$ respectively. However Ramanujan wrote few details of his proofs and the first rigorous deductions were made by the Borwein brothers in 1985, see [6]. Other general proofs based on the modular theory are for example in [1, 2, 6, 3, 7, 17], and we believe that the method that we use in this paper is an interesting alternative one. Other kind of proofs are in [14, 8, 9, 10, 13], and two beautiful surveys are [3] and [17].

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2. A method for proving Ramanujan series for $1/\pi$

We will use the following notation:

$$F_s(\alpha) = {}_2F_1\left(\begin{matrix} \frac{1}{s}, 1-\frac{1}{s} \\ 1 \end{matrix} \,\middle|\, \alpha \right), \quad G_s(\alpha) = \alpha \frac{dF_s(\alpha)}{d\alpha},$$

and the following version of the Legendre's relation:

$$\alpha F_s(\alpha) G_s(\beta) + \beta F_s(\beta) G_s(\alpha) = \frac{1}{\pi} \sin \frac{\pi}{s}, \qquad \beta = 1 - \alpha. \tag{6}$$

We will show that this relation explains why π appears in the Ramanujan series. The other ingredients we need to prove the series are: a transformation of modular origin and the known Clausen's identity

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{s}\right)_n \left(1-\frac{1}{s}\right)_n}{(1)_n^3} z^n = F_s(\alpha) F_s(\alpha), \qquad z = 4\alpha(1-\alpha). \tag{7}$$

Our method consist in a variant of a James Wan's original idea [16]. We explained it in [11] and [12] proving some new Ramanujan-Orr series for $1/\pi$. In this paper we show how to apply it to prove Ramanujan series in a simple way when we know the required transformation, or what is equivalent: when we know the required modular equation, because the multiplier is given by the formula (23). We explain our technique with two examples: one corresponding to a series of positive terms and the other one to an alternating series.

2.1. **Example for series of positive terms.** We reprove below the following series for $1/\pi$ of level $\ell = 2$ ($s = 4$) due to Ramanujan:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} \frac{1}{3^{4n}} (10n+1) = \frac{9\sqrt{2}}{4\pi}. \tag{8}$$

We begin with the following theorem:

Theorem 1. *For $s = 4$ (level $\ell = 2$), we have*

$$F_4(\alpha_0) = \frac{\sqrt{5}}{5} F_4(\beta_0), \tag{9}$$

$$G_4(\alpha_0) = \frac{16\sqrt{5} - 36}{5} F_4(\beta_0) + \frac{161\sqrt{5} - 360}{5} G_4(\beta_0), \tag{10}$$

where

$$\alpha_0 = \frac{1}{2} - \frac{2\sqrt{5}}{9}, \quad \beta_0 = 1 - \alpha_0 = \frac{1}{2} + \frac{2\sqrt{5}}{9}.$$

Proof. In [9, page 608] we see the following transformation of level 2 and degree $1/d$ with $d = 5$:

$$F_4(\alpha) = m F_4(\beta) \tag{11}$$

where

$$\alpha = \frac{64x^5(1+x)}{(1+4x^2)(1-2x-4x^2)^2}, \quad \beta = \frac{64x(1+x)^5}{(1+4x^2)(1+22x-4x^2)^2},$$

and

$$m = \frac{\sqrt{1-2x-4x^2}}{\sqrt{1+22x-4x^2}}.$$

If we take $\beta = 1-\alpha$, then we get the following solution

$$x_0 = \frac{\sqrt{5}-2}{2}, \quad \alpha_0 = \frac{1}{2} - \frac{2\sqrt{5}}{9}, \quad \beta_0 = \frac{1}{2} + \frac{2\sqrt{5}}{9}, \quad m_0 = \frac{1}{\sqrt{5}}.$$

In addition, we have

$$\alpha_0' = \frac{\sqrt{5}+2}{27}, \quad \beta_0' = \frac{\sqrt{5}+2}{27}, \quad m_0' = \frac{-8\sqrt{5}-16}{15}.$$

Differentiating (11) with respect to α we have

$$G_4(\alpha) = \alpha \frac{m'}{\alpha'} F_4(\beta) + \alpha \frac{m}{\beta} \frac{\beta'}{\alpha'} G_4(\beta), \tag{12}$$

where the $'$ stands for the derivative with respect to x. Finally, substituting $x = x_0$, in (11) and (12) we arrive at the results stated by the theorem. ☐

We are ready to prove (8).

Proof. Applying to both sides of (7) with $s = 4$ the operator

$$\frac{2}{9} + \frac{20}{9} z \frac{d}{dz}\bigg|_{z=z_0} = \frac{2}{9} + \frac{20}{9} \frac{z}{z'} \frac{d}{d\alpha}\bigg|_{\alpha=\alpha_0}$$

where here the $'$ means the derivative with respect to α, we obtain

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)}{(1)_n^3} \frac{1}{3^{4n}} \left(\frac{20}{9} n + \frac{2}{9}\right) \tag{13}$$

$$= \frac{2}{9} F_4(\alpha_0) F_4(\alpha_0) + \left(\frac{\sqrt{5}}{2} + \frac{10}{9}\right) F_4(\alpha_0) G_4(\alpha_0) + \left(\frac{\sqrt{5}}{2} + \frac{10}{9}\right) G_4(\alpha_0) F_4(\alpha_0).$$

Observe that we have intentionally repeated two equal terms without simplifying the sum. Then, if we use the relations (9) and (10) to replace one factor $F_4(\alpha_0)$ of the two first terms and to replace $G_4(\alpha_0)$ in the last term, we arrive at

$$\alpha_0 F_4(\alpha_0) G_4(\beta_0) + \beta_0 F_4(\beta_0) G_4(\alpha_0),$$

which in view of (6) is equal to

$$\frac{1}{\pi} \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2\pi},$$

and we are done. ☐

2.2. **Example for alternating series.** Here we prove with our method the following alternating series of level $\ell = 2$ due to Ramanujan:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} \left(\frac{-1}{48}\right)^n (28n+3) = \frac{16\sqrt{3}}{3\pi}. \tag{14}$$

Theorem 2. *For $s = 4$ (level $\ell = 2$), we have*

$$F_{_4}(\alpha_0) = \frac{(3+i)\sqrt{2}}{10} F_{_4}(\beta_0), \tag{15}$$

and

$$\begin{aligned} G_{_4}(\alpha_0) = & \left(\frac{-63}{20}\sqrt{2} + \frac{9\sqrt{6}}{5}\right) F_{_4}(\beta_0) \\ & + \left[\left(\frac{-291}{10}\sqrt{2} + \frac{84}{5}\sqrt{6}\right) + \left(\frac{-28}{5}\sqrt{6} + \frac{97}{10}\sqrt{2}\right) i\right] G_{_4}(\beta_0), \end{aligned} \tag{16}$$

where

$$\alpha_0 = \frac{1}{2} - \frac{7\sqrt{3}}{24}, \quad \beta_0 = \frac{1}{2} + \frac{7\sqrt{3}}{24}.$$

Proof. We use again the transformation of degree $1/d$ with $d = 5$:

$$F_4(\alpha) = mF_4(\beta), \tag{17}$$

where

$$\alpha = \frac{64x^5(1+x)}{(1+4x^2)(1-2x-4x^2)^2}, \quad \beta = \frac{64x(1+x)^5}{(1+4x^2)(1+22x-4x^2)^2},$$

and

$$m = \frac{\sqrt{1-2x-4x^2}}{\sqrt{1+22x-4x^2}}.$$

Another solution of $\beta = 1 - \alpha$ is the following one:

$$x_0 = \frac{2\sqrt{3}-3}{4} - \frac{2-\sqrt{3}}{4} i, \quad \alpha_0 = \frac{1}{2} - \frac{7\sqrt{3}}{24}, \quad \beta_0 = \frac{1}{2} + \frac{7\sqrt{3}}{24}, \quad m_0 = \frac{(3+i)\sqrt{2}}{10}.$$

It is interesting to note that $|m_0| = 1/\sqrt{5}$. We also get

$$m_0' = \left(\frac{-27}{40}\sqrt{2} - \frac{69}{200}\sqrt{6}\right) - \left(\frac{33}{200}\sqrt{6} + \frac{9}{40}\sqrt{2}\right) i, \tag{18}$$

$$\alpha_0' = \left(\frac{-23}{240} - \frac{\sqrt{3}}{16}\right) - \left(\frac{\sqrt{3}}{48} + \frac{11}{240}\right) i, \tag{19}$$

$$\beta_0' = \left(\frac{-5}{48} - \frac{\sqrt{3}}{16}\right) + \left(\frac{1}{48} + \frac{\sqrt{3}}{48}\right) i. \tag{20}$$

Differentiating (17) with respect to α we have

$$G_4(\alpha) = \alpha \frac{m'}{\alpha'} F_4(\beta) + \alpha \frac{m}{\beta} \frac{\beta'}{\alpha'} G_4(\beta), \tag{21}$$

where the $'$ stands for the derivative with respect to x. Finally, substituting $x = x_0$, in (17) and (21) we arrive at the results stated by the theorem. □

We are ready to prove (14).

Proof. Applying to both sides of (7) with $s = 4$ the operator

$$\frac{3}{32}\sqrt{6} + \frac{28}{32}\sqrt{6} z \frac{d}{dz}\bigg|_{z=z_0} = \frac{3}{32}\sqrt{6} + \frac{28}{32}\sqrt{6} \frac{z}{z'} \frac{d}{d\alpha}\bigg|_{\alpha=\alpha_0}$$

where here the $'$ means the derivative with respect to α, we obtain

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)}{(1)_n^3} \left(\frac{-1}{48}\right)^n \left(\frac{28}{32}\sqrt{6}n + \frac{3}{32}\sqrt{6}\right) = \frac{3\sqrt{6}}{32} F_4(\alpha_0) F_4(\alpha_0) \tag{22}$$

$$+(1-Ci)\left(\frac{3\sqrt{2}}{4} + \frac{7}{16}\sqrt{6}\right) F_4(\alpha_0) G_4(\alpha_0) + (1+Ci)\left(\frac{3\sqrt{2}}{4} + \frac{7}{16}\sqrt{6}\right) G_4(\alpha_0) F_4(\alpha_0).$$

Observe that we have intentionally introduced the factors $(1 - Ci)$ and $(1 + Ci)$, and that the expression holds for all values of C because the terms with C cancels. Then, we use the relations (15) and (16) to replace one factor $F_4(\alpha_0)$ of the two first terms and to replace $G_4(\alpha_0)$ in the last term. Finally, if we choose $C = 1/3$, we arrive at

$$\alpha_0 F_4(\alpha_0) G_4(\beta_0) + \beta_0 F_4(\beta_0) G_4(\alpha_0),$$

which in view of (6) is equal to

$$\frac{1}{\pi} \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2\pi},$$

and we are done. □

3. Explicit formulas

We already know that $z = 4\alpha(1 - \alpha)$. Here, we give explicit formulas for b and a, and also for the modular variable q.

3.1. **On modular equations and multipliers.** Transformations of modular origin can be used to prove Ramanujan-type series for $1/\pi$. Those proved in [9] and [8] are written like the one used in this paper. For the great quantity of them given by Ramanujan, see [4, Chapters 19, 20] and [5, Chapters 33, 36]. For the modular equations $P_s(\alpha, \beta) = 0$ corresponding to $s = 2, 3, 4, 6$ (levels $\ell = 4, 3, 2, 1$), the multiplier is given by the following formula:

$$m_s(\alpha, \beta) = \frac{1}{\sqrt{d}} \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)} \frac{d\alpha}{d\beta}\right)^{1/2}, \tag{23}$$

where $1/d$ is the degree of the modular equation. Hence the associated transformation of level ℓ and degree $1/d$ reads

$$F_s(\alpha) = m_s(\alpha,\beta) F_s(\beta), \qquad P_s(\alpha,\beta)=0.$$

You can see a proof of (23) for the case $s=2$ (level $\ell=4$) in [4, Entry 24 (vi)], but a similar proof can be given for the four hypergeometric levels.

3.2. **Explicit formulas.** To get the explicit formulas for a and b we will use our method and the identity

$$\frac{\beta_0'}{\alpha_0'} = \frac{1}{d\, m_0^2}, \tag{24}$$

which comes from (23). First, observe that the level is related to s in the following way:

$$\ell = 4 \sin^2 \frac{\pi}{s}. \tag{25}$$

It is not a coincidence [14, eq. 27]. Using our method for the case of general series and cases $z_0>0$ or $z_0<0$, that is, applying the operator

$$a + bz\frac{d}{dz}\bigg|_{z_0},$$

to both sides of (7), using the substitutions

$$F_s(\alpha_0) = mF_s(\beta_0), \qquad G_s(\alpha_0) = \alpha_0 \frac{m_0'}{\alpha_0'} F_s(\beta_0) + \frac{\alpha_0}{d\, m_0\, \beta_0} G_s(\beta_0),$$

in the way that we have explained in the examples, and equating the coefficients of $F(\alpha_0)F(\beta_0)$, $F(\alpha_0)G(\beta_0)$ and $F(\beta_0)G(\alpha_0)$ to 0, α_0 and β_0 respectively, we deduce the following explicit formulas:

$$b = (1-2\alpha_0)\, \frac{\operatorname{Re}(m_0)}{\sin \frac{\pi}{s}}\, d, \tag{26}$$

and

$$a = -(1+Ci)\, \frac{\alpha_0\beta_0}{\alpha_0'}\, \frac{m_0'}{m_0}\, \frac{b}{1-2\alpha_0}, \qquad C = \frac{\operatorname{Im}(m_0)}{\operatorname{Re}(m_0)}. \tag{27}$$

For the case $z>0$, we have $m_0 = 1/\sqrt{d}$. Hence, we can write the above formulas in the way

$$b = 2(1-2\alpha_0)\sqrt{\frac{d}{\ell}}, \qquad a = -2\alpha_0\beta_0 \frac{m_0'}{\alpha_0'}\, \frac{d}{\sqrt{\ell}}. \tag{28}$$

For the case $z<0$ (alternating series), we have observed experimentally that

$$m_0 \stackrel{?}{=} \frac{\sqrt{4d-\ell}}{2d} + \frac{\sqrt{\ell}}{2d}\, i, \tag{29}$$

Hence, assuming it, we can write (26) and (27) in the following form:

$$b = 2\,(1-2\alpha_0)\sqrt{\frac{d}{\ell}-\frac{1}{4}}, \qquad a = -2\alpha_0\beta_0 \frac{m_0'}{\alpha_0'}\,\frac{d}{\sqrt{\ell}}. \tag{30}$$

Finally, we relate the modular variable q with d and ℓ assuming (29). Let $q = e^{-2\pi\sqrt{r}}$ and $q = -e^{-2\pi\sqrt{r}}$, the modular variable corresponding to the cases $z > 0$ and $z < 0$, respectively. From the known formula

$$4r = \frac{b^2}{1-z} = \frac{b^2}{(1-2\alpha)^2},$$

we deduce that

$$r = \frac{d}{4\sin^2\frac{\pi}{s}} = \frac{d}{\ell}, \quad r = \frac{d}{4\sin^2\frac{\pi}{s}} - \frac{1}{4} = \frac{d}{\ell} - \frac{1}{4},$$

for the cases $z > 0$ and $z < 0$, respectively. Hence,

$$q = e^{-2\pi\sqrt{\frac{d}{\ell}}}, \qquad q = -e^{-2\pi\sqrt{\frac{d}{\ell}-\frac{1}{4}}}, \tag{31}$$

for all the series of positive terms and for all the alternating series respectively.

3.3. **An experimental test.** The test which consist of evaluating numerically

$$\frac{F_s(\alpha_0)}{F_s(\beta_0)} = m_0, \quad \beta_0 = 1-\alpha_0, \tag{32}$$

has been very useful in discovering that $|m_0^2|$ but not m_0^2 (algebraic) is a positive rational number, and that $|m_0^2| = 1/d$. For example, for the series of level $\ell = 3$ $(s = 3)$ and $z_0 = -1/500^2$, as $z_0 = 4\alpha_0(1-\alpha_0)$ we obtain

$$\alpha_0 = \frac{1}{2} - \frac{53\sqrt{89}}{1000}, \quad \beta_0 = 1-\alpha_0 = \frac{1}{2} + \frac{53\sqrt{89}}{1000},$$

and evaluating numerically (32), we get with an approximation of 20 digits that

$$\left|\frac{F_s(\alpha_0)}{F_s(\beta_0)}\right| \approx 0.20851441405707476267,$$

which we identify as $1/\sqrt{23}$. Hence, for proving with our method that alternating series, we need a transformation of degree $1/d$ with $d = 23$ for the level $\ell = 3$, and with such a transformation we can prove it rigorously. See the tables at the end of the paper.

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d	a	b	$z<0$	d	a	b	$z>0$
3	$\frac{1}{2}$	2	-1	3	$\frac{1}{4}$	$\frac{6}{4}$	$\frac{1}{4}$
5	$\frac{1}{2\sqrt{2}}$	$\frac{6}{2\sqrt{2}}$	$-\frac{1}{8}$	7	$\frac{5}{16}$	$\frac{42}{16}$	$\frac{1}{64}$

TABLE 1. Rational Ramanujan-type series of $\ell = 4$ for $1/\pi$

d	a	b	$z<0$	d	a	b	$z>0$
3	$\frac{3}{8}$	$\frac{20}{8}$	$-\frac{1}{4}$	2	$\frac{2}{9}$	$\frac{14}{9}$	$\frac{32}{81}$
4	$\frac{8}{9\sqrt{7}}$	$\frac{65}{9\sqrt{7}}$	$-\frac{16^2}{63^2}$	3	$\frac{1}{2\sqrt{3}}$	$\frac{8}{2\sqrt{3}}$	$\frac{1}{9}$
5	$\frac{3\sqrt{3}}{16}$	$\frac{28\sqrt{3}}{16}$	$-\frac{1}{48}$	5	$\frac{4}{9\sqrt{2}}$	$\frac{40}{9\sqrt{2}}$	$\frac{1}{81}$
7	$\frac{23}{72}$	$\frac{260}{72}$	$-\frac{1}{18^2}$	9	$\frac{27}{49\sqrt{3}}$	$\frac{360}{49\sqrt{3}}$	$\frac{1}{7^4}$
13	$\frac{41\sqrt{5}}{288}$	$\frac{644\sqrt{5}}{288}$	$-\frac{1}{5\cdot 72^2}$	11	$\frac{19}{18\sqrt{11}}$	$\frac{280}{18\sqrt{11}}$	$\frac{1}{99^2}$
19	$\frac{1123}{3528}$	$\frac{21460}{3528}$	$-\frac{1}{882^2}$	29	$\frac{4412}{9801\sqrt{2}}$	$\frac{105560}{9801\sqrt{2}}$	$\frac{1}{99^4}$

TABLE 2. Rational Ramanujan-type series of $\ell = 2$ for $1/\pi$

d	a	b	$z<0$	d	a	b	$z>0$
3	$\frac{\sqrt{3}}{4}$	$\frac{5\sqrt{3}}{4}$	$-\frac{9}{16}$	2	$\frac{1}{3\sqrt{3}}$	$\frac{6}{3\sqrt{3}}$	$\frac{1}{2}$
5	$\frac{7}{12\sqrt{3}}$	$\frac{51}{12\sqrt{3}}$	$-\frac{1}{16}$	4	$\frac{8}{27}$	$\frac{60}{27}$	$\frac{2}{27}$
7	$\frac{\sqrt{15}}{12}$	$\frac{9\sqrt{15}}{12}$	$-\frac{1}{80}$	5	$\frac{8}{15\sqrt{3}}$	$\frac{66}{15\sqrt{3}}$	$\frac{4}{125}$
11	$\frac{106}{192\sqrt{3}}$	$\frac{1230}{192\sqrt{3}}$	$-\frac{1}{2^{10}}$				
13	$\frac{26\sqrt{7}}{216}$	$\frac{330\sqrt{7}}{216}$	$-\frac{1}{3024}$				
23	$\frac{827}{1500\sqrt{3}}$	$\frac{14151}{1500\sqrt{3}}$	$-\frac{1}{500^2}$				

TABLE 3. Rational Ramanujan-type series of $\ell = 3$ for $1/\pi$

d	a	b	$z<0$	d	a	b	$z>0$
2	$\frac{8}{5\sqrt{15}}$	$\frac{63}{5\sqrt{15}}$	$-\frac{4^3}{5^3}$	2	$\frac{3}{5\sqrt{5}}$	$\frac{28}{5\sqrt{5}}$	$\frac{3^3}{5^3}$
3	$\frac{15}{32\sqrt{2}}$	$\frac{154}{32\sqrt{2}}$	$-\frac{3^3}{8^3}$	3	$\frac{6}{5\sqrt{15}}$	$\frac{66}{5\sqrt{15}}$	$\frac{4}{5^3}$
5	$\frac{25}{32\sqrt{6}}$	$\frac{342}{32\sqrt{6}}$	$-\frac{1}{8^3}$	4	$\frac{20}{11\sqrt{33}}$	$\frac{252}{11\sqrt{33}}$	$\frac{2^3}{11^3}$
7	$\frac{279}{160\sqrt{30}}$	$\frac{4554}{160\sqrt{30}}$	$-\frac{9}{40^3}$	7	$\frac{144\sqrt{3}}{85\sqrt{85}}$	$\frac{2394\sqrt{3}}{85\sqrt{85}}$	$\frac{4^3}{85^3}$
11	$\frac{526\sqrt{15}}{80^2}$	$\frac{10836\sqrt{15}}{80^2}$	$-\frac{1}{80^3}$				
17	$\frac{10177\sqrt{330}}{3\cdot 440^2}$	$\frac{261702\sqrt{330}}{3\cdot 440^2}$	$-\frac{1}{440^3}$				
41	$\frac{27182818\sqrt{10005}}{3\cdot 53360^2}$	$\frac{1090280268\sqrt{10005}}{3\cdot 53360^2}$	$-\frac{1}{53360^3}$				

TABLE 4. Rational Ramanujan-type series of $\ell = 1$ for $1/\pi$