DEFORMATIONS OF LOG CANONICAL AND F-PURE SINGULARITIES

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ABSTRACT. We introduce a lifting property for local cohomology, which leads to a unified treatment of the dualizing complex for flat morphisms with semi-log-canonical, Du Bois or F-pure fibers. As a consequence we obtain that, in all 3 cases, the cohomology sheaves of the relative dualizing complex are flat and commute with base change. We also derive several consequences for deformations of semi-log-canonical, Du Bois and F-pure singularities.

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1. INTRODUCTION

One of the difficulties of higher dimensional birational geometry and moduli theory is that the occuring singularities are frequently not Cohen-Macaulay.

On a proper Cohen-Macaulay scheme we have a dualizing *sheaf* ω_X and Serre duality. By contrast, on an arbitrary proper scheme we have a dualizing *complex* ω_X^{\bullet} and the isomorphism of Serre duality is replaced by a spectral sequence of Grothendieck duality.

The "most important" cohomology sheaf of the dualizing complex ω_X^{\bullet} is

$$\mathsf{h}^{-\dim X}(\omega_X^{\bullet}) \simeq \omega_{X_X}$$

and X is Cohen-Macaulay if and only if the other cohomology sheaves $h^{-i}(\omega_X^{\bullet})$ are all zero, cf. [Con00, 3.5.1]. Thus these $h^{-i}(\omega_X^{\bullet})$ measure "how far" X is from being Cohen-Macaulay; see Proposition 8.1 for a more precise claim. Our main result implies that in flat families $X \to B$ with log canonical or F-pure fibers, the cohomology sheaves $h^{-i}(\omega_{X/B}^{\bullet})$ are flat over B and commute with base change. In particular, being Cohen-Macaulay is a deformation invariant

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property for such singularities. Note that for flat families Cohen-Macaulay is always an open condition but usually not a closed one.

One of the puzzles of higher dimensional singularity theory is that while the singularities of the Minimal Model Program (log terminal, log canonical, Du Bois, etc.) and of positive characteristic commutative algebra (*F*-pure, *F*-injective, etc.) are very closely related, the methods to study them are completely different. Here we isolate the following quite powerful common property for some of these classes.

Definition 1.1. Let A be a noetherian ring, and (T, \mathfrak{n}) a noetherian local A-algebra. We say that T has *liftable local cohomology over* A if for any noetherian local A-algebra (R, \mathfrak{m}) and nilpotent ideal $I \subset R$ such that $R/I \simeq T$, the natural morphism on local cohomology

$$H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{n}}(T)$$

is surjective for all i.

We say that T has *liftable local cohomology* if it has liftable local cohomology over \mathbb{Z} .

Remark 1.2. Notice that, using the above notation, if $\phi : A' \to A$ is a ring homomorphism from another noetherian ring A' then if T has liftable local cohomology over A', then it also has liftable local cohomology over A. In particular, if T has liftable local cohomology over \mathbb{Z} , then it has liftable local cohomology over any noetherian ring A justifying the above terminology.

Furthermore, if A = k is a field of characteristic 0 then the notions of having liftable local cohomology over k and over \mathbb{Z} are equivalent. This follows in one direction by the above and in the other by the Cohen structure theorem [StacksProject, Tag 032A].

Remark 1.3. A closely related notion, a ring being *cohomologically full*, is defined in [DSM18]. This notion and results of this article are used in [CV18] to settle a conjecture by Herzog on ideals with square-free initial ideals.

We prove in Theorem 6.1 that Du Bois singularities have liftable local cohomology. On the Frobenius side, the right concept seems to be F-anti-nilpotent singularities, a notion introduced in [EH08], that lies between F-pure and F-injective [Ma14, MQ17]. We are very grateful to L. Ma and K. Schwede for pointing out that, by a result of Ma–Schwede–Shimomoto [MSS17], F-anti-nilpotent singularities also have liftable local cohomology over their ground field. We discuss this in Proposition 7.2.

With this definition (cf. Definition 4.2), our main technical theorem is the following.

Theorem 1.4 = Theorem 5.14. Let $f : X \to B$ be a flat morphism of schemes that is essentially of finite type and let $b \in B$ such that X_b has liftable local cohomology over B. Then there exists an open neighborhood $X_b \subset U \subset X$ such that $h^{-i}(\omega_{U/B}^{\bullet})$ is flat over B and commutes with base change for each $i \in \mathbb{Z}$.

For applications the following consequences are especially important.

Corollary 1.5. Let $f : X \to B$ be a flat morphism of schemes, essentially of finite type over a field k. Let $b \in B$ be a point. Assume that

- (i) either char k = 0 and X_b is DuBois, e.g., semi-log-canonical,
- (ii) or char k > 0 and X_b is *F*-anti-nilpotent, e.g., *F*-pure.

Then there exists an open neighborhood $X_b \subset U \subset X$ such that $h^{-i}(\omega_{U/B}^{\bullet})$ is flat over B and commutes with base change for each $i \in \mathbb{Z}$.

Corollary 1.5(i) can be viewed as a generalization of the following Corollary 1.6(i), proved in [KK10] for projective morphisms and in [MSS17] in general (cf. [KS16b]).

Corollary 1.6. Let (X, x) be a local scheme, essentially of finite type over a field and assume that

(i) either char k = 0 and X is DuBois, e.g., semi-log-canonical,

(ii) or char k > 0 and X is F-anti-nilpotent, e.g., F-pure.

If (X, x) admits a flat deformation whose generic fiber is Cohen-Macaulay then (X, x) is also Cohen-Macaulay.

In many cases this is quite sharp, see Example 2.1 and Theorem 8.5 for some stronger versions. This also gives the following immediate corollary.

Corollary 1.7 (cf. Corollary 8.7). Let X be an abelian variety of dimension at least 2 defined over a field k. If char k > 0 assume that X is ordinary. Then the cone over an arbitrary projective embedding of X is not smoothable.

Note that a special case of Corollary 1.7 over \mathbb{C} , the non-smoothability of the projective cone over an abelian variety of dimension at least 2, was proved in [Som79]. See Corollary 8.7 for a stronger version.

1.8. THE ORGANIZATION OF THE PAPER. In Section 2 we give examples and show some applications of the main results. In Section 6 we prove that a Du Bois local scheme has *liftable local cohomology*. In Section 7 we recall a few basic notions about singularities defined by the behaviour of the Frobenius morphism in positive characteristic and recall that F-anti-nilpotent singularities have liftable local cohomology over their ground field. In Section 3 and Section 4 we study infinitesimal deformations of schemes with liftable local cohomology and prove the main result for families over Artinian bases. In Section 5 we prove a rather general flatness and base change criterion, see Theorem 5.12, which may be of independent interest and derive Theorem 1.4 as relatively easy consequences of this and the results of Section 4. In Section 8 we prove a criterion for S_n singularities in terms of the dualizing complex, see Proposition 8.1, and use this and Theorem 1.4 to prove Corollary 1.6.

1.9. DUALIZING COMPLEX AND ITS RELATIVES. The *(normalized)* dualizing complex of X is denoted by ω_X^{\bullet} and if X is of pure dimension n the canonical sheaf of X is defined as $\omega_X := h^{-n}(\omega_X^{\bullet})$. Note that if X is not normal, then this is not necessarily the push-forward of the canonical sheaf from the non-singular locus.

We will work with *three* closely related, but generally different *objects*:

- the dualizing complex; ω_X^{\bullet} ,
- the canonical sheaf; $\omega_X = \mathsf{h}^{-n}(\omega_X^{\bullet})$, and

• the object defined by $\underline{\omega}_X^{\bullet} := \mathcal{RHom}_X(\underline{\Omega}_X^0, \omega_X^{\bullet}).$ (See Section 6 for a description of $\underline{\Omega}_X^0$). Note that one has a natural morphism $\underline{\omega}_X^{\bullet} \to \omega_X^{\bullet}$ dual to $\eta : \mathscr{O}_X \to \underline{\Omega}_X^0$.

For a morphism $f: X \to B$, the *(normalized) relative dualizing complex* of f will be denoted by $\omega_{X/B}^{\bullet}$ and if f has equidimensional fibers of dimension n, then the *relative canonical sheaf* of f is $\omega_{X/B} := h^{-n}(\omega_{X/B}^{\bullet})$. If B consists of a single (closed) point, then these notions reduce to the ones discussed above. For more details on relative dualizing complexes see [StacksProject, Tag 0E2S]

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2. EXAMPLES

2.A. Characteristic zero

In this section we review rational, Cohen-Macaulay, and DuBois singularities of cones in characteristic zero and demonstrate some consequences of the main results.

Example 2.1 Deformations of cones. Let X be a projective variety and \mathscr{L} an ample line bundle on X. Assume for simplicity that X has rational singularities. Let

$$C_a(X,\mathscr{L}) := \operatorname{Spec}_k \bigoplus_{r=0}^{\infty} H^0(X,\mathscr{L}^r)$$

be the affine cone over X with conormal bundle \mathscr{L} and vertex v. Then the singularity $v \in C_a(X, \mathscr{L})$ is

(2.1.1) rational $\Leftrightarrow H^i(X, \mathscr{L}^r) = 0$ for every $i > 0, r \ge 0$,

(2.1.2) Cohen-Macaulay $\Leftrightarrow H^i(X, \mathscr{L}^r) = 0$ for every dim $X > i > 0, r \ge 0$ and

(2.1.3) Du Bois $\Leftrightarrow H^i(X, \mathscr{L}^r) = 0$ for every i > 0, r > 0;

see [Kol13, 3.11, 3.13] and [GK14, 2.5] for proofs.

Let $D \subset X$ be an effective divisor with rational singularities such that $\mathscr{L} \simeq \mathscr{O}_X(D)$. Set $\mathscr{L}_D := \mathscr{L}|_D$. There is a natural morphism $C_a(D, \mathscr{L}_D) \to C_a(X, \mathscr{L})$ which is an embedding if and only if $H^0(X, \mathscr{L}^r) \to H^0(D, \mathscr{L}_D^r)$ is surjective for every $r \ge 0$, equivalently, if and only if $H^1(X, \mathscr{L}^r) \to H^1(X, \mathscr{L}^{r+1})$ is injective for every $r \ge 0$. As in [Kol13, 3.10], using Serre vanishing we get that

 $(2.1.4) \ C_a(D,\mathscr{L}_D) \text{ is a Cartier divisor of } C_a(X,\mathscr{L}) \Leftrightarrow H^1(X,\mathscr{L}^r) = 0 \text{ for every } r \geq 0.$

If this holds then $C_a(D, \mathscr{L}_D)$ has a deformation whose generic fiber is $X \setminus D$. So $C_a(D, \mathscr{L}_D)$ is smoothable if $X \setminus D$ is smooth.

By looking at the cohomology of the sequences

$$(2.1.5) 0 \to \mathscr{L}^{r-1} \to \mathscr{L}^r \to \mathscr{L}^r_D \to 0$$

we see that

(2.1.6) $C_a(D, \mathscr{L}_D)$ is Du Bois $\Leftrightarrow H^1(X, \mathscr{L}^r) \twoheadrightarrow H^1(X, \mathscr{L}^{r+1})$ is surjective for every $r \ge 0$ and $H^i(X, \mathscr{L}^r) = 0$ for every $i > 1, r \ge 0$.

Putting all these together we see that if $C_a(D, \mathscr{L}_D)$ is Du Bois and has a flat deformation to $X \setminus D$ then $v \in C_a(X, \mathscr{L})$ is a rational singularity. In particular, $C_a(D, \mathscr{L}_D)$ is Cohen-Macaulay.

This is actually stronger than Corollary 1.6, but here we also assumed that $X \setminus D$ has rational singularities. See also [KS16b] for closely related results.

By Corollary 1.6, if a local, Du Bois scheme (X, x) is smoothable, then it is Cohen-Macaulay. The next example shows that this is close to being optimal for some cones. **Example 2.2.** Let (S, H) be a polarized K3 surface and set $X := S \times \mathbb{P}^2$. Fix $a, b \ge 1$ and set $\mathscr{L}(a, b) := \pi_1^* \mathscr{O}_S(aH) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^2}(b)$ and let $D(a, b) \subset X$ be a smooth member of the associated linear system.

The affine cone $C_a(D(a, b), \mathscr{L}(a, b)|_{D(a, b)})$ is a hyperplane section of the cone $C_a(X, \mathscr{L}(a, b))$, hence smoothable. It is not Cohen-Macaulay since $H^2(D(a, b), \mathscr{O}_{D(a, b)}) = 1$ and also not Du Bois since $H^1(D(a, b), \mathscr{L}(a, b)|_{D(a, b)}) = 1$.

However, for any a' > a, b' > b the cone $C_a(D(a, b), \mathscr{L}(a', b')|_{D(a,b)})$ is DuBois but still not Cohen-Macaulay. Thus the cones $C_a(D(a, b), \mathscr{L}(a', b')|_{D(a,b)})$ are not smoothable.

More generally, one gets similar examples starting with any smooth variety X for which $H^1(X, \mathscr{O}_X) = 0$ but $H^i(X, \mathscr{O}_X) \neq 0$ for some $2 \leq i \leq \dim X - 2$.

Example 2.3 Singularities of cones I. Let X be a smooth, projective variety such that $K_X \equiv 0$. Kodaira vanishing and (2.1.2–3) show that $C_a(X, \mathscr{L})$ is DuBois. It is Cohen-Macaulay iff $H^i(X, \mathscr{O}_X) = 0$ for $0 < i < \dim X$. This and Corollary 1.6 imply that if $C_a(X, \mathscr{L})$ is smoothable then $H^i(X, \mathscr{O}_X) = 0$ for $0 < i < \dim X$. In particular, if X is an abelian variety then $C_a(X, \mathscr{L})$ is not smoothable. In Corollary 8.7 we prove that if X is an abelian variety, then $C_a(X, \mathscr{L})$ cannot be deformed even to an S_3 scheme.

2.B. All characteristics

Example 2.4 Singularities of cones II. Let us use the notation introduced in **Example 2.1** and first note that (2.1.1) and (2.1.2) remain true in all characteristics:

(2.4.1) $C_a(X, \mathscr{L})$ is rational $\Leftrightarrow H^i(X, \mathscr{L}^r) = 0$ for every $i > 0, r \ge 0$, and

(2.4.2) $C_a(X, \mathscr{L})$ is Cohen-Macaulay $\Leftrightarrow H^i(X, \mathscr{L}^r) = 0$ for every dim $X > i > 0, r \ge 0$.

For future reference we add a more sophisticated version of (2.1.2):

(2.4.3) $C_a(X, \mathscr{L})$ is S_n for some $n \in \mathbb{N} \Leftrightarrow H^i(X, \mathscr{L}^r) = 0$ for every $n - 1 > i > 0, r \ge 0$. The reader may find a proof, for instance, in [Pat13, 4.3], cf. [Kol13, 3.11].

2.C. Positive characteristic

For the definition of *F*-singularities appearing in this section, please refer to Section 7.

Example 2.5. Let k be a field of characteristic p > 0 and let Y be the curve coinsisting of the three coordinate axes in \mathbb{A}^3_k , i.e., let $Y = \operatorname{Spec} k[x, y, z]/(xy, xz, yz)$. Then Y is F-pure by [HR76, 5.38] and hence it is also F-anti-nilpotent by [Ma14].

A frequently used way to show that a class of singularities is invariant under small deformation is to show the following two conditions:

- (i) The class of singularities in question satisfies an *inversion of adjunction type* property, i.e., if a Cartier divisor $Y \subseteq X$ belongs to this class, then so does X.
- (ii) The class of singularities in question satisfies a *Bertini type* property, i.e., if X belongs to this class and $Y \subseteq X$ is a general member of a very ample linear system, then Y also belongs to this class.

It is easy to see that these two conditions imply that if in a flat family a fiber belongs to the given class of singularities, then so do nearby fibers.

This method indeed proves that \mathbb{Q} -Gorenstein *F*-pure singularities defined over an algebraically closed field are invariant under small deformation. Property (i), the inversion of

adjunction type property, for Gorenstein F-pure singularities holds by [Fed83, 3.4(2)] and property (ii), the Bertini type property, at least over an algebraically closed field holds by [SZ13]. The Q-Gorenstein case of (i) can be proved using [Sch09a, 7.2].

Similarly, *F*-injective singularities with liftable local cohomology satisfy (i) [HMS14].

Without the Q-Gorenstein assumption F-pure singularities do not satisfy (i) [Fed83, Sin99]. In contrast, F-anti-nilpotent singularities satisfy (i) [MQ17], but it is not known at the moment whether they satisfy (ii).

The fact that F-anti-nilpotent singularities satisfy (i), but F-pure singularities in general do not, leads to simple examples of F-anti-nilpotent singularities that are not F-pure:

Example 2.6. [Fed83, Sin99, QS17, MQ17] Let $X = \operatorname{Spec} k[x, y, z, t]/(xy, xz, y(z - t^2))$ and $Y = (t = 0) \subseteq X$. Then $Y \simeq \operatorname{Spec} k[x, y, z]/(xy, xz, yz)$ and hence it is *F*-pure by **Example 2.5.** Furthermore, then X is also *F*-anti-nilpotent by [MQ17, 4.2], but it is not *F*-pure by [Sin99, 3.2].

Example 2.7 Singularities of cones III. We have seen in Example 2.3 that in characteristic 0 a cone over an abelian variety has Du Bois singularities. We have a similar statement in positive characteristic: Let X be an ordinary abelian variety over a field of positive characteristic and \mathscr{L} an ample line bundle on X. Then $C_a(X, \mathscr{L})$ has F-pure singularities by [MS87, Lemma 1.1] and hence F-anti-nilpotent singularities by [Ma14].

3. FILTRATIONS ON MODULES OVER ARTINIAN LOCAL RINGS

We will use the following notation throughout.

3.1. MAXIMAL FILTRATIONS. Let (S, \mathfrak{m}, k) be an Artinian local ring and N a finite S-module with a filtration $N = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_q \supseteq N_{q+1} = 0$ such that $N_j / N_{j+1} \simeq k$ as S-modules for each $j = 0, \ldots, q$. Further let $f : (X, x) \to (\operatorname{Spec} S, \mathfrak{m})$ be a flat local morphism and denote the fiber of f over \mathfrak{m} by $X_{\mathfrak{m}}$. It follows that then for each $j = 0, \ldots, q$,

(3.1.1)
$$f^*\left(\frac{N_j}{N_{j+1}}\right) \simeq \mathscr{O}_{X_{\mathfrak{m}}}.$$

$$0 \longrightarrow I_j / I_{j+1} \longrightarrow S_{j+1} \longrightarrow S_j \longrightarrow 0,$$

we obtain another short exact sequence of S-modules:

(3.3.1)
$$0 \longrightarrow \omega_{S_j} \xrightarrow{\varsigma_j} \omega_{S_{j+1}} \longrightarrow \operatorname{Hom}_S(k, \omega_S) \simeq k \longrightarrow 0.$$

Therefore we obtain a filtration of $N = \omega_S$ by the submodules $N_j := \omega_{S_{q+1-j}}$ as in (3.1) where $q + 1 = \text{length}_S(S) = \text{length}_S(\omega_S)$. The composition of the embeddings in (3.3.1) will be denoted by $\varsigma := \varsigma_q \circ \cdots \circ \varsigma_1 : \omega_{S_1} \hookrightarrow \omega_{S_{q+1}} = \omega_S$.

Recall that the *socle* of a module M over a local ring (S, \mathfrak{m}, k) is

(3.3.2)
$$\operatorname{Soc} M := (0:\mathfrak{m})_M = \{x \in M \mid \mathfrak{m} \cdot x = 0\} \simeq \operatorname{Hom}_S(k, M).$$

Soc *M* is naturally a *k*-vector space and $\dim_k \operatorname{Soc} \omega_S = 1$ by the definition of the canonical module. In particular, $\operatorname{Soc} \omega_S \simeq k$ and this is the only *S*-submodule of ω_S isomorphic to *k*. Lemma 3.4. Using the notation from (3.2) and (3.3), we have that

(3.4.1)
$$\operatorname{im} \varsigma = \operatorname{Soc} \omega_S = I_q \omega_S.$$

Remark 3.4.2. Note that we are not simply stating that these modules in (3.4.1) are isomorphic, but that they are equal as submodules of ω_S .

Proof. Since $S_1 \simeq S/\mathfrak{m} \simeq k$, and hence $\omega_{S_1} \simeq k$, it follows that the image of the embedding $\varsigma: \omega_{S_1} \hookrightarrow \omega_S$ maps ω_{S_1} isomorphically onto Soc ω_S :

$$(3.4.3) im \varsigma = Soc \,\omega_S.$$

As ω_S is a dualizing sheaf, $I_q \omega_S \neq 0$, and since $I_q \simeq S/_{\mathfrak{m}}$ it follows that

$$0 \neq I_q \omega_S \subseteq (0:\mathfrak{m})_{\omega_S} = \operatorname{Soc} \omega_S \simeq k.$$

Since k is a simple S-module, this implies that $I_q \omega_s = \operatorname{Soc} \omega_s$ which proves (3.4.1).

4. FAMILIES OVER ARTINIAN LOCAL RINGS

We will frequently use the following notation.

Notation 4.1. Let A be a noetherian ring, (R, \mathfrak{m}) a noetherian local A-algebra, $I \subset R$ a nilpotent ideal and $(T, \mathfrak{n}) := (R/I, \mathfrak{m}/I)$ with natural morphism $\alpha : R \twoheadrightarrow T$.

Definition 4.2. Recall from **Definition 1.1** that we say that (T, \mathfrak{n}) has *liftable local cohomology* over A if for any (R, \mathfrak{m}) as in Notation 4.1, the induced homomorphism on local cohomology $H^i_{\mathfrak{m}}(R) \twoheadrightarrow H^i_{\mathfrak{n}}(T)$ is surjective for all *i*.

We extend this definition to schemes: Let (X, x) be a local scheme over a noetherian ring A. Then we say that (X, x) has *liftable local cohomology over* A if $\mathcal{O}_{X,x}$ has liftable local cohomology over A. If $f: X \to B$ is a morphism of schemes then we say that X has *liftable local cohomology over* B if (X, x) has liftable local cohomology over A for each $x \in X$ and for each Spec $A \subseteq B$ open affine neighbourhood of $f(x) \in B$.

Remark 4.3. A simple consequence of the definition is that if X has liftable local cohomology over a scheme Z, then for any morphism $g: X \to B$ of Z-schemes X has liftable local cohomology over B as well. In particular, if X has liftable local cohomology over a field k, then it has liftable local cohomology over any other k-scheme to which it admits a map. In addition if char k = 0, then X has liftable local cohomology by **Remark 1.2**.

Next we need a simple lemma regarding liftable local cohomology:

Lemma 4.4. Using Notation 4.1 let M be an R-module such that there exists a surjective morphism $M \twoheadrightarrow T$. Assume that the induced natural homomorphism $H^i_{\mathfrak{m}}(R) \twoheadrightarrow H^i_{\mathfrak{n}}(T)$ is surjective for some $i \in \mathbb{N}$. Then the induced homomorphism on local cohomology

(4.4.1)
$$H^{i}_{\mathfrak{m}}(M) \longrightarrow H^{i}_{\mathfrak{m}}(T) \simeq H^{i}_{\mathfrak{n}}(T)$$

is surjective for the same i. In particular, if (T, \mathfrak{n}) has liftable local cohomology over A, then the homomorphism in (4.4.1) is surjective for every $i \in \mathbb{N}$.

Proof. Let $t \in M$ be such that $\alpha(t) = 1 \in T$ and let $\beta : R \to M$ be defined by $1 \mapsto t$. Then $\alpha \circ \beta = \alpha : R \to T$ is the natural quotient morphism, hence the surjective morphism $H^i_{\mathfrak{m}}(R) \twoheadrightarrow H^i_{\mathfrak{m}}(T)$ factors through $H^i_{\mathfrak{m}}(M)$ which proves the statement. \Box

Proposition 4.5. Let (S, \mathfrak{m}, k) be an Artinian local ring and $f : (X, x) \to (\operatorname{Spec} S, \mathfrak{m})$ a flat local morphism. Let N be a finite S-module with a filtration as in (3.1) and assume that $(X_{\mathfrak{m}}, x)$ has liftable local cohomology over S. Then for each i, j, the natural sequence of morphisms induced by the embeddings $N_{j+1} \hookrightarrow N_j$ forms a short exact sequence,

$$0 \longrightarrow H^i_x(f^*N_{j+1}) \longrightarrow H^i_x(f^*N_j) \longrightarrow H^i_x\left(f^*\left(N_j \middle/ N_{j+1}\right)\right) \simeq H^i_x(\mathscr{O}_{X_{\mathfrak{m}}}) \longrightarrow 0.$$

Proof. Since $ann\left(\frac{N_j}{N_{j+1}}\right) = \mathfrak{m}$, there is a natural surjective morphism

$$f^*N_j \otimes \mathscr{O}_{X_{\mathfrak{m}}} \twoheadrightarrow f^*\left(N_j \middle/ N_{j+1}\right)$$

By Lemma 4.4 and (3.1.1), the natural homomorphism

(4.5.1)
$$H_x^i(f^*N_j) \longrightarrow H_x^i\left(f^*\left(\frac{N_j}{N_{j+1}}\right)\right) \simeq H_x^i(\mathscr{O}_{X_{\mathfrak{m}}})$$

is surjective for all *i*. Since *f* is flat, we have a short exact sequence for every j > 0:

$$0 \longrightarrow f^* N_{j+1} \longrightarrow f^* N_j \longrightarrow f^* \left(\frac{N_j}{N_{j+1}} \right) \longrightarrow 0,$$

and hence the statement follows from (4.5.1).

4.6. THE EXCEPTIONAL INVERSE IMAGE OF THE STRUCTURE SHEAVES. Let (S, \mathfrak{m}, k) be an Artinian local ring with a filtration by ideals as in (3.2). Further let $f: X \to \operatorname{Spec} S$ be a flat morphism that is essentially of finite type and $f_j = f|_{X_j}: X_j := X \times_{\operatorname{Spec} S} \operatorname{Spec} S_j \to \operatorname{Spec} S_j$ where $S_j = S/I_j$ as defined in (3.2), e.g., $X_{q+1} = X$ and $X_1 = X_{\mathfrak{m}}$, the fiber of f over the closed point of S. By a slight abuse of notation we will denote $\omega_{\operatorname{Spec} S}$ with ω_S as well, but it will be clear from the context which one is meant at any given time.

Using the description of the exceptional inverse image functor via the residual/dualizing complexes [Con00, (3.3.6)] (cf.[R&D, 3.4(a)], [StacksProject, Tag 0E9L]):

(4.6.1)
$$f^{!} = \mathcal{RHom}_{X}(\mathcal{L}f^{*}\mathcal{RHom}_{S}(\underline{\quad},\omega_{S}^{\bullet}),\omega_{X}^{\bullet})$$

and the facts that S is Artinian and f is flat, we have that

$$\omega_{X_j/S_j}^{\bullet} \simeq f_j^! \mathscr{O}_{\operatorname{Spec} S_j} \simeq \mathscr{RHom}_{X_j}(f_j^* \omega_{S_j}, \omega_{X_j}^{\bullet}).$$

By Grothendieck duality

$$\mathcal{RHom}_{X_j}(f_j^*\omega_{S_j},\omega_{X_j}^{\bullet})\simeq \mathcal{RHom}_X(f_j^*\omega_{S_j},\omega_X^{\bullet}),$$

and as $f_j^* \omega_{S_j} = f^* \omega_{S_j}$ and $\omega_{S_j} \simeq \operatorname{Hom}_S(S_j, \omega_S) \simeq \mathcal{RHom}_S(\mathscr{O}_{\operatorname{Spec} S_j}, \omega_S^{\bullet})$ we obtain that (4.6.2) $\omega_{X_j/S_j}^{\bullet} \simeq \mathcal{RHom}_X(f^* \omega_{S_j}, \omega_X^{\bullet}) \simeq f^! \mathscr{O}_{\operatorname{Spec} S_j},$

in particular, that

(4.6.3)
$$\omega_{X_{\mathfrak{m}}}^{\bullet} \simeq f^{!}k \simeq \mathcal{RHom}_{X}(f^{*}\operatorname{Hom}_{S}(k,\omega_{S}),\omega_{X}^{\bullet}) \simeq \mathcal{RHom}_{X}(\mathscr{O}_{X_{\mathfrak{m}}},\omega_{X}^{\bullet}).$$

4.7. NATURAL MORPHISMS OF DUALIZING COMPLEXES. We will continue using the notation from (4.6). Applying $f^!$ to the natural surjective morphism $S_{j+1} \longrightarrow S_j$ gives a natural morphism

(4.7.1)
$$\varrho_j: \omega_{X_{j+1}/S_{j+1}}^{\bullet} \longrightarrow \omega_{X_j/S_j}^{\bullet}.$$

Notice that ρ_j is Grothendieck dual to $f^*\varsigma_j$ defined in (3.2). Indeed, ς_j is obtained by applying $\operatorname{Hom}_S(\underline{\ }, \omega_S)$ to the morphism $S_{j+1} \longrightarrow S_j$, and then ρ_j is obtained by applying f^* and then $\mathcal{RHom}_X(\underline{\ }, \omega_X^{\bullet})$. Notice further that $h^{-i}(\rho_j)$ factors through the natural base change morphism of Proposition 5.3 for each $i \in \mathbb{Z}$.

The composition of the surjective morphisms $S_{j+1} \longrightarrow S_j$ for all j is the natural surjective morphism $S \longrightarrow S/\mathfrak{m} \simeq k$, and hence the composition of the ϱ_j 's gives the natural morphism

(4.7.2)
$$\varrho := \varrho_1 \circ \cdots \circ \varrho_q : \omega_{X/S}^{\bullet} \longrightarrow \omega_{X_1/S_1}^{\bullet} = \omega_{X_{\mathfrak{m}}}^{\bullet},$$

which is then Grothendieck dual to $f^*\varsigma := f^*(\varsigma_q \circ \cdots \circ \varsigma_1)$ and $h^{-i}(\varrho_j)$ factors through the natural base change morphism of **Proposition 5.3** for each $i \in \mathbb{Z}$.

In the rest of this section we will use the following notation and assumptions.

Assumptions 4.8. Let (S, \mathfrak{m}, k) be an Artinian local ring and $f : (X, x) \to (\operatorname{Spec} S, \mathfrak{m})$ a flat local morphism that is essentially of finite type. Assume that $(X_{\mathfrak{m}}, x)$, where $X_{\mathfrak{m}}$ is the fiber of f over the closed point of $\operatorname{Spec} S$, has liftable local cohomology over S. Note that by definition $x \in X_{\mathfrak{m}}$ and that we will keep using the notation introduced in (4.7.1) and (4.7.2). Theorem 4.9. For each $i, j \in \mathbb{N}$,

- (i) the natural morphism $h^{-i}(\varrho_j) : h^{-i}(\omega_{X_{j+1}/S_{j+1}}^{\bullet}) \longrightarrow h^{-i}(\omega_{X_j/S_j}^{\bullet})$ is surjective,
- (ii) the natural morphism $\mathbf{h}^{-i}(\varrho) : \mathbf{h}^{-i}(\omega_{X/S}^{\bullet}) \longrightarrow \mathbf{h}^{-i}(\omega_{X_{\mathfrak{m}}}^{\bullet})$ is surjective,

(iii) the natural morphisms induced by ϱ_i form a short exact sequence,

$$0 \longrightarrow \mathsf{h}^{-i}(\omega_{X_{\mathfrak{m}}}^{\bullet}) \longrightarrow \mathsf{h}^{-i}(\omega_{X_{j+1}/S_{j+1}}^{\bullet}) \xrightarrow{\mathsf{h}^{-i}(\varrho_j)} \mathsf{h}^{-i}(\omega_{X_j/S_j}^{\bullet}) \longrightarrow 0.$$

(iv) ker $\mathsf{h}^{-i}(\varrho_j) = I_j \mathsf{h}^{-i}(\omega_{X_{j+1}/S_{j+1}}^{\bullet}) \simeq I_j \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}) / I_{j+1} \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}),$

(v)
$$\mathsf{h}^{-i}(\omega_{X_j/S_j}^{\bullet}) \simeq \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}) / I_j \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}) \simeq \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}) \otimes_{\mathscr{O}_X} \mathscr{O}_{X_j}$$
, and

(vi) ker $\mathsf{h}^{-i}(\varrho) = \mathfrak{m}\mathsf{h}^{-i}(\omega_{X/S}^{\bullet}).$

Proof. Let $N = \omega_S$ and consider the filtration on N given by $\omega_{S_j} = N_{q+1-j}$, cf. (3.3), (3.3.1). Further let ()[^] denote the completion at x (the closed point of X). Then by Proposition 4.5, for all $i, j \in \mathbb{N}$, there exists a short exact sequence

$$(4.9.1) \qquad 0 \longrightarrow H^i_x(f^*\omega_{S_j}) \longrightarrow H^i_x(f^*\omega_{S_{j+1}}) \longrightarrow H^i_x\left(f^*\left(\omega_{S_{j+1}}/\omega_{S_j}\right)\right) \longrightarrow 0.$$

Applying local duality [R&D, Corollary V.6.5] to (4.9.1) gives the short exact sequence

$$0 \longrightarrow \mathcal{E} \operatorname{xt}_X^{-i} \left(f^* \left(\omega_{S_{j+1}} / \omega_{S_j} \right), \omega_X^* \right)^{\widehat{}} \longrightarrow \mathcal{E} \operatorname{xt}_X^{-i} (f^* \omega_{S_{j+1}}, \omega_X^*)^{\widehat{}} \longrightarrow \mathcal{E} \operatorname{xt}_X^{-i} (f^* \omega_{S_j}, \omega_X^*)^{\widehat{}} \longrightarrow 0.$$

for all $i, j \in \mathbb{N}$. Since completion is faithfully flat [StacksProject, Tag 00MC], this implies that there are short exact sequences

$$(4.9.2) \qquad \begin{array}{c} 0 \longrightarrow \mathcal{E} \chi t_X^{-i} \left(f^* \left(\omega_{S_{j+1}} \middle/ \omega_{S_j} \right), \omega_X^{\bullet} \right) \longrightarrow \\ \longrightarrow \mathcal{E} \chi t_X^{-i} \left(f^* \omega_{S_{j+1}}, \omega_X^{\bullet} \right) \longrightarrow \mathcal{E} \chi t_X^{-i} \left(f^* \omega_{S_j}, \omega_X^{\bullet} \right) \longrightarrow 0. \end{array}$$

Recall that $\mathcal{E}_{\mathcal{X}} t_X^{-i} \left(f^* \omega_{S_j}, \omega_X^{\bullet} \right) \simeq \mathsf{h}^{-i} (\omega_{X_j/S_j}^{\bullet})$ for each i, j, by (4.6.2). Further observe that the surjective morphism in (4.9.2) is the $-i^{\text{th}}$ cohomology sheaf of the Grothendieck dual of $f^*\varsigma_j$ and hence via the above isomorphisms, it corresponds to $\mathsf{h}^{-i}(\varrho_j)$. Therefore (4.9.2) implies (i). By (3.3.1) $f^* \left(\omega_{S_{j+1}} / \omega_{S_j} \right) \simeq \mathscr{O}_{X_{\mathfrak{m}}}$, and hence $\mathcal{E}_{\mathcal{X}} t_X^{-i} \left(f^* \left(\omega_{S_{j+1}} / \omega_{S_j} \right), \omega_X^{\bullet} \right) \simeq \mathsf{h}^{-i}(\omega_{X_{\mathfrak{m}}}^{\bullet})$, so (4.9.2) also implies (iii). Composing the surjective morphisms in (4.9.2) for all j implies that the natural morphism

$$\mathsf{h}^{-i}(\omega_{X/S}^{\bullet}) \simeq \mathcal{E} \mathsf{X} t_X^{-i}\left(f^* \omega_S, \omega_X^{\bullet}\right) \xrightarrow{\mathsf{h}^{-i}(\varrho)} \mathcal{E} \mathsf{X} t_X^{-i}\left(f^* \omega_{S_q}, \omega_X^{\bullet}\right) \simeq \mathsf{h}^{-i}(\omega_{X_{\mathfrak{m}}}^{\bullet})$$

is surjective and hence (ii) follows as well.

Similarly, composing the injective maps in (4.9.1) for all j shows that the embedding $\varsigma : \omega_{S_1} \hookrightarrow \omega_S$ induces an embedding on local cohomology:

(4.9.3)
$$H_x^i(f^*\omega_{S_1}) \subseteq H_x^i(f^*\omega_S)$$

Next we prove (iv) for j = q first. Since $h^{-i}(\omega_{X_q/S_q}^{\bullet})$ is supported on X_q it follows that

$$I_q \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}) \subseteq K := \ker \mathsf{h}^{-i}(\varrho_q)$$

Recall from (3.2) that there exists a $t_q \in I_q$ such that $I_q = St_q \simeq S/\mathfrak{m}$ and from Lemma 3.4 that $I_q \omega_S = \operatorname{Soc} \omega_S$. It follows that for j = q the short exact sequence of (3.3.1) takes the form

$$(4.9.4) 0 \longrightarrow \omega_{S_q} \longrightarrow \omega_S \xrightarrow{\tau} \operatorname{Soc} \omega_S \longrightarrow 0,$$

where $\tau : \omega_S \to \operatorname{Soc} \omega_S \subset \omega_S$ may be identified with multiplication by t_q on ω_S . Applying f^* and taking local cohomology we obtain the short exact sequence

$$(4.9.5) \qquad 0 \longrightarrow H^i_x(f^*\omega_{S_q}) \longrightarrow H^i_x(f^*\omega_S) \xrightarrow{H^i_x(\tau)} H^i_x(f^*\operatorname{Soc}\omega_S) \longrightarrow 0,$$

which is of course just (4.9.1) for j = q. Clearly, the morphism $H_x^i(\tau)$ may also be identified with multiplication by t_q on $H_x^i(f^*\omega_S)$. By Lemma 3.4 and (4.9.3), the natural morphism $H_x^i(\varsigma) : H_x^i(f^* \operatorname{Soc} \omega_S) = H_x^i(I_q f^*\omega_S) = H_x^i(f^*\omega_{S_1}) \to H_x^i(f^*\omega_S)$ is injective. Since $H_x^i(\tau)$, i.e., multiplication by t_q on $H_x^i(f^*\omega_S)$, is surjective onto $H_x^i(f^* \operatorname{Soc} \omega_S)$, it follows that

(4.9.6)
$$H_x^i(f^*\operatorname{Soc}\omega_S) \xrightarrow{\simeq}_{H_x^i(\varsigma)} \operatorname{im} H_x^i(\varsigma) = I_q H_x^i(f^*\omega_S) \xrightarrow{\leftarrow} H_x^i(f^*\omega_S)$$

i.e., $H_x^i(f^* \operatorname{Soc} \omega_S)$ coincides with $I_q H_x^i(f^* \omega_S)$ as submodules of $H_x^i(f^* \omega_S)$. Next let E be an injective hull of $\kappa(x) = \mathscr{O}_{X,x}/\mathfrak{m}_{X,x}$ and consider a morphism $\phi : H_x^i(f^* \operatorname{Soc} \omega_S) \to E$. As E is injective, ϕ extends to a morphism $\widetilde{\phi} : H_x^i(f^* \omega_S) \to E$. If $a \in H_x^i(f^* \omega_S)$, then $t_q a \in I_q H_x^i(f^* \omega_S) = H_x^i(f^* \operatorname{Soc} \omega_S)$, so

$$t_q \phi(a) = \phi(t_q a) = \phi(t_q a) = \left(\phi \circ H_x^i(\tau)\right)(a)$$

Therefore, $\phi \circ H_x^i(\tau) = t_q \widetilde{\phi}$. Similarly, if $\psi : H_x^i(f^*\omega_S) \to E$ is an arbitrary morphism, then setting $\phi = \psi|_{H_x^i(f^* \operatorname{Soc} \omega_S)} : H_x^i(f^* \operatorname{Soc} \omega_S) \to E$ and applying the same computation as above, with $\widetilde{\phi}$ replaced by ψ , shows that $\phi \circ H_x^i(\tau) = t_q \psi$. It follows that the embedding induced by $H_x^i(\tau)$,

(4.9.7)
$$\alpha : \operatorname{Hom}_{\mathscr{O}_{X,x}}(H^i_x(f^*\operatorname{Soc}\omega_S), E) \hookrightarrow \operatorname{Hom}_{\mathscr{O}_{X,x}}(H^i_x(f^*\omega_S), E)$$

identifies $\operatorname{Hom}_X(H^i_x(f^*\operatorname{Soc}\omega_S), E)$ with $I_q \operatorname{Hom}_X(H^i_x(f^*\omega_S), E)$. By local duality this implies that

$$\left(\ker\left[\mathsf{h}^{-i}(\varrho_q):\mathsf{h}^{-i}(\omega_{X/S}^{\bullet})\twoheadrightarrow\mathsf{h}^{-i}(\omega_{X_q/S_q}^{\bullet})\right]\Big/I_q\mathsf{h}^{-i}(\omega_{X/S}^{\bullet})\right)\otimes\widehat{\mathscr{O}}_{X,x}=0$$

and hence, since completion is faithfully flat, this implies (iv) in the case j = q. Running through the same argument with S replaced by S_{j+1} gives the equality in (iv) for all j. In addition, (iv) for j = q also implies (v) for $j \ge q$. Assuming that (v) holds for j = r + 1implies the isomorphism in (iv) for j = r. In turn, the entire (iv) for j = r, combined with (v) for j = r+1, implies (v) for j = r. Therefore, (iv) and (v) follow by descending induction on j and then (vi) follows from (iv) and the definition of ρ .

Next we need a simple lemma.

Lemma 4.10. Let R be a ring. M an R-module, $t \in R$ and $J = (t) \subseteq R$. Assume that $(0:J)_M = (0:J)_R \cdot M$. Then the natural morphism $J \otimes_R M \xrightarrow{\simeq} JM$ is an isomorphism.

Proof. This natural morphism is always surjective. Suppose $m \in M$ is such that $t \otimes m \mapsto 0$ via this morphism. In other words such that tm = 0. This means, by definition, that $m \in (0:J)_M$ and hence by assumption there exist $y \in (0:J)_R \subseteq R$ and $m' \in M$ such that m = ym'. Then $t \otimes m = t \otimes ym' = yt \otimes m' = 0$, since yt = 0. This proves the claim. \Box

Proposition 4.11. Using the same notation as above,

(i)
$$I_j \otimes \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}) \simeq I_j \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}),$$

(ii) for any
$$l \in \mathbb{N}$$
, $I_j / I_{j+l} \otimes \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}) \simeq I_j \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}) / I_{j+l} \mathsf{h}^{-i}(\omega_{X/S}^{\bullet})$, and

(iii) for any
$$l \in \mathbb{N}$$
, $\mathfrak{m}^l / \mathfrak{m}^{l+1} \otimes \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}) \simeq \mathfrak{m}^l \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}) / \mathfrak{m}^{l+1} \mathsf{h}^{-i}(\omega_{X/S}^{\bullet})$

Proof. Notice that since $H_x^i(f^* \operatorname{Soc} \omega_S)$ is both a quotient and a submodule of $H_x^i(f^* \omega_S)$, there are two natural maps between $\operatorname{Hom}_{\mathscr{O}_{X,x}}(H_x^i(f^* \operatorname{Soc} \omega_S), E)$ and $\operatorname{Hom}_{\mathscr{O}_{X,x}}(H_x^i(f^* \omega_S), E)$. Regarding $H_x^i(f^* \operatorname{Soc} \omega_S)$ a quotient module via $H_x^i(\tau)$ we get the embedding $\alpha = (\underline{}) \circ H_x^i(\tau)$ in (4.9.7), and considering it a submodule the restriction map

$$\beta : \operatorname{Hom}_{\mathscr{O}_{X,x}}(H^i_x(f^*\omega_S), E) \longrightarrow \operatorname{Hom}_{\mathscr{O}_{X,x}}(H^i_x(f^*\operatorname{Soc}\omega_S), E).$$
$$\phi \longmapsto \phi|_{H^i_x(f^*\operatorname{Soc}\omega_S)}$$

These maps are of course not inverses to each other. In fact, we have already established (cf. (4.9.7)) that $\phi_{|_{H^i_x(f^* \operatorname{Soc} \omega_S)}} \circ H^i_x(\tau) = t_q \phi$ and hence the composition $\alpha \circ \beta$ is just multiplication by t_q :

$$(4.11.1) \qquad \qquad \phi \in \operatorname{Hom}_{\mathscr{O}_{X,x}}(H^{i}_{x}(f^{*}\omega_{S}), E) \xrightarrow{\beta} \operatorname{Hom}_{\mathscr{O}_{X,x}}(H^{i}_{x}(f^{*}\operatorname{Soc}\omega_{S}), E) \xrightarrow{\alpha \circ \beta} \simeq \downarrow \alpha \xrightarrow{\gamma} t_{q} \phi \in I_{q} \operatorname{Hom}_{\mathscr{O}_{X,x}}(H^{i}_{x}(f^{*}\omega_{S}), E).$$

This implies, (cf. (4.9.3) and (4.9.6)), that $h^{-i}(\varrho)$ may be identified with multiplication by t_q on $h^{-i}(\omega_{X/S}^{\bullet})$. Together with Theorem 4.9(vi) this implies that

$$(0:I_q)_{\mathsf{h}^{-i}(\omega_{X/S}^{\bullet})} = \ker \mathsf{h}^{-i}(\varrho) = \mathfrak{m}\mathsf{h}^{-i}(\omega_{X/S}^{\bullet}) = (0:I_q)_S \cdot \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}),$$

and hence the natural morphism

(4.11.2)
$$I_q \otimes_S \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}) \xrightarrow{\simeq} I_q \mathsf{h}^{-i}(\omega_{X/S}^{\bullet})$$

is an isomorphism by Lemma 4.10. Now assume, by induction, that (i) holds for S_q in place of S. In particular, keeping in mind that $S_q = S/I_q$, the natural map

(4.11.3)
$$I_j / I_q \otimes_{S_q} \mathsf{h}^{-i}(\omega_{X_q/S_q}^{\bullet}) \xrightarrow{\simeq} \left(I_j / I_q \right) \mathsf{h}^{-i}(\omega_{X_q/S_q}^{\bullet})$$

is an isomorphism for all j. Consider the short exact sequence (cf. Theorem 4.9(v)),

$$0 \longrightarrow I_q \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}) \longrightarrow \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}) \longrightarrow \mathsf{h}^{-i}(\omega_{X_q/S_q}^{\bullet}) \longrightarrow 0$$

and apply $I_j/I_q \otimes_S (_)$. The image of $I_j/I_q \otimes_S I_q h^{-i}(\omega_{X/S}^{\bullet})$ in $I_j/I_q \otimes_S h^{-i}(\omega_{X/S}^{\bullet})$ is 0 and hence by (4.11.3) the natural map

$$\underbrace{I_j / I_q \otimes_S \mathsf{h}^{-i}(\omega_{X/S}^{\bullet})}_{\simeq} \simeq \underbrace{I_j / I_q \otimes_{S_q} \mathsf{h}^{-i}(\omega_{X_q/S_q}^{\bullet}) \xrightarrow{\simeq} \left(I_j / I_q \right) \mathsf{h}^{-i}(\omega_{X_q/S_q}^{\bullet}) \simeq}_{\simeq \left(I_j / I_q \right) \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}) / I_q \mathsf{h}^{-i}(\omega_{X/S}^{\bullet})} \simeq \underbrace{I_j \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}) / I_q \mathsf{h}^{-i}(\omega_{X/S}^{\bullet})}_{= I_j \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}) / I_q \mathsf{h}^{-i}(\omega_{X/S}^{\bullet})} \simeq \underbrace{I_j \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}) / I_q \mathsf{h}^{-i}(\omega_{X/S}^{\bullet})}_{= I_j \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}) / I_q \mathsf{h}^{-i}(\omega_{X/S}^{\bullet})} \simeq \underbrace{I_j \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}) / I_q \mathsf{h}^{-i}(\omega_{X/S}^{\bullet})}_{= I_j \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}) / I_q \mathsf{h}^{-i}(\omega_{X/S}^{\bullet})} \simeq \underbrace{I_j \mathsf{h}^{-i}(\omega_{X/S}^{\bullet}) / I_q \mathsf{h}^{-i}(\omega_{X/S}^{\bullet})}_{= I_j \mathsf{h}^{-i}(\omega_{X/S}^{\bullet})}$$

is an isomorphism. This, combined with (4.11.2) and the 5-lemma, implies (i). Then (ii) is a direct consequence of (i) and the fact that tensor product is right exact.

Finally, recall, that the choice of filtration in (3.2) was fairly unrestricted. In particular, we may assume that the filtration I_{\bullet} of S is chosen so that for all $l \in \mathbb{N}$, there exists a j(l) such that $I_{j(l)} = \mathfrak{m}^{l}$. Applying (ii) for this filtration implies (iii).

The following theorem is an easy combination of the results of this section.

Theorem 4.12. Let (S, \mathfrak{m}, k) be an Artinian local ring and $f : (X, x) \to \operatorname{Spec} S$ a flat local morphism that is essentially of finite type. If $(X_{\mathfrak{m}}, x)$ has liftable local cohomology over S, then $h^{-i}(\omega_{X/S}^{\bullet})$ is flat over $\operatorname{Spec} S$ for each i.

Proof. This follows directly from Proposition 4.11(iii) and [StacksProject, Tag 0AS8].

5. FLATNESS AND BASE CHANGE

In this section we prove a rather general flatness and base change theorem for the cohomology sheaves of the relative dualizing complex. The main essential assumption is that the relative dualizing complex exists.

Definitions and notation 5.1. For morphisms $f: X \to B$ and $\vartheta: Z \to B$, the symbol X_Z will denote $X \times_B Z$ and $f_Z: X_Z \to Z$ the induced morphism. In particular, for $b \in B$ we write $X_b = f^{-1}(b)$.

Let $f: X \to B$ be a morphism of locally noetherian schemes. Then f is *embeddable into* a smooth morphism of dimension N if there exists a smooth morphism $\pi: P \to B$ of pure relative dimension N over B and a closed embedding $j: X \hookrightarrow P$ such that $f = \pi \circ j$. Furthermore, f is *locally embeddable into a smooth morphism* if each $x \in X$ has a neighbourhood $x \in U_x \subseteq X$ such that $f|_{U_x}$ is embeddable into a smooth morphism of dimension N for some $N \in \mathbb{N}$.

Note that if $f: X \to B$ is a flat morphism that is essentially of finite type then it is locally embeddable into a smooth morphism and that if f is flat and locally embeddable into a smooth morphism then it admits a relative dualizing complex by [StacksProject, Tag 0E2X]. Lemma 5.2. Let (B, b) be a local scheme and $f: X \to B$ a flat morphism embeddable into a smooth morphism $P \to B$ of relative dimension N. Then

$$\mathsf{h}^{-i}(\omega_{X/B}^{\bullet}) \simeq \mathfrak{E}_{\mathcal{X}} t_P^{N-i}(\mathscr{O}_X, \omega_{P/B}) \quad and \quad \mathsf{h}^{-i}(\omega_{X_b}^{\bullet}) \simeq \mathfrak{E}_{\mathcal{X}} t_{P_b}^{N-i}(\mathscr{O}_{X_b}, \omega_{P_b}).$$

Proof. Since P/B is an N-dimensional smooth morphism, $\omega_{P/B}^{\bullet} = \omega_{P/B}[N]$ is a relative dualizing complex. By Grothendieck duality [R&D, VII.3.4],

$$\mathsf{h}^{-i}(\omega_{X/B}^{\bullet}) \simeq \mathsf{h}^{-i}(\mathcal{RHom}_{P}(\mathcal{O}_{X}, \omega_{P/B}[N])) \simeq \mathcal{Ext}_{P}^{N-i}(\mathcal{O}_{X}, \omega_{P/B}).$$

The same argument implies the equivalent statement for $h^{-i}(\omega_{X_h}^{\bullet})$.

The following statement is standard. We include it for ease of reference.

Proposition 5.3. Let $f : X \to B$ be a flat morphism of schemes that admits a relative dualizing complex and let $Z \to B$ be a morphism. Then for each $i \in \mathbb{Z}$ there exists a natural base change morphism,

$$\varrho_Z^{-i}:\mathsf{h}^{-i}(\omega_{X/B}^{\bullet})\otimes_B\mathscr{O}_Z\longrightarrow\mathsf{h}^{-i}(\omega_{X_Z/Z}^{\bullet})$$

Proof. For any complex A[•], tensoring with an object induces a natural morphism,

$$h^i(A^{\bullet}) \otimes M \longrightarrow h^i(A^{\bullet} \overset{L}{\otimes} M).$$

Applying this to the dualizing complex gives a natural map

$$\varrho_Z^{-i}:\mathsf{h}^{-i}(\omega_{X/B}^{\bullet})\otimes_B\mathscr{O}_Z\longrightarrow\mathsf{h}^{-i}\big(\omega_{X/B}^{\bullet}\overset{L}{\otimes}_B\mathscr{O}_Z\big).$$

But $\omega_{X/B}^{\bullet} \otimes_B \mathscr{O}_Z \simeq \omega_{X_Z/Z}^{\bullet}$ by the base change property of dualizing complexes [StacksProject, Tag 0E2Y], so the statement follows.

Terminology 5.4. Let $f: X \to B$ be a flat morphism of schemes and $\vartheta: Z \to B$ a morphism. Then for an $i \in \mathbb{Z}$, we will say that $h^{-i}(\omega_{X/B}^{\bullet})$ commutes with base change to Z if the natural base change morphism ϱ_Z^{-i} of **Proposition 5.3** is an isomorphism.

Remark 5.5. Since the base change morphism is defined naturally, it can be checked locally whether it is an isomorphism. In other words, if $h^{-i}(\omega_{X/B}^{\bullet})$ commutes with base change to Z locally on X, then it commutes with base change to Z.

Remark 5.6. A simple case when the condition in (5.4) holds is if $f: X \to B$ has Cohen-Macaulay fibers. In that case the only non-zero cohomology sheaf of the relative dualizing complex is $h^{-m}(\omega_{X/B}^{\bullet}) \simeq \omega_{X/B}$ where $m = \dim X - \dim B$ by [Con00, 3.5.1] and it commutes with base change by [Con00, 3.6.1]. In moduli theory typically one has to deal with non-Cohen-Macaulay fibers. The next example shows that for these not even $\omega_{X/B}$ commutes with base change. However, we see in Lemma 5.9 that the $h^{-i}(\omega_{X/B}^{\bullet})$ commute with inverse limits.

Example 5.7. Let Y be a normal quasi-projective threefold with isolated singularities and a trivial canonical divisor. Assume that Y is S_2 , but not S_3 . For instance, a non-Cohen-Macaulay normal threefold such as a cone over an abelian surface in characteristic 0 satisfies these conditions cf. Example 2.3. Consider a general projection of Y to a line and resolve the indeterminacies of the projection map. Let X denote the blow-up of Y on which this rational map becomes a morphism and let $\pi : X \to \mathbb{A}^1$ denote the resulting morphism. Note that since the projection was general we may assume that the birational morphism $X \to Y$ is locally isomorphic near their singular points. In particular, we may assume that X is a normal affine threefold with isolated singularities and a trivial canonical divisor, which is S_2 , but not S_3 . Observe that then $h^{-2}(\omega_{X/\mathbb{A}^1}^{\bullet}) \simeq \omega_{X/\mathbb{A}^1} \simeq \mathcal{O}_X$ by construction. Next let $z \in \mathbb{A}^1$ be the image of a non- S_3 point of X. Then X_z and hence \mathcal{O}_{X_z} is not S_2 , since otherwise X would be S_3 along X_z . At the same time $h^{-2}(\omega_{X_z/\{z\}}^{\bullet}) \simeq \omega_{X_z}$ is an S_2 sheaf (cf.[KM98, 5.69], [Kov17, 3.7.5]) and hence cannot be isomorphic to \mathcal{O}_{X_z} . This implies that $h^{-2}(\omega_{X/\mathbb{A}^1}^{\bullet})$ does not commute with base change for the morphism $\pi : X \to \mathbb{A}^1$.

Notation 5.8. Let $f : (X, x) \to (B, b) = (\operatorname{Spec} S, \mathfrak{m})$ be a local morphism. Let $q \in \mathbb{N}$, $S_q := S/\mathfrak{m}^q, \mathfrak{m}_q = \mathfrak{m}/\mathfrak{m}^q$ its (unique) maximal ideal, $B_q = \operatorname{Spec} S_q, X_q := X \times_B B_q$, and $f_q : (X_q, x) \to (B_q, b)$ the induced local morphism. Further let $\widehat{B} := \operatorname{Spec}(\lim_{\leftarrow} S_q)$, the completion of B at b and $\widehat{X} := X \times_B \widehat{B}$.

Lemma 5.9. Let $f: (X, x) \to (B, b)$ be a flat local morphism that admits a relative dualizing complex. Fix an $i \in \mathbb{Z}$ and assume that the inverse system $\left(\mathsf{h}^{-i-1}(\omega_{X_q/B_q}^{\bullet})\right)$ satisfies the Mittag-Leffler condition [StacksProject, Tag 0595]. Then the natural base change morphism

(cf. Proposition 5.3) induces an isomorphism:

$$\lim_{\longleftarrow} \left(\mathsf{h}^{-i}(\omega_{X/B}^{\bullet}) \otimes_X \mathscr{O}_{X_q} \right) \xrightarrow{\simeq} \lim_{\longleftarrow} \mathsf{h}^{-i}(\omega_{X_q/B_q}^{\bullet}),$$

Remark 5.10. If the local scheme (X_b, x) has liftable local cohomology over B, then the inverse system $\left(\mathsf{h}^{-i-1}(\omega_{X_q/B_q}^{\bullet})\right)$ satisfies the Mittag-Leffler condition by **Theorem 4.9**(i).

Proof. By the base change property of dualizing complexes [StacksProject, Tag 0E2Y] there exist natural restricting morphisms,

$$\omega_{X_{q+1}/B_{q+1}}^{\bullet} \longrightarrow \omega_{X_{q+1}/B_{q+1}}^{\bullet} \overset{L}{\otimes}_{X_{q+1}} \mathscr{O}_{X_q} \simeq \omega_{X_q/B_q}^{\bullet},$$

so $(\omega_{X_q/B_q}^{\boldsymbol{\cdot}})$ forms an inverse system in $D^b(X)$ and hence $\mathcal{R}\lim\omega_{X_q/B_q}^{\boldsymbol{\cdot}}$, the derived limit of the inverse system $(\omega_{X_q/B_q}^{\boldsymbol{\cdot}})$, exists [StacksProject, Tag 0CQD]. Since the inverse system $(\mathsf{h}^{-i-1}(\omega_{X_q/B_q}^{\boldsymbol{\cdot}}))$ satisfies the Mittag-Leffler condition, $\mathcal{R}^1 \lim \mathsf{h}^{-i-1}(\omega_{X_q/B_q}^{\boldsymbol{\cdot}}) = 0$ by [StacksProject, Tag 091D(3)]. Combined with [StacksProject, Tag 0CQE] this implies that the natural base change morphism of Proposition 5.3 induces an isomorphism

$$\mathsf{h}^{-i}(\mathcal{R}\lim \omega^{\bullet}_{X_q/B_q}) \xrightarrow{\simeq} \lim_{\longleftarrow} \mathsf{h}^{-i}(\omega^{\bullet}_{X_q/B_q}).$$

The base change property of dualizing complexes also applies to $\omega_{X/B}^{\bullet}$ and hence the natural restricting morphisms induce isomorphisms,

$$\omega_{X/B}^{\bullet} \overset{L}{\otimes}_{X} \mathscr{O}_{X_{q}} \simeq \omega_{X_{q}/B_{q}}^{\bullet}.$$

Then the derived completion of $\omega_{X/B}^{\bullet}$ with respect to the ideal $\mathscr{J} := f^* \mathfrak{m}_{B,b} = \mathscr{I}_{X_b \subseteq X} \subseteq \mathscr{O}_{X,x}$ [StacksProject, Tag 0BKH] is isomorphic to $\mathscr{R} \lim \omega_{X_q/B_q}^{\bullet}$ constructed above. Then the statement follows by [StacksProject, Tag 0A06].

Remark 5.11. In the proof above it is important to consider the derived limit $\mathscr{R}\lim \omega^{\bullet}_{X_q/B_q}$ as a derived completion over X and not over B, because for the cited results the $h^{-i}(\omega^{\bullet}_{X/B})$ need to be finite modules. They are finite over $\mathscr{O}_{X,x}$ but not necessarily over $\mathscr{O}_{B,b}$.

Next we prove our main flatness and base change statement.

Theorem 5.12. Let $X \to B$ be a flat morphism locally embeddable into a smooth morphism. Fix an $i \in \mathbb{Z}$ and assume that for any Artinian scheme Z and morphism $Z \to B$, the sheaf $h^{-i}(\omega_{XZ/Z}^{\bullet})$ is flat over Z and commutes with any base change to a closed subscheme of Z. Then $h^{-i}(\omega_{X/B}^{\bullet})$ is flat over B and commutes with arbitrary base change.

Proof. Since the base change morphism ϱ_Z^{-i} is natural, the statement is local on B, so we may replace B with a local scheme (B, b). Furthermore, since both flatness and whether or not ϱ_Z^{-i} is an isomorphism can be tested locally on X, we may also replace X with a local scheme (X, x), use the notation established in Notation 5.8, assume that $f: (X, x) \to (B, b)$ is embeddable into a smooth morphism and apply Lemma 5.2.

Let $M_q := \mathsf{h}^{-i}(\omega_{X_q/B_q}^{\bullet})$. Then by assumption M_q is flat over B_q for every $q \in \mathbb{N}$ and the natural base change morphism is an isomorphism:

$$M_{q+1} \otimes_{B_{q+1}} \mathscr{O}_{B_q} \xrightarrow{\simeq} M_q.$$

In particular, the induced natural morphism $M_{q+1} \to M_q$ is surjective and hence (M_q) satisfies the Mittag-Leffler condition and $\lim_{\leftarrow} M_q$ is flat over B by the first statement of [StacksProject, Tag 0912]. Furthermore, let $Q = \mathcal{O}_{X_j}$ for a fixed $j \in \mathbb{N}$. Then $M_q \otimes_X Q \simeq M_j$ for any $q \geq j$ by assumption and hence $\lim_{\leftarrow} (M_q \otimes_X Q) \simeq M_j$. Then by the second statement of [StacksProject, Tag 0912],

(5.12.1)
$$(\lim_{\longleftarrow} M_q) \otimes_X \mathscr{O}_{X_j} = (\lim_{\longleftarrow} M_q) \otimes_X Q \simeq \lim_{\longleftarrow} (M_q \otimes_X Q) \simeq M_j.$$

On the other hand, $\lim_{\leftarrow} M_q \simeq \lim_{\leftarrow} \left(\mathsf{h}^{-i}(\omega_{X/B}^{\bullet}) \otimes_X \mathscr{O}_{X_q} \right)$ by Lemma 5.9 and so by [StacksProject, Tag 031C],

(5.12.2)
$$(\lim_{\longleftarrow} M_q) \otimes_X \mathscr{O}_{X_j} \simeq \mathsf{h}^{-i}(\omega_{X/B}^{\bullet}) \otimes_X \mathscr{O}_{X_j}.$$

Comparing (5.12.1) and (5.12.2) shows that $h^{-i}(\omega_{X/B}^{\bullet})$ commutes with base change to B_q for every $q \in \mathbb{N}$ and then $h^{-i}(\omega_{X/B}^{\bullet})$ commutes with arbitrary base change by Lemma 5.2 and [AK80, 1.9]. Using that $M_q = h^{-i}(\omega_{X_q/B_q}^{\bullet})$ is flat over B_q for every $q \in \mathbb{N}$, it follows that $h^{-i}(\omega_{X/B}^{\bullet})$ is flat over B by [StacksProject, Tag 0523].

Corollary 5.13. Let $f : (X, x) \to (B, b)$ be a flat local morphism that is essentially of finite type. If (X_b, x) has liftable local cohomology over B then $h^{-i}(\omega_{X/B}^{\bullet})$ is flat over B and commutes with arbitrary base change for each $i \in \mathbb{Z}$.

Proof. By Theorem 4.12, Theorem 4.9(i) and Theorem 4.9(v) f satisfies the assumptions of Theorem 5.12 and hence the statement follows from Theorem 5.12.

Now we are ready to prove Theorem 1.4.

Theorem 5.14 = Theorem 1.4. Let $f : X \to B$ be a flat morphism of schemes that is essentially of finite type and let $b \in B$ such that X_b has liftable local cohomology over B. Then there exists an open neighborhood $X_b \subset U \subset X$ such that $h^{-i}(\omega_{U/B}^{\bullet})$ is flat over B and commutes with base change for each $i \in \mathbb{Z}$.

Proof. Let $x \in X_b$ and temporarily replace $f : X \to B$ with the induced local morphism $(X, x) \to (B, b)$. Then $h^{-i}(\omega_{X/B}^{\bullet})$ is flat over B and commutes with arbitrary base change by Corollary 5.13.

Since localization is an exact functor, we obtain that for the original $f: X \to B$ and any $x \in X_b$ the localized cohomology sheaves $h^{-i}(\omega_{X/B})_x$ are flat over B and commute with base change for each $i \in \mathbb{Z}$. Both of these properties are open on X and hence there is an open neighbourhood of x where they hold. The union of these neighbourhoods for all $x \in X_b$ provide an open neighbourhood of X_b where these properties hold. \Box

Now Theorem 5.14 and Theorem 6.1 implies Corollary 1.5(i) and Theorem 5.14 and Proposition 7.2 implies Corollary 1.5(ii).

6. DU BOIS SINGULARITIES

In characteristic 0, the optimal setting for deformation invariance of cohomology seems to be the class of Du Bois singularities, introduced by Steenbrink [Ste83]. For a proper complex

variety with Du Bois singularities the natural morphism

$$(\star) \qquad \qquad H^i(X, \mathbb{C}_X) \longrightarrow H^i(X, \mathscr{O}_X)$$

is surjective, and one should think of Du Bois singularities as the largest class for which this holds, cf. [Kov12]. This surjectivity enables one to use topological arguments to control the sheaf cohomology groups $H^i(X, \mathscr{O}_X)$ in flat families as in [DJ74].

The proof of Corollary 1.6 for projective morphisms in [KK10] very much relied on global duality, hence properness. Our first hope was that one can localize the proofs by replacing (\star) with the analogous map between local cohomology groups

$$H^i_x(X, \mathbb{C}_X) \longrightarrow H^i_x(X, \mathscr{O}_X).$$

However, this turned out to be too simplistic, one needs to consider instead the map

$$H^i_x(X, \mathbb{C}_X) \longrightarrow \mathbb{H}^i_x(X, \underline{\Omega}^0_X),$$

where $\underline{\Omega}_X^0$ denotes the 0th associated graded Du Bois complex of X. For the construction of the Du Bois complex see [DB81, GNPP88] and for its relevance to higher dimensional geometry see [Kol13, §6]. The surjectivity in (*) seems simple, but it is a key element of Kodaira type vanishing theorems [Kol87], [Kol95, §12], [Kov00], [KSS10] and leads to various results on deformations of Du Bois schemes [DJ74, KK10, KS16b]. Eventually we understood that for our purposes the key property is liftable local cohomology.

Theorem 6.1. Let X be a scheme, essentially of finite type over a field of characteristic 0. Assume that X is DuBois. Then X has liftable local cohomology.

For the definition of Du Bois singularities the reader is referred to [Kol13, §6]. A scheme defined over a field of characteristic 0 is said to have Du Bois singularities if its base extension to \mathbb{C} does. In addition to the properties mentioned above recall that Du Bois singularities are invariant under small deformation by [KS16a, 4.1].

Let us start the proof of Theorem 6.1 by recalling the following statement.

Theorem 6.2 [KS16a, KS16b, MSS17]. Let X be a scheme, essentially of finite type over a field of characteristic 0. Then the natural morphism

$$\mathsf{h}^{\imath}(\underline{\omega}_{X}^{\bullet}) \hookrightarrow \mathsf{h}^{\imath}(\omega_{X}^{\bullet})$$

is injective for every $i \in \mathbb{Z}$.

Remark 6.3. Theorem 6.2 was first proved in [KS16a, Theorem 3.3]. A version for pairs, essentially with the same proof, was given in [KS16b, Theorem B]. Both of these were stated for reduced schemes even though the proof does not need that assumption. This was noticed and carefully confirmed in [MSS17, Lemma 3.2] where the proof is carried out in detail for the not-necessarily-reduced case.

Corollary 6.4. Let X be a scheme, essentially of finite type over a field of characteristic 0 and $x \in X$ a closed point. Then the natural morphism

$$H^i_x(\mathscr{O}_X) \longrightarrow \mathbb{H}^i_x(\underline{\Omega}^0_X)$$

is surjective for each $i \in \mathbb{Z}$.

Proof. Let *E* be an injective hull of the residue field $\kappa(x)$ as an $\mathscr{O}_{X,x}$ -module. Then by local duality [R&D, Theorem V.6.2] there exists a commutative diagram where the vertical maps are isomorphisms:

(6.4.1)
$$\begin{array}{c} \mathcal{R}\Gamma_{x}(\mathcal{O}_{X}) & \longrightarrow \mathcal{R}\Gamma_{x}(\underline{\Omega}_{X}^{0}) \\ \simeq & \downarrow & \downarrow \simeq \\ \mathcal{R}\mathrm{Hom}_{\mathscr{O}_{X,x}}(\omega_{X}^{\star}, E) & \longrightarrow \mathcal{R}\mathrm{Hom}_{\mathscr{O}_{X,x}}(\underline{\omega}_{X}^{\star}, E) \end{array}$$

Since E is injective, the functor $\operatorname{Hom}_{\mathscr{O}_{X,x}}(\underline{\ }, E)$ is exact and hence it commutes with taking cohomology. Thus one has that

$$\mathsf{h}^{i}(\mathcal{R}\mathrm{Hom}_{\mathscr{O}_{X,x}}(\omega_{X}^{\bullet}, E)) \simeq \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathsf{h}^{i}(\omega_{X}^{\bullet}), E)$$

and

$$h^{i}(\mathcal{R}Hom_{\mathscr{O}_{X,x}}(\underline{\omega}_{X}^{\bullet}, E)) \simeq Hom_{\mathscr{O}_{X,x}}(h^{i}(\underline{\omega}_{X}^{\bullet}), E).$$

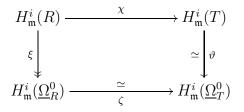
It follows that by taking cohomology of the diagram in (6.4.1) one obtains for each i the commutative diagram

Again, since $\operatorname{Hom}_{\mathscr{O}_{X,x}}(\underline{\ }, E)$ is exact, it follows from Theorem 6.2 that the bottom homomorphism is surjective which implies the desired statement.

Remark 6.5. An important aspect of **Corollary 6.4** is that the local cohomology of \mathscr{O}_X depends on the non-reduced structure, while that of $\underline{\Omega}^0_X$ does not. Essentially, the left hand side reflects the algebraic structure, while the right hand side behaves as if it only depended on the topology (this is not entirely true!).

This behavior allows us to prove Theorem 6.1. The proof is based on the interplay between the non-reduced and reduced data. It is similar in spirit to the proofs of [DJ74, Lemme 1], [KS16b, Theorem 5.1], and [MSS17, Lemma 3.3].

Proof of Theorem 6.1. Using Notation 4.1 assume that A = k is a field of characteristic 0 and that $X = \operatorname{Spec} T$ has Du Bois singularities. We need to prove that the induced morphism on local cohomology $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(T)$ is surjective for each *i*. Consider the following diagram:



Using the notation of the diagram, one has that ξ is surjective by Corollary 6.4, ζ is an isomorphism because $\underline{\Omega}_R^0 \simeq \underline{\Omega}_T^0$, and ϑ is an isomorphism, because Spec *T* has Du Bois singularities. It follows then that χ is surjective.

7. *F*-PURE SINGULARITIES

There is an intriguing correspondence between singularities of the minimal model program in characteristic 0 and singularities defined by the action of the Frobenius morphism in positive characteristic. For more on this correspondence the reader may consult [ST12, App. C] or [Kol13, §8.4]. Our goal here is to show that F-pure, or more generally F-anti-nilpotent singularities have liftable local cohomology over their ground field.

Definition 7.1. Let (R, \mathfrak{m}) be a noetherian local ring of characteristic p > 0 with the Frobenius endomorphism $F : R \to R$; $x \mapsto x^p$.

Recall that a homomorphism of R-modules $M \to M'$ is called *pure* if for every R-module N the induced homomorphism $M \otimes_R N \to M' \otimes_R N$ is injective. R is called F-pure if the Frobenius endomorphism is pure. R is called F-finite if R is a finitely generated R-module via the Frobenius endomorphism F. For instance, if R is essentially of finite type over a field, then it is F-finite. Further note that if R is F-finite or complete then it is F-pure if and only if the Frobenius endomorphism $F: R \to R$ has a left inverse [HR76, 5.3].

R is called *F*-injective if the induced Frobenius action on $H^i_{\mathfrak{m}}(R)$ is injective for all $i \in \mathbb{N}$. This holds for example if *R* is *F*-pure by [HR76, 2.2] and if *R* is Gorenstein then it is *F*-pure if and only if it is *F*-injective [Fed83, 3.3].

A strengthening of the notion of F-injective was recently introduced in [EH08]: Consider the induced Frobenius action $F : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$. A submodule $M \subseteq H^i_{\mathfrak{m}}(R)$ is called F-stable if $F(M) \subseteq M$ and R is called F-anti-nilpotent if for any F-stable submodule $M \subseteq H^i_{\mathfrak{m}}(R)$, the induced Frobenius action on the quotient $H^i_{\mathfrak{m}}(R)/M$ is injective. If Ris F-anti-nilpotent, then it is F-injective, since $\{0\} \subseteq H^i_{\mathfrak{m}}(R)$ is an F-stable submodule. Furthermore, if R is F-pure, then it is F-anti-nilpotent by [Ma14, 3.8]. So we have the following implications:

(7.1.1)
$$F$$
-pure \implies F -anti-nilpotent \implies F -injective.

Let (X, x) be a local scheme. Then we say that X has *F*-pure, resp. *F*-anti-nilpotent, resp. *F*-injective singularities if the local ring $\mathcal{O}_{X,x}$ has the corresponding property. An arbitrary scheme X of equicharacteristic p > 0 has *F*-pure, resp. *F*-anti-nilpotent, resp. *F*-injective singularities if the local scheme (X, x) has the corresponding property for each $x \in X$.

These singularities are related to the singularities of the minimal model program. Normal \mathbb{Q} -Gorenstein *F*-pure singularities are log canonical by [HW02] and it is conjectured that in some form the converse also holds. Similarly, *F*-injective singularities correspond to Du Bois singularities: If X is essentially of finite type over a field of characteristic 0 and its reduction mod p is *F*-injective for infinitely many p's, then X has Du Bois singularities by [Sch09b] and the converse of this is also conjectured to hold. So the (outside) implication in (7.1.1) is analogous to that log canonical singularities are Du Bois [KK10].

Curiously, we have this additional notion, F-anti-nilpotent, in between the more familiar F-pure and F-injective notions. It turns out that F-anti-nilpotent, and hence F-pure, rings have liftable local cohomology, but F-injective in general do not. This suggests that possibly F-anti-nilpotent is a better analog of Du Bois singularities in positive characteristic than F-injective. Of course, this is far from conclusive evidence, and this issue will not be settled here.

These singularities, defined by the action of Frobenius, have been studied extensively through their local cohomology. So it is no surprise that the fact that F-anti-nilpotent singularities have liftable local cohomology is a relatively simple consequence of known results. The following statement is essentially proved in [MSS17, Remark 3.4], although their statement is slightly different, so we include a proof for completeness.

Proposition 7.2 (Ma-Schwede-Shimomoto). *F*-anti-nilpotent singularities have liftable local co-homology over their ground field.

Proof. (Following the argument in [MSS17, Remark 3.4]). Using Notation 4.1 assume that A = k is a field of characteristic p > 0 and that (R, \mathfrak{m}) has *F*-anti-nilpotent singularities. We need to prove that the induced morphism on local cohomology $H^i_{\mathfrak{m}}(R) \twoheadrightarrow H^i_{\mathfrak{m}}(T)$ is surjective for each *i* (cf. Definition 1.1).

Since the statement is about local cohomology we may assume that R is complete and hence $R \simeq R'/J$ where R' is a complete regular local ring and $J \subseteq R'$ is an ideal. Note that denoting the pre-image of $I \subset R$ in R' by I', we have that $T \simeq R'/I'$ is also a quotient of R'.

Let $M := \operatorname{im} [H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(T)]$, which is an *F*-stable submodule of $H^i_{\mathfrak{m}}(T)$. Then *M* contains $F^e(H^i_{\mathfrak{m}}(T))$ for some e > 0 by [Lyu06, Lemma 2.2] and hence the Frobenius action on $H^i_{\mathfrak{m}}(T)/M$ is nilpotent. In particular it is injective only if this quotient is 0. Therefore, if *R* is *F*-anti-nilpotent, then $H^i_{\mathfrak{m}}(R) \twoheadrightarrow H^i_{\mathfrak{m}}(T)$ is surjective as desired. \Box

Corollary 7.3 (Ma). F-pure singularities have liftable local cohomology over their ground field.

Proof. F-pure singularities are *F*-anti-nilpotent by [Ma14, 3.8], so this is a direct consequence of Proposition 7.2.

8. DEGENERATIONS OF COHEN-MACAULAY SINGULARITIES WITH LIFTABLE LOCAL COHOMOLOGY

Proposition 8.1. Let Z be a scheme that admits a dualizing complex ω_Z^{\bullet} , $z \in Z$ a (not-necessarily-closed) point, and $n \in \mathbb{N}$. Then Z is S_n at z if and only if for all $i \in \mathbb{Z}$,

(8.1.1)
$$\mathbf{h}^{-i}(\omega_Z^{\bullet})_z = 0 \text{ for } i < \min(n, \dim_z Z) + \dim z$$

In particular, if Z is equidimensional, then Z is S_n if and only if for all $i \in \mathbb{Z}$, $i < \dim Z$,

(8.1.2)
$$\dim \operatorname{supp} \mathbf{h}^{-i}(\omega_Z^{\bullet}) \le i - n.$$

(In this statement we take $\dim \emptyset = -\infty$).

Proof. Since $h^{-i}(\omega_Z^{\bullet}) = \mathcal{E}_{\mathcal{X}} t_Z^{-i}(\mathscr{O}_Z, \omega_Z^{\bullet})$, (8.1.1) follows directly from [Kov11, Prop 3.2].

Next, let $i \in \mathbb{Z}$, $i < \dim Z$ be such that $h^{-i}(\omega_Z^{\bullet}) \neq 0$ and let $z \in \operatorname{supp} h^{-i}(\omega_Z^{\bullet})$ be a general point such that $\dim z = \dim \operatorname{supp} h^{-i}(\omega_Z^{\bullet})$. If Z is S_n , then $i \geq \min(n, \dim_z Z) + \dim z$ by (8.1.1) and hence, since $i < \dim Z$, we must have $\min(n, \dim_z Z) = n$ (this is where Z being equidimensional is used), so indeed dim $\operatorname{supp} h^{-i}(\omega_Z^{\bullet}) \leq i - n$.

In order to prove the other implication let $i \in \mathbb{Z}$, $i < \dim Z$ be again such that $h^{-i}(\omega_Z^{\bullet}) \neq 0$, but now choose an arbitrary point $z \in \operatorname{supp} h^{-i}(\omega_Z^{\bullet})$. In this case we only have that $\dim z \leq \dim \operatorname{supp} h^{-i}(\omega_Z^{\bullet})$, but this will be enough. If $\dim \operatorname{supp} h^{-i}(\omega_Z^{\bullet}) \leq i - n < \dim Z - n$, then $n < \dim Z - \dim \operatorname{supp} h^{-i}(\omega_Z^{\bullet}) \leq \dim Z - \dim z = \dim_z Z$, i.e., $\min(n, \dim_z Z) = n$. It also follows that $i \geq n + \dim \operatorname{supp} h^{-i}(\omega_Z^{\bullet}) \geq \min(n, \dim_z Z) + \dim z$, and hence Z is S_n at z by (8.1.1). We obtain that for all $i < \dim Z$, $\operatorname{supp} h^{-i}(\omega_Z^{\bullet})$ is contained in the S_n -locus of Z. However, Z is Cohen-Macaulay and hence S_n at every point in $Z \setminus \bigcup_{i < \dim Z} \operatorname{supp} h^{-i}(\omega_Z^{\bullet})$, which proves (8.1.2).

Corollary 8.2. Let Z be an equidimensional scheme that admits a dualizing complex ω_Z^{\bullet} . If Z is S_n for some $n \in \mathbb{N}$, then $h^{-i}(\omega_Z^{\bullet}) = 0$ for i < n.

Theorem 8.3. Let $f : X \to B$ be a flat morphism with equidimensional fibers that is locally embeddable into a smooth morphism. Assume that there exists a $b_0 \in B$ such that X_{b_0} has liftable local cohomology over B. If X_{b_0} is not S_n then there exists an open subset $b_0 \in V \subseteq B$ such that X_b is not S_n for each $b \in V$.

Proof. By Theorem 5.14 there exists an open neighborhood $X_b \subset U \subset X$ such that $\mathsf{h}^{-i}(\omega_{U/B}^{\bullet})$ is flat over B and commutes with base change for each $i \in \mathbb{Z}$. Then dim supp $\mathsf{h}^{-i}(\omega_{U_b}^{\bullet})$ is a locally constant function on the set $\{b \in B \mid \mathsf{h}^{-i}(\omega_{X_b}^{\bullet}) \neq 0\}$, so the claim follows from (8.1.2).

8.A. Deformations of local schemes

Definition 8.4. Let $(A, \mathfrak{m}_1, \ldots, \mathfrak{m}_r)$ be a semi-local ring. Then (X, x_1, \ldots, x_r) is called a *semi-local scheme* where X = Spec A and $x_1 = \mathfrak{m}_1, \ldots, x_r = \mathfrak{m}_r \in X$. If r = 1 and A is a local ring then (X, x_1) is a *local scheme*.

A family of semi-local schemes consists of a pair (X, χ) where $\chi \subseteq X$ is a closed subscheme and a flat morphism $f: X \to B$ that is essentially of finite type such that $f|_{\chi}: \chi \to B$ is a dominant finite morphism and for any $b \in B$, $(X_b, \operatorname{red}(\chi_b))$ is an equidimensional semi-local scheme. By a slight abuse of notation this family of semi-local schemes will be denoted by $f: (X, \chi) \to B$.

Let P be a local property of a scheme such as being Du Bois, F-pure, F-anti-nilpotent, S_n , or Cohen-Macaulay. We will say that a semi-local scheme (X, x_1, \ldots, x_r) has property P if X is P at x_1, \ldots, x_r . In particular, we will say that (X, x_1, \ldots, x_r) is a Du Bois semi-local scheme, etc. Similarly for "local scheme" in place of "semi-local scheme".

Theorem 8.5. Let $f: (X, \chi) \to B$ be a family of semi-local schemes. Assume that

(i) B is irreducible,

(ii) the fibers of f are equidimensional,

(iii) there exists a $b_0 \in B$ such that X_{b_0} has liftable local cohomology over B, and

(iv) there exists a $b_1 \in B$ and an $n \in \mathbb{N}$ such that X_{b_1} is S_n (resp. Cohen-Macaulay).

Then there exists an open set $b_0 \in U \subseteq B$ such that X_b is S_n (resp. Cohen-Macaulay) for each $b \in U$. In particular, X_{b_0} is S_n (resp. Cohen-Macaulay).

Proof. It is enough to prove the statement for the S_n property. Since $f|_{\chi}$ is proper, the set $U := \{b \in B \mid X_b \text{ is } S_n\}$ is open in B by [EGA-IV/3, 12.1.6]. By (iv) it is non-empty and hence it is dense in B. Then it must contain b_0 by Theorem 8.3, which proves the statement. \Box

Remark 8.6. If χ_{b_0} has Du Bois singularities then we may even choose U such that χ_b is S_n and has Du Bois singularities for all $b \in U$ by [KS16a, 4.1]. As we mentioned earlier, it is not known whether small deformations of F-anti-nilpotent singularities remain F-anti-nilpotent. It is also an interesting question whether the condition of having liftable local cohomology is invariant under small deformations.

It follows that Theorem 8.5 applies to families of semi-local schemes with DuBois or F-anti-nilpotent singularities by Theorem 6.1 and Proposition 7.2 (cf. Remark 4.3). As a simple consequence we obtain a generalization of Example 2.3.

Corollary 8.7. Let Z be a normal projective variety over a field k such that K_Z is \mathbb{Q} -Cartier and numerically equivalent to 0. Let \mathscr{L} be an ample line bundle on Z and $X = C_a(Z, \mathscr{L})$ the affine cone over Z with conormal bundle \mathscr{L} . If X has liftable local cohomology over k and admits an S_n deformation for some $n \in \mathbb{N}$, then $H^i(Z, \mathscr{O}_Z) = 0$ for 0 < i < n - 1. In particular, a cone over an abelian variety (ordinary, if char k > 0) of dimension at least 2 does not admit an S_3 deformation.

Proof. If X admits an S_n deformation for some $n \in \mathbb{N}$, then X itself is S_n by Theorem 8.5 and the first statement follows from (2.4.3). Then the second statement follows from Example 2.3 and Theorem 6.1 in characteristic 0 and from Example 2.7 and Corollary 7.3 in positive characteristic.

Remark 8.8. As noted in the introduction, it is proved in [Som79] that the projective cone over an abelian variety of dimension at least 2 over \mathbb{C} is not smoothable inside the ambient projective space. This is a strict special case of Corollary 8.7. In general, it is possible that a non-smoothable projective variety is locally smoothable. An example of that is given in [CS16, 2.18]. Furthermore, Corollary 8.7 is also valid in positive characteristic.

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