MULTIPLICITY AND CONCENTRATION RESULTS FOR FRACTIONAL SCHRÖDINGER-POISSON EQUATIONS WITH MAGNETIC FIELDS AND CRITICAL GROWTH

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Abstract. We deal with the following fractional Schrödinger-Poisson equation with magnetic field

$$\varepsilon^{2s}(-\Delta)^s_{A/\varepsilon}u+V(x)u+\varepsilon^{-2t}(|x|^{2t-3}*|u|^2)u=f(|u|^2)u+|u|^{2^*_s-2}u \quad \text{in } \mathbb{R}^3,$$

where $\varepsilon>0$ is a small parameter, $s\in(\frac{3}{4},1)$, $t\in(0,1)$, $2^*_s=\frac{6}{3-2s}$ is the fractional critical exponent, $(-\Delta)^s_A$ is the fractional magnetic Laplacian, $V:\mathbb{R}^3\to\mathbb{R}$ is a positive continuous potential, $A:\mathbb{R}^3\to\mathbb{R}^3$ is a smooth magnetic potential and $f:\mathbb{R}\to\mathbb{R}$ is a subcritical nonlinearity. Under a local condition on the potential V, we study the multiplicity and concentration of nontrivial solutions as $\varepsilon\to 0$. In particular, we relate the number of nontrivial solutions with the topology of the set where the potential V attains its minimum.

1. introduction

In this paper we are concerned with the following fractional nonlinear Schrödinger-Poisson equation

$$\varepsilon^{2s}(-\Delta)^s_{A/\varepsilon}u+V(x)u+\varepsilon^{-2t}(|x|^{2t-3}*|u|^2)u=f(|u|^2)u+|u|^{2^*_s-2}u \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where $\varepsilon>0$ is a parameter, $s\in(\frac{3}{4},1)$, $t\in(0,1)$, $2^*_s=\frac{6}{3-2s}$ is the fractional critical exponent, $V\in C(\mathbb{R}^3,\mathbb{R})$ and $A\in C^{0,\alpha}(\mathbb{R}^3,\mathbb{R}^3)$, $\alpha\in(0,1]$, are the electric and magnetic potentials respectively. Here the fractional magnetic Laplacian $(-\Delta)^s_A$ is defined, whenever $u\in C^\infty_c(\mathbb{R}^3,\mathbb{C})$, as

$$(-\Delta)^s_A u(x):=c_{3,s}\lim_{r\to 0}\int_{B^c_r(x)}\frac{u(x)-e^{\imath(x-y)\cdot A(\frac{x+y}{2})}u(y)}{|x-y|^{3+2s}}dy, \quad c_{3,s}:=\frac{4^s\Gamma\left(\frac{3+2s}{2}\right)}{\pi^{3/2}|\Gamma(-s)|}, \tag{1.2}$$

and it has been recently considered in [24]. The motivations that led to its introduction are mainly analyzed in [24, 37] and rely essentially on the Lévy-Khintchine formula for the generator of a general Lévy process. As stated in [54], this operator can be seen as the fractional counterpart of the magnetic Laplacian $-\Delta_A:=\left(\frac{1}{\imath}\nabla-A\right)^2$ given by

$$-\Delta_A u=-\Delta u-\frac{2}{\imath}A(x)\cdot\nabla u+|A(x)|^2u-\frac{1}{\imath}u\,\mathrm{div}(A(x));$$

see [40, 42, 50] for more details. We recall that the magnetic Laplacian arises in the study of the following Schrödinger equation with magnetic field

$$-\Delta_A u+V(x)u=f(x,|u|^2)u \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

for which a lot of interesting existence and multiplicity results have been established; see for instance [2, 3, 12, 18, 20, 27, 29, 39] and references therein.

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In the nonlocal framework, only few and recent works deal with fractional magnetic Schrödinger equations like

$$\varepsilon^{2s}(-\Delta)^s_A u + V(x)u = f(x,|u|^2)u \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

For instance, d'Avenia and Squassina [24] studied the existence of ground state to (1.4) when $\varepsilon = 1$, V is constant and f is a subcritical or critical nonlinearity. Fiscella et al. [32] proved the multiplicity of nontrivial solutions for a fractional magnetic problem with homogeneous boundary conditions. Zhang et al. [60] obtained the existence of mountain pass solutions which tend to the trivial solution as $\varepsilon \to 0$ for a fractional magnetic Schrödinger equation involving critical frequency and critical growth. In [10] the author and d'Avenia dealt with the existence and the multiplicity of solutions to (1.4) for small $\varepsilon > 0$ when the potential V satisfies (1.6) and f has a subcritical growth.

In absence of magnetic field (that is $A \equiv 0$), the fractional magnetic Laplacian $(-\Delta)^s_A$ reduces to the well-known fractional Laplacian $(-\Delta)^s$ which has achieved a tremendous popularity in these last twenty years due to its great applications in several contexts such as phase transitions, quasi-geostrophic flows, game theory, population dynamics, quantum mechanics and so on; see [17, 26, 44] for more details. From a mathematical point of view, several contributions [7–9, 21, 30, 31, 52] have been given in the investigation of fractional Schrödinger equations like

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(x,u) \text{ in } \mathbb{R}^N, \tag{1.5}$$

which plays a crucial role in fractional quantum mechanics; see [41] and the appendix in [22] for a more detailed physical interpretation. In particular way, a special attention has been devoted to concentration phenomena of solutions to (1.5) as $\varepsilon \to 0$. For instance, Dávila et al. [23], via a Lyapunov-Schmidt variational reduction, studied solutions to (1.5) with a spike pattern concentrating around a finite number of points in space as $\varepsilon \to 0$, when V is a bounded sufficiently smooth potential and $f(u) = u^p$ with $p \in (1, 2^*_s - 1)$. Shang and Zhang [53] dealt with the existence and multiplicity of solutions for a critical fractional Schrödinger equation requiring that the involved potential V verifies the following condition due to Rabinowitz [49]:

$$\liminf_{|x|\to\infty} V(x) > \inf_{x\in\mathbb{R}^N} V(x). \tag{1.6}$$

Dipierro et al. [28] combined the Mountain Pass Theorem [6] and Concentration-Compactness Lemma to provide a multiplicity result for a fractional elliptic problem with critical growth. Alves and Miyagaki [4] (see also [7, 9, 11]) used a penalization argument to study the existence and concentration of positive solutions of (1.5) when f has a subcritical growth and V verifies the following assumptions due to del Pino and Felmer [25]:

(V_1) $\inf_{x\in\mathbb{R}^3} V(x) = V_0 > 0$;

(V_2) there exists a bounded domain $\Lambda \subset \mathbb{R}^3$ such that

$$V_0 < \min_{\partial\Lambda} V \quad \text{and} \quad M = \{x \in \Lambda : V(x) = V_0\} \neq \emptyset. \tag{1.7}$$

On the other hand, in these last years, some interesting papers appeared dealing with fractional Schrödinger-Poisson systems like

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)\phi u = g(x,u) & \text{in } \mathbb{R}^3 \\ \varepsilon^{2t}(-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.8}$$

which can be considered as the nonlocal counterpart of the well-known Schrödinger-Poisson system which describes systems of identical charged particles interacting each other in the case that effects of magnetic field could be ignored and its solution represents, in particular, a standing wave for such a system; see [15]. In the classical case $s = t = 1$, we refer to [5, 13, 51, 61] and [34, 35, 56, 58] in which several results for unperturbed (i.e. $\varepsilon = 1$) and perturbed (i.e. $\varepsilon > 0$ small) Schrödinger-Poisson systems and in absence of magnetic fields have been established, and [16, 48, 62] for some existence, uniqueness and multiplicity results when $A \not\equiv 0$.

Concerning (1.8), the first result is probably due to Giammetta [33], who studied the local and global well-posedness of a fractional Schrödinger-Poisson system in which the fractional diffusion appears only in the second equation in (1.8). In [59] Zhang et al. used a perturbation approach to prove the existence of positive solutions to (1.8) with $\varepsilon = 1$, $V(x) = \mu > 0$ and g is a general nonlinearity having subcritical or critical growth. Murcia and Siciliano [46] showed that, for suitably small ε, the number of positive solutions to a doubly singularly perturbed fractional Schrödinger-Poisson system is estimated below by the Ljusternick-Schnirelmann category of the set of minima of the potential. Teng [55] investigated the existence of ground state solutions for a critical unperturbed fractional Schrödinger-Poisson system. Liu and Zhang [43] studied multiplicity and concentration of solutions to (1.8) involving the fractional critical exponent and a potential V satisfying global condition (1.6). To the best of our knowledge, fractional magnetic Schrödinger-Poisson equations like (1.1) have not ever been considered until now. Particularly motivated by this fact and by the works [2, 4, 10, 43], in the present paper we investigate the multiplicity and concentration of nontrivial solutions to (1.1) when $\varepsilon \to 0$, under assumptions (V_1)-(V_2) on the continuous potential V, and $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function satisfying the following conditions:

(f_1) $f(t) = 0$ for $t \leq 0$ and $\displaystyle\lim_{t\to 0} \frac{f(t)}{t} = 0$;

(f_2) there exist $q, \nu \in (4, 2^*_s)$ and $\mu > 0$ such that

$$f(t) \geq \mu t^{\frac{\nu-2}{2}} \quad \forall t > 0 \text{ and } \lim_{t\to\infty} \frac{f(t)}{t^{\frac{q-2}{2}}} = 0;$$

(f_3) there exists $\theta \in (4, q)$ such that $0 < \frac{\theta}{2} F(t) \leq t f(t)$ for any $t > 0$, where $F(t) = \int_0^t f(\tau)d\tau$;

(f_4) $t \mapsto \frac{f(t)}{t}$ is increasing for $t > 0$.

A typical example of function verifying (f_1)-(f_4) is given by $f(t) = \sum_{i=1}^k \alpha_i (t^+)^{\frac{q_i-2}{2}}$, with $\alpha_i \geq 0$ not all null and $q_i \in [\theta, 2^*_s)$ for all $i \in \{1, \dots, k\}$.

Our main result can be stated as follows:

Theorem 1.1. *Assume that (V_1)-(V_2) and (f_1)-(f_4) hold. Then, for any $\delta > 0$ such that*

$$M_\delta = \{x \in \mathbb{R}^3 : dist(x, M) \leq \delta\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, problem (1.1) has at least $cat_{M_\delta}(M)$ nontrivial solutions. Moreover, if u_ε denotes one of these solutions and x_ε is a global maximum point of $|u_\varepsilon|$, then we have

$$\lim_{\varepsilon\to 0} V(x_\varepsilon) = V_0$$

and

$$|u_\varepsilon(x)| \leq \frac{C\,\varepsilon^{3+2s}}{C\,\varepsilon^{3+2s} + |x - x_\varepsilon|^{3+2s}} \quad \forall x \in \mathbb{R}^3.$$

Remark 1.1. *Let us note that if $s, t \in (0, 1)$ are such that $4s+2t \geq 3$, then $H^s(\mathbb{R}^3, \mathbb{R}) \subset L^{\frac{12}{3+2t}}(\mathbb{R}^3, \mathbb{R})$ and $\phi^t_{|u|}$ is well-defined; see Section 2 below. Therefore, $s \in (\frac{3}{4}, 1)$ and $t \in (0, 1)$ are admissible exponents. Moreover, the restriction $s \in (\frac{3}{4}, 1)$ is related to the growth assumptions on f (in fact, we have that $2^*_s > 4$) which allow us to apply variational arguments, use the Nehari manifold and verify the Palais-Smale condition; see Sections 3 and 4 below. For what concerns the dimension $N = 3$, we suspect that our results can be extended only in low dimensions such that $N \leq 4s + 2t$ (see for instance [46]) and considering more general nonlinearities such that $\frac{F(u)}{u^2} \to \infty$ as $u \to \infty$ and that do not verify (f_4). Anyway, the three dimensional case is relevant for the physical meaning of the fractional Schrödinger-Poisson system.*

The proof of Theorem 1.1 relies on suitable variational methods and Ljusternik-Schnirelmann theory inspired by [1] and [2] in which the authors dealt with classical Schrödinger equations with critical growth and $A \equiv 0$ and subcritical growth and $A \not\equiv 0$ respectively. First of all we note that, using the change of variable $x \mapsto \varepsilon x$, problem (1.1) is equivalent to the following one

$$(-\Delta)^s_{A_\varepsilon} u + V_\varepsilon(x) u + \phi^t_{|u|} u = f(|u|^2)u + |u|^{2^*_s-2}u \text{ in } \mathbb{R}^3, \tag{1.9}$$

where $A_\varepsilon(x) = A(\varepsilon x)$, $V_\varepsilon(x) = V(\varepsilon x)$ and $\phi^t_{|u|} = |x|^{2t-3} * |u|^2$. Since we do not have any information on the behavior of V at infinity, we adapt the penalization argument developed by del Pino and Felmer in [25], which consists in modifying the nonlinearity f in a special way and to consider an auxiliary problem. More precisely, as in [2], we fix $k > \frac{\theta}{\theta-2}$ and $a > 0$ such that $f(a) + a^{\frac{2^*_s-2}{2}} = \frac{V_0}{k}$, and we consider the function

$$\hat{f}(t) := \begin{cases} f(t) + (t^+)^{\frac{2^*_s-2}{2}} & \text{if } t \le a \\ \frac{V_0}{k} & \text{if } t > a. \end{cases}$$

Let $t_a, T_a > 0$ such that $t_a < a < T_a$ and take $\xi \in C^\infty_c(\mathbb{R}, \mathbb{R})$ such that
(ξ_1) $\xi(t) \le \hat{f}(t)$ for all $t \in [t_a, T_a]$,
(ξ_2) $\xi(t_a) = \hat{f}(t_a)$, $\xi(T_a) = \hat{f}(T_a)$, $\xi'(t_a) = \hat{f}'(t_a)$ and $\xi'(T_a) = \hat{f}'(T_a)$,
(ξ_3) the map $t \mapsto \frac{\xi(t)}{t}$ is increasing for all $t \in [t_a, T_a]$.
Then we define $\tilde{f} \in C^1(\mathbb{R}, \mathbb{R})$ as follows:

$$\tilde{f}(t) := \begin{cases} \hat{f}(t) & \text{if } t \notin [t_a, T_a] \\ \xi(t) & \text{if } t \in [t_a, T_a]. \end{cases}$$

Finally, we introduce the following penalized nonlinearity $g : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ by setting

$$g(x,t) = \chi_\Lambda(x)(f(t) + (t^+)^{\frac{2^*_s-2}{2}}) + (1 - \chi_\Lambda(x))\tilde{f}(t),$$

where χ_Λ is the characteristic function on Λ, and we set $G(x,t) = \int_0^t g(x,\tau)\, d\tau$. From assumptions ($f_1$)-($f_4$) and ($\xi_1$)-($\xi_3$), it follows that g verifies the following properties:
(g_1) $\displaystyle\lim_{t\to 0} \frac{g(x,t)}{t} = 0$ uniformly in $x \in \mathbb{R}^3$;
(g_2) $g(x,t) \le f(t) + t^{\frac{2^*_s-2}{2}}$ for any $x \in \mathbb{R}^3$ and $t > 0$;
(g_3) (i) $0 < \frac{\theta}{2} G(x,t) \le g(x,t)t$ for any $x \in \Lambda$ and $t > 0$,
(ii) $0 \le G(x,t) \le g(x,t)t \le \frac{V(x)}{k} t$ and $0 \le g(x,t) \le \frac{V(x)}{k}$ for any $x \in \Lambda^c$ and $t > 0$;
(g_4) $t \mapsto \frac{g(x,t)}{t}$ is increasing for all $x \in \Lambda$ and $t > 0$.
Then we consider the following modified problem

$$(-\Delta)^s_{A_\varepsilon} u + V_\varepsilon(x) u + \phi^t_{|u|} u = g_\varepsilon(x, |u|^2)u \text{ in } \mathbb{R}^3, \tag{1.10}$$

where $g_\varepsilon(x,t) := g(\varepsilon x, t)$. Let us note that if u is a solution of (1.10) such that

$$|u(x)| \le t_a \text{ for all } x \in \Lambda^c_\varepsilon, \tag{1.11}$$

where $\Lambda_\varepsilon := \{x \in \mathbb{R}^3 : \varepsilon x \in \Lambda\}$, then u is indeed a solution of the original problem (1.9).

Since we want to find nontrivial solutions to (1.9), we look for critical points of the following functional associated with (1.9):

$$J_\varepsilon(u) = \frac{c_{3,s}}{2} \iint_{\mathbb{R}^6} \frac{|u(x) - e^{\imath(x-y)\cdot A_\varepsilon(\frac{x+y}{2})} u(y)|^2}{|x-y|^{3+2s}}\, dxdy + \frac{1}{2}\int_{\mathbb{R}^3} V_\varepsilon(x)|u|^2\, dx$$
$$+ \frac{1}{4}\int_{\mathbb{R}^3} \phi^t_{|u|}|u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} G_\varepsilon(x, |u|^2)\, dx$$

defined on the fractional Sobolev space

$$H^s_\varepsilon = \left\{ u \in \mathcal{D}^s_{A_\varepsilon}(\mathbb{R}^3, \mathbb{C}) : \int_{\mathbb{R}^3} V_\varepsilon(x)|u|^2\, dx < \infty \right\};$$

see Section 2 for more details. The main difficulty in the study of J_ε is related to verify a local Palais-Smale compactness condition at any level $c < c_* := \frac{s}{3} S_*^{\frac{3}{2s}}$, where S_* is the best Sobolev constant of the embedding $H^s(\mathbb{R}^3, \mathbb{R})$ in $L^{2^*_s}(\mathbb{R}^3, \mathbb{R})$. Indeed, the appearance of the magnetic field, the critical exponent, the convolution term $|x|^{2t-3} * |u|^2$ and the nonlocal nature of the fractional magnetic Laplacian, make our analysis much more complicated and delicate with respect to [1,2,4,10,43]. We circumvent these issues proving some careful estimates and using the Concentration-Compactness Lemma for the fractional Laplacian [11,28,47]; see Lemma 3.2. The Hölder regularity assumption on the magnetic field A and the fractional diamagnetic inequality established in [24] will be used to show that the mountain pass minimax level c_ε of J_ε is less than c_* for $\varepsilon > 0$ small enough. In order to obtain multiple solutions for the modified problem, we use some techniques developed by Benci and Cerami in [14], which are based on suitable comparisons between the category of some sublevel sets of the modified functional and the category of the set M. After that, we need to prove that if u_ε is a solution of modified problem (1.10), then $|u_\varepsilon|$ satisfies (1.11) for ε small enough. In order to achieve our goal, we aim to show that the (translated) sequence (u_n) verifies the property $|u_n(x)| \to 0$ as $|x| \to \infty$ uniformly with respect to $n \in \mathbb{N}$. In the case $A = 0$ (see for instance [4,11]), this is proved by using some fundamental estimates established in [31] concerning the Bessel operator. When $A \not\equiv 0$, we do not have similar informations for the following fractional equation

$$(-\Delta)^s_A u + V_0 u = h(|u|^2)u \text{ in } \mathbb{R}^3.$$

To overcome this difficulty, we use a clever approximation argument which allows us to deduce that if u is a solution to (1.10), then $|u|$ is a subsolution to

$$(-\Delta)^s u + V_0 u = g_\varepsilon(x, |u|^2)|u| \text{ in } \mathbb{R}^3;$$

see Lemma 5.1. We recall that in the case $s = 1$, it is clear that if u is a solution to

$$-\Delta_A u + V_0 u = h(|u|^2)u \text{ in } \mathbb{R}^3,$$

then $|u|$ is a subsolution to

$$-\Delta|u| + V_0|u| = h(|u|^2)|u| \text{ in } \mathbb{R}^3,$$

in view of the Kato's inequality [38]

$$-\Delta|u| \le \Re(sign(u)(-\Delta_A u)),$$

and then we can apply standard arguments to prove that $|u(x)| \to 0$ as $|x| \to \infty$ (the decay is exponential); see for instance [39]. Unfortunately, in our setting, even if we suspect that a distributional Kato's inequality for (1.2) holds true (see for instance [10] in which a pointwise fractional magnetic Kato's inequality is used), we are not able to prove it. We point out that in [36], the authors obtained a Kato's inequality for magnetic relativistic Schrödinger operators

$$H^\beta_{A,m} = [(-\imath\nabla - A(x))^2 + m^2]^{\beta/2}$$

with $m \ge 0$ and $\beta \in (0, 1]$, which include (1.2) when $\beta = 1$ and $m = 0$, that is $H^1_{A,0} = (-\Delta)^{1/2}_A$. On the other hand, due to the nonlocal character of (1.2), we cannot adapt in our framework the arguments developed in [2] to prove that $|u_n(x)| \to 0$ as $|x| \to \infty$ uniformly with respect to $n \in \mathbb{N}$. For the above reasons, in this work we develop some new ideas needed to achieve our claim. Roughly speaking, we will show that a Kato's inequality holds for the modified problem (1.10). More precisely, we first show that each $|u_n|$ is bounded in $L^\infty(\mathbb{R}^3,\mathbb{R})$-norm uniformly in $n\in\mathbb{N}$, by means of a Moser iteration argument [45]. At this point, we prove that each $|u_n|$ verifies

$$(-\Delta)^s|u_n| + V_0|u_n| \leq g_\varepsilon(x,|u_n|^2)|u_n| \text{ in } \mathbb{R}^3,$$

by using $\frac{u_n}{u_{\delta,n}}\varphi$ as test function in the modified problem, where $u_{\delta,n}=\sqrt{|u_n|^2+\delta^2}$ and φ is a real smooth nonnegative function with compact support in $\mathbb{R}^3$, and then we pass to the limit as $\delta\to 0$. This fact combined with a comparison argument and the results in [4, 31], allows us to deduce that $|u_n(x)|\to 0$ as $|x|\to\infty$ uniformly with respect to $n\in\mathbb{N}$; see Lemma 5.1. Finally, we give a decay estimate of modulus $|u_\varepsilon|$ of solutions u_ε to (1.1).

As far as we know, this is the first time that penalization methods jointly with Ljusternik-Schnirelmann theory are used to obtain multiple solutions for a fractional magnetic Schrödinger-Poisson equation with critical growth.

The paper is structured as follows. In Section 2 we recall some properties on the involved fractional Sobolev spaces. In Section 3 we prove some compactness properties for the modified functional. In Section 4 we introduce the barycenter map which will be a fundamental tool to obtain a multiplicity result for problem (1.10) via Ljusternick-Schnirelmann theory. In the last section we give the proof of Theorem 1.1.

2. Preliminaries

In this section we collect some notations and technical lemmas which will be used along the paper.

We define $H^s(\mathbb{R}^3,\mathbb{R})$ as the fractional Sobolev space

$$H^s(\mathbb{R}^3,\mathbb{R}) = \{u\in L^2(\mathbb{R}^3,\mathbb{R}) : [u]<\infty\}$$

where

$$[u]^2 = \iint_{\mathbb{R}^6} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}dxdy.$$

We recall that the embedding $H^s(\mathbb{R}^3,\mathbb{R})\subset L^q(\mathbb{R}^3,\mathbb{R})$ is continuous for all $q\in[2,2^*_s)$ and locally compact for all $q\in[1,2^*_s)$; see [26, 44] for more details on this topic.

Let $L^2(\mathbb{R}^3,\mathbb{C})$ be the space of complex-valued functions such that $\int_{\mathbb{R}^3}|u|^2\,dx<\infty$ endowed with the inner product $\langle u,v\rangle_{L^2}=\Re\int_{\mathbb{R}^3}u\bar{v}\,dx$, where the bar denotes complex conjugation.

Let us denote by

$$[u]^2_A := \frac{c_{3,s}}{2}\iint_{\mathbb{R}^6}\frac{|u(x)-e^{\imath(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{3+2s}}\,dxdy,$$

and consider

$$D^s_A(\mathbb{R}^3,\mathbb{C}) := \left\{u\in L^{2^*_s}(\mathbb{R}^3,\mathbb{C}) : [u]^2_A<\infty\right\}.$$

Then we introduce the Hilbert space

$$H^s_\varepsilon := \left\{u\in D^s_{A_\varepsilon}(\mathbb{R}^3,\mathbb{C}) : \int_{\mathbb{R}^3}V_\varepsilon(x)|u|^2\,dx<\infty\right\}$$

endowed with the scalar product

$$\begin{aligned}\langle u,v\rangle_\varepsilon &= \Re\int_{\mathbb{R}^3}V_\varepsilon(x)u\bar{v}dx\\ &\quad+\frac{c_{3,s}}{2}\Re\iint_{\mathbb{R}^6}\frac{(u(x)-e^{\imath(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}u(y))\overline{(v(x)-e^{\imath(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}v(y))}}{|x-y|^{3+2s}}dxdy\end{aligned}$$

and let

$$\|u\|_\varepsilon := \sqrt{\langle u,u\rangle_\varepsilon} = \sqrt{[u]^2_{A_\varepsilon}+\|\sqrt{V_\varepsilon}|u|\|^2_{L^2(\mathbb{R}^3)}}.$$

The space H^s_ε satisfies the following fundamental properties; see [10, 24] for more details.

Lemma 2.1. *[10, 24] The space H^s_ε is complete and $C^\infty_c(\mathbb{R}^3,\mathbb{C})$ is dense in H^s_ε.*

Lemma 2.2. *[24] If $u\in H^s_A(\mathbb{R}^3,\mathbb{C})$ then $|u|\in H^s(\mathbb{R}^3,\mathbb{R})$ and we have*

$$[|u|]\leq [u]_A.$$

Theorem 2.1. *[24] The space H^s_ε is continuously embedded in $L^r(\mathbb{R}^3,\mathbb{C})$ for $r\in[2,2^*_s]$, and compactly embedded in $L^r_{\rm loc}(\mathbb{R}^3,\mathbb{C})$ for $r\in[1,2^*_s)$.*

Lemma 2.3. *[10] If $u\in H^s(\mathbb{R}^3,\mathbb{R})$ and u has compact support, then $w=e^{\imath A(0)\cdot x}u\in H^s_\varepsilon$.*

We also recall the following vanishing lemma [31]:

Lemma 2.4. *[31] Let $q\in[2,2^*_s)$. If (u_n) is a bounded sequence in $H^s(\mathbb{R}^3,\mathbb{R})$ and if*

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_R(y)}|u_n|^q dx=0$$

for some $R>0$, then $u_n\to 0$ in $L^r(\mathbb{R}^3,\mathbb{R})$ for all $r\in(2,2^*_s)$.

Now, let $s,t\in(0,1)$ such that $4s+2t\geq 3$. Using the embedding $H^s(\mathbb{R}^3,\mathbb{R})\subset L^q(\mathbb{R}^3,\mathbb{R})$ for all $q\in[2,2^*_s)$, we can see that

$$H^s(\mathbb{R}^3,\mathbb{R})\subset L^{\frac{12}{3+2t}}(\mathbb{R}^3,\mathbb{R}). \tag{2.1}$$

For any $u\in H^s_\varepsilon$, we get $|u|\in H^s(\mathbb{R}^3,\mathbb{R})$ by Lemma 2.2, and the linear functional $\mathcal{L}_{|u|}:D^{t,2}(\mathbb{R}^3,\mathbb{R})\to\mathbb{R}$ given by

$$\mathcal{L}_{|u|}(v)=\int_{\mathbb{R}^3}|u|^2 v\,dx$$

is well defined and continuous in view of Hölder inequality and (2.1). Indeed, we can see that

$$|\mathcal{L}_{|u|}(v)|\leq\left(\int_{\mathbb{R}^3}|u|^{\frac{12}{3+2t}}dx\right)^{\frac{3+2t}{6}}\left(\int_{\mathbb{R}^3}|v|^{2^*_t}dx\right)^{\frac{1}{2^*_t}}\leq C\|u\|^2_{D^{s,2}}\|v\|_{D^{t,2}}, \tag{2.2}$$

where

$$\|v\|^2_{D^{t,2}}=\iint_{\mathbb{R}^6}\frac{|v(x)-v(y)|^2}{|x-y|^{3+2t}}dxdy.$$

Then, by the Lax-Milgram Theorem there exists a unique $\phi^t_{|u|}\in D^{t,2}(\mathbb{R}^3,\mathbb{R})$ such that

$$(-\Delta)^t\phi^t_{|u|}=|u|^2 \text{ in } \mathbb{R}^3. \tag{2.3}$$

Therefore we obtain the following t-Riesz formula

$$\phi^t_{|u|}(x)=c_t\int_{\mathbb{R}^3}\frac{|u(y)|^2}{|x-y|^{3-2t}}\,dy\quad(x\in\mathbb{R}^3),\qquad c_t=\pi^{-\frac{3}{2}}2^{-2t}\frac{\Gamma(3-2t)}{\Gamma(t)}. \tag{2.4}$$

We note that the above integral is convergent at infinity since $|u|^2\in L^{\frac{6}{3+2t}}(\mathbb{R}^3,\mathbb{R})$.
In the sequel, we will omit the constants $c_{3,s}$ and c_t in order to lighten the notation. We conclude this section giving some properties on the convolution term.

Lemma 2.5. *Let us assume that $4s+2t\geq 3$ and $u\in H^s_\varepsilon$. Then we have:*

(1) $\phi^t_{|u|}:H^s(\mathbb{R}^3,\mathbb{R})\to D^{t,2}(\mathbb{R}^3,\mathbb{R})$ is continuous and maps bounded sets into bounded sets,
(2) if $u_n\rightharpoonup u$ in H^s_ε then $\phi^t_{|u_n|}\rightharpoonup\phi^t_{|u|}$ in $D^{t,2}(\mathbb{R}^3,\mathbb{R})$,
(3) $\phi^t_{|ru|}=r^2\phi^t_{|u|}$ for all $r\in\mathbb{R}$ and $\phi^t_{|u(\cdot+y)|}(x)=\phi^t_{|u|}(x+y)$,

(4) $\phi^t_{|u|} \geq 0$ *for all* $u \in H^s_\varepsilon$*, and we have*

$$\|\phi^t_{|u|}\|_{D^{t,2}} \leq C\|u\|^2_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)} \leq C\|u\|^2_\varepsilon \text{ and } \int_{\mathbb{R}^3} \phi^t_{|u|}|u|^2 dx \leq C\|u\|^4_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)} \leq C\|u\|^4_\varepsilon.$$

Proof. (1) Since $\phi^t_{|u|} \in D^{t,2}(\mathbb{R}^3, \mathbb{R})$ satisfies (2.3), that is

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{t}{2}} \phi^t_{|u|} (-\Delta)^{\frac{t}{2}} v \, dx = \int_{\mathbb{R}^3} |u|^2 v \, dx \tag{2.5}$$

for all $v \in D^{t,2}(\mathbb{R}^3, \mathbb{R})$, we can see that $\mathcal{L}_{|u|}$ is such that $\|\mathcal{L}_{|u|}\|_{\mathcal{L}(D^{t,2},\mathbb{R})} = \|\phi^t_{|u|}\|_{D^{t,2}}$ for all $u \in H^s_\varepsilon$. Hence, in order to prove the continuity of $\phi^t_{|u|}$, it is enough to show that the map $u \in H^s_\varepsilon \mapsto \mathcal{L}_{|u|} \in \mathcal{L}(D^{t,2}, \mathbb{R})$ is continuous. Let $u_n \to u$ in H^s_ε. Using Lemma 2.2 and Theorem 2.1 we deduce that $|u_n| \to |u|$ in $L^{\frac{12}{3+2t}}(\mathbb{R}^3)$. Hence, for all $v \in D^{t,2}(\mathbb{R}^3, \mathbb{R})$ we have

$$\begin{aligned}
|\mathcal{L}_{|u_n|}(v) - \mathcal{L}_{|u|}(v)| &= \left| \int_{\mathbb{R}^3} (|u_n|^2 - |u|^2) v \, dx \right| \\
&\leq \left(\int_{\mathbb{R}^3} ||u_n|^2 - |u|^2|^{\frac{6}{3+2t}} \, dx \right)^{\frac{3+2t}{6}} \|v\|_{L^{\frac{6}{3-2t}}(\mathbb{R}^3)} \\
&\leq C \left[\left(\int_{\mathbb{R}^3} ||u_n| - |u||^{\frac{12}{3+2t}} \, dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} ||u_n| + |u||^{\frac{12}{3+2t}} \, dx \right)^{\frac{1}{2}} \right]^{\frac{3+2t}{6}} \|v\|_{D^{t,2}} \\
&\leq C \||u_n| - |u|\|_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)} \|v\|_{D^{t,2}}
\end{aligned}$$

which implies that $\|\phi^t_{|u_n|} - \phi^t_{|u|}\|_{D^{t,2}} = \|\mathcal{L}_{|u_n|} - \mathcal{L}_{|u|}\|_{\mathcal{L}(D^{t,2},\mathbb{R})} \to 0$ as $n \to \infty$.
(2) If $u_n \rightharpoonup u$ in H^s_ε, then Lemma 2.2 and Theorem 2.1 yield $|u_n| \to |u|$ in $L^q_{loc}(\mathbb{R}^3, \mathbb{R})$ for all $q \in [1, 2^*_s)$. Hence, for all $v \in C^\infty_c(\mathbb{R}^3, \mathbb{R})$ we get

$$\begin{aligned}
\langle \phi^t_{|u_n|} - \phi^t_{|u|}, v \rangle &= \int_{\mathbb{R}^3} (|u_n|^2 - |u|^2) v \, dx \\
&\leq \left(\int_{supp(v)} ||u_n| - |u||^2 \, dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} ||u_n| + |u||^2 \, dx \right)^{\frac{1}{2}} \|v\|_{L^\infty(\mathbb{R}^3)} \\
&\leq C \||u_n| - |u|\|_{L^2(supp(v))} \|v\|_{L^\infty(\mathbb{R}^3)} \to 0.
\end{aligned}$$

(3) is obtained by the definition of $\phi^t_{|u|}$.
(4) It is clear that $\phi^t_{|u|} \geq 0$. Using (2.5) with $v = \phi^t_{|u|}$, Hölder inequality and (2.1) we have

$$\|\phi^t_{|u|}\|^2_{D^{t,2}} \leq \|u\|^2_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)} \|\phi^t_{|u|}\|_{L^{2^*_t}(\mathbb{R}^3)} \leq C\|u\|^2_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)} \|\phi^t_{|u|}\|_{D^{t,2}} \leq C\|u\|^2_\varepsilon \|\phi^t_{|u|}\|_{D^{t,2}}.$$

On the other hand, in view of (2.4), Hardy-Littlewood-Sobolev inequality [42] and (2.1) we get

$$\int_{\mathbb{R}^3} \phi^t_{|u|}|u|^2 dx \leq C\||u|^2\|^2_{L^{\frac{6}{3+2t}}(\mathbb{R}^3)} = C\|u\|^4_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)} \leq C\|u\|^4_\varepsilon.$$

□

3. Variational framework for the modified functional

It is standard to check that weak solutions to (1.10) can be found as critical points of the Euler-Lagrange functional

$$J_\varepsilon(u) = \frac{1}{2}\|u\|^2_\varepsilon + \frac{1}{4}\int_{\mathbb{R}^3} \phi^t_{|u|}|u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} G_\varepsilon(x, |u|^2) \, dx,$$

We also consider the autonomous problem associated to (1.10), that is

$$(-\Delta)^s u + V_0 u + \phi^t_{|u|} u = f(u^2)u + |u|^{2^*_s-2}u \text{ in } \mathbb{R}^3, \tag{3.1}$$

and we introduce the corresponding energy functional $J_{V_0} : H^s(\mathbb{R}^3, \mathbb{R}) \to \mathbb{R}$ given by

$$\begin{aligned} J_{V_0}(u) &= \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 + V_0|u|^2\, dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi^t_{|u|} u^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} F(u^2)\, dx - \frac{1}{2^*_s}\int_{\mathbb{R}^3} |u|^{2^*_s}\, dx \\ &= \frac{1}{2}\|u\|^2_{V_0} + \frac{1}{4}\int_{\mathbb{R}^3} \phi^t_{|u|} u^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} F(u^2)\, dx - \frac{1}{2^*_s}\int_{\mathbb{R}^3} |u|^{2^*_s}\, dx \end{aligned}$$

where we used the notation $\|\cdot\|_{V_0}$ to denote the $H^s(\mathbb{R}^3, \mathbb{R})$-norm (equivalent to the standard one). We also denote by J_μ the functional associated to the problem (3.1) replacing V_0 by μ.

Now, let us introduce the Nehari manifold associated to (1.9), that is

$$\mathcal{N}_\varepsilon := \{u \in H^s_\varepsilon \setminus \{0\} : \langle J'_\varepsilon(u), u\rangle = 0\},$$

and we denote by $\mathcal{N}_{V_0}$ the Nehari manifold associated to (3.1). Using the growth conditions of g, we can show that there exists $r > 0$ independent of u such that

$$\|u\|_\varepsilon \geq r \text{ for all } u \in \mathcal{N}_\varepsilon. \tag{3.2}$$

Indeed, fixed $u \in \mathcal{N}_\varepsilon$, we get

$$\begin{aligned} 0 &= \|u\|^2_\varepsilon + \int_{\mathbb{R}^3} \phi^t_{|u|}|u|^2\, dx - \int_{\mathbb{R}^3} g_\varepsilon(x, |u|^2)|u|^2\, dx \\ &\geq \|u\|^2_\varepsilon - \frac{1}{k}\int_{\mathbb{R}^3} V_\varepsilon(x)|u|^2\, dx - C\|u\|^{2^*_s}_{L^{2^*_s}(\mathbb{R}^3)} \\ &\geq \frac{k-1}{k}\|u\|^2_\varepsilon - C\|u\|^{2^*_s}_\varepsilon. \end{aligned}$$

In what follows, we show that J_ε possesses a mountain pass geometry [6].

Lemma 3.1. *(i)* $J_\varepsilon(0) = 0$;
(ii) there exists $\alpha, \rho > 0$ *such that* $J_\varepsilon(u) \geq \alpha$ *for any* $u \in H^s_\varepsilon$ *such that* $\|u\|_\varepsilon = \rho$;
(iii) there exists $e \in H^s_\varepsilon$ *with* $\|e\|_\varepsilon > \rho$ *such that* $J_\varepsilon(e) < 0$.

Proof. Using (g_1), (g_2), and Theorem 2.1 we can see that for any $\delta > 0$ there exists $C_\delta > 0$ such that

$$J_\varepsilon(u) \geq \frac{1}{2}\|u\|^2_\varepsilon - \delta C\|u\|^4_\varepsilon - C_\delta\|u\|^{2^*_s}_\varepsilon.$$

Choosing $\delta > 0$ sufficiently small, we can see that (i) holds. Regarding (ii), we can note that in view of (f_3) and Lemma 2.5, we have for any $u \in H^s_\varepsilon \setminus \{0\}$ with $supp(u) \subset \Lambda_\varepsilon$ and $T > 1$

$$\begin{aligned} J_\varepsilon(Tu) &\leq \frac{T^2}{2}\|u\|^2_\varepsilon + \frac{T^4}{4}\int_{\mathbb{R}^3} \phi^t_{|u|}|u|^2 dx - \frac{1}{2}\int_{\Lambda_\varepsilon} F(T^2|u|^2)\, dx \\ &\leq \frac{T^4}{2}\left(\|u\|^2_\varepsilon + \int_{\mathbb{R}^3} \phi^t_{|u|}|u|^2 dx\right) - CT^\theta \int_{\Lambda_\varepsilon} |u|^\theta\, dx + C \end{aligned}$$

which together with $\theta > 4$, implies that $J_\varepsilon(Tu) \to -\infty$ as $T \to \infty$. □

In view of Lemma 3.1, we can define the minimax level

$$c_\varepsilon = \inf_{\gamma\in\Gamma_\varepsilon} \max_{t\in[0,1]} J_\varepsilon(\gamma(t)) \quad \text{where} \quad \Gamma_\varepsilon = \{\gamma \in C([0,1], H^s_\varepsilon) : \gamma(0) = 0 \text{ and } J_\varepsilon(\gamma(1)) < 0\}.$$

It is standard to verify that c_ε can be characterized as follows:

$$c_\varepsilon = \inf_{u\in H^s_\varepsilon\setminus\{0\}} \sup_{t\geq 0} J_\varepsilon(tu) = \inf_{u\in\mathcal{N}_\varepsilon} J_\varepsilon(u);$$

see [57] for more details. Using a version of the Mountain Pass Theorem without (PS) condition (see [57]), we can deduce the existence of a Palais-Smale sequence sequence (u_n) at the level c_ε.

Now, we show that J_ε verifies a compactness condition which is related to the best constant S_* of the Sobolev embedding $H^s(\mathbb{R}^3,\mathbb{R}) \subset L^{2^*_s}(\mathbb{R}^3,\mathbb{R})$ (see [26]). More precisely:

Lemma 3.2. *Let $c < c_* = \frac{s}{3} S_*^{\frac{3}{2s}}$. Then J_ε satisfies the Palais-Smale condition at the level c.*

Proof. Let $(u_n) \subset H^s_\varepsilon$ be a $(PS)_c$-sequence of J_ε, that is

$$J_\varepsilon(u_n) \to c < \frac{s}{3} S_*^{\frac{3}{2s}} \text{ and } J'_\varepsilon(u_n) \to 0.$$

We divide the proof into three steps.

Step 1 The sequence (u_n) is bounded in H^s_ε. Indeed, using (g_3) we can see that

$$\begin{aligned}
c + o_n(1)\|u_n\|_\varepsilon &= J_\varepsilon(u_n) - \frac{1}{\theta}\langle J'_\varepsilon(u_n), u_n\rangle \\
&= \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2_\varepsilon + \left(\frac{1}{4} - \frac{1}{\theta}\right)\int_{\mathbb{R}^3} \phi^t_{|u_n|}|u_n|^2 dx \\
&+ \frac{1}{\theta}\int_{\mathbb{R}^3}\left[g_\varepsilon(x,|u_n|^2)|u_n|^2 - \frac{\theta}{2}G_\varepsilon(x,|u_n|^2)\right] dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2_\varepsilon + \left(\frac{2-\theta}{2\theta}\right)\int_{\Lambda^c_\varepsilon} G_\varepsilon(x,|u_n|^2)\, dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2_\varepsilon + \left(\frac{2-\theta}{2\theta k}\right)\int_{\Lambda^c_\varepsilon} V_\varepsilon(x)|u_n|^2\, dx \\
&\geq \left(\frac{\theta-2}{2\theta}\right)\left(1 - \frac{1}{k}\right)\|u_n\|^2_\varepsilon.
\end{aligned}$$

Then, recalling that $k > \frac{\theta}{\theta-2} > 1$, we get the thesis.

Step 2 For any $\xi > 0$ there exists $R = R_\xi > 0$ such that $\Lambda_\varepsilon \subset B_R$ and

$$\limsup_{n\to\infty} \int_{B^c_R}\int_{\mathbb{R}^3} \frac{|u_n(x) - u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}|^2}{|x-y|^{3+2s}}\, dxdy + \int_{B^c_R} V_\varepsilon(x)|u_n|^2\, dx \leq \xi. \tag{3.3}$$

Let $\eta_R \in C^\infty(\mathbb{R}^3,\mathbb{R})$ such that $0 \leq \eta_R \leq 1$, $\eta_R = 0$ in $B_{\frac{R}{2}}$, $\eta_R = 1$ in B^c_R and $|\nabla \eta_R| \leq \frac{C}{R}$ for some $C > 0$ independent of R. Since $\langle J'_\varepsilon(u_n), \eta_R u_n\rangle = o_n(1)$ we have

$$\begin{aligned}
&\Re\left(\iint_{\mathbb{R}^6} \frac{(u_n(x) - u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)})\overline{(u_n(x)\eta_R(x) - u_n(y)\eta_R(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)})}}{|x-y|^{3+2s}}\, dxdy\right) \\
&+ \int_{\mathbb{R}^3} \phi^t_{|u_n|}|u_n|^2\eta_R dx + \int_{\mathbb{R}^3} V_\varepsilon(x)\eta_R|u_n|^2\, dx = \int_{\mathbb{R}^N} g_\varepsilon(x,|u_n|^2)|u_n|^2\eta_R\, dx + o_n(1).
\end{aligned}$$

Let us note that

$$\begin{aligned}
&\Re\left(\iint_{\mathbb{R}^6} \frac{(u_n(x) - u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)})\overline{(u_n(x)\eta_R(x) - u_n(y)\eta_R(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)})}}{|x-y|^{3+2s}}\, dxdy\right) \\
&= \Re\left(\iint_{\mathbb{R}^6} \overline{u_n(y)}e^{-\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)} \frac{(u_n(x) - u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)})(\eta_R(x) - \eta_R(y))}{|x-y|^{3+2s}}\, dxdy\right) \\
&+ \iint_{\mathbb{R}^6} \eta_R(x)\frac{|u_n(x) - u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}|^2}{|x-y|^{3+2s}}\, dxdy,
\end{aligned}$$

so, using (g_3)-(ii) and Lemma 2.5 we obtain

$$\begin{aligned}
&\iint_{\mathbb{R}^6} \eta_R(x)\frac{|u_n(x)-u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}|^2}{|x-y|^{3+2s}}\,dxdy+\int_{\mathbb{R}^3} V_\varepsilon(x)\eta_R|u_n|^2\,dx\\
&\le -\Re\left(\iint_{\mathbb{R}^6}\overline{u_n(y)}e^{-\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}\frac{(u_n(x)-u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)})(\eta_R(x)-\eta_R(y))}{|x-y|^{3+2s}}\,dxdy\right)\\
&+\frac{1}{k}\int_{\mathbb{R}^3} V_\varepsilon(x)\eta_R|u_n|^2\,dx+o_n(1).
\end{aligned}\tag{3.4}$$

From the Hölder inequality and the boundedness of (u_n) in H^s_ε it follows that

$$\begin{aligned}
&\left|\Re\left(\iint_{\mathbb{R}^6}\overline{u_n(y)}e^{-\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}\frac{(u_n(x)-u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)})(\eta_R(x)-\eta_R(y))}{|x-y|^{3+2s}}\,dxdy\right)\right|\\
&\le\left(\iint_{\mathbb{R}^6}\frac{|u_n(x)-u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}|^2}{|x-y|^{3+2s}}\,dxdy\right)^{\frac12}\left(\iint_{\mathbb{R}^6}|\overline{u_n(y)}|^2\frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}\,dxdy\right)^{\frac12}\\
&\le C\left(\iint_{\mathbb{R}^6}|u_n(y)|^2\frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}\,dxdy\right)^{\frac12}.
\end{aligned}\tag{3.5}$$

Arguing as in Lemma 4.3 in [11] (see formula (42) there) or Lemma 2.1 in [9], we can prove that

$$\limsup_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^6}|u_n(y)|^2\frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}\,dxdy=0.\tag{3.6}$$

Then, in view of (3.4), (3.5) and (3.6) we can conclude that

$$\limsup_{R\to\infty}\limsup_{n\to\infty}\left(1-\frac{1}{k}\right)\int_{B_R^c}\int_{\mathbb{R}^3}\frac{|u_n(x)-u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}|^2}{|x-y|^{3+2s}}\,dxdy+\int_{B_R^c}V_\varepsilon(x)|u_n|^2\,dx=0$$

that is (3.3) is satisfied.

Step 3: Up to subsequence, u_n strongly converges in H^s_ε.
Using $u_n\rightharpoonup u$ in H^s_ε, Theorem 2.1 and (g_1)-(g_2), it is easy to see that

$$(u_n,\psi)_\varepsilon\to(u,\psi)_\varepsilon \text{ and } \Re\left(\int_{\mathbb{R}^3}g_\varepsilon(x,|u_n|^2)u_n\bar{\psi}dx\right)\to\Re\left(\int_{\mathbb{R}^3}g_\varepsilon(x,|u|^2)u\bar{\psi}dx\right).\tag{3.7}$$

Moreover, using (3.3) and Theorem 2.1 we can see that for all $\xi>0$ there exists $R=R_\xi>0$ such that for any n large enough

$$\begin{aligned}
\|u_n-u\|_{L^q(\mathbb{R}^3)}&=\|u_n-u\|_{L^q(B_R)}+\|u_n-u\|_{L^q(B_R^c)}\\
&\le\|u_n-u\|_{L^q(B_R)}+(\|u_n\|_{L^q(B_R^c)}+\|u\|_{L^q(B_R^c)})\\
&\le\xi+2C\xi,
\end{aligned}$$

where $q\in[2,2^*_s)$, which gives

$$u_n\to u \text{ in } L^q(\mathbb{R}^3,\mathbb{C})\quad\forall q\in[2,2^*_s).\tag{3.8}$$

Since $||u_n|-|u||\le|u_n-u|$ and $\frac{12}{3+2t}\in(2,2^*_s)$, we also have $|u_n|\to|u|$ in $L^{\frac{12}{3+2t}}(\mathbb{R}^3,\mathbb{R})$.
Then, recalling that $\phi_{|u|}:L^{\frac{12}{3+2t}}(\mathbb{R}^3,\mathbb{R})\to D^{t,2}(\mathbb{R}^3,\mathbb{R})$ is continuous (see Lemma 2.5) we can deduce that

$$\phi^t_{|u_n|}\to\phi^t_{|u|} \text{ in } D^{t,2}(\mathbb{R}^3,\mathbb{R}).\tag{3.9}$$

Putting together (3.8), (3.9), Hölder inequality and Theorem 2.1 we obtain

$$\begin{aligned}\Re\left(\int_{\mathbb{R}^3}(\phi^t_{|u_n|}u_n-\phi^t_{|u|}u)\bar\psi dx\right)&=\Re\left(\int_{\mathbb{R}^3}\phi^t_{|u_n|}(u_n-u)\bar\psi+\int_{\mathbb{R}^3}(\phi^t_{|u_n|}-\phi^t_{|u|})u\bar\psi dx\right)\\&\leq\|\phi^t_{|u_n|}\|_{L^{\frac{6}{3+2t}}(\mathbb{R}^3)}\|u_n-u\|_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)}\|\psi\|_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)}\\&+\|\phi^t_{|u_n|}-\phi^t_{|u|}\|_{\frac{6}{3+2t}}\|u\|_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)}\|\psi\|_{\frac{12}{3+2t}}\\&\leq C\|u_n-u\|_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)}+C\|\phi^t_{|u_n|}-\phi^t_{|u|}\|_{D^{t,2}}\to 0.\end{aligned}\tag{3.10}$$

Therefore, using $\langle J'_\varepsilon(u_n),\psi\rangle=o_n(1)$ for all $\psi\in C^\infty_c(\mathbb{R}^3,\mathbb{C})$, and taking into account (3.7), (3.9) and (3.10), we can check that $J'_\varepsilon(u)=0$. In particular

$$\|u\|^2_\varepsilon+\int_{\mathbb{R}^3}\phi^t_{|u|}|u|^2dx=\int_{\mathbb{R}^3}g_\varepsilon(x,|u|^2)|u|^2\,dx.\tag{3.11}$$

On the other hand, we know that $\langle J'_\varepsilon(u_n),u_n\rangle=o_n(1)$ implies that

$$\|u_n\|^2_\varepsilon+\int_{\mathbb{R}^3}\phi^t_{|u_n|}|u_n|^2dx=\int_{\mathbb{R}^3}g_\varepsilon(x,|u_n|^2)|u_n|^2\,dx+o_n(1),\tag{3.12}$$

Now, we show that

$$\int_{\mathbb{R}^3}\phi^t_{|u_n|}|u_n|^2dx\to\int_{\mathbb{R}^3}\phi^t_{|u|}|u|^2dx.\tag{3.13}$$

Let us begin by proving that

$$|\mathbb{D}(u_n)-\mathbb{D}(u)|\leq\sqrt{\mathbb{D}(||u_n|^2-|u|^2|^{1/2})}\sqrt{\mathbb{D}(||u_n|^2+|u|^2|^{1/2})},$$

where

$$\mathbb{D}(u)=\iint_{\mathbb{R}^6}|x-y|^{-(3-2t)}|u(x)|^2|u(y)|^2dxdy.$$

Indeed, taking into account that $|x|^{-(3-2t)}$ is even and Theorem 9.8 in [42] (see the remark after Theorem 9.8 and recall that $-3<-(3-2t)<0$) we have

$$\begin{aligned}|\mathbb{D}(u_n)-\mathbb{D}(u)|&=\left|\iint_{\mathbb{R}^6}|x-y|^{-(3-2t)}|u_n(x)|^2|u_n(y)|^2dxdy-\iint_{\mathbb{R}^6}|x-y|^{-(3-2t)}|u(x)|^2|u(y)|^2dxdy\right|\\&=\Big|\iint_{\mathbb{R}^6}|x-y|^{-(3-2t)}|u_n(x)|^2|u_n(y)|^2dxdy+\iint_{\mathbb{R}^6}|x-y|^{-(3-2t)}|u_n(x)|^2|u(y)|^2dxdy\\&-\iint_{\mathbb{R}^6}|x-y|^{-(3-2t)}|u(x)|^2|u_n(y)|^2dxdy-\iint_{\mathbb{R}^6}|x-y|^{-(3-2t)}|u(x)|^2|u(y)|^2dxdy\Big|\\&=\left|\iint_{\mathbb{R}^6}|x-y|^{-(3-2t)}(|u_n(x)|^2-|u(x)|^2|)(|u_n(y)|^2+|u(y)|^2)dxdy\right|\\&\leq\iint_{\mathbb{R}^6}|x-y|^{-(3-2t)}||u_n(x)|^2-|u(x)|^2|||u_n(y)|^2+|u(y)|^2|dxdy\\&\leq C\sqrt{\mathbb{D}(||u_n|^2-|u|^2|^{1/2})}\sqrt{\mathbb{D}(||u_n|^2+|u|^2|^{1/2})}.\end{aligned}$$

Thus, using Hardy-Littlewood-Sobolev inequality (see Theorem 4.3 in [42]), Hölder inequality, the boundedness of $(|u_n|)$ in $H^s(\mathbb{R}^3,\mathbb{R})$ and $|u_n|\to|u|$ in $L^{\frac{12}{3+2t}}(\mathbb{R}^3,\mathbb{R})$ we can see that

$$\begin{aligned}|\mathbb{D}(u_n)-\mathbb{D}(u)|^2&\leq C\|||u_n|^2-|u|^2||^{1/2}\|^4_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)}\|||u_n|^2+|u|^2||^{1/2}\|^4_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)}\\&\leq C\||u_n|-|u|\|^2_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)}\to 0.\end{aligned}$$

Finally we show that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} g_\varepsilon(x,|u_n|^2)|u_n|^2\,dx=\int_{\mathbb{R}^3} g_\varepsilon(x,|u|^2)|u|^2\,dx. \tag{3.14}$$

Using (f_1), (f_2), (g_2) and Theorem 2.1 we get

$$\int_{\mathbb{R}^3\setminus B_R} g_\varepsilon(x,|u_n|^2)|u_n|^2\,dx\leq C(\delta+\delta^{\frac{q}{2}}+\delta^{\frac{2^*_s}{2}}), \tag{3.15}$$

for any n big enough. On the other hand, choosing R large enough, we may assume that

$$\int_{\mathbb{R}^3\setminus B_R} g_\varepsilon(x,|u|^2)|u|^2\,dx\leq \delta. \tag{3.16}$$

From the arbitrariness of $\delta>0$, we can see that (3.15) and (3.16) yield

$$\int_{\mathbb{R}^3\setminus B_R} g_\varepsilon(x,|u_n|^2)|u_n|^2\,dx\to\int_{\mathbb{R}^3\setminus B_R} g_\varepsilon(x,|u|^2)|u|^2\,dx \tag{3.17}$$

as $n\to\infty$. Now, we note that from the definition of g we know that

$$g_\varepsilon(x,|u_n|^2)|u_n|^2\leq f(|u_n|^2)|u_n|^2+|u_n|^{2^*_s}+\frac{V_0}{K}|u_n|^2 \text{ in } \mathbb{R}^3\setminus\Lambda_\varepsilon.$$

Since $B_R\cap(\mathbb{R}^3\setminus\Lambda_\varepsilon)$ is bounded, we can use (f_1), (f_2), (g_2), the Dominated Convergence Theorem and the strong convergence in $L^q_{loc}(\mathbb{R}^3,\mathbb{R})$ to see that

$$\int_{B_R\cap(\mathbb{R}^3\setminus\Lambda_\varepsilon)} g_\varepsilon(x,|u_n|^2)|u_n|^2\,dx\to\int_{B_R\cap(\mathbb{R}^3\setminus\Lambda_\varepsilon)} g_\varepsilon(x,|u|^2)|u|^2\,dx \tag{3.18}$$

as $n\to\infty$.
At this point, we show that

$$\lim_{n\to\infty}\int_{\Lambda_\varepsilon}|u_n|^{2^*_s}\,dx=\int_{\Lambda_\varepsilon}|u|^{2^*_s}\,dx. \tag{3.19}$$

Indeed, if we assume that (3.19) is true, from Theorem 2.1, (g_2), (f_1), (f_2) and the Dominated Convergence Theorem, we can see that

$$\int_{B_R\cap\Lambda_\varepsilon} g_\varepsilon(x,|u_n|^2)|u_n|^2\,dx\to\int_{B_R\cap\Lambda_\varepsilon} g_\varepsilon(x,|u|^2)|u|^2\,dx. \tag{3.20}$$

Putting together (3.17), (3.18) and (3.20), we can conclude that (3.14) holds. Taking into account (3.11), (3.12), (3.13) and (3.14) we can deduce that

$$\lim_{n\to\infty}\|u_n\|^2_\varepsilon=\|u\|^2_\varepsilon.$$

In what follows we prove that (3.19) is satisfied. From (3.3) and Lemma 2.2 we can see that $(|u_n|)$ is tight in $H^s(\mathbb{R}^3,\mathbb{R})$, so by Concentration-Compactness Lemma [11,28,47], we can find an at most countable index set I, sequences $(x_i)\subset\mathbb{R}^3$, $(\mu_i),(\nu_i)\subset(0,\infty)$ such that

$$\mu\geq|(-\Delta)^{\frac{s}{2}}|u||^2+\sum_{i\in I}\mu_i\delta_{x_i},$$

$$\nu=|u|^{2^*_s}+\sum_{i\in I}\nu_i\delta_{x_i}\quad\text{ and } S_*\nu_i^{\frac{2}{2^*_s}}\leq\mu_i \tag{3.21}$$

for any $i\in I$, where δ_{x_i} is the Dirac mass at the point x_i. Let us show that $(x_i)_{i\in I}\cap\Lambda_\varepsilon=\emptyset$. Assume by contradiction that $x_i\in\Lambda_\varepsilon$ for some $i\in I$. For any $\rho>0$, we define $\psi_\rho(x)=\psi(\frac{x-x_i}{\rho})$ where $\psi\in C^\infty_0(\mathbb{R}^N,[0,1])$ is such that $\psi=1$ in B_1, $\psi=0$ in $\mathbb{R}^3\setminus B_2$ and $\|\nabla\psi\|_{L^\infty(\mathbb{R}^3)}\leq 2$. We suppose that $\rho > 0$ is such that $supp(\psi_\rho) \subset \Lambda_\varepsilon$. Since $(\psi_\rho u_n)$ is bounded in H^s_ε, we can see that $\langle J'_\varepsilon(u_n), \psi_\rho u_n\rangle = o_n(1)$, so, using the pointwise diamagnetic inequality [24], we get

$$\begin{aligned}
&\iint_{\mathbb{R}^6} \psi_\rho(y) \frac{||u_n(x)| - |u_n(y)||^2}{|x-y|^{3+2s}}\, dxdy \\
&\quad \leq -\Re\left(\iint_{\mathbb{R}^6} \frac{(\psi_\rho(x)-\psi_\rho(y))(u_n(x)-u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)})}{|x-y|^{3+2s}} \overline{u_n(y)} e^{-\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)} dxdy\right) \\
&\quad\quad + \int_{\mathbb{R}^3} \psi_\rho f(|u_n|^2)|u_n|^2\, dx + \int_{\mathbb{R}^3} \psi_\rho |u_n|^{2^*_s}\, dx + o_n(1). \qquad (3.22)
\end{aligned}$$

Due to the fact that f has subcritical growth and ψ_ρ has compact support, we can see that

$$\lim_{\rho\to 0}\lim_{n\to\infty} \int_{\mathbb{R}^3} \psi_\rho f(|u_n|^2)|u_n|^2\, dx = \lim_{\rho\to 0}\int_{\mathbb{R}^3} \psi_\rho f(|u|^2)|u|^2\, dx = 0. \qquad (3.23)$$

Now, we show that

$$\lim_{\rho\to 0}\lim_{n\to\infty} \Re\left(\iint_{\mathbb{R}^6} \frac{(\psi_\rho(x)-\psi_\rho(y))(u_n(x)-u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)})}{|x-y|^{3+2s}} \overline{u_n(y)} e^{-\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)} dxdy\right) = 0. \qquad (3.24)$$

Using Hölder inequality and the fact that (u_n) is bounded in H^s_ε, we can see that

$$\begin{aligned}
&\left|\Re\left(\iint_{\mathbb{R}^6} \frac{(\psi_\rho(x)-\psi_\rho(y))(u_n(x)-u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)})}{|x-y|^{3+2s}} \overline{u_n(y)} e^{-\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)} dxdy\right)\right| \\
&\leq C\left(\iint_{\mathbb{R}^6} |u_n(y)|^2 \frac{|\psi_\rho(x)-\psi_\rho(y)|^2}{|x-y|^{3+2s}}\, dxdy\right)^{\frac{1}{2}}.
\end{aligned}$$

Arguing as in Lemma 4.3 in [11] (see formula (53) there) we can deduce that

$$\lim_{\rho\to 0}\lim_{n\to\infty} \iint_{\mathbb{R}^6} |u_n(x)|^2 \frac{|\psi_\rho(x)-\psi_\rho(y)|^2}{|x-y|^{3+2s}}\, dxdy = 0 \qquad (3.25)$$

which implies that (3.24) holds. Therefore, from (3.21) and taking the limit as $n\to\infty$ and $\rho\to 0$ in (3.22) we can deduce that (3.23) and (3.24) yield $\nu_i \geq \mu_i$ for all $i \in I$. In view of the last statement in (3.21), we have $\nu_i \geq S^{\frac{3}{2s}}$, and using Lemma 2.2 and (g_3) we can deduce that

$$\begin{aligned}
c &= J_\varepsilon(u_n) - \frac{1}{4}\langle J'_\varepsilon(u_n), u_n\rangle + o_n(1) \\
&\geq \frac{1}{4}\|u_n\|^2_\varepsilon + \frac{1}{2}\int_{\mathbb{R}^3\setminus\Lambda_\varepsilon} \left[\frac{1}{2} g_\varepsilon(x, |u_n|^2)|u_n|^2 - G_\varepsilon(x, |u_n|^2)\right] dx + \frac{4s-3}{12}\int_{\Lambda_\varepsilon} |u_n|^{2^*_s}\, dx + o_n(1) \\
&\geq \left[\frac{1}{4}\int_{\Lambda_\varepsilon} \psi_\rho |(-\Delta)^{\frac{s}{2}}|u_n||^2 dx + \frac{1}{4}\int_{\mathbb{R}^3\setminus\Lambda_\varepsilon} V_\varepsilon(x)|u_n|^2 dx\right] - \frac{1}{4}\int_{\mathbb{R}^3\setminus\Lambda_\varepsilon} G_\varepsilon(x, |u_n|^2) dx \\
&\quad + \frac{4s-3}{12}\int_{\Lambda_\varepsilon} |u_n|^{2^*_s}\, dx + o_n(1) \\
&\geq \frac{1}{4}\int_{\Lambda_\varepsilon} \psi_\rho |(-\Delta)^{\frac{s}{2}}|u_n||^2 dx + \left(\frac{1}{4} - \frac{1}{4k}\right)\int_{\mathbb{R}^3\setminus\Lambda_\varepsilon} V_\varepsilon(x)|u_n|^2 dx + \frac{4s-3}{12}\int_{\Lambda_\varepsilon} |u_n|^{2^*_s}\, dx + o_n(1) \\
&\geq \frac{1}{4}\int_{\Lambda_\varepsilon} \psi_\rho |(-\Delta)^{\frac{s}{2}}|u_n||^2 dx + \frac{4s-3}{12}\int_{\Lambda_\varepsilon} \psi_\rho |u_n|^{2^*_s}\, dx + o_n(1).
\end{aligned}$$

Then, in view of (3.21), $\nu_i \geq S^{\frac{3}{2s}}$ and taking the limit as $n \to \infty$, we find

$$
\begin{aligned}
c &\geq \frac{1}{4} \sum_{\{i\in I: x_i \in \Lambda_\varepsilon\}} \psi_\rho(x_i)\mu_i + \frac{4s-3}{12} \sum_{\{i\in I: x_i\in\Lambda_\varepsilon\}} \psi_\rho(x_i)\nu_i \\
&\geq \frac{1}{4} \sum_{\{i\in I: x_i \in \Lambda_\varepsilon\}} \psi_\rho(x_i) S_* \nu_i^{2/2^*_s} + \frac{4s-3}{12} \sum_{\{i\in I: x_i\in\Lambda_\varepsilon\}} \psi_\rho(x_i)\nu_i \\
&\geq \frac{1}{4} S_*^{\frac{3}{2s}} + \frac{4s-3}{12} S_*^{\frac{3}{2s}} = \frac{1}{3} S_*^{\frac{3}{2s}},
\end{aligned}
$$

which gives a contradiction. This means that (3.19) holds and we can conclude the proof. □

In view of Lemma 3.1, Lemma 3.2 and that $c_\varepsilon < c_*$ for $\varepsilon > 0$ small enough (see Lemma 3.4 below), one can apply the Mountain Pass Theorem [6] to deduce the existence of a nontrivial solution to (1.10) for small ε. Nevertheless, to obtain multiple critical points, we need to work with the functional J_ε constrained to $\mathcal{N}_\varepsilon$. Therefore, it is fundamental to prove the following compactness result:

Proposition 3.1. *Let $c \in \mathbb{R}$ be such that $c < c_* = \frac{s}{3} S_*^{\frac{3}{2s}}$. Then, the functional J_ε restricted to $\mathcal{N}_\varepsilon$ satisfies the $(PS)_c$ condition at the level c.*

Proof. Let $(u_n) \subset \mathcal{N}_\varepsilon$ be such that $J_\varepsilon(u_n) \to c$ and $\|J'_\varepsilon(u_n)_{|\mathcal{N}_\varepsilon}\|_* = o_n(1)$. Then there exists $(\lambda_n) \subset \mathbb{R}$ such that

$$
J'_\varepsilon(u_n) = \lambda_n T'_\varepsilon(u_n) + o_n(1) \tag{3.26}
$$

where $T_\varepsilon : H^s_\varepsilon \to \mathbb{R}$ is given by

$$
T_\varepsilon(u) = \|u\|^2_\varepsilon + \int_{\mathbb{R}^3} \phi^t_{|u|} |u|^2 \, dx - \int_{\mathbb{R}^3} g_\varepsilon(x, |u|^2)|u|^2 \, dx.
$$

Then, using $\langle J'_\varepsilon(u_n), u_n\rangle = 0$, the definition of g and the monotonicity of η we can see that

$$
\begin{aligned}
&\langle T'_\varepsilon(u_n), u_n\rangle \\
&= 2\|u_n\|^2_\varepsilon + 4\int_{\mathbb{R}^3} \phi^t_{|u_n|}|u_n|^2 \, dx - 2\int_{\mathbb{R}^3} g'_\varepsilon(x, |u_n|^2)|u_n|^4 \, dx - 2\int_{\mathbb{R}^3} g_\varepsilon(x, |u_n|^2)|u_n|^2 \, dx \\
&= -2\|u_n\|^2_\varepsilon + 2\int_{\mathbb{R}^3} g_\varepsilon(x, |u_n|^2)|u_n|^2 \, dx - 2\int_{\mathbb{R}^3} g'_\varepsilon(x, |u_n|^2)|u_n|^4 \, dx \\
&= -2\|u_n\|^2_\varepsilon + 2\int_{\Lambda_\varepsilon \cup \{|u_n|^2 < t_a\}} \left[g_\varepsilon(x, |u_n|^2)|u_n|^2 - g'_\varepsilon(x, |u_n|^2)|u_n|^4\right] dx \\
&\quad + 2\int_{\Lambda^c_\varepsilon \cap \{t_a \leq |u_n|^2 \leq T_a\}} \left[g_\varepsilon(x, |u_n|^2)|u_n|^2 - g'_\varepsilon(x, |u_n|^2)|u_n|^4\right] dx \\
&\quad + 2\int_{\Lambda^c_\varepsilon \cap \{|u_n|^2 > T_a\}} \left[g_\varepsilon(x, |u_n|^2)|u_n|^2 - g'_\varepsilon(x, |u_n|^2)|u_n|^4\right] dx \\
&\leq -2\|u_n\|^2_\varepsilon + \frac{2}{k}\int_{\Lambda^c_\varepsilon \cap \{|u_n|^2 > T_a\}} V_\varepsilon(x)|u_n|^2 \, dx \\
&\quad + 2\int_{\Lambda_\varepsilon \cup \{|u_n|^2 < t_a\}} \left[g_\varepsilon(x, |u_n|^2)|u_n|^2 - g'_\varepsilon(x, |u_n|^2)|u_n|^4\right] dx \\
&\quad + 2\int_{\Lambda^c_\varepsilon \cap \{t_a \leq |u_n|^2 \leq T_a\}} \left[g_\varepsilon(x, |u_n|^2)|u_n|^2 - g'_\varepsilon(x, |u_n|^2)|u_n|^4\right] dx \leq 0
\end{aligned}
$$

where we used $f'(t)t - f(t) \geq 0$ for any $t > 0$ in view of (f_4), condition (ξ_3), $f', \tilde{f}' \in C(\mathbb{R}^3)$ and recalling the definition of g we know that

$$
g'(x,t) = 0 \quad \forall t \geq T_a \quad \text{ and } \quad g'(x,t) = \tilde{f}'(t) \quad \forall x \in \mathbb{R}^3 \setminus \Lambda.
$$

Indeed, we obtain

$$\begin{aligned}\langle T'_\varepsilon(u_n),u_n\rangle &\le \left(\frac{2}{k}-2\right)\|u_n\|^2_\varepsilon + 2\int_{\Lambda_\varepsilon\cup\{|u_n|^2<t_a\}} \left[f(|u_n|^2)|u_n|^2 - f'(|u_n|^2)|u_n|^4\right]\,dx \\ &\quad - \int_{\Lambda_\varepsilon\cup\{|u_n|^2<t_a\}} (2^*_s-4)|u_n|^{2^*_s}\,dx \\ &\quad + 2\int_{\Lambda^c_\varepsilon\cap\{t_a\le|u_n|^2\le T_a\}} \left[\xi(|u_n|^2)|u_n|^2 - \xi'(|u_n|^2)|u_n|^4\right]\,dx \\ &\le \left(\frac{2}{k}-2\right)\|u_n\|^2_\varepsilon - (2^*_s-4)\int_{\Lambda_\varepsilon\cup\{|u_n|^2<t_a\}} |u_n|^{2^*_s}\,dx \\ &\le -(2^*_s-4)\int_{\Lambda_\varepsilon} |u_n|^{2^*_s}\,dx.\end{aligned}$$

Taking into account the above fact and the boundedness of (u_n) in H^s_ε, we can see that $\langle T'_\varepsilon(u_n),u_n\rangle \to \ell \le 0$. If $\ell=0$ we can use $\langle J'_\varepsilon(u_n),u_n\rangle=0$ to deduce that

$$0\le \left(1-\frac{1}{k}\right)\|u_n\|^2_\varepsilon \le o_n(1),$$

that is $\|u_n\|_\varepsilon\to 0$, which is impossible due to (3.2). As a consequence, $\ell<0$ and taking into account (3.26) we get $\lambda_n\to 0$, that is u_n is a $(PS)_c$ sequence for the unconstrained functional. The result follows from Lemma 3.2. □

As a consequence of the previous result we can see that

Corollary 3.1. *The critical points of the functional J_ε on $\mathcal{N}_\varepsilon$ are critical points of J_ε.*

In what follows, we recall the following useful compactness result for the autonomous problem (3.1) whose proof can be obtained arguing as in Proposition 3.4 in [43].

Lemma 3.3. *Let $(u_n)\subset\mathcal{N}_\mu$ be a sequence satisfying $J_\mu(u_n)\to c<\frac{s}{3}S_*^{\frac{3}{2s}}$. Then, up to subsequences, the following alternatives holds:*
- *(i) (u_n) strongly converges in $H^s(\mathbb{R}^3,\mathbb{R})$,*
- *(ii) there exists a sequence $(\tilde{y}_n)\subset\mathbb{R}^3$ such that, up to a subsequence, $v_n(x)=u_n(x+\tilde{y}_n)$ converges strongly in $H^s(\mathbb{R}^3,\mathbb{R})$.*

In particular, there exists a minimizer $w\in H^s(\mathbb{R}^3,\mathbb{R})$ for J_μ with $J_\mu(w)=c$.

Finally, we prove the following interesting relation between c_ε and c_{V_0}.

Lemma 3.4. *The numbers c_ε and c_{V_0} satisfy the following inequality*

$$\limsup_{\varepsilon\to 0} c_\varepsilon \le c_{V_0} < c_*.$$

Proof. Firstly, we note that $c_{V_0}<\frac{s}{3}S_*^{\frac{3}{2s}}=c_*$ by Lemma 3.1 in [43]. Now, in view of Lemma 3.3, there exists a positive ground state $w\in H^s(\mathbb{R}^3,\mathbb{R})$ to the autonomous problem (3.1), so that $J'_{V_0}(w)=0$ and $J_{V_0}(w)=c_{V_0}$. Moreover, we know (see Proposition 3.4 in [43]) that $w\in C^{1,\gamma}(\mathbb{R}^3,\mathbb{R})\cap L^\infty(\mathbb{R}^3,\mathbb{R})$, for some $\gamma>0$. Therefore, $|w(x)|\to 0$ as $|x|\to\infty$, and we can find $R>0$ such that $(-\Delta)^s w+\frac{V_0}{2}w\le 0$ in $|x|>R$. Using Lemma 4.3 in [31] we know that there exists a positive continuous function $\tilde{w}$ such that for $|x|>R$ (taking R larger if it is necessary), it holds $(-\Delta)^s\tilde{w}+\frac{V_0}{2}\tilde{w}=0$ and $\tilde{w}(x)=\frac{C_0}{|x|^{3+2s}}$. In view of the continuity of w and $\tilde{w}$ there exists some constant $C_1>0$ such that

$z = w - C_1\tilde{w} \leq 0$ on $|x| = R$. Moreover, we can see that $(-\Delta)^s z + \frac{V_0}{2} z \geq 0$ in $|x| \geq R$. Using the maximum principle we can deduce that $z \leq 0$ in $|x| \geq R$, that is

$$0 < w(x) \leq \frac{C}{|x|^{3+2s}} \quad \text{for } |x| >> 1. \tag{3.27}$$

Let $\eta \in C_c^\infty(\mathbb{R}^3, [0,1])$ be a cut-off function such that $\eta = 1$ in a neighborhood of zero $B_{\frac{\delta}{2}}$ and $\mathrm{supp}(\eta) \subset B_\delta \subset \Lambda$ for some $\delta > 0$. Let us define $w_\varepsilon(x) := \eta_\varepsilon(x) w(x) e^{\imath A(0)\cdot x}$, with $\eta_\varepsilon(x) = \eta(\varepsilon x)$ for $\varepsilon > 0$, and we observe that $|w_\varepsilon| = \eta_\varepsilon w$ and $w_\varepsilon \in H^s_\varepsilon$ in view of Lemma 2.3. Now we prove that

$$\lim_{\varepsilon\to 0} \|w_\varepsilon\|_\varepsilon^2 = \|w\|_{V_0}^2 \in (0, \infty). \tag{3.28}$$

Since it is clear that $\int_{\mathbb{R}^3} V_\varepsilon(x)|w_\varepsilon|^2 dx \to \int_{\mathbb{R}^3} V_0|w|^2 dx$, we only need to show that

$$\lim_{\varepsilon\to 0} [w_\varepsilon]^2_{A_\varepsilon} = [w]^2. \tag{3.29}$$

Using Lemma 5 in [47] we know that

$$[\eta_\varepsilon w] \to [w] \text{ as } \varepsilon \to 0. \tag{3.30}$$

On the other hand

$$\begin{aligned}
[w_\varepsilon]^2_{A_\varepsilon} &= \iint_{\mathbb{R}^6} \frac{|e^{\imath A(0)\cdot x}\eta_\varepsilon(x)w(x) - e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)} e^{\imath A(0)\cdot y}\eta_\varepsilon(y)w(y)|^2}{|x-y|^{3+2s}} dxdy \\
&= [\eta_\varepsilon w]^2 + \iint_{\mathbb{R}^6} \frac{\eta_\varepsilon^2(y)w^2(y)|e^{\imath[A_\varepsilon(\frac{x+y}{2}) - A(0)]\cdot(x-y)} - 1|^2}{|x-y|^{3+2s}} dxdy \\
&\quad + 2\Re \iint_{\mathbb{R}^6} \frac{(\eta_\varepsilon(x)w(x) - \eta_\varepsilon(y)w(y))\eta_\varepsilon(y)w(y)(1 - e^{-\imath[A_\varepsilon(\frac{x+y}{2}) - A(0)]\cdot(x-y)})}{|x-y|^{3+2s}} dxdy \\
&=: [\eta_\varepsilon w]^2 + X_\varepsilon + 2Y_\varepsilon.
\end{aligned}$$

Then, in view of $|Y_\varepsilon| \leq [\eta_\varepsilon w]\sqrt{X_\varepsilon}$ and (3.30), it is suffices to prove that $X_\varepsilon \to 0$ as $\varepsilon \to 0$ to deduce that (3.29) holds.
Let us note that for $0 < \beta < \alpha/(1 + \alpha - s)$,

$$\begin{aligned}
X_\varepsilon &\leq \int_{\mathbb{R}^3} w^2(y) dy \int_{|x-y|\geq \varepsilon^{-\beta}} \frac{|e^{\imath[A_\varepsilon(\frac{x+y}{2}) - A(0)]\cdot(x-y)} - 1|^2}{|x-y|^{3+2s}} dx \\
&\quad + \int_{\mathbb{R}^3} w^2(y) dy \int_{|x-y|<\varepsilon^{-\beta}} \frac{|e^{\imath[A_\varepsilon(\frac{x+y}{2}) - A(0)]\cdot(x-y)} - 1|^2}{|x-y|^{3+2s}} dx \\
&=: X^1_\varepsilon + X^2_\varepsilon.
\end{aligned} \tag{3.31}$$

Using $|e^{\imath t} - 1|^2 \leq 4$ and $w \in H^s(\mathbb{R}^3, \mathbb{R})$, we get

$$X^1_\varepsilon \leq C \int_{\mathbb{R}^3} w^2(y) dy \int_{\varepsilon^{-\beta}}^\infty \rho^{-1-2s} d\rho \leq C\, \varepsilon^{2\beta s} \to 0. \tag{3.32}$$

Since $|e^{\imath t}-1|^2\leq t^2$ for all $t\in\mathbb{R}$, $A\in C^{0,\alpha}(\mathbb{R}^3,\mathbb{R}^3)$ for $\alpha\in(0,1]$, and $|x+y|^2\leq 2(|x-y|^2+4|y|^2)$, we have

$$\begin{aligned} X_\varepsilon^2 &\leq \int_{\mathbb{R}^3} w^2(y)dy \int_{|x-y|<\varepsilon^{-\beta}} \frac{|A_\varepsilon\left(\frac{x+y}{2}\right)-A(0)|^2}{|x-y|^{3+2s-2}}dx \\ &\leq C\,\varepsilon^{2\alpha}\int_{\mathbb{R}^3} w^2(y)dy \int_{|x-y|<\varepsilon^{-\beta}} \frac{|x+y|^{2\alpha}}{|x-y|^{3+2s-2}}dx \\ &\leq C\,\varepsilon^{2\alpha}\Bigg(\int_{\mathbb{R}^3} w^2(y)dy \int_{|x-y|<\varepsilon^{-\beta}} \frac{1}{|x-y|^{3+2s-2-2\alpha}}dx \\ &\qquad + \int_{\mathbb{R}^3} |y|^{2\alpha}w^2(y)dy \int_{|x-y|<\varepsilon^{-\beta}} \frac{1}{|x-y|^{3+2s-2}}dx\Bigg) \\ &=: C\,\varepsilon^{2\alpha}(X_\varepsilon^{2,1}+X_\varepsilon^{2,2}). \end{aligned} \tag{3.33}$$

Then

$$X_\varepsilon^{2,1}=C\int_{\mathbb{R}^3} w^2(y)dy\int_0^{\varepsilon^{-\beta}}\rho^{1+2\alpha-2s}d\rho\leq C\,\varepsilon^{-2\beta(1+\alpha-s)}\,. \tag{3.34}$$

On the other hand, using (3.27), we infer that

$$\begin{aligned} X_\varepsilon^{2,2} &\leq C\int_{\mathbb{R}^3}|y|^{2\alpha}w^2(y)dy\int_0^{\varepsilon^{-\beta}}\rho^{1-2s}d\rho \\ &\leq C\,\varepsilon^{-2\beta(1-s)}\left[\int_{B_1(0)} w^2(y)dy+\int_{B_1^c(0)}\frac{1}{|y|^{2(3+2s)-2\alpha}}dy\right] \\ &\leq C\,\varepsilon^{-2\beta(1-s)}\,. \end{aligned} \tag{3.35}$$

Taking into account (3.31), (3.32), (3.33), (3.34) and (3.35) we can conclude that $X_\varepsilon\to 0$. Therefore (3.28) holds. Moreover, by (3.30), the Dominated Convergence Theorem, and the fact that H^s_ε is a Hilbert space, we can see that $|w_\varepsilon|=\eta_\varepsilon w$ strongly converges to w in $H^s(\mathbb{R}^3,\mathbb{R})$, so we deduce that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}\phi^t_{|w_\varepsilon|}|w_\varepsilon|^2dx=\int_{\mathbb{R}^3}\phi^t_w w^2dx. \tag{3.36}$$

Now, let $t_\varepsilon>0$ be the unique number such that

$$J_\varepsilon(t_\varepsilon w_\varepsilon)=\max_{t\geq 0}J_\varepsilon(tw_\varepsilon).$$

Then t_ε verifies

$$t_\varepsilon^2\|w_\varepsilon\|_\varepsilon^2+t_\varepsilon^4\int_{\mathbb{R}^3}\phi^t_{|w_\varepsilon|}|w_\varepsilon|^2dx=\int_{\mathbb{R}^3}g_\varepsilon(x,t_\varepsilon^2|w_\varepsilon|^2)|t_\varepsilon w_\varepsilon|^2dx=\int_{\mathbb{R}^3}f(t_\varepsilon^2|w_\varepsilon|^2)|t_\varepsilon w_\varepsilon|^2+|t_\varepsilon w_\varepsilon|^{2^*_s}dx \tag{3.37}$$

where we used $supp(\eta)\subset\Lambda$ and $g(x,t)=f(t)+t^{\frac{2^*_s-2}{2}}$ on Λ.
Let us prove that $t_\varepsilon\to 1$ as $\varepsilon\to 0$. Using that $\eta=1$ in $B_{\frac{\delta}{2}}$, that w is a continuous positive function, that $\frac{f(t^2)}{t^2}\geq 0$ for $t>0$ and that $2^*_s-4=\frac{2(4s-3)}{3-2s}>0$ we can see that

$$\frac{1}{t_\varepsilon^2}\|w_\varepsilon\|_\varepsilon^2+\int_{\mathbb{R}^3}\phi^t_{|w_\varepsilon|}|w_\varepsilon|^2dx\geq t_\varepsilon^{\frac{2(4s-3)}{3-2s}}\alpha_0^{2^*_s}|B_{\frac{\delta}{2}}|$$

where $\alpha_0=\min_{\bar{B}_{\frac{\delta}{2}}}w>0$. So, if $t_\varepsilon\to\infty$ as $\varepsilon\to 0$ then we can use (3.28) and (3.36) to deduce that $\int_{\mathbb{R}^3}\phi^t_w w^2dx=\infty$ which gives a contradiction. On the other hand, if $t_\varepsilon\to 0$ as $\varepsilon\to 0$ we can use (3.37), the growth assumptions on g, (3.28), (3.36) to infer that $\|w\|_0^2=0$ which is impossible. In conclusion $t_\varepsilon \to t_0 \in (0,\infty)$ as $\varepsilon \to 0$. Now, taking the limit as $\varepsilon \to 0$ in (3.37) and using (3.36), (3.28), we can see that

$$\frac{1}{t_0^2}\|w\|_{V_0}^2 + \int_{\mathbb{R}^3} \phi_w^t w^2 dx = \int_{\mathbb{R}^3} \frac{f(t_0^2 w^2)}{(t_0^2 w^2)} w^4 dx + t_0^{2_s^*-4} \int_{\mathbb{R}^3} |w_0|^{2_s^*} dx.$$

By $w \in \mathcal{N}_0$ it follows that

$$\left(\frac{1}{t_0^2} - 1\right) \|w\|_{V_0}^2 + \int_{\mathbb{R}^3} \phi_w^t w^2 dx = \int_{\mathbb{R}^3} \left(\frac{f(t_0^2 w^2)}{(t_0^2 w^2)} - \frac{f(w^2)}{w^2}\right) w^4 dx + (t_0^{2_s^*-4} - 1) \int_{\mathbb{R}^3} |w_0|^{2_s^*} dx,$$

and in view of (f_4), we can deduce that $t_0 = 1$. Then, applying the Dominated Convergence Theorem, we obtain that $\lim_{\varepsilon\to 0} J_\varepsilon(t_\varepsilon w_\varepsilon) = J_{V_0}(w) = c_{V_0}$. Since $c_\varepsilon \leq \max_{t\geq 0} J_\varepsilon(t w_\varepsilon) = J_\varepsilon(t_\varepsilon w_\varepsilon)$, we can conclude that $\limsup_{\varepsilon\to 0} c_\varepsilon \leq c_{V_0}$. □

4. Multiple solutions for the modified problem

This section is devoted to apply the Ljusternik-Schnirelmann category theory to prove a multiplicity result for the problem (1.10). We begin proving the following technical results.

Lemma 4.1. *Let $\varepsilon_n \to 0$ and $(u_n) \subset \mathcal{N}_{\varepsilon_n}$ be such that $J_{\varepsilon_n}(u_n) \to c_{V_0}$. Then there exists $(\tilde{y}_n) \subset \mathbb{R}^3$ such that $v_n(x) = |u_n|(x + \tilde{y}_n)$ has a convergent subsequence in $H^s(\mathbb{R}^3, \mathbb{R})$. Moreover, up to a subsequence, $y_n = \varepsilon_n \tilde{y}_n \to y_0$ for some $y_0 \in M$.*

Proof. Taking into account that $\langle J'_{\varepsilon_n}(u_n), u_n\rangle = 0$, that $J_{\varepsilon_n}(u_n) = c_{V_0} + o_n(1)$, Lemma 3.4 and arguing as in the first part of Lemma 3.2, it is easy to see that there exists $C > 0$ (independent of n) such that $\|u_n\|_{\varepsilon_n} \leq C$ for all $n \in \mathbb{N}$. Moreover, from Lemma 2.2, we also know that $(|u_n|)$ is bounded in $H^s(\mathbb{R}^3, \mathbb{R})$. Now, we prove that there exist a sequence $(\tilde{y}_n) \subset \mathbb{R}^3$, and constants $R > 0$ and $\gamma > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(\tilde{y}_n)} |u_n|^2 \, dx \geq \gamma > 0. \tag{4.1}$$

If by contradiction (4.1) does not hold, then for all $R > 0$ we get

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_n|^2 \, dx = 0.$$

From the boundedness $(|u_n|)$ and Lemma 2.4 we can see that $|u_n| \to 0$ in $L^q(\mathbb{R}^3, \mathbb{R})$ for any $q \in (2, 2_s^*)$. This fact combined with (f_1) and (f_2) gives

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} f(|u_n|^2)|u_n|^2 \, dx = 0 = \lim_{n\to\infty} \int_{\mathbb{R}^3} F(|u_n|^2) \, dx. \tag{4.2}$$

Moreover $|u_n| \to 0$ in $L^{\frac{12}{3+2t}}(\mathbb{R}^3, \mathbb{R})$, so using (4)-Lemma 2.5 we deduce that

$$\int_{\mathbb{R}^3} \phi_{|u_n|}^t |u_n|^2 dx \to 0. \tag{4.3}$$

Therefore

$$\int_{\mathbb{R}^3} G_{\varepsilon_n}(x, |u_n|^2) \, dx \leq \frac{1}{2_s^*} \int_{\Lambda_\varepsilon \cup \{|u_n|^2 \leq t_a\}} |u_n|^{2_s^*} \, dx + \frac{V_0}{2k} \int_{\Lambda_\varepsilon^c \cap \{|u_n|^2 > T_a\}} |u_n|^2 \, dx + o_n(1) \tag{4.4}$$

and

$$\int_{\mathbb{R}^3} g_{\varepsilon_n}(x, |u_n|^2)|u_n|^2 \, dx = \int_{\Lambda_\varepsilon \cup \{|u_n|^2 \leq t_a\}} |u_n|^{2_s^*} \, dx + \frac{V_0}{k} \int_{\Lambda_\varepsilon^c \cap \{|u_n|^2 > T_a\}} |u_n|^2 \, dx + o_n(1). \tag{4.5}$$

Using (4.3), (4.5) and $\langle J'_{\varepsilon_n}(u_n), u_n\rangle = 0$ we can deduce that

$$\|u_n\|^2_{\varepsilon_n} - \frac{V_0}{k}\int_{\Lambda^c_\varepsilon \cap\{|u_n|^2>T_a\}} |u_n|^2\,dx = \int_{\Lambda_\varepsilon\cup\{|u_n|^2\leq t_a\}} |u_n|^{2^*_s}\,dx. \tag{4.6}$$

Let $\ell \geq 0$ be such that

$$\|u_n\|^2_{\varepsilon_n} - \frac{V_0}{k}\int_{\Lambda^c_\varepsilon \cap\{|u_n|^2>T_a\}} |u_n|^2\,dx \to \ell.$$

If $\ell = 0$, then $u_n \to 0$ in H^s_ε so that $J_{\varepsilon_n}(u_n) \to 0$ which contradicts $c_{V_0} > 0$. Then $\ell > 0$. In view of (4.6) we can see that $\int_{\Lambda_\varepsilon\cup\{|u_n|^2\leq t_a\}} |u_n|^{2^*_s}\,dx \to \ell$. Taking into account $J_{\varepsilon_n}(u_n) \to c_{V_0}$, (4.4) and $\langle J'_{\varepsilon_n}(u_n), u_n\rangle = 0$ we can deduce that $\ell \leq \frac{3}{s}c_{V_0}$. From Lemma 2.2 and the definition of S_*, we know that

$$\|u_n\|^2_{\varepsilon_n} - \frac{V_0}{k}\int_{\Lambda^c_\varepsilon \cap\{|u_n|^2>T_a\}} |u_n|^2\,dx \geq S_*\left(\int_{\Lambda_\varepsilon\cup\{|u_n|^2\leq t_a\}} |u_n|^{2^*_s}\,dx\right)^{2/2^*_s},$$

and letting the limit as $n \to \infty$ we find $\ell \geq S_*\ell^{2/2^*_s}$ which combined with $\ell \leq \frac{3}{s}c_{V_0}$ implies that $c_{V_0} \geq \frac{s}{3}S_*^{\frac{3}{2s}}$ which is impossible in view of Lemma 3.4. Therefore (4.1) holds.

Now, we set $v_n(x) = |u_n|(x+\tilde{y}_n)$. Then (v_n) is bounded in $H^s(\mathbb{R}^3,\mathbb{R})$, and we may assume that $v_n \rightharpoonup v \not\equiv 0$ in $H^s(\mathbb{R}^3,\mathbb{R})$ as $n\to\infty$. Fix $t_n > 0$ such that $\tilde{v}_n = t_n v_n \in \mathcal{N}_{V_0}$. Using Lemma 2.2, we can see that

$$c_{V_0} \leq J_{V_0}(\tilde{v}_n) \leq \max_{t\geq 0} J_{\varepsilon_n}(tv_n) = J_{\varepsilon_n}(u_n)$$

which together with Lemma 3.4 implies that $J_{V_0}(\tilde{v}_n) \to c_{V_0}$. In particular, $\tilde{v}_n \nrightarrow 0$ in $H^s(\mathbb{R}^3,\mathbb{R})$. Since (v_n) and $(\tilde{v}_n)$ are bounded in $H^s(\mathbb{R}^3,\mathbb{R})$ and $\tilde{v}_n \nrightarrow 0$ in $H^s(\mathbb{R}^3,\mathbb{R})$, we deduce that $t_n \to t^* \geq 0$. Indeed $t^* > 0$ since $\tilde{v}_n \nrightarrow 0$ in $H^s(\mathbb{R}^3,\mathbb{R})$. From the uniqueness of the weak limit, we can deduce that $\tilde{v}_n \rightharpoonup \tilde{v} = t^* v \not\equiv 0$ in $H^s(\mathbb{R}^3,\mathbb{R})$. This combined with Lemma 3.3 yields

$$\tilde{v}_n \to \tilde{v} \text{ in } H^s(\mathbb{R}^3,\mathbb{R}). \tag{4.7}$$

As a consequence, $v_n \to v$ in $H^s(\mathbb{R}^3,\mathbb{R})$ as $n\to\infty$.

Now, we set $y_n = \varepsilon_n \tilde{y}_n$ and we show that (y_n) admits a subsequence, still denoted by y_n, such that $y_n \to y_0$ for some $y_0 \in \Lambda$ such that $V(y_0) = V_0$. Firstly, we prove that (y_n) is bounded. Assume by contradiction that, up to a subsequence, $|y_n| \to \infty$ as $n\to\infty$. Take $R > 0$ such that $\Lambda \subset B_R(0)$. Since we may suppose that $|y_n| > 2R$, we have that for any $z \in B_{R/\varepsilon_n}$

$$|\varepsilon_n z + y_n| \geq |y_n| - |\varepsilon_n z| > R.$$

Now, using $(u_n) \subset \mathcal{N}_{\varepsilon_n}$, (V_1), Lemma 2.2, Lemma 2.5, the definition of g and the change of variable $x \mapsto z + \tilde{y}_n$ we observe that

$$\begin{aligned}
[v_n]^2 + \int_{\mathbb{R}^3} V_0 v_n^2\,dx &\leq [v_n]^2 + \int_{\mathbb{R}^3} V_0 v_n^2\,dx + \int_{\mathbb{R}^3} \phi^t_{|v_n|}|v_n|^2 dx \\
&\leq \int_{\mathbb{R}^3} g(\varepsilon_n x + y_n, |v_n|^2)|v_n|^2\,dx \\
&\leq \int_{B_{\frac{R}{\varepsilon_n}}(0)} \tilde{f}(|v_n|^2)|v_n|^2\,dx + \int_{\mathbb{R}^3\setminus B_{\frac{R}{\varepsilon_n}}(0)} f(|v_n|^2)|v_n|^2 + |v_n|^{2^*_s}\,dx \\
&\leq \frac{V_0}{k}\int_{\mathbb{R}^3} |v_n|^2\,dx.
\end{aligned}$$

which implies that $v_n \to 0$ in $H^s(\mathbb{R}^3,\mathbb{R})$, that is a contradiction. Therefore, (y_n) is bounded and we may assume that $y_n \to y_0 \in \mathbb{R}^3$. If $y_0 \notin \overline{\Lambda}$, then we can argue as before to infer that $v_n \to 0$ in $H^s(\mathbb{R}^3,\mathbb{R})$, which is impossible. Hence $y_0 \in \overline{\Lambda}$. Now, suppose by contradiction that $V(y_0) > V_0$.

Then, using (4.7), Fatou's Lemma, the invariance of $\mathbb{R}^3$ by translations, Lemma 2.2 and Lemma 3.4, we get

$$\begin{aligned}
c_{V_0} = J_{V_0}(\tilde{v}) &< \frac{1}{2}[\tilde{v}]^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(y_0)\tilde{v}^2\, dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi^t_{|\tilde{v}|}\tilde{v}^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} F(|\tilde{v}|^2) + \frac{1}{2^*_s}\int_{\mathbb{R}^3} |\tilde{v}|^{2^*_s}\, dx \\
&\le \liminf_{n\to\infty}\Big[\frac{1}{2}[\tilde{v}_n]^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon_n x + y_n)|\tilde{v}_n|^2\, dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi^t_{|\tilde{v}_n|}|\tilde{v}_n|^2 dx \\
&- \frac{1}{2}\int_{\mathbb{R}^3} F(|\tilde{v}_n|^2) + \frac{1}{2^*_s}\int_{\mathbb{R}^3} |\tilde{v_n}|^{2^*_s}\, dx\Big] \\
&\le \liminf_{n\to\infty}\Big[\frac{t_n^2}{2}[|u_n|]^2 + \frac{t_n^2}{2}\int_{\mathbb{R}^3} V(\varepsilon_n z)|u_n|^2\, dz + \frac{t_n^4}{4}\int_{\mathbb{R}^3} \phi^t_{|u_n|}|u_n|^2 dx \\
&- \frac{1}{2}\int_{\mathbb{R}^3} F(|t_n u_n|^2) + \frac{t_n^{2^*_s}}{2^*_s}\int_{\mathbb{R}^3} |u_n|^{2^*_s}\, dz\Big] \\
&\le \liminf_{n\to\infty} J_{\varepsilon_n}(t_n u_n) \le \liminf_{n\to\infty} J_{\varepsilon_n}(u_n) = c_{V_0}
\end{aligned}$$

which gives a contradiction. Hence, $y_0 \in M$ and this ends the proof of lemma. □

Now, we aim to relate the number of positive solutions of (1.9) to the topology of the set Λ. For this reason, we take $\delta > 0$ such that

$$M_\delta = \{x \in \mathbb{R}^3 : \operatorname{dist}(x, M) \le \delta\} \subset \Lambda,$$

and we consider $\eta \in C_0^\infty(\mathbb{R}_+, [0,1])$ such that $\eta(t) = 1$ if $0 \le t \le \frac{\delta}{2}$ and $\eta(t) = 0$ if $t \ge \delta$.
For any $y \in \Lambda$, we introduce (see [10])

$$\Psi_{\varepsilon,y}(x) = \eta(|\,\varepsilon\, x - y|) w\left(\frac{\varepsilon\, x - y}{\varepsilon}\right) e^{\imath\tau_y\left(\frac{\varepsilon\, x - y}{\varepsilon}\right)},$$

where $\tau_y(x) = \sum_{j=1}^3 A_j(x)x_j$ and $w \in H^s(\mathbb{R}^3)$ is a positive ground state solution to the autonomous problem (3.1) (such a solution exists in view of Lemma 3.3).

Let $t_\varepsilon > 0$ be the unique number such that

$$\max_{t\ge 0} J_\varepsilon(t\Psi_{\varepsilon,y}) = J_\varepsilon(t_\varepsilon\Psi_{\varepsilon,y}).$$

Finally, we consider $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ defined by setting

$$\Phi_\varepsilon(y) = t_\varepsilon \Psi_{\varepsilon,y}.$$

Lemma 4.2. *The functional Φ_ε satisfies the following limit*

$$\lim_{\varepsilon\to 0} J_\varepsilon(\Phi_\varepsilon(y)) = c_{V_0} \text{ uniformly in } y \in M.$$

Proof. Assume by contradiction that there exist $\delta_0 > 0$, $(y_n) \subset M$ and $\varepsilon_n \to 0$ such that

$$|J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - c_{V_0}| \ge \delta_0. \tag{4.8}$$

Let us observe that by Lemma 4.1 in [10] and the Dominated Convergence Theorem we get

$$\begin{aligned}
&\|\Psi_{\varepsilon_n,y_n}\|^2_{\varepsilon_n} \to \|w\|^2_{V_0} \in (0,\infty) \text{ and } \int_{\mathbb{R}^3} \phi^t_{|\Psi_{\varepsilon_n,y_n}|}|\Psi_{\varepsilon_n,y_n}|^2 dx \to \int_{\mathbb{R}^3} \phi^t_w w^2 dx \\
&\|\Psi_{\varepsilon_n,y_n}\|_{L^{2^*_s}(\mathbb{R}^3)} \to \|w\|_{L^{2^*_s}(\mathbb{R}^3)}.
\end{aligned} \tag{4.9}$$

Concerning the second limit in (4.9), we note that $|\Psi_{\varepsilon,y}| = \eta(|\,\varepsilon\, x - y|)w\left(\frac{\varepsilon\, x-y}{\varepsilon}\right)$ converges strongly to w in $H^s(\mathbb{R}^3,\mathbb{R})$, so we use the following property (see (6) of Lemma 2.3 in [55]):

$$\text{if } u_n \to u \text{ in } H^s(\mathbb{R}^3,\mathbb{R}) \text{ then } \int_{\mathbb{R}^3} \phi^t_{u_n} u_n^2\, dx \to \int_{\mathbb{R}^3} \phi^t_u u^2\, dx.$$

On the other hand, since $\langle J'_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)), \Phi_{\varepsilon_n}(y_n)\rangle = 0$ and using the change of variable $z = \dfrac{\varepsilon_n\, x - y_n}{\varepsilon_n}$ it follows that

$$\begin{aligned} &t^2_{\varepsilon_n}\|\Psi_{\varepsilon_n,y_n}\|^2_{\varepsilon_n} + t^4_{\varepsilon_n}\int_{\mathbb{R}^3} \phi^t_{|\Psi_{\varepsilon_n,y_n}|}|\Psi_{\varepsilon_n,y_n}|^2 dz \\ &= \int_{\mathbb{R}^3} g(\varepsilon_n\, z + y_n, |t_{\varepsilon_n}\eta(|\,\varepsilon_n\, z|)w(z)|^2)|t_{\varepsilon_n}\eta(|\,\varepsilon_n\, z|)w(z)|^2 dz. \end{aligned}$$

If $z \in B_{\frac{\delta}{\varepsilon_n}}(0) \subset M_\delta \subset \Lambda$, then $\varepsilon_n\, z + y_n \in B_\delta(y_n) \subset M_\delta \subset \Lambda_\varepsilon$. Thus, being $g(x,t) = f(t) + t^{\frac{2^*_s-2}{2}}$ for all $x \in \Lambda$ and $\eta(t) = 0$ for $t \geq \delta$, we get

$$\begin{aligned} &t^2_{\varepsilon_n}\|\Psi_{\varepsilon_n,y_n}\|^2_{\varepsilon_n} + t^4_{\varepsilon_n}\int_{\mathbb{R}^3} \phi^t_{|\Psi_{\varepsilon_n,y_n}|}|\Psi_{\varepsilon_n,y_n}|^2 dz \\ &= \int_{\mathbb{R}^3} f(|t_{\varepsilon_n}\eta(|\,\varepsilon_n\, z|)w(z)|^2)|t_{\varepsilon_n}\eta(|\,\varepsilon_n\, z|)w(z)|^2 + |t_{\varepsilon_n}\eta(|\,\varepsilon_n\, z|)w(z)|^{2^*_s} dz. \end{aligned} \tag{4.10}$$

Since $\eta = 1$ in $B_{\frac{\delta}{2}}(0) \subset B_{\frac{\delta}{\varepsilon_n}}(0)$ for all n large enough, we get from (4.10)

$$\begin{aligned} &\frac{1}{t^2_{\varepsilon_n}}\|\Psi_{\varepsilon_n,y_n}\|^2_{\varepsilon} + \int_{\mathbb{R}^3} \phi^t_{|\Psi_{\varepsilon_n,y_n}|}\Psi^2_{\varepsilon_n,y_n} dx \\ &= \int_{\mathbb{R}^3} \frac{f(|t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}|^2) + |t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}|^{2^*_s-2}}{|t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}|^2}|\Psi_{\varepsilon_n,y_n}|^4 dx \\ &\geq t^{2^*_s-4}_{\varepsilon_n}\int_{B_{\frac{\delta}{2}}(0)} |w(z)|^{2^*_s}\, dz \\ &\geq t^{\frac{2(4s-3)}{3-2s}}_{\varepsilon_n} w(\hat{z})^{2^*_s}|B_{\frac{\delta}{2}}(0)|, \end{aligned} \tag{4.11}$$

where

$$w(\hat{z}) = \min_{z\in B_{\frac{\delta}{2}}} w(z) > 0.$$

Now, assume by contradiction that $t_{\varepsilon_n} \to \infty$. So, using $t_{\varepsilon_n} \to \infty$, $s \in (\frac{3}{4}, 1)$, (4.9) and (4.11) we obtain

$$\int_{\mathbb{R}^3} \phi^t_w w^2 dx = \infty,$$

that is a contradiction. Therefore (t_{ε_n}) is bounded and, up to subsequence, we may assume that $t_{\varepsilon_n} \to t_0$ for some $t_0 \geq 0$. Let us prove that $t_0 > 0$. Suppose by contradiction that $t_0 = 0$. Then, taking into account (4.9) and assumptions (g_1) and (g_2), we can see that (4.10) yields

$$\|t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}\|^2_{\varepsilon_n} \to 0$$

which is impossible because of (3.2). Hence $t_0 > 0$. Thus, letting the limit as $n \to \infty$ in (4.10), we deduce that

$$\frac{1}{t_0^2}\|w\|^2_{V_0} + \int_{\mathbb{R}^3} \phi^t_w w^2 dx = \int_{\mathbb{R}^3} \frac{f((t_0 w)^2) + (t_0 w)^{2^*_s-2}}{(t_0 w)^2}\, w^4\, dx.$$

Taking into account that $w \in \mathcal{N}_{V_0}$ and condition (f_4) we can infer that $t_0 = 1$. Then, letting the limit as $n \to \infty$ and using that $t_{\varepsilon_n} \to 1$ we can conclude that

$$\lim_{n\to\infty} J_{\varepsilon_n}(\Phi_{\varepsilon_n,y_n}) = J_{V_0}(w) = c_{V_0},$$

which contradicts (4.8). □

At this point, we are in the position to define the barycenter map. For any $\delta > 0$, we take $\rho = \rho(\delta) > 0$ such that $M_\delta \subset B_\rho$, and we consider $\Upsilon : \mathbb{R}^3 \to \mathbb{R}^3$ defined by setting

$$\Upsilon(x) = \begin{cases} x & \text{if } |x| < \rho \\ \frac{\rho x}{|x|} & \text{if } |x| \geq \rho. \end{cases}$$

We define the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^3$ as follows

$$\beta_\varepsilon(u) = \frac{\displaystyle\int_{\mathbb{R}^3} \Upsilon(\varepsilon\, x)|u(x)|^4\, dx}{\displaystyle\int_{\mathbb{R}^3} |u(x)|^4\, dx}.$$

Arguing as Lemma 4.3 in [10], it is easy to see that the function β_ε verifies the following limit:

Lemma 4.3.

$$\lim_{\varepsilon\to 0} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \textit{ uniformly in } y \in M.$$

At this point, we introduce a subset $\widetilde{\mathcal{N}}_\varepsilon$ of $\mathcal{N}_\varepsilon$ by taking a function $h_1 : \mathbb{R}^+ \to \mathbb{R}^+$ such that $h_1(\varepsilon) \to 0$ as $\varepsilon \to 0$, and setting

$$\widetilde{\mathcal{N}}_\varepsilon = \{u \in \mathcal{N}_\varepsilon : J_\varepsilon(u) \leq c_{V_0} + h_1(\varepsilon)\}.$$

Fixed $y \in M$, from Lemma 4.2 follows that $h_1(\varepsilon) = |J_\varepsilon(\Phi_\varepsilon(y)) - c_{V_0}| \to 0$ as $\varepsilon \to 0$. Therefore $\Phi_\varepsilon(y) \in \widetilde{\mathcal{N}}_\varepsilon$, and $\widetilde{\mathcal{N}}_\varepsilon \neq \emptyset$ for any $\varepsilon > 0$. Moreover, proceeding as in Lemma 4.5 in [10], we have:

Lemma 4.4.

$$\lim_{\varepsilon\to 0} \sup_{u\in\widetilde{\mathcal{N}}_\varepsilon} \operatorname{dist}(\beta_\varepsilon(u), M_\delta) = 0.$$

We conclude this section giving the proof of our multiplicity result for (1.10).

Theorem 4.1. *For any $\delta > 0$ such that $M_\delta \subset \Lambda$, there exists $\tilde{\varepsilon}_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, problem (1.10) has at least $cat_{M_\delta}(M)$ nontrivial solutions.*

Proof. Given $\delta > 0$ such that $M_\delta \subset \Lambda$, we can use Lemma 4.3, Lemma 4.2, Lemma 4.4 and argue as in [19] to deduce the existence of $\tilde{\varepsilon}_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the following diagram

$$M \overset{\Phi_\varepsilon}{\to} \widetilde{\mathcal{N}}_\varepsilon \overset{\beta_\varepsilon}{\to} M_\delta$$

is well defined and $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopically equivalent to the embedding $\iota : M \to M_\delta$. Thus $cat_{\widetilde{\mathcal{N}}_\varepsilon}(\widetilde{\mathcal{N}}_\varepsilon) \geq cat_{M_\delta}(M)$. It follows from Proposition 3.1 and standard Ljusternik-Schnirelmann theory that J_ε possesses at least $cat_{\widetilde{\mathcal{N}}_\varepsilon}(\widetilde{\mathcal{N}}_\varepsilon)$ critical points on $\mathcal{N}_\varepsilon$. Using Corollary 3.1 we can obtain $cat_{M_\delta}(M)$ nontrivial solutions for (1.10). □

5. Proof of Theorem 1.1

In this last section we provide the proof of our main result. Firstly, we develop a Moser iteration scheme [45] which will be the main key to deduce that the solutions to (1.9) are indeed solutions to (1.1).

Lemma 5.1. *Let $\varepsilon_n \to 0$ and $u_n \in \widetilde{\mathcal{N}}_{\varepsilon_n}$ be a solution to (1.10). Then $v_n = |u_n|(\cdot + \tilde{y}_n)$ satisfies $v_n \in L^\infty(\mathbb{R}^3, \mathbb{R})$ and there exists $C > 0$ such that*

$$\|v_n\|_{L^\infty(\mathbb{R}^3)} \leq C \text{ for all } n \in \mathbb{N},$$

where $\tilde{y}_n$ is given by Lemma 4.1. Moreover

$$\lim_{|x|\to\infty} v_n(x) = 0 \text{ uniformly in } n \in \mathbb{N}.$$

Proof. For any $L > 0$ we define $u_{L,n} := \min\{|u_n|, L\} \geq 0$ and we set $v_{L,n} = u_{L,n}^{2(\beta-1)} u_n$ where $\beta > 1$ will be chosen after (5.9). Taking $v_{L,n}$ as a test function in (1.10) we can see that

$$\begin{aligned}
&\Re\left(\iint_{\mathbb{R}^6} \frac{(u_n(x) - u_n(y)e^{\imath A(\frac{x+y}{2})\cdot(x-y)})}{|x-y|^{3+2s}} \overline{(u_n u_{L,n}^{2(\beta-1)}(x) - u_n u_{L,n}^{2(\beta-1)}(y) e^{\imath A(\frac{x+y}{2})\cdot(x-y)})}\, dxdy\right) \\
&= -\int_{\mathbb{R}^3} \phi^t_{|u_n|} |u_n|^2 u_{L,n}^{2(\beta-1)} dx + \int_{\mathbb{R}^3} g_{\varepsilon_n}(x, |u_n|^2)|u_n|^2 u_{L,n}^{2(\beta-1)}\, dx - \int_{\mathbb{R}^3} V_{\varepsilon_n}(x)|u_n|^2 u_{L,n}^{2(\beta-1)}\, dx. \qquad (5.1)
\end{aligned}$$

Let us note that

$$\begin{aligned}
&\Re\left[(u_n(x) - u_n(y)e^{\imath A(\frac{x+y}{2})\cdot(x-y)})\overline{(u_n u_{L,n}^{2(\beta-1)}(x) - u_n u_{L,n}^{2(\beta-1)}(y) e^{\imath A(\frac{x+y}{2})\cdot(x-y)})}\right] \\
&= \Re\Big[|u_n(x)|^2 v_L^{2(\beta-1)}(x) - u_n(x)\overline{u_n(y)} u_{L,n}^{2(\beta-1)}(y) e^{-\imath A(\frac{x+y}{2})\cdot(x-y)} - u_n(y)\overline{u_n(x)} u_{L,n}^{2(\beta-1)}(x) e^{\imath A(\frac{x+y}{2})\cdot(x-y)} \\
&+ |u_n(y)|^2 u_{L,n}^{2(\beta-1)}(y)\Big] \\
&\geq (|u_n(x)|^2 u_{L,n}^{2(\beta-1)}(x) - |u_n(x)||u_n(y)| u_{L,n}^{2(\beta-1)}(y) - |u_n(y)||u_n(x)| u_{L,n}^{2(\beta-1)}(x) + |u_n(y)|^2 u_{L,n}^{2(\beta-1)}(y) \\
&= (|u_n(x)| - |u_n(y)|)(|u_n(x)| u_{L,n}^{2(\beta-1)}(x) - |u_n(y)| u_{L,n}^{2(\beta-1)}(y)),
\end{aligned}$$

so we have

$$\begin{aligned}
&\Re\left(\iint_{\mathbb{R}^6} \frac{(u_n(x) - u_n(y)e^{\imath A(\frac{x+y}{2})\cdot(x-y)})}{|x-y|^{3+2s}} \overline{(u_n u_{L,n}^{2(\beta-1)}(x) - u_n u_{L,n}^{2(\beta-1)}(y) e^{\imath A(\frac{x+y}{2})\cdot(x-y)})}\, dxdy\right) \\
&\geq \iint_{\mathbb{R}^6} \frac{(|u_n(x)| - |u_n(y)|)}{|x-y|^{3+2s}} (|u_n(x)| u_{L,n}^{2(\beta-1)}(x) - |u_n(y)| u_{L,n}^{2(\beta-1)}(y))\, dxdy. \qquad (5.2)
\end{aligned}$$

For all $t \geq 0$, let us define

$$\gamma(t) = \gamma_{L,\beta}(t) = t t_L^{2(\beta-1)}$$

where $t_L = \min\{t, L\}$. Since γ is an increasing function, we have

$$(a-b)(\gamma(a) - \gamma(b)) \geq 0 \quad \text{for any } a, b \in \mathbb{R}.$$

Let us define the functions

$$\Lambda(t) = \frac{|t|^2}{2} \quad \text{and} \quad \Gamma(t) = \int_0^t (\gamma'(\tau))^{\frac{1}{2}} d\tau.$$

and we note that

$$\Lambda'(a-b)(\gamma(a) - \gamma(b)) \geq |\Gamma(a) - \Gamma(b)|^2 \text{ for any } a, b \in \mathbb{R}. \qquad (5.3)$$

Indeed, for any $a, b \in \mathbb{R}$ such that $a < b$, the Jensen inequality yields

$$\begin{aligned}\Lambda'(a-b)(\gamma(a)-\gamma(b)) &= (a-b)\int_b^a \gamma'(t)dt \\ &= (a-b)\int_b^a (\Gamma'(t))^2 dt \\ &\geq \left(\int_b^a \Gamma'(t)dt\right)^2 \\ &= (\Gamma(a)-\Gamma(b))^2.\end{aligned}$$

In similar fashion we can prove that if $a \geq b$ then $\Lambda'(a-b)(\gamma(a)-\gamma(b)) \geq (\Gamma(b)-\Gamma(a))^2$ that is (5.3) holds. Then, in view of (5.3), we can see that

$$|\Gamma(|u_n(x)|)-\Gamma(|u_n(y)|)|^2 \leq (|u_n(x)|-|u_n(y)|)((|u_n|u_{L,n}^{2(\beta-1)})(x)-(|u_n|u_{L,n}^{2(\beta-1)})(y)). \tag{5.4}$$

Taking into account (5.2) and (5.4), we obtain

$$\Re\left(\iint_{\mathbb{R}^6} \frac{(u_n(x)-u_n(y)e^{\imath A(\frac{x+y}{2})\cdot(x-y)})}{|x-y|^{3+2s}}\overline{(u_n u_{L,n}^{2(\beta-1)}(x)-u_n u_{L,n}^{2(\beta-1)}(y)e^{\imath A(\frac{x+y}{2})\cdot(x-y)})}\,dxdy\right) \geq [\Gamma(|u_n|)]^2. \tag{5.5}$$

Since $\Gamma(|u_n|) \geq \frac{1}{\beta}|u_n|u_{L,n}^{\beta-1}$ and using the fractional Sobolev embedding $\mathcal{D}^{s,2}(\mathbb{R}^3,\mathbb{R}) \subset L^{2_s^*}(\mathbb{R}^3,\mathbb{R})$ (see [26]), we deduce that

$$[\Gamma(|u_n|)]^2 \geq S_*\|\Gamma(|u_n|)\|^2_{L^{2_s^*}(\mathbb{R}^3)} \geq \left(\frac{1}{\beta}\right)^2 S_*\||u_n|u_{L,n}^{\beta-1}\|^2_{L^{2_s^*}(\mathbb{R}^3)}. \tag{5.6}$$

Putting together (5.1), (5.5), (5.6) and using (4) of Lemma 2.5, we can infer that

$$\left(\frac{1}{\beta}\right)^2 S_*\||u_n|u_{L,n}^{\beta-1}\|^2_{L^{2_s^*}(\mathbb{R}^3)} + \int_{\mathbb{R}^3} V_{\varepsilon_n}(x)|u_n|^2 u_{L,n}^{2(\beta-1)}dx \leq \int_{\mathbb{R}^3} g_{\varepsilon_n}(x,|u_n|^2)|u_n|^2 u_{L,n}^{2(\beta-1)}dx. \tag{5.7}$$

On the other hand, from assumptions (g_1) and (g_2), for any $\xi > 0$ there exists $C_\xi > 0$ such that

$$g_{\varepsilon_n}(x,t^2)t^2 \leq \xi|t|^2 + C_\xi|t|^{2_s^*} \text{ for all } t \in \mathbb{R}. \tag{5.8}$$

Taking $\xi \in (0, V_0)$ and using (5.7) and (5.8) we can see that

$$\|w_{L,n}\|^2_{L^{2_s^*}(\mathbb{R}^3)} \leq C\beta^2 \int_{\mathbb{R}^3} |u_n|^{2_s^*} u_{L,n}^{2(\beta-1)}, \tag{5.9}$$

where $w_{L,n} := |u_n|u_{L,n}^{\beta-1}$.

Now, we take $\beta = \frac{2_s^*}{2}$ and fix $R > 0$. Recalling that $0 \leq u_{L,n} \leq |u_n|$ and applying Hölder inequality we have

$$\begin{aligned}\int_{\mathbb{R}^3} |u_n|^{2_s^*} u_{L,n}^{2(\beta-1)}dx &= \int_{\mathbb{R}^3} |u_n|^{2_s^*-2}|u_n|^2 u_{L,n}^{2_s^*-2}dx \\ &= \int_{\mathbb{R}^3} |u_n|^{2_s^*-2}(|u_n|u_{L,n}^{\frac{2_s^*-2}{2}})^2dx \\ &\leq \int_{\{|u_n|<R\}} R^{2_s^*-2}|u_n|^{2_s^*}dx + \int_{\{|u_n|>R\}} |u_n|^{2_s^*-2}(|u_n|u_{L,n}^{\frac{2_s^*-2}{2}})^2dx \\ &\leq \int_{\{|u_n|<R\}} R^{2_s^*-2}|u_n|^{2_s^*}dx + \left(\int_{\{|u_n|>R\}} |u_n|^{2_s^*}dx\right)^{\frac{2_s^*-2}{2_s^*}} \left(\int_{\mathbb{R}^3} (|u_n|u_{L,n}^{\frac{2_s^*-2}{2}})^{2_s^*}dx\right)^{\frac{2}{2_s^*}}.\end{aligned} \tag{5.10}$$

Since $(|u_n|)$ is bounded in $H^s(\mathbb{R}^3,\mathbb{R})$, we can see that for any R sufficiently large

$$\left(\int_{\{|u_n|>R\}} |u_n|^{2^*_s} dx\right)^{\frac{2^*_s-2}{2^*_s}} \leq \frac{1}{2\beta^2}. \tag{5.11}$$

Putting together (5.9), (5.10) and (5.11) we get

$$\left(\int_{\mathbb{R}^3} (|u_n| u_{L,n}^{\frac{2^*_s-2}{2}})^{2^*_s}\right)^{\frac{2}{2^*_s}} \leq C\beta^2 \int_{\mathbb{R}^3} R^{2^*_s-2}|u_n|^{2^*_s} dx < \infty$$

and taking the limit as $L\to\infty$ we obtain $|u_n|\in L^{\frac{(2^*_s)^2}{2}}(\mathbb{R}^3,\mathbb{R})$.

Now, using $0\leq u_{L,n}\leq |u_n|$ and passing to the limit as $L\to\infty$ in (5.9) we have

$$\|u_n\|^{2\beta}_{L^{\beta 2^*_s}(\mathbb{R}^3)} \leq C\beta^2 \int_{\mathbb{R}^3} |u_n|^{2^*_s+2(\beta-1)},$$

from which we deduce that

$$\left(\int_{\mathbb{R}^3} |u_n|^{\beta 2^*_s} dx\right)^{\frac{1}{(\beta-1)2^*_s}} \leq C\beta^{\frac{1}{\beta-1}} \left(\int_{\mathbb{R}^3} |u_n|^{2^*_s+2(\beta-1)}\right)^{\frac{1}{2(\beta-1)}}.$$

For $m\geq 1$ we define β_{m+1} inductively so that $2^*_s+2(\beta_{m+1}-1)=2^*_s\beta_m$ and $\beta_1=\frac{2^*_s}{2}$. Then we have

$$\left(\int_{\mathbb{R}^3} |u_n|^{\beta_{m+1} 2^*_s} dx\right)^{\frac{1}{(\beta_{m+1}-1)2^*_s}} \leq C\beta_{m+1}^{\frac{1}{\beta_{m+1}-1}} \left(\int_{\mathbb{R}^3} |u_n|^{2^*_s\beta_m}\right)^{\frac{1}{2^*_s(\beta_m-1)}}.$$

Let us define

$$D_m = \left(\int_{\mathbb{R}^3} |u_n|^{2^*_s\beta_m}\right)^{\frac{1}{2^*_s(\beta_m-1)}}.$$

Using an iteration argument, we can find $C_0>0$ independent of m such that

$$D_{m+1} \leq \prod_{k=1}^{m} C\beta_{k+1}^{\frac{1}{\beta_{k+1}-1}} D_1 \leq C_0 D_1.$$

Taking the limit as $m\to\infty$ we get

$$\|u_n\|_{L^\infty(\mathbb{R}^3)} \leq C_0 D_1 =: K \text{ for all } n\in\mathbb{N}. \tag{5.12}$$

Moreover, by interpolation, $(|u_n|)$ strongly converges in $L^r(\mathbb{R}^3,\mathbb{R})$ for all $r\in(2,\infty)$, and in view of the growth assumptions on g, also $g(\varepsilon\, x,|u_n|^2)|u_n|$ strongly converges in the same Lebesgue spaces. Now, we aim to prove that $|u_n|$ is a weak subsolution to

$$\begin{cases} (-\Delta)^s v + V_0 v = g_{\varepsilon_n}(x,v^2)v & \text{in } \mathbb{R}^3 \\ v\geq 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{5.13}$$

In some sense, we are going to prove that a Kato's inequality holds for the modified problem (1.10). Fix $\varphi\in C^\infty_c(\mathbb{R}^3,\mathbb{R})$ such that $\varphi\geq 0$, and we take $\psi_{\delta,n}=\frac{u_n}{u_{\delta,n}}\varphi$ as test function in (1.9), where we set $u_{\delta,n}=\sqrt{|u_n|^2+\delta^2}$ for $\delta>0$. We note that $\psi_{\delta,n}\in H^s_{\varepsilon_n}$ for all $\delta>0$ and $n\in\mathbb{N}$. Indeed

$\int_{\mathbb{R}^3} V_{\varepsilon_n}(x)|\psi_{\delta,n}|^2 dx \leq \int_{\mathrm{supp}(\varphi)} V_{\varepsilon_n}(x)\varphi^2 dx < \infty$. On the other hand, we can observe

$$\begin{aligned}\psi_{\delta,n}(x)-\psi_{\delta,n}(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)} &= \left(\frac{u_n(x)}{u_{\delta,n}(x)}\right)\varphi(x) - \left(\frac{u_n(y)}{u_{\delta,n}(y)}\right)\varphi(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)} \\ &= \left[\left(\frac{u_n(x)}{u_{\delta,n}(x)}\right) - \left(\frac{u_n(y)}{u_{\delta,n}(x)}\right)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}\right]\varphi(x) \\ &+ [\varphi(x)-\varphi(y)]\left(\frac{u_n(y)}{u_{\delta,n}(x)}\right)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)} \\ &+ \left(\frac{u_n(y)}{u_{\delta,n}(x)} - \frac{u_n(y)}{u_{\delta,n}(y)}\right)\varphi(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}\end{aligned}$$

which implies that

$$\begin{aligned}&|\psi_{\delta,n}(x)-\psi_{\delta,n}(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}|^2 \\ &\leq \frac{4}{\delta^2}|u_n(x)-u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}|^2\|\varphi\|^2_{L^\infty(\mathbb{R}^3)} + \frac{4}{\delta^2}|\varphi(x)-\varphi(y)|^2\||u_n|\|^2_{L^\infty(\mathbb{R}^3)} \\ &+ \frac{4}{\delta^4}\||u_n|\|^2_{L^\infty(\mathbb{R}^3)}\|\varphi\|^2_{L^\infty(\mathbb{R}^3)}|u_{\delta,n}(y)-u_{\delta,n}(x)|^2 \\ &\leq \frac{4}{\delta^2}|u_n(x)-u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}|^2\|\varphi\|^2_{L^\infty(\mathbb{R}^3)} + \frac{4K^2}{\delta^2}|\varphi(x)-\varphi(y)|^2 \\ &+ \frac{4K^2}{\delta^4}\|\varphi\|^2_{L^\infty(\mathbb{R}^3)}||u_n(y)|-|u_n(x)||^2\end{aligned}$$

where we used $|z+w+k|^2 \leq 4(|z|^2+|w|^2+|k|^2)$ for all $z,w,k\in\mathbb{C}$, $|e^{\imath t}|=1$ for all $t\in\mathbb{R}$, $u_{\delta,n}\geq\delta$, $|\frac{u_n}{u_{\delta,n}}|\leq 1$, (5.12) and $|\sqrt{|z|^2+\delta^2}-\sqrt{|w|^2+\delta^2}| \leq ||z|-|w||$ for all $z,w\in\mathbb{C}$.
Since $u_n\in H^s_{\varepsilon_n}$, $|u_n|\in H^s(\mathbb{R}^3,\mathbb{R})$ (by Lemma 2.2) and $\varphi\in C^\infty_c(\mathbb{R}^3,\mathbb{R})$, we deduce that $\psi_{\delta,n}\in H^s_{\varepsilon_n}$.
Then we have

$$\begin{aligned}&\Re\left[\iint_{\mathbb{R}^6}\frac{(u_n(x)-u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)})}{|x-y|^{3+2s}}\left(\frac{\overline{u_n(x)}}{u_{\delta,n}(x)}\varphi(x) - \frac{\overline{u_n(y)}}{u_{\delta,n}(y)}\varphi(y)e^{-\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}\right)dxdy\right] \\ &+\int_{\mathbb{R}^3}V_{\varepsilon_n}(x)\frac{|u_n|^2}{u_{\delta,n}}\varphi dx + \int_{\mathbb{R}^3}\phi^t_{|u_n|}\frac{|u_n|^2}{u_{\delta,n}}\varphi dx = \int_{\mathbb{R}^3}g_{\varepsilon_n}(x,|u_n|^2)\frac{|u_n|^2}{u_{\delta,n}}\varphi dx. \qquad (5.14)\end{aligned}$$

Now, using $\Re(z)\leq|z|$ for all $z\in\mathbb{C}$ and $|e^{\imath t}|=1$ for all $t\in\mathbb{R}$, we have

$$\begin{aligned}&\Re\left[(u_n(x)-u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)})\left(\frac{\overline{u_n(x)}}{u_{\delta,n}(x)}\varphi(x) - \frac{\overline{u_n(y)}}{u_{\delta,n}(y)}\varphi(y)e^{-\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}\right)\right] \\ &=\Re\left[\frac{|u_n(x)|^2}{u_{\delta,n}(x)}\varphi(x) + \frac{|u_n(y)|^2}{u_{\delta,n}(y)}\varphi(y) - \frac{u_n(x)\overline{u_n(y)}}{u_{\delta,n}(y)}\varphi(y)e^{-\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)} - \frac{u_n(y)\overline{u_n(x)}}{u_{\delta,n}(x)}\varphi(x)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}\right] \\ &\geq\left[\frac{|u_n(x)|^2}{u_{\delta,n}(x)}\varphi(x) + \frac{|u_n(y)|^2}{u_{\delta,n}(y)}\varphi(y) - |u_n(x)|\frac{|u_n(y)|}{u_{\delta,n}(y)}\varphi(y) - |u_n(y)|\frac{|u_n(x)|}{u_{\delta,n}(x)}\varphi(x)\right]. \qquad (5.15)\end{aligned}$$

Let us note that

$$\begin{aligned}
&\frac{|u_n(x)|^2}{u_{\delta,n}(x)}\varphi(x)+\frac{|u_n(y)|^2}{u_{\delta,n}(y)}\varphi(y)-|u_n(x)|\frac{|u_n(y)|}{u_{\delta,n}(y)}\varphi(y)-|u_n(y)|\frac{|u_n(x)|}{u_{\delta,n}(x)}\varphi(x)\\
&=\frac{|u_n(x)|}{u_{\delta,n}(x)}(|u_n(x)|-|u_n(y)|)\varphi(x)-\frac{|u_n(y)|}{u_{\delta,n}(y)}(|u_n(x)|-|u_n(y)|)\varphi(y)\\
&=\left[\frac{|u_n(x)|}{u_{\delta,n}(x)}(|u_n(x)|-|u_n(y)|)\varphi(x)-\frac{|u_n(x)|}{u_{\delta,n}(x)}(|u_n(x)|-|u_n(y)|)\varphi(y)\right]\\
&+\left(\frac{|u_n(x)|}{u_{\delta,n}(x)}-\frac{|u_n(y)|}{u_{\delta,n}(y)}\right)(|u_n(x)|-|u_n(y)|)\varphi(y)\\
&=\frac{|u_n(x)|}{u_{\delta,n}(x)}(|u_n(x)|-|u_n(y)|)(\varphi(x)-\varphi(y))+\left(\frac{|u_n(x)|}{u_{\delta,n}(x)}-\frac{|u_n(y)|}{u_{\delta,n}(y)}\right)(|u_n(x)|-|u_n(y)|)\varphi(y)\\
&\geq\frac{|u_n(x)|}{u_{\delta,n}(x)}(|u_n(x)|-|u_n(y)|)(\varphi(x)-\varphi(y)) \qquad (5.16)
\end{aligned}$$

where in the last inequality we used the fact that

$$\left(\frac{|u_n(x)|}{u_{\delta,n}(x)}-\frac{|u_n(y)|}{u_{\delta,n}(y)}\right)(|u_n(x)|-|u_n(y)|)\varphi(y)\geq 0$$

because

$$h(t)=\frac{t}{\sqrt{t^2+\delta^2}} \text{ is increasing for } t\geq 0 \quad \text{and} \quad \varphi\geq 0 \text{ in } \mathbb{R}^3.$$

Since

$$\frac{|\frac{|u_n(x)|}{u_{\delta,n}(x)}(|u_n(x)|-|u_n(y)|)(\varphi(x)-\varphi(y))|}{|x-y|^{3+2s}}\leq\frac{||u_n(x)|-|u_n(y)||}{|x-y|^{\frac{3+2s}{2}}}\frac{|\varphi(x)-\varphi(y)|}{|x-y|^{\frac{3+2s}{2}}}\in L^1(\mathbb{R}^6),$$

and $\frac{|u_n(x)|}{u_{\delta,n}(x)}\to 1$ a.e. in $\mathbb{R}^3$ as $\delta\to 0$, we can use (5.15), (5.16) and the Dominated Convergence Theorem to deduce that

$$\begin{aligned}
&\limsup_{\delta\to 0}\Re\left[\iint_{\mathbb{R}^6}\frac{(u_n(x)-u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)})}{|x-y|^{3+2s}}\left(\frac{\overline{u_n(x)}}{u_{\delta,n}(x)}\varphi(x)-\frac{\overline{u_n(y)}}{u_{\delta,n}(y)}\varphi(y)e^{-\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}\right)dxdy\right]\\
&\geq\limsup_{\delta\to 0}\iint_{\mathbb{R}^6}\frac{|u_n(x)|}{u_{\delta,n}(x)}(|u_n(x)|-|u_n(y)|)(\varphi(x)-\varphi(y))\frac{dxdy}{|x-y|^{3+2s}}\\
&=\iint_{\mathbb{R}^6}\frac{(|u_n(x)|-|u_n(y)|)(\varphi(x)-\varphi(y))}{|x-y|^{3+2s}}dxdy. \qquad (5.17)
\end{aligned}$$

On the other hand, from the Dominated Convergence Theorem again (we recall that $\frac{|u_n|^2}{u_{\delta,n}}\leq|u_n|$), Fatou's Lemma and $\varphi\in C^\infty_c(\mathbb{R}^3,\mathbb{R})$ we can see that

$$\lim_{\delta\to 0}\int_{\mathbb{R}^3}V_{\varepsilon_n}(x)\frac{|u_n|^2}{u_{\delta,n}}\varphi dx=\int_{\mathbb{R}^3}V_{\varepsilon_n}(x)|u_n|\varphi dx\geq\int_{\mathbb{R}^3}V_0|u_n|\varphi dx \qquad (5.18)$$

$$\liminf_{\delta\to 0}\int_{\mathbb{R}^3}\phi^t_{|u_n|}\frac{|u_n|^2}{u_{\delta,n}}\varphi dx\geq\int_{\mathbb{R}^3}\phi^t_{|u|}|u|\varphi dx\geq 0 \qquad (5.19)$$

and

$$\lim_{\delta\to 0}\int_{\mathbb{R}^3}g_{\varepsilon_n}(x,|u_n|^2)\frac{|u_n|^2}{u_{\delta,n}}\varphi dx=\int_{\mathbb{R}^3}g_{\varepsilon_n}(x,|u_n|^2)|u_n|\varphi dx. \qquad (5.20)$$

Putting together (5.14), (5.17), (5.19), (5.18) and (5.20) we can deduce that

$$\iint_{\mathbb{R}^6} \frac{(|u_n(x)|-|u_n(y)|)(\varphi(x)-\varphi(y))}{|x-y|^{3+2s}}dxdy + \int_{\mathbb{R}^3} V_0|u_n|\varphi dx \leq \int_{\mathbb{R}^3} g_{\varepsilon_n}(x,|u_n|^2)|u_n|\varphi dx$$

for any $\varphi \in C_c^\infty(\mathbb{R}^3,\mathbb{R})$ such that $\varphi \geq 0$, that is $|u_n|$ is a weak subsolution to (5.13). Now, it is clear that $v_n = |u_n|(\cdot + \tilde{y}_n)$ solves

$$(-\Delta)^s v_n + V_0 v_n \leq g(\varepsilon_n\, x + \varepsilon_n\, \tilde{y}_n, v_n^2)v_n \text{ in } \mathbb{R}^3. \tag{5.21}$$

Let us denote by $z_n \in H^s(\mathbb{R}^3,\mathbb{R})$ the unique solution to

$$(-\Delta)^s z_n + V_0 z_n = g_n \text{ in } \mathbb{R}^3, \tag{5.22}$$

where

$$g_n := g(\varepsilon_n\, x + \varepsilon_n\, \tilde{y}_n, v_n^2)v_n \in L^r(\mathbb{R}^3,\mathbb{R}) \quad \forall r \in [2,\infty].$$

Since (5.12) yields $\|v_n\|_{L^\infty(\mathbb{R}^3)} \leq C$ for all $n \in \mathbb{N}$, by interpolation we know that $v_n \to v$ strongly converges in $L^r(\mathbb{R}^3,\mathbb{R})$ for all $r \in (2,\infty)$, for some $v \in L^r(\mathbb{R}^3,\mathbb{R})$, and from the growth assumptions on f, we can see that also $g_n \to f(v^2)v$ in $L^r(\mathbb{R}^3,\mathbb{R})$ and $\|g_n\|_{L^\infty(\mathbb{R}^3)} \leq C$ for all $n \in \mathbb{N}$. In view of [31], we deduce that $z_n = \mathcal{K} * g_n$, where $\mathcal{K}$ is the Bessel kernel, and arguing as in [4], we obtain that $|z_n(x)| \to 0$ as $|x| \to \infty$ uniformly with respect to $n \in \mathbb{N}$. Since v_n satisfies (5.21) and z_n solves (5.22), by comparison it is easy to see that $0 \leq v_n \leq z_n$ a.e. in $\mathbb{R}^3$ and for all $n \in \mathbb{N}$. Then we can conclude that $v_n(x) \to 0$ as $|x| \to \infty$ uniformly with respect to $n \in \mathbb{N}$. □

Now we are ready to give the proof of the main result of this paper.

Proof of Theorem 1.1. Let $\delta > 0$ be such that $M_\delta \subset \Lambda$, and we show that there exists $\hat{\varepsilon}_\delta > 0$ such that for any $\varepsilon \in (0,\hat{\varepsilon}_\delta)$ and any solution $u_\varepsilon \in \widetilde{\mathcal{N}}_\varepsilon$ of (1.10) we have

$$\|u_\varepsilon\|_{L^\infty(\mathbb{R}^3\setminus\Lambda_\varepsilon)} < t_a. \tag{5.23}$$

Assume by contradiction that for some sequence $\varepsilon_n \to 0$ we can obtain $u_n := u_{\varepsilon_n} \in \widetilde{\mathcal{N}}_{\varepsilon_n}$ such that

$$\|u_n\|_{L^\infty(\mathbb{R}^3\setminus\Lambda_\varepsilon)} \geq t_a. \tag{5.24}$$

Since $J_{\varepsilon_n}(u_n) \leq c_{V_0} + h_1(\varepsilon_n)$, we can argue as in the first part of Lemma 4.1 to see that $J_{\varepsilon_n}(u_n) \to c_{V_0}$. Using Lemma 4.1 there exists $(\tilde{y}_n) \subset \mathbb{R}^3$ such that $\varepsilon_n\, \tilde{y}_n \to y_0$ for some $y_0 \in M$. Now, we can find $r > 0$ such that, for some subsequence still denoted by itself, we obtain $B_r(\tilde{y}_n) \subset \Lambda$ for all $n \in \mathbb{N}$. Therefore $B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n) \subset \Lambda_{\varepsilon_n}$ $n \in \mathbb{N}$. As a consequence

$$\mathbb{R}^3 \setminus \Lambda_{\varepsilon_n} \subset \mathbb{R}^3 \setminus B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n) \text{ for any } n \in \mathbb{N}.$$

In view of Lemma 5.1, there exists $R > 0$ such that

$$v_n(x) < t_a \text{ for } |x| \geq R, n \in \mathbb{N},$$

where $v_n(x) = |u_n|(x + \tilde{y}_n)$. Hence $|u_n(x)| < t_a$ for any $x \in \mathbb{R}^3 \setminus B_R(\tilde{y}_n)$ and $n \in \mathbb{N}$. Then there exists $\nu \in \mathbb{N}$ such that for any $n \geq \nu$ and $r/\varepsilon_n > R$ it holds

$$\mathbb{R}^3 \setminus \Lambda_{\varepsilon_n} \subset \mathbb{R}^3 \setminus B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n) \subset \mathbb{R}^3 \setminus B_R(\tilde{y}_n).$$

Then $|u_n(x)| < t_a$ for any $x \in \mathbb{R}^3 \setminus \Lambda_{\varepsilon_n}$ and $n \geq \nu$, and this contradicts (5.24).

Let $\tilde{\varepsilon}_\delta > 0$ be given by Theorem 4.1 and we set $\varepsilon_\delta = \min\{\tilde{\varepsilon}_\delta, \hat{\varepsilon}_\delta\}$. Applying Theorem 4.1 we obtain $cat_{M_\delta}(M)$ nontrivial solutions to (1.10). If $u \in H^s_\varepsilon$ is one of these solutions, then $u \in \widetilde{\mathcal{N}}_\varepsilon$, and in view of (5.23) and the definition of g we can infer that u is also a solution to (1.10). Observing that $\hat{u}_\varepsilon(x) = u_\varepsilon(x/\varepsilon)$ is a solution to (1.1), we can deduce that (1.1) has at least $cat_{M_\delta}(M)$ nontrivial solutions. Finally, we study the behavior of the maximum points of $|\hat{u}_n|$. Take $\varepsilon_n \to 0$ and (u_n) a sequence of solutions to (1.10). In view of (g_1), there exists $\gamma \in (0, t_a)$ such that

$$g_\varepsilon(x, t^2)t^2 \leq \frac{V_0}{2}t^2, \text{ for all } x \in \mathbb{R}^3, |t| \leq \gamma. \tag{5.25}$$

Using a similar discussion as above, we can take $R > 0$ such that

$$\|u_n\|_{L^\infty(B^c_R(\tilde{y}_n))} < \gamma. \tag{5.26}$$

Up to a subsequence, we may also assume that

$$\|u_n\|_{L^\infty(B_R(\tilde{y}_n))} \geq \gamma. \tag{5.27}$$

Indeed, if (5.27) is not true, we get $\|u_n\|_{L^\infty(\mathbb{R}^3)} < \gamma$, and it follows from $J'_{\varepsilon_n}(u_n) = 0$, (5.25) and Lemma 2.2 that

$$\begin{aligned} [|u_n|]^2 + \int_{\mathbb{R}^3} V_0|u_n|^2 dx &\leq \|u_n\|^2_{\varepsilon_n} + \int_{\mathbb{R}^3} \phi^t_{|u_n|}|u_n|^2 dx \\ &= \int_{\mathbb{R}^3} g_{\varepsilon_n}(x, |u_n|^2)|u_n|^2 \, dx \\ &\leq \frac{V_0}{2} \int_{\mathbb{R}^3} |u_n|^2 \, dx \end{aligned}$$

which implies that $\||u_n|\|_{H^s(\mathbb{R}^3)} = 0$, that is a contradiction. Then (5.27) holds.
Using (5.26) and (5.27), we can infer that the maximum points p_n of $|u_n|$ belong to $B_R(\tilde{y}_n)$, that is $p_n = \tilde{y}_n + q_n$ for some $q_n \in B_R$. Recalling that the associated solution of (1.1) is of the form $\hat{u}_n(x) = u_n(x/\varepsilon_n)$, we can see that a maximum point η_{ε_n} of $|\hat{u}_n|$ is $\eta_{\varepsilon_n} = \varepsilon_n \tilde{y}_n + \varepsilon_n q_n$. Since $q_n \in B_R$, $\varepsilon_n \tilde{y}_n \to y_0$ and $V(y_0) = V_0$, from the continuity of V we can conclude that

$$\lim_{n\to\infty} V(\eta_{\varepsilon_n}) = V_0.$$

Finally, we give an estimate on the decay of $|\hat{u}_n|$. Invoking Lemma 4.3 in [31], we can find a function w such that

$$0 < w(x) \leq \frac{C}{1 + |x|^{3+2s}}, \tag{5.28}$$

and

$$(-\Delta)^s w + \frac{V_0}{2} w \geq 0 \text{ in } \mathbb{R}^3 \setminus B_{R_1} \tag{5.29}$$

for some suitable $R_1 > 0$. Using Lemma 5.1, we know that $v_n(x) \to 0$ as $|x| \to \infty$ uniformly in $n \in \mathbb{N}$, so there exists $R_2 > 0$ such that

$$h_n = g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, v_n^2)v_n \leq \frac{V_0}{2} v_n \text{ in } B^c_{R_2}. \tag{5.30}$$

Let us denote by w_n the unique solution to

$$(-\Delta)^s w_n + V_0 w_n = h_n \text{ in } \mathbb{R}^3.$$

Then $w_n(x) \to 0$ as $|x| \to \infty$ uniformly in $n \in \mathbb{N}$, and by comparison $0 \leq v_n \leq w_n$ in $\mathbb{R}^3$. Moreover, in view of (5.30) and $\phi^t_{|w_n|} \geq 0$, it holds

$$(-\Delta)^s w_n + \frac{V_0}{2} w_n \leq h_n - \frac{V_0}{2} w_n \leq 0 \text{ in } B^c_{R_2}.$$

Choose $R_3 = \max\{R_1, R_2\}$ and we set

$$c = \inf_{B_{R_3}} w > 0 \text{ and } \tilde{w}_n = (b+1)w - c w_n. \tag{5.31}$$

where $b = \sup_{n\in\mathbb{N}} \|w_n\|_{L^\infty(\mathbb{R}^3)} < \infty$. Our goal is to show that

$$\tilde{w}_n \geq 0 \text{ in } \mathbb{R}^3. \tag{5.32}$$

Firstly, we observe that

$$\lim_{|x|\to\infty} \sup_{n\in\mathbb{N}} \tilde{w}_n(x) = 0, \tag{5.33}$$

$$\tilde{w}_n \geq bc + w - bc > 0 \text{ in } B_{R_3}, \tag{5.34}$$

$$(-\Delta)^s \tilde{w}_n + \frac{V_0}{2}\tilde{w}_n \geq 0 \text{ in } \mathbb{R}^3 \setminus B_{R_3}. \tag{5.35}$$

Now, we argue by contradiction, and we assume that there exists a sequence $(\bar{x}_{j,n}) \subset \mathbb{R}^3$ such that

$$\inf_{x\in\mathbb{R}^3} \tilde{w}_n(x) = \lim_{j\to\infty} \tilde{w}_n(\bar{x}_{j,n}) < 0. \tag{5.36}$$

From (5.33), we can deduce that $(\bar{x}_{j,n})$ is bounded, and, up to subsequence, we may assume that there exists $\bar{x}_n \in \mathbb{R}^3$ such that $\bar{x}_{j,n} \to \bar{x}_n$ as $j \to \infty$. Thus, (5.36) yields

$$\inf_{x\in\mathbb{R}^3} \tilde{w}_n(x) = \tilde{w}_n(\bar{x}_n) < 0. \tag{5.37}$$

Using the minimality of $\bar{x}_n$ and the representation formula for the fractional Laplacian (see Lemma 3.2 in [26]), we can see that

$$(-\Delta)^s \tilde{w}_n(\bar{x}_n) = \frac{c_{3,s}}{2} \int_{\mathbb{R}^3} \frac{2\tilde{w}_n(\bar{x}_n) - \tilde{w}_n(\bar{x}_n + \xi) - \tilde{w}_n(\bar{x}_n - \xi)}{|\xi|^{3+2s}} d\xi \leq 0. \tag{5.38}$$

Taking into account (5.34) and (5.36), we obtain that $\bar{x}_n \in \mathbb{R}^3 \setminus B_{R_3}$. This together with (5.37) and (5.38) imply

$$(-\Delta)^s \tilde{w}_n(\bar{x}_n) + \frac{V_0}{2}\tilde{w}_n(\bar{x}_n) < 0,$$

which contradicts (5.35). Thus (5.32) holds, and using (5.28) and $v_n \leq w_n$ we get

$$0 \leq v_n(x) \leq w_n(x) \leq \frac{(b+1)}{c} w(x) \leq \frac{\tilde{C}}{1+|x|^{3+2s}} \text{ for all } n \in \mathbb{N}, x \in \mathbb{R}^3,$$

for some constant $\tilde{C} > 0$. Therefore, recalling the definition of v_n, we can see that

$$\begin{aligned}
|\hat{u}_n|(x) = |u_n|\left(\frac{x}{\varepsilon_n}\right) &= v_n\left(\frac{x}{\varepsilon_n} - \tilde{y}_n\right) \\
&\leq \frac{\tilde{C}}{1 + |\frac{x}{\varepsilon_n} - \tilde{y}_n|^{3+2s}} \\
&= \frac{\tilde{C}\, \varepsilon_n^{3+2s}}{\varepsilon_n^{3+2s} + |x - \varepsilon_n\, \tilde{y}_n|^{3+2s}} \\
&\leq \frac{\tilde{C}\, \varepsilon_n^{3+2s}}{\varepsilon_n^{3+2s} + |x - \eta_{\varepsilon_n}|^{3+2s}}.
\end{aligned}$$

This ends the proof of Theorem 1.1. □

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