

## Logarithmic vanishing theorems for effective $q$ -ample divisors

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Abstract. Let  $X$  be a compact Kähler manifold and  $D$  be a simple normal crossing divisor. If  $D$  is the support of some effective  $k$ -ample divisor, we show

$$H^q(X, \Omega_X^p(\log D)) = 0, \quad \text{for } p + q > n + k.$$

### 1. Introduction

The classical Cartan–Serre–Grothendieck theorem says that a line bundle  $L$  over a compact complex manifold  $X$  is *ample* if and only if for every coherent sheaf  $\mathcal{F}$  on  $X$ , there exists a positive integer  $m_0 = m_0(X, \mathcal{F}, L)$  such that

$$(1.1) \quad H^q(X, \mathcal{F} \otimes L^{\otimes m}) = 0, \quad \text{for } q > 0, \quad m \geq m_0.$$

On the other hand, on compact complex manifolds, the ampleness is also equivalent to the existence of a smooth metric with positive curvature, thanks to the celebrated Kodaira embedding theorem. Hence, the asymptotic vanishing theorem (1.1) can imply the absolute Akizuki–Kodaira–Nakano vanishing theorem:

$$(1.2) \quad H^q(X, \Omega_X^p \otimes L) = 0, \quad \text{for } p + q > \dim X.$$

It is well-known that the implication of (1.2) from (1.1) is very deep and requires many fantastic techniques in complex geometry and algebraic geometry.

The notion of ampleness has a very natural generalization along the line of Cartan–Serre–Grothendieck criterion.

**Definition 1.1.** A line bundle  $L$  over a compact complex manifold  $X$  is called  *$k$ -ample*, if for any coherent sheaf  $\mathcal{F}$  on  $X$  there exists a positive integer  $m_0 = m_0(X, L, \mathcal{F}) > 0$  such that

$$(1.3) \quad H^q(X, \mathcal{F} \otimes L^m) = 0, \quad \text{for } q > k, \quad m \geq m_0.$$

As analogous to the ample line bundle case, one may wonder whether the asymptotic vanishing theorem (1.3) can imply certain Akizuki–Kodaira–Nakano type vanishing theorems. With no doubt, it is a very challenging problem, since in this general context, the ambient manifold  $X$  is not necessarily projective.

Actually, even if  $X$  is projective, the asymptotic vanishing theorem (1.3) can not imply the Kodaira type vanishing

$$(1.4) \quad H^q(X, K_X \otimes L) = 0, \quad \text{for } q > k.$$

Indeed, a counter-example is discovered by Ottem in [11, Section 9]: there exist a projective threefold  $X$  and a 1-ample line bundle  $L$  such that  $H^2(X, K_X \otimes L) \neq 0$ .

It is well-known that there is a differential geometric proof of the vanishing theorem (1.2) by using positive metrics on ample line bundles. There is also a natural generalization for positive line bundles. A line bundle  $L$  over a compact complex manifold  $X$  is called  $k$ -positive, if there exists a smooth Hermitian metric  $h$  on  $L$  such that the Chern curvature  $R^{(L,h)} = -\sqrt{-1}\partial\bar{\partial}\log h$  has at least  $(n - k)$  positive eigenvalues at any point on  $X$ . In 1962, Andreotti and Grauert proved in [1, Théorème 14] that  $k$ -positive line bundles are always  $k$ -ample (see also [7, Proposition 2.1]). In [7], Demailly, Peternell and Schneider proposed a converse to the Andreotti-Grauert theorem and asked whether  $k$ -ample line bundles are  $k$ -positive. In dimension two, Demailly [6] proved an asymptotic version of a converse to the Andreotti-Grauert theorem using tools related to asymptotic cohomology; subsequently, Matsumura [10] gave a positive answer to the question for surfaces. Recently, the third author proved in [12] that  $(\dim X - 1)$ -ample line bundles are  $(\dim X - 1)$ -positive when  $X$  is a projective manifold. However, there exist higher dimensional counterexamples in the range  $\frac{\dim X}{2} - 1 < q < \dim X - 2$ , constructed by Ottem [11]. In summary, it should not be a reasonable approach to establishing Akizuki-Kodaira-Nakano type vanishing theorems for  $k$ -ample line bundles by using Hermitian metrics.

The main result of our paper is

**Theorem 1.2.** *Let  $X$  be a compact Kähler manifold with  $\dim X = n$  and  $D$  be a simple normal crossing divisor. If  $D$  is the support of some effective  $k$ -ample divisor, then*

$$(1.5) \quad H^q(X, \Omega_X^p(\log D)) = 0, \quad \text{for } p + q > n + k.$$

*In particular,*

$$(1.6) \quad H^q(X, K_X \otimes \mathcal{O}_X(D)) = 0, \quad \text{for } q > k.$$

Note that, when  $X$  is projective and  $D$  is an ample divisor (i.e.  $D$  is 0-ample), Theorem 1.2 is obtained by Deligne in [3]. For a general  $k$ -ample line bundle, the special case (1.6) is obtained by Greb and Küronya in [8, Theorem 3.4] when  $X$  is projective. On a projective toric variety  $X$ , Broomhead, Ottem and Prendergast-Smith proved in [2, Theorem 7.1] that the  $k$ -ampleness of a line bundle  $L$  can imply the Kodaira type vanishing theorem  $H^q(X, K_X \otimes L) = 0$  for  $q > k$ .

## 2. Logarithmic vanishing theorems

Let  $X$  be a compact complex manifold with  $\dim_{\mathbb{C}} X = n$ , and  $D = \sum_{i=1}^r D_i$  be a simple normal crossing divisor, i.e. every irreducible component  $D_i$  is smooth and all intersections are transverse. That is, for every  $p \in X$ , we can choose local coordinates  $z_1, \dots, z_n$  such that  $D = (\prod_{i=1}^k z_i = 0)$  in a neighborhood of  $p$ .

The sheaf of germs of differential  $p$ -forms on  $X$  with at most logarithmic poles along  $D$ , denoted  $\Omega_X^p(\log D)$  (introduced by Deligne in [3]) is the sheaf whose sections on an open subset  $V$  of  $X$  are

$$\Gamma(V, \Omega_X^p(\log D)) := \{\alpha \in \Gamma(V, \Omega_X^p \otimes \mathcal{O}_X(D)) \text{ and } d\alpha \in \Gamma(V, \Omega_X^{p+1} \otimes \mathcal{O}_X(D))\}.$$

If  $D'$  is an effective divisor with  $\text{Supp}(D') = D$ , we denote by

$$\Omega_X^p(*D') = \bigcup_{k \geq 0} \Omega_X^p(kD'),$$

which is the sheaf of meromorphic  $p$ -forms that are holomorphic on  $X - D$  and have poles on of arbitrary (finite) order on  $D$ . Hence

$$(2.1) \quad \Omega_X^p(*D) = \Omega_X^p(*D').$$

A divisor  $D'$  is called  $k$ -ample if  $\mathcal{O}_X(D')$  is a  $k$ -ample line bundle. In the following result, we see clearly how to use the  $k$ -ample condition in the proof of Theorem 1.2.

**Lemma 2.1.** *Let  $X$  be a compact complex manifold with  $\dim_{\mathbb{C}} X = n$  and  $D'$  be an effective  $k$ -ample divisor. Then*

$$H^q(X, \Omega_X^p(*D')) = 0, \quad \text{for } q > k.$$

*Proof.* We use the notations  $[\bullet]$  and  $[\bullet]_{\ell}$  to denote a class in  $H^q(X, \Omega_X^p(*D'))$  and  $H^q(X, \Omega_X^p(\ell D'))$  respectively. If  $[\alpha] \in H^q(X, \Omega_X^p(*D'))$ , from the definition of  $*D'$ , then there exists some  $\ell_0 > 0$  such that

$$(2.2) \quad [\alpha]_{\ell} \in H^q(X, \Omega_X^p(\ell D')), \quad \text{for } \ell \geq \ell_0.$$

Since  $\mathcal{O}_X(D')$  is  $k$ -ample, by definition, there exists some  $\ell_1 > 0$  such that

$$(2.3) \quad H^q(X, \Omega_X^p(\ell D')) = 0, \quad \text{for } q > k, \quad \ell \geq \ell_1.$$

By (2.2) and (2.3), for  $\ell \geq \max\{\ell_0, \ell_1\}$  and  $q > k$ , we know

$$\alpha = \bar{\partial}\beta$$

for some  $\beta \in A^{0,q}(X, \Omega_X^p(\ell D')) \subset A^{0,q}(X, \Omega^p(*D'))$ . Therefore,

$$H^q(X, \Omega_X^p(*D')) = 0$$

for  $q > k$ . □

The proof of Theorem 1.2. There is an exact sequence:

$$H^q(X, \Omega_X^0(*D)) \xrightarrow{d} H^q(X, \Omega_X^1(*D)) \xrightarrow{d} \cdots \xrightarrow{d} H^q(X, \Omega_X^n(*D)).$$

The associated cohomology is denoted by

$$E_2^{p,q} = H_d^p(H^q(X, \Omega_X^*(D))).$$

By Lemma 2.1 and formula (2.1), one has

$$(2.4) \quad E_2^{p,q} = 0, \quad \text{for } q > k.$$

Let  $\mathbb{H}^*(X, \Omega^*(D))$  be the hypercohomology of the complex. Hence, by (2.4), one has

$$(2.5) \quad \mathbb{H}^s(X, \Omega^*(D)) = 0, \quad \text{for } s > n + k.$$

On the other hand, it is well-known (e.g. [9, Page 453]) that

$$(2.6) \quad \mathbb{H}^s(X, \Omega^*(D)) \cong H^s(X - D, \mathbb{C}).$$

Hence, by (2.5), one has

$$(2.7) \quad H^s(X - D, \mathbb{C}) = 0, \quad \text{for } s > n + k.$$

On the other hand, one has (e.g. [4]) the following identity when  $X$  is Kähler:

$$(2.8) \quad \dim_{\mathbb{C}} H^s(X - D, \mathbb{C}) = \sum_{p+q=s} \dim_{\mathbb{C}} H^q(X, \Omega_X^p(\log D)).$$

Combining (2.7) with (2.8), we obtain

$$H^q(X, \Omega_X^p(\log D)) = 0,$$

for  $p + q > n + k$ . □

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