A GEOMETRIC MODEL OF AN ARBITRARY DIFFERENTIALLY CLOSED FIELD OF CHARACTERISTIC ZERO

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Abstract. We give an elementary construction of an arbitrary differentially closed field and of a universal extension of a differential field in terms of Nash function fields. We also give a characterization of any Archimedean ordered differentially closed field in terms of Nash functions.

Introduction

The study of differential algebras was started in the first half of the twentieth century by J. F. Ritt [21, 22], and continued by E. R. Kolchin and J. F. Ritt [23] (see also [12, 13]), I. Kaplansky [11] and others (see for instance [10, 16, 19, 26, 27, 28, 29, 32, 33]). The investigation of these algebras in the context of model theory was initiated by A. Robinson [25]. Despite a fairly long period of study of differential algebras, it is difficult to indicate papers where natural examples of differentially closed fields are given. By A. Seidenberg's embedding theorem [32, 33] we only know that: *Every countable ordinary differential field of characteristic zero F is differentially isomorphic over F to a differential subfield of the field of germs of meromorphic functions in one variable at the origin*. L. Harrington [9] proved that *if a complete and model complete decidable theory T has the finite basis property and every quantifier-free constrained formula (in the language of T) is complete, then T has a recursively presentable prime model*. He used this model-theoretic result to construct the differential closure of any given recursively presentable differential field.

The aims of this article are: to give models of ordinary differentially closed fields of characteristic zero (Theorems 3.3 and 3.7); to construct a universal extension of an ordinary differential field (Theorem 3.12); and to construct an Archimedean ordinary ordered differentially closed field (Theorem 4.8). To this end we present a

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construction, in terms of Nash functions, of all algebraically closed fields of characteristic zero, i.e., the algebraic closure of the rational function field $k = \mathbb{Q}(\Lambda_T)$ in the system of independent variables $\Lambda_T = (\Lambda_t : t \in T)$, $T \neq \emptyset$, with coefficients in $\mathbb{Q}$ (see [37]). It suffices to consider such fields, because any ordinary differentially closed field K of characteristic zero is differentially isomorphic to the algebraic closure of some field $\mathbb{Q}(\Lambda_T)$ (Theorem 3.3). If $T = \emptyset$ then $\mathbb{Q}(\Lambda_T) = \mathbb{Q}$ and so the differential closure of $\mathbb{Q}(\Lambda_T)$ is contained in the algebraic closure of $\mathbb{Q}(\Lambda_{\mathbb{N}})$, i.e., one can take $T = \mathbb{N}$ (Proposition 1.1). We assume the Kuratowski-Zorn Lemma (and indirectly the axiom of choice, see [14]), so the set T can be well-ordered if $T \neq \emptyset$.

The construction of any differentially closed field will be based on the construction of some family Ω of open connected semialgebraic subsets of $\mathbb{C}^T$, called a c-filter (see Section 2) and the rings $\mathcal{N}(U)$ of complex Nash functions on sets $U \in \Omega$. The algebraic closure of $\mathbb{Q}(\Lambda_t : t \in T)$ will be constructed as the set of equivalence classes of the following relation in $\bigcup_{U\in\Omega} \mathcal{N}(U)$:

$$(f_1 : U_1 \to \mathbb{C}) \sim (f_2 : U_2 \to \mathbb{C}) \text{ iff } f_1|_{U_3} = f_2|_{U_3} \text{ for some } U_3 \in \Omega.$$

Then the set $\mathcal{N}_\Omega$ of equivalence classes of "$\sim$" with the usual operations of addition and multiplication is a field, which is the algebraic closure of $\mathbb{Q}(\Lambda_T)$ (see Proposition 2.5, cf. [38, Theorem 2.4 and Corollary 2.5]). Whenever the space $\mathbb{C}^T$ is infinite-dimensional, we will construct a derivation δ on $\mathcal{N}_\Omega$ such that for each pair $p, q \in \mathcal{N}_\Omega\{y\}$ of differential polynomials such that $\operatorname{ord} q < \operatorname{ord} p$, $q \neq 0$, there is some $f \in \mathcal{N}_\Omega$ with $p(f) = 0$ and $q(f) \neq 0$ (see Theorem 3.7), i.e., δ satisfies the L. Blum [3] definition of an ordinary differential closed field of characteristic zero.

To build various kinds of differentially closed fields we construct two c-filters, $\Omega^{\mathbb{K}}_{\mathbf{x_0}}$ and $\mathcal{W}^{\mathbb{K}}_T$, in $\mathbb{K}^T$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ (see Section 2.2 and Proposition 2.10 in Section 2.5). All sets in those c-filters will be simply connected. Moreover, each $U \in \mathcal{W}^{\mathbb{C}}_T$ is dense in $\mathbb{C}^T$. This enables us to construct, for any ordinary differential Nash field, a differentially closed extension of that field of the same cardinality (see Corollary 3.9) and to construct a universal extension of an ordinary differential field (Theorem 3.12). We also construct, in terms of real Nash functions on $U \in \Omega^{\mathbb{R}}_{\mathbf{x_0}}$, a model of an arbitrary ordinary Archimedean ordered differentially closed field (see Section 4).

1. Differential fields

In this section we will collect some fact concerning differential fields. For more detailed information on this topic see for instance [11, 12, 16, 19, 22, 28, 29].

1.1. **Differential algebras.** Let k be a commutative ring with unity and let A be a k-algebra, i.e., a left k-module with multiplication. A k-linear mapping $\delta : A \to A$

is called a *derivation* on A if $\delta(ab) = \delta(a)b + a\delta(b)$ for any $a, b \in A$. Then obviously all elements of k are necessarily constants, i.e., $\delta(\lambda) = 0$ for $\lambda \in k$.

A *k-differential algebra* (A, Δ) is defined as a k-algebra A with a nonempty set Δ of derivations on A such that $\delta\delta'(a) = \delta'\delta(a)$ for all $a \in A$ and $\delta, \delta' \in \Delta$. If the k-algebra A is a ring, an integral domain or a field, then we call the k-differential algebra (A, Δ) a *k-differential ring*, a *k-differential domain* or a *k-differential field* respectively. If $k = \mathbb{Q}$, we will write "differential" instead of "k-differential". If $m = \operatorname{card} \Delta = 1$, the k-differential algebra (respectively ring, domain or field) is called *ordinary*; if $m > 1$, it is called *partial.* If $\Delta = \{\delta\}$, the k-differential algebra (A, Δ) is denoted by (A, δ).

Let (A, Δ) be a k-differential algebra and let Θ be the set of *k-derivative operators* on A generated by Δ, i.e., the free commutative semigroup generated by Δ. Then any $\theta \in \Theta$ can be uniquely expressed in the form of a product $\theta = \prod_{\delta\in\Delta} \delta^{e(\delta)}$, where $e(\delta) \in \mathbb{N}$ (we assume that $0 \in \mathbb{N}$). The number $s = \sum_{\delta\in\Delta} e(\delta)$ is called the *order* of θ and is denoted by $\operatorname{ord}\theta$.

1.2. **Differential polynomials.** Let (R, Δ) be a k-differential ring, Θ the set of k-derivative operators on R generated by Δ, and J a nonempty set. We denote by $R\{y_j : j \in J\}$ the *ring of k-differential polynomials*, i.e., the ring of polynomials

$$R\{y_j : j \in J\} := R[Y_{J,\Theta}]$$

with coefficients in R, in the system of variables $Y_{J,\Theta} = (y_{j,\theta} : j \in J,\ \theta \in \Theta)$, where we assume that $\theta_1(y_{j,\theta_2}) = y_{j,\theta_1\theta_2}$ for $\theta_1, \theta_2 \in \Theta$ and $y_j = y_{j,\theta}$ for $\theta \in \Theta$ of order 0. The ring $R\{y_j : j \in J\}$ has the structure of a k-differential ring with the set of k-derivations Δ if we set $\delta(y_{j,\theta}) = y_{j,\delta\theta}$ for $j \in J$, $\delta \in \Delta$ and $\theta \in \Theta$. If $\operatorname{card} J = 1$, we will write $R\{y\}$ instead of $R\{y_j : j \in J\}$.

Take any k-differential polynomial $p \in R\{y_j : j \in J\}$. Then there are $d, n \in \mathbb{N}$, $n > 0$, and $(j_1, \theta_1), \ldots, (j_n, \theta_n) \in J \times \Theta$ such that

$$p(Y_{J,\Theta}) = \sum_{i_1+\cdots+i_n \le d} r_{i_1,\ldots,i_n} (y_{j_1,\theta_1})^{i_1} \cdots (y_{j_n,\theta_n})^{i_n},$$

where $i_1, \ldots, i_n \in \mathbb{N}$ and $r_{i_1,\ldots,i_n} \in R$ for $i_1 + \cdots + i_n \le d$. The number

$$\max\{i_1 + \cdots + i_n : r_{i_1,\ldots,i_n} \neq 0,\ i_1 + \cdots + i_n \le d\}$$

is called the *degree* of p and denoted by $\deg p$ ($\deg p = -\infty$ if $p = 0$). The number

$$\max \bigcup_{1\le s\le n} \{\operatorname{ord}\theta_s : r_{i_1,\ldots,i_n} \neq 0,\ i_1 + \cdots + i_n \le d,\ i_s > 0\}$$

is called the *order* of p and denoted by $\operatorname{ord} p$ (we set $\max\emptyset = -1$, and then $\operatorname{ord} p = -1$ if $p \in R$).

If $\operatorname{card} J = 1$ and $\Delta = \{\delta\}$, then for any polynomial $p \in R\{y\}$, we denote by p^* the unique polynomial from $R[x_0, \dots, x_n]$, where $n = \operatorname{ord} p$, such that

$$p(y) = p^*(y_{\delta^0}, \dots, y_{\delta^n}).$$

Then for any $a \in R$ we have

$$p(a) = p^*(a, \delta(a), \dots, \delta^n(a)).$$

1.3. **Differentially closed fields.** A field K of characteristic zero equipped with $m > 0$ commuting derivations is called *differentially closed* (or *partial differentially closed* for $m > 1$) if every system of differential polynomial equations and inequations in several variables with a solution in some differential extension of K has a solution in K. If $m = 1$, then the ordinary differential field K is called *ordinary differentially closed*.

In this paper we will use the following (equivalent) definition of ordinary differentially closed fields, due to L. Blum [3]:

An ordinary differential field (K, δ) of characteristic zero is called *differentially closed* if for each pair $p, q \in K\{y\}$ of differential polynomials such that $\operatorname{ord} q < \operatorname{ord} p$ and $q \neq 0$, there is some $a \in K$ with $p(a) = 0$ and $q(a) \neq 0$.

From [34, Lemma 4] we obtain the following fact (cf. [30, 31, 32, 33]).

Proposition 1.1. *Assume that (K, δ) is a differentially closed field of characteristic zero. Then the transcendence degree $\operatorname{trdeg}_{\mathbb{Q}} K$ of K over $\mathbb{Q}$ is infinite.*

Proof. We will write y' for y_δ and y for y_0. Consider the differential polynomials $p_0 = (1 + y)y' - y$, $q_0 = y$. Then $0 \leq \operatorname{ord} q_0 < \operatorname{ord} p_0$. So, there exists $\varphi_0 \in K$ such that $p_0(\varphi_0) = 0$ and $q_0(\varphi_0) \neq 0$. Consider a sequence of differential polynomials $p_j = p_0$ and $q_j = q_{j-1}(y - \varphi_{j-1})$, where $\varphi_{j-1} \in K$, $p_{j-1}(\varphi_{j-1}) = 0$ and $q_{j-1}(\varphi_{j-1}) \neq 0$ for $j \in \mathbb{N}$, $j > 0$. Since (K, δ) is a differentially closed field, such sequences exist. Consequently, there exist an infinite number of distinct nonzero solutions of the equation $p_0 = 0$. Then [34, Lemma 4] yields the assertion. □

Let the differential field $(\mathcal{U}, \delta^*)$ be a differential extension of a differential field (K, δ) of characteristic zero, i.e., K is a subfield of $\mathcal{U}$ and $\delta^*(a) = \delta(a)$ for $a \in K$. We say that the extension $(\mathcal{U}, \delta^*)$ of (K, δ) is *finitely generated* if $\mathcal{U}$ has a finite subset A such that $(\mathcal{U}, \delta^*)$ is the smallest differential extension of (K, δ) in $(\mathcal{U}, \delta^*)$ that contains A. The set A is called the *set of generators* of the extension $(\mathcal{U}, \delta^*)$ over (K, δ). The extension $(\mathcal{U}, \delta^*)$ over (K, δ) is called *simply generated* if it has a set of generators consisting of one element.

After E.R. Kolchin [12, 13], we say that $(\mathcal{U}, \delta^*)$ is a *semiuniversal extension* of (K, δ) if every finitely generated differential extension of (K, δ) differentially embeds over K in $(\mathcal{U}, \delta^*)$. We say that $(\mathcal{U}, \delta^*)$ is a *universal extension* of (K, δ) if $(\mathcal{U}, \delta^*)$ is semiuniversal over every finitely generated differential extension of (K, δ). A universal extension of the field $\mathbb{Q}$ is called a *universal differential field*.

1.4. **Ordered differentially closed fields.** We will use the following definition of ordered ordinary differentially closed fields, due to M. Singer [35].

Let R be a real field with an ordering $\succ$ and a derivation δ. The field (R, δ) is called an *ordered* (or *real*) *ordinary differentially closed field*, or briefly *ordered differentially closed*, if R is real closed and for any $p, q_1, \ldots, q_m \in R\{y\}$ such that $n = \operatorname{ord} p \geq \operatorname{ord} q_i$ for $1 \leq i \leq m$, and any $a_0, \ldots, a_n \in R$ such that $p^*(a_0, \ldots, a_n) = 0$ with $\frac{\partial p^*}{\partial x_n}(a_0, \ldots, a_n) \neq 0$ and $q_i^*(a_0, \ldots, a_n) \succ 0$ for $1 \leq i \leq m$, there exists $a \in R$ such that $p(a) = 0$ and $q_i(a) \succ 0$ for $0 \leq i \leq m$.

Remark 1.2. From the definition we immediately see that if (R, δ) is an ordered differentially closed field, then for any $n \in \mathbb{N}$ the space

$$V = \{(a, \delta(a), \ldots, \delta^n(a)) \in R^{n+1} : a \in R\}$$

is dense in R^{n+1}, i.e., for any polynomials $g_1, \ldots, g_m \in R[x_0, \ldots, x_n]$, if

$$X = \{(a_0, \ldots, a_n) \in R^{n+1} : g_i(a_0, \ldots, a_n) \succ 0,\ 1 \leq i \leq m\} \neq \emptyset,$$

then $V \cap X \neq \emptyset$.

The following is known (see [36]):

Proposition 1.3. *If (R, δ) is an ordered differentially closed field, then $(R(i), \delta^*)$, where $i^2 = -1$ and $\delta^*(f_1 + if_2) = \delta(f_1) + i\delta(f_2)$ for $f_1, f_2 \in R$, is a differentially closed field (of characteristic 0).*

From the above and Proposition 1.1 we have

Corollary 1.4. *Assume that (R, δ) is an ordered differentially closed field. Then the transcendence degree $\operatorname{trdeg}_{\mathbb{Q}} R$ of R over $\mathbb{Q}$ is infinite.*

Let R be a real closed field ordered by $\succ$. For $x = (x_1, \ldots, x_n) \in R^n$ we denote $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$. The *Euclidean topology* in R^n is the topology for which the open balls $B(x, r) := \{y \in R^n : \|x - y\| \prec r\}$, $x \in R^n$, $r \in R$, $r \succ 0$, form a basis of open subsets (see [4, Definition 2.1.9]). Polynomials are continuous with respect to the Euclidean topology.

Proposition 1.5. *Let (R, δ) be an ordered differentially closed field, ordered by $\succ$. Then for any $p \in R\{y\}$ and any $a, b \in R$ such that $a \prec b$ and $p(a)p(b) \prec 0$ there exists $c \in R$ such that $a \prec c \prec b$ and $p(c) = 0$. Moreover, if $n = \operatorname{ord} p$ then c can be chosen in such a way that $\mathbf{c} \in B(\mathbf{s}, \|\mathbf{s} - \mathbf{a}\|)$, where $\mathbf{c} = (c, \delta(c), \ldots, \delta^n(c))$, $\mathbf{s} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ and $\mathbf{a} = (a, \delta(a), \ldots, \delta^n(a))$, $\mathbf{b} = (b, \delta(b), \ldots, \delta^n(b))$.*

Proof. Take any $p \in R\{y\}$, $n = \operatorname{ord} p$, and $a, b \in R$ as in the assumption. Let $p^* \in R[x_0, \dots, x_n]$ be the unique polynomial such that $p(y) = p^*(y_{\delta^0}, y_\delta, \dots, y_{\delta^n})$, and let $\mathbf{a} = (a, \delta(a), \dots, \delta^n(a))$, $\mathbf{b} = (b, \delta(b), \dots, \delta^n(b))$, $\mathbf{s} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$.

One can assume that the polynomial p^* is irreducible in $R[x_0, \dots, x_n]$ and its degree with respect to x_n is positive. Since $p^*(\mathbf{a})p^*(\mathbf{b}) \prec 0$, the sign of the polynomial p^* changes in R^{n+1} and by [4, Theorem 4.5.1], the ideal $(p^*) \subset R[x_0, \dots, x_n]$ is real, prime and it is the ideal of polynomials vanishing on the hypersurface $V = \{x \in R^{n+1} : p^*(x) = 0\}$. Moreover, $\dim V = n$. Consequently, $\frac{\partial p^*}{\partial x_n} \notin (p^*)$ and $\dim W < n$, where $W = \{x \in V : \frac{\partial p^*}{\partial x_n}(x) = 0\}$.

Take polynomials $q_1, q_2 \in R\{y\}$ defined by $q_1(y) = \|\mathbf{s}-\mathbf{a}\|^2 - \|(y_{\delta^0}, \dots, y_{\delta^n}) - \mathbf{s}\|^2$, $q_2(y) = (y - a)(b - y)$. Then $\operatorname{ord} q_1 = n$, $\operatorname{ord} q_2 = 0$ and $q_1(y) = q_1^*(y_{\delta^0}, y_\delta, \dots, y_{\delta^n})$ for $q_1^*(x_0, \dots, x_n) = \|\mathbf{s} - \mathbf{a}\|^2 - \|(x_0, \dots, x_n) - \mathbf{s}\|^2$, and $q_2(y) = q_2^*(y_{\delta^0}, y_\delta, \dots, y_{\delta^n})$ for $q_2^*(x_0, \dots, x_n) = (x_0 - a)(b - x_0)$, and $B(\mathbf{s}, \|\mathbf{s} - \mathbf{a}\|) = \{x \in R^{n+1} : q_1^*(x) \succ 0\}$.

Take a polynomial $r \in R[t]$ defined by $r(t) = p^*(t\mathbf{a} + (1 - t)\mathbf{b})$. By the assumptions, $r(0)r(1) \prec 0$. Since R is real closed, there exists $t_0 \in R$ with $0 \prec t_0 \prec 1$ such that $r(t_0) = 0$ (see [4, Proposition 1.2.4]). We may assume that r changes sign at the point t_0 (i.e., $r(t_0) = 0$ and r takes positive and negative values in any neighbourhood of t_0). Put $\mathbf{a}_0 = t_0\mathbf{a} + (1 - t_0)\mathbf{b}$. Then $\mathbf{a}_0 \in B(\mathbf{s}, \|\mathbf{s} - \mathbf{a}\|)$, $p^*(\mathbf{a}_0) = 0$ and $q_2^*(\mathbf{a}_0) \succ 0$. So, there exists $\varepsilon \succ 0$ such that $B(\mathbf{a}_0, \varepsilon) \subset B(\mathbf{s}, \|\mathbf{s} - \mathbf{a}\|)$ and $q_2^*(x) \succ 0$ for $x \in B(\mathbf{a}_0, \varepsilon)$. Set $B = B(\mathbf{a}_0, \varepsilon)$.

Let $U_1 = \{x \in R^{n+1} : p^*(x) \succ 0\}$ and $U_2 = \{x \in R^{n+1} : p^*(x) \prec 0\}$. Since p^* is irreducible and changes sign in B, we have $U_1 \cap B \neq \emptyset$ and $U_2 \cap B \neq \emptyset$. So, by [4, Lemma 3.4.2], $\dim(V \cap B) = \dim(B \setminus (U_1 \cup U_2)) = n$. Since $\dim W < n$, there exists $\mathbf{a}_1 \in (V \cap B) \setminus W$, and so $p^*(\mathbf{a}_1) = 0$, $\frac{\partial p^*}{\partial x_n}(\mathbf{a}_1) \neq 0$ and $q_j^*(\mathbf{a}_1) \succ 0$, and obviously $\operatorname{ord} q_j \leq \operatorname{ord} p$ for $j = 1, 2$. Then by definition of ordered differentially closed field, there exists $c \in R$ such that $p(c) = 0$ and $q_j(c) \succ 0$, $j = 1, 2$. Consequently, $\mathbf{c} \in B$ and $a \prec c \prec b$, which completes the proof. □

J. van der Hoeven [10] proved that the field $\mathbb{T}$ of transseries satisfies the first part ot the assertion of Proposition 1.5. It is not clear whether the converse of the van der Hoeven result holds for real differential fields. Nevertheless, we have the converse of the “moreover” part of Proposition 1.5.

Corollary 1.6. *Let (R, δ) be an ordered differential field, ordered by $\succ$. Assume that R is real closed. Then the following conditions are equivalent:*

(a) (R, δ) *is an ordered differentially closed field.*

(b) *For any* $p \in R\{y\}$, $n = \operatorname{ord} p \geq 0$, *and any* $\mathbf{a}, \mathbf{b} \in R^{n+1}$ *with* $p^*(\mathbf{a})p^*(\mathbf{b}) \prec 0$, *there exists* $c \in R$ *such that* $p(c) = 0$ *and* $\mathbf{c} \in B(\mathbf{s}, \|\mathbf{s} - \mathbf{a}\|)$, *where* $\mathbf{c} = (c, \delta(c), \ldots, \delta^n(c))$ *and* $\mathbf{s} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$.

(c) *For any* $p \in R\{y\}$, $n = \operatorname{ord} p \geq 0$, *any* $\mathbf{a} \in R^{n+1}$ *at which* p^* *changes sign, and any ball* $B(\mathbf{a}, r)$, $r \succ 0$, *there exists* $c \in R$ *such that* $p(c) = 0$ *and* $\mathbf{c} \in B(\mathbf{a}, r)$, *where* $\mathbf{c} = (c, \delta(c), \ldots, \delta^n(c))$.

(d) *For any* $p \in R\{y\}$, $n = \operatorname{ord} p \geq 0$, *any* $\mathbf{a} \in R^{n+1}$ *with* $p^*(\mathbf{a}) = 0$, $\frac{\partial p^*}{\partial x_n}(\mathbf{a}) \neq 0$, *and any ball* $B(\mathbf{a}, r)$, $r \succ 0$, *there exists* $c \in R$ *such that* $p(c) = 0$ *and* $\mathbf{c} \in B(\mathbf{a}, r)$, *where* $\mathbf{c} = (c, \delta(c), \ldots, \delta^n(c))$.

Proof. The implication (a) $\Rightarrow$ (b) is proved similarly to Proposition 1.5. The implication (c) $\Rightarrow$ (d) is obvious.

If $p \in R\{y\}$ with $n = \operatorname{ord} p \geq 0$ changes sign at $\mathbf{a} \in R^{n+1}$, then for any ball $B(\mathbf{a}, r)$ with $r \succ 0$ there are $\mathbf{a}_1, \mathbf{b}_1 \in B(\mathbf{a}, r)$ such that $p^*(\mathbf{a}_1)p^*(\mathbf{b}_1) \prec 0$ and $B(\mathbf{s}, \|\mathbf{s} - \mathbf{a}_1\|) \subset B(\mathbf{a}, r)$, where $\mathbf{s} = \frac{1}{2}(\mathbf{a}_1 + \mathbf{b}_1)$. Thus, (b) gives (c).

Take any $p, q_1, \ldots, q_m \in R\{y\}$ with $n = \operatorname{ord} p \geq \operatorname{ord} q_j$, $1 \leq j \leq m$, and $p^*(\mathbf{a}) = 0$, $\frac{\partial p^*}{\partial x_n}(\mathbf{a}) \neq 0$, and $q_j(\mathbf{a}) \succ 0$, $1 \leq j \leq m$, for some $\mathbf{a} \in R^{n+1}$. Since polynomials are continuous in the Euclidean topology in R^{n+1}, there exists a ball $B(\mathbf{a}, r)$, $r \succ 0$, such that $q_i(x) \succ 0$ for $x \in B(\mathbf{a}, r)$, $1 \leq j \leq m$. Thus (d) gives (a). $\square$

2. Semialgebraic preliminaries

2.1. **$\mathbb{Q}$-algebraic and semialgebraic sets.** Let us recall some facts from [37] concerning algebraic and semialgebraic sets in a space of infinite dimensions.

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let T be a nonempty set. We denote by $\Lambda_T = (\Lambda_t : t \in T)$ a system of independent variables, and by $\mathbb{K}[\Lambda_T]$ and $\mathbb{K}(\Lambda_T)$ the ring of polynomials in the variables of Λ_T over $\mathbb{K}$ and its quotient field, respectively. More precisely, for any $P \in \mathbb{K}(\Lambda_T)$ we have $P \in \mathbb{K}(\Lambda_{t_1}, \ldots, \Lambda_{t_m})$ for some finitely many $t_1, \ldots, t_m \in T$.

We denote by $\mathbb{K}^T$ the set of all functions $T \to \mathbb{K}$ equipped with the product topology. Then all projections $\mathbb{K}^T \ni x \mapsto x(t) \in \mathbb{K}$, $t \in T$, are continuous.

A subset of $\mathbb{K}^T$ is called $\mathbb{Q}$-*algebraic* when it is defined by a finite system of equations $P = 0$, where $P \in \mathbb{Q}[\Lambda_T]$. Any $\mathbb{Q}$-algebraic set in $\mathbb{K}^T$ is of the form $\{x \in \mathbb{K}^T : (x(t_1), \ldots, x(t_m)) \in V\}$, where $m \in \mathbb{N}$, $t_1, \ldots, t_m \in T$ and $V \subset \mathbb{K}^m$ is a $\mathbb{Q}$-algebraic subset of $\mathbb{K}^m$ (a complex $\mathbb{Q}$-algebraic set if $\mathbb{K} = \mathbb{C}$).

Note that any $\mathbb{Q}$-algebraic subset of $\mathbb{C}^T$ is also $\mathbb{Q}$-algebraic in $(\mathbb{R}^T)^2$. However, a $\mathbb{Q}$-algebraic set in $(\mathbb{R}^T)^2$ is also $\mathbb{Q}$-algebraic in $\mathbb{C}^T$ only if it is $\mathbb{Q}$-algebraic as a complex algebraic set.

A subset of $\mathbb{K}^T$ (if $\mathbb{K} = \mathbb{C}$ we identify $\mathbb{C}^T$ with $(\mathbb{R}^T)^2$) is called $\mathbb{Q}$-*semialgebraic* when it is defined by a finite boolean combination of inequalities $P > 0$ or $P \geq 0$, where $P \in \mathbb{Q}[\Lambda_T]$ ($P \in \mathbb{Q}[\Lambda_T, \Lambda'_T]$, where $\Lambda'_T = (\Lambda'_t : t \in T)$ is a system of independent variables and Λ_T, Λ'_T represent real and imaginary parts of complex numbers, if $\mathbb{K} = \mathbb{C}$). Analogously to the above, any $\mathbb{Q}$-semialgebraic set in $\mathbb{K}^T$ is of the form $\{x \in \mathbb{K}^T : (x(t_1), \ldots, x(t_m)) \in X\}$, where $m \in \mathbb{N}$, $t_1, \ldots, t_m \in T$ and $X \subset \mathbb{K}^m$ is a $\mathbb{Q}$-semialgebraic subset of $\mathbb{K}^m$.

From the basic properties of algebraic and semialgebraic sets in finite-dimensional real vector spaces (see [2], [4], [5], [20]) we obtain

Proposition 2.1. (a) *The family of* $\mathbb{Q}$*-algebraic sets in* $\mathbb{K}^T$ *is closed with respect to finite unions and intersections.*

(b) *The family of* $\mathbb{Q}$*-semialgebraic sets in* $\mathbb{K}^T$ *is closed with respect to complements and finite unions and intersections.*

(c) (Tarski-Seidenberg). *Let* $\pi_{t_1,\ldots,t_m} : \mathbb{R}^T \ni x \mapsto (x(t_1), \ldots, x(t_m)) \in \mathbb{R}^m$, *where* $t_1, \ldots, t_m \in T$. *If* $X \subset \mathbb{R}^T$ *and* $Y \subset \mathbb{R}^m$ *are* $\mathbb{Q}$*-semialgebraic, then so are* $\pi_{t_1,\ldots,t_m}(X)$ *and* $\pi^{-1}_{t_1,\ldots,t_m}(Y)$.

(d) *For any* $\mathbb{Q}$*-semialgebraic set* $X \subset \mathbb{R}^T$, *the interior* $\operatorname{Int} X$, *closure* $\overline{X}$ *and the boundary* $\operatorname{Fr} X$ *are* $\mathbb{Q}$*-semialgebraic.*

(e) *Every connected component of a* $\mathbb{Q}$*-semialgebraic subset of* $\mathbb{R}^T$ *is* $\mathbb{Q}$*-semialgebraic.*

2.2. ***c*-filters.** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. A family Ω of subsets of $\mathbb{K}^T$ satisfying the following conditions:

(i) any $U \in \Omega$ is a nonempty open connected $\mathbb{Q}$-semialgebraic set,

(ii) for any $\mathbb{Q}$-algebraic set $V \subsetneq \mathbb{K}^T$ there exists $U \in \Omega$ such that $V \cap U = \emptyset$,

(iii) for any $U_1, U_2 \in \Omega$ there exists $U_3 \in \Omega$ such that $U_3 \subset U_1 \cap U_2$,

(iv) for any $U \in \Omega$ there exist an open connected and simply connected $\mathbb{Q}$-semialgebraic set $U_0 \subset \mathbb{K}^T$ and a set $U' \in \Omega$ such that $U' \subset U_0 \subset U$,

will be called a *c-filter* in $\mathbb{K}^T$ (cf. [37]).

Condition (iv) in the definition of *c*-filter is necessary in the construction of the $\mathbb{C}$-field of Nash functions (see Section 2.3), because the construction is based on the monodromy theorem (see [38, Theorem 2.4] and [37, Remark 5.6]). In [37] we used *c*-filters only in the real space $\mathbb{R}^T$ and we omitted condition (iv), because it was unnecessary. In the real case a $\mathbb{Q}$-Nash function $f : U \to \mathbb{R}$ such that $P(\lambda, f(\lambda)) = 0$, where $P \in \mathbb{Q}[\Lambda_T, Z]$, is defined by $f(\lambda) = \xi_i(\lambda)$, $\lambda \in U$, for fixed $1 \leq i \leq m$, where $\xi_1(\lambda) < \cdots < \xi_m(\lambda)$ are roots of the polynomial $P(\lambda, Z)$, provided the resultant of P with respect to Z has no zeros in U. However, in the real case we can use the results from [37], because then condition (iv) follows from the others. Namely we have

Proposition 2.2. *Let Ω be a family of subsets of $\mathbb{R}^T$ satisfying conditions* (i), (ii) and (iii) *in the definition of c-filter. Then Ω also satisfies condition* (iv)*.*

Proof. Take any $U \in \Omega$. Then there exist $m \in \mathbb{N}$ and $t_1, \dots, t_m \in T$ such that $U = \{x \in \mathbb{R}^T : (x(t_1), \dots, x(t_m)) \in X\}$ for some open connected $\mathbb{Q}$-semialgebraic set $X \subset \mathbb{R}^m$. Take a cylindrical decomposition $S_1, \dots, S_\nu$ of $\mathbb{R}^m$ into $\mathbb{Q}$-semialgebraic sets adapted to the set X (see [1, Theorem 5.6 and Algorithm 11.15]). We may assume that $X = \bigcup_{j=1}^n S_j$ for some $n \leq \nu$, and $\operatorname{Int} S_1, \dots, \operatorname{Int} S_\ell \neq \emptyset$, while $\operatorname{Int} S_{\ell+1} = \dots = \operatorname{Int} S_n = \emptyset$. Then $\overline{X} = \bigcup_{j=1}^\ell \overline{S_j}$, and $S_1, \dots, S_\ell$ are open connected and simply connected $\mathbb{Q}$-semialgebraic sets. Moreover, $\bigcup_{j=1}^\ell \operatorname{Fr} S_j$ is contained in some proper $\mathbb{Q}$-algebraic subset W of $\mathbb{R}^m$, where $\operatorname{Fr} S_j$ denotes the boundary of S_j. Set

$$U_j = \{x \in \mathbb{R}^T : (x(t_1), \dots, x(t_m)) \in S_j\}, \quad j = 1, \dots, \ell,$$
$$V = \{x \in \mathbb{R}^T : (x(t_1), \dots, x(t_m)) \in W\}.$$

Then $U \setminus V \subset \sum_{j=1}^\ell U_j \subset U$, and by conditions (ii) and (iii) in the definition of c-filter, there exists $U' \in \Omega$ such that $U' \subset U \setminus V$, and by (i), $U' \subset U_j$ for some $j \in \{1, \dots, \ell\}$. Then taking $U_0 = U_j$ we deduce the assertion. □

In the real case we have the following property of c-filters.

Proposition 2.3 ([37, Proposition 2.1])**.** *For any c-filter Ω of subsets of $\mathbb{R}^T$, the set $\partial\Omega := \bigcap_{U \in \Omega} \overline{U}$ has at most one point.*

The assertion of Proposition 2.3 fails in the complex case (see Remark 2.15 in Section 2.5). However, it does hold for some c-filters in $\mathbb{C}^T$. Namely, let $T \subset \mathbb{R}$ be a set algebraically independent over $\mathbb{Q}$, and let $\mathbf{x}_0 \in \mathbb{R}^T$ be defined by $\mathbf{x}_0(t) = t$ for $t \in T$. Then there exists a c-filter $\Omega_{\mathbf{x}_0}^{\mathbb{K}}$ of subsets of $\mathbb{K}^T$ of the form

$$U = \{x \in \mathbb{K}^T : |x(t_j) - \tilde{x}_j| < \varepsilon,\ j = 1, \dots, m\}, \tag{1}$$

for any $t_1, \dots, t_m \in T$ with $t_1 < \dots < t_m$ and $\varepsilon \in \mathbb{Q}_+$ and $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_m) \in \mathbb{Q}^m$ such that $|t_j - \tilde{x}_j| < \varepsilon$ for $j = 1, \dots, m$ (and so $\mathbf{x}_0 \in U$).

A c-filter Ω in $\mathbb{K}^T$ such that $\mathbf{x}_0 \in U$ for any $U \in \Omega$ will be called *centered* at $\mathbf{x}_0$.

2.3. **Field of Nash functions.** A function $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^T$ is an open $\mathbb{Q}$-semialgebraic set, is called a $\mathbb{Q}$-*Nash function* if f is real analytic and there exists a nonzero polynomial $P \in \mathbb{Q}[\Lambda_T, Z]$ such that $P(\lambda, f(\lambda)) = 0$ for $\lambda \in U$. In fact f

depends on a finite number of variables, so the analyticity of f is clear. The ring of $\mathbb{Q}$-Nash functions in U is denoted by $\mathcal{N}^{\mathbb{R}}(U)$.

A function $f : U \to \mathbb{C}$, where $U \subset \mathbb{C}^T$ is an open $\mathbb{Q}$-semialgebraic set (as a subset of $\mathbb{R}^T \times \mathbb{R}^T$), is called a $\mathbb{C}$-$\mathbb{Q}$-*Nash function* if f is holomorphic and there exists a nonzero polynomial $P \in \mathbb{Q}[\Lambda_T, Z]$ such that $P(\lambda, f(\lambda)) = 0$ for $\lambda \in U$. The ring of $\mathbb{C}$-$\mathbb{Q}$-Nash functions in U is denoted by $\mathcal{N}^{\mathbb{C}}(U)$.

Any nonzero polynomial $P \in \mathbb{Q}[\Lambda_T, Z]$ determines at most $\deg_Z P$ $\mathbb{Q}$-Nash functions in a nonempty open connected $\mathbb{Q}$-semialgebraic set $U \subset \mathbb{R}^T$ (respectively $\mathbb{C}$-$\mathbb{Q}$-Nash functions in a nonempty open connected $\mathbb{Q}$-semialgebraic set $U \subset \mathbb{C}^T$).

For the basic properties of Nash functions and semialgebraic sets in finite-dimensional vector spaces see for instance [2], [4], [5], [18]. From these properties we immediately obtain:

Proposition 2.4. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and let $U \subset \mathbb{K}^T$ be a nonempty open connected $\mathbb{Q}$-semialgebraic set. Then the ring $\mathcal{N}^{\mathbb{K}}(U)$ is a domain.*

By Proposition 2.4, for any c-filter Ω in $\mathbb{K}^T$ and any $U \in \Omega$, the ring $\mathcal{N}^{\mathbb{K}}(U)$ of $\mathbb{Q}$-Nash functions if $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$-$\mathbb{Q}$-Nash functions if $\mathbb{K} = \mathbb{C}$ on U is a domain. In $\bigcup_{U\in\Omega} \mathcal{N}^{\mathbb{K}}(U)$ we introduce an equivalence relation by

$$(f_1 : U_1 \to \mathbb{K}) \sim (f_2 : U_2 \to \mathbb{K}) \quad \text{iff} \quad f_1|_{U_3} = f_2|_{U_3} \text{ for some } U_3 \in \Omega.$$

The $\sim$-equivalence class of $f : U \to \mathbb{R}$ will be denoted by $[f]_\Omega$ or simply by f, and the set of all such classes by $\mathcal{N}^{\mathbb{K}}_\Omega$. The set $\mathcal{N}^{\mathbb{K}}_\Omega$, together with the usual operations of addition and multiplication

$$[f_1]_\Omega + [f_2]_\Omega = \left[f_1|_U + f_2|_U\right]_\Omega, \quad [f_1]_\Omega \cdot [f_2]_\Omega = \left[f_1|_U f_2|_U\right]_\Omega,$$

where $f_1 \in \mathcal{N}^{\mathbb{K}}(U_1)$, $f_2 \in \mathcal{N}^{\mathbb{K}}(U_2)$ and $U \in \Omega$, $U \subset U_1 \cap U_2$, is a field, called the $\mathbb{K}$-*field of Nash functions*.

From [37, Theorems 5.2 and Remark 5.6] we obtain the following proposition.

Proposition 2.5. *Let Ω be a c-filter in $\mathbb{K}^T$.*

(a) *If $\mathbb{K} = \mathbb{R}$, then the field $\mathcal{N}^{\mathbb{K}}_\Omega$ is a real closure of the field $\mathbb{Q}(\Lambda_T)$, where the c-filter Ω determines a linear ordering $\succ_\Omega$ in $\mathcal{N}^{\mathbb{K}}_\Omega$ by (see Section 2.4)*

$$f \succ_\Omega g \quad \text{iff there exists } U \in \Omega \text{ such that } f(x) > g(x) \text{ for all } x \in U.$$

(b) *If $\mathbb{K} = \mathbb{C}$, then the field $\mathcal{N}^{\mathbb{K}}_\Omega$ is the algebraic closure of the field $\mathbb{Q}(\Lambda_T)$.*

Note that in Proposition 2.5 (b), the existence of solutions of any equation $P(Z) = 0$, where $P \in \mathcal{N}^{\mathbb{C}}_\Omega[Z]$, $\deg P > 0$, follows from the monodromy theorem and the condition (iv) in the definition of c-filter (cf. [38, proof of Theorem 2.4]).

2.4. **Orderings in fields of real Nash functions.** Let us fix a c-filter Ω in $\mathbb{R}^T$.

Recall that by $\partial\Omega$ we denote the set $\bigcap_{U\in\Omega}\overline{U}$. Recall also that Ω *determines an ordering* $\succ$ in $\mathcal{N}_{\Omega}^{\mathbb{R}}$ (see Proposition 2.5), i.e., a total ordering satisfying:

$$f \succ g \;\Rightarrow\; f+h \succ g+h \quad \text{and} \quad f \succ 0 \wedge g \succ 0 \;\Rightarrow\; fg \succ 0$$

such that $f \succ 0$ iff $f > 0$ on some $U \in \Omega$. If $f \succ g$ then we also write $g \prec f$.

From [37, Theorem 3.1, Remark 3.2 and Corollary 5.4] we have

Theorem 2.6. *The following conditions are equivalent:*

(a) *The field* $(\mathcal{N}_{\Omega}^{\mathbb{R}}, \succ)$ *is Archimedean.*

(b) *There exists* $x_{\succ} \in \partial\Omega$ *whose coordinates are pairwise different and the set of these coordinates is algebraically independent over* $\mathbb{Q}$.

(c) *There exists* $x_{\succ} \in \partial\Omega$ *such that* $x_{\succ} \in U$ *for any* $U \in \Omega$.

(d) *There exists* $x_{\succ} \in \partial\Omega$ *such that* $f \succ 0$ *iff* $f(x_{\succ}) > 0$*, provided* $f \in \mathcal{N}_{\Omega}^{\mathbb{R}}$.

Remark 2.7. If $(\mathcal{N}_{\Omega}^{\mathbb{R}}, \succ)$ is an Archimedean field, where Ω is a c-filter in $\mathbb{R}^T$, then one can assume that $T \subset \mathbb{R}$ and it is algebraically independent over $\mathbb{Q}$ and ordered in such a way that for $t_1, t_2 \in T$ we have $\Lambda_{t_1} \succ \Lambda_{t_2}$ iff $t_1 > t_2$. In fact, according to Theorem 2.6, it suffices to take the set of coordinates of $x_{\succ}$ as the set T.

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $T \subset \mathbb{R}$ be an infinite set algebraically independent over $\mathbb{Q}$. Let $\mathbf{x}_0 \in \mathbb{R}^T$ be defined by $\mathbf{x}_0(t) = t$ for $t \in T$. Take the c-filter $\Omega_{\mathbf{x}_0}^{\mathbb{K}}$ centered at $\mathbf{x}_0$ defined in Section 2.2. The field $\mathcal{N}_{\Omega_{\mathbf{x}_0}^{\mathbb{K}}}^{\mathbb{K}}$ will be denoted by $\mathcal{N}_{\mathbf{x}_0}^{\mathbb{K}}$. Then Theorem 2.6 gives

Corollary 2.8. *The field* $\mathcal{N}_{\mathbf{x}_0}^{\mathbb{R}}$ *is an Archimedean, real closed field which is the real closure of* $\mathbb{Q}(\Lambda_T)$. *Moreover the function* $\mathcal{N}_{\mathbf{x}_0}^{\mathbb{R}} \ni f \mapsto f(\mathbf{x}_0) \in \mathbb{R}$ *is an order preserving monomorphism.*

It is easy to prove that $\Omega_{\mathbf{x}_0}^{\mathbb{R}} = \{U \cap \mathbb{R}^T : U \in \Omega_{\mathbf{x}_0}^{\mathbb{C}}\}$. Since any analytic function $f : U \to \mathbb{C}$, where $U \subset \mathbb{R}^T$ is an open set, has a unique holomorphic extension $\tilde{f} : \tilde{U} \to \mathbb{C}$ onto some open set $\tilde{U} \subset \mathbb{C}^T$ with $U \subset \tilde{U}$, by Proposition 2.5 we immediately obtain

Corollary 2.9. *The field* $\mathcal{N}_{\mathbf{x}_0}^{\mathbb{C}}$ *is the algebraic closure of* $\mathbb{Q}(\Lambda_T)$ *and of* $\mathcal{N}_{\mathbf{x}_0}^{\mathbb{R}}$. *Moreover, the mapping*

$$\Psi : \mathcal{N}_{\mathbf{x}_0}^{\mathbb{C}} \ni f \mapsto \operatorname{Re} f|_{\mathbb{R}^T} + i \operatorname{Im} f|_{\mathbb{R}^T} \in \mathcal{N}_{\mathbf{x}_0}^{\mathbb{R}}(i)$$

is an isomorphism of fields, where $i^2 = -1$.

2.5. **Another c-filter on $\mathbb{K}^T$.** Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. Let m be a fixed positive integer and Λ a system of m variables $\Lambda_1,\dots,\Lambda_m$.

Take any nonzero $P\in\mathbb{Q}[\Lambda]$. Set

$$\Gamma_P=\{(\lambda_1,\dots,\lambda_m)\in\mathbb{K}^m:\ P(\lambda_1,\dots,\lambda_{m-1},\lambda_m+\gamma)=0 \text{ for some } \gamma\in[0,\infty)\}.$$

We define a polynomial $\omega(P)\in\mathbb{Q}[\Lambda_1,\dots,\Lambda_{m-1}]\setminus\{0\}$ (or a number $\omega(P)\in\mathbb{Q}\setminus\{0\}$ if $m=1$) by $\omega(P)=P_0$, where

$$P=P_0\Lambda_m^d+P_1\Lambda_m^{d-1}+\cdots+P_d,$$

and $P_i\in\mathbb{Q}[\Lambda_1,\dots,\Lambda_{m-1}]$ (or $P_i\in\mathbb{Q}$ if $m=1$) for $i=0,\dots,d$, $P_0\neq 0$.

We now define sets $W_P\subset\mathbb{K}^m$, $P\in\mathbb{Q}[\Lambda]\setminus\{0\}$, by induction on m:

$$W_P=\mathbb{K}\setminus\Gamma_P\subset\mathbb{K}\quad\text{if}\quad m=1,$$
$$W_P=(\mathbb{K}^m\setminus\Gamma_P)\cap(W_{\omega(P)}\times\mathbb{K})\subset\mathbb{K}^m\quad\text{if}\quad m>1.$$

By the Tarski-Seidenberg Theorem (see [31, 39]), the sets W_P for $P\in\mathbb{Q}[\Lambda]\setminus\{0\}$ are $\mathbb{Q}$-semialgebraic. Indeed, this is clear for $\mathbb{K}=\mathbb{R}$. Let us explain it in the case when $\mathbb{K}=\mathbb{C}$. Then $P(\lambda_1,\dots,\lambda_m)=P(x_1+iy_1,\dots,x_m+iy_m)=u(x_1,\dots,x_m,y_1,\dots,y_m)+iv(x_1,\dots,x_m,y_1,\dots,y_m)$, where $i^2=-1$ and $u,v\in\mathbb{Q}[x_1,\dots,x_m,y_1,\dots,y_m]$ are the real and imaginary parts of P respectively. So,

$$\begin{aligned}\Gamma_P=\{(x_1,\dots,x_m,y_1,\dots,y_m)\in\mathbb{R}^{2m}:&\ u(x_1,\dots,x_m+\gamma,y_1,\dots,y_m)\\ &=v(x_1,\dots,x_m+\gamma,y_1,\dots,y_m)=0 \text{ for some } \gamma\in[0,\infty)\}\end{aligned}$$

is $\mathbb{Q}$-semialgebraic in $\mathbb{R}^{2m}$, and consequently W_P is $\mathbb{Q}$-semialgebraic in $\mathbb{C}^m=\mathbb{R}^{2m}$.

An argument analogous to the proof of [38, Theorem 1.1] gives the following

Proposition 2.10. *The family $\mathcal{W}=\{W_P: P\in\mathbb{Q}[\Lambda],\ P\neq 0\}$ is a c-filter and satisfies the following conditions:*

R_0. $W_P\subset\{\lambda\in\mathbb{K}^m:\ P(\lambda)\neq 0\}$,

R_1. $W_P\cap W_Q=W_{PQ}$,

R_2. *W_P is an unbounded subset of $\mathbb{K}^m$,*

R_3. *W_P is open, connected and simply connected,*

R_4. *for $\mathbb{K}=\mathbb{C}$, W_P is a dense subset of $\mathbb{C}^m$.*

R_5. $W_P=\mathbb{K}^m$ *for* $P=\text{const}$, $P\neq 0$.

We have (cf. [37, Lemma 4.2])

Lemma 2.11. *Let $1\le i_1<\cdots<i_m\le n$, and let $P\in\mathbb{Q}[\Lambda_{i_1},\dots,\Lambda_{i_m}]\setminus\{0\}$. Let $Q\in\mathbb{Q}[\Lambda_1,\dots,\Lambda_n]$ be a polynomial of the form*

$$Q(x_1,\dots,x_n)=P(x_{i_1},\dots,x_{i_m}),\quad (x_1,\dots,x_n)\in\mathbb{K}^n.$$

Then $W_P \subset \mathbb{K}^m$, $W_Q \subset \mathbb{K}^n$, *and*

$$W_Q = \{(x_1, \dots, x_n) \in \mathbb{K}^n : (x_{i_1}, \dots, x_{i_m}) \in W_P\}.$$

Let T be a nonempty linearly ordered set with ordering $\succ$. For any $t_1, \dots, t_m \in T$ with $t_1 \prec \cdots \prec t_m$ we consider the projection map

$$\pi_{t_1,\dots,t_m} : \mathbb{K}^T \ni x \mapsto (x(t_1), \dots, x(t_m)) \in \mathbb{K}^m.$$

We define a family $\mathcal{W}_T^{\mathbb{K}}$ of $\mathbb{Q}$-semialgebraic subsets U of $\mathbb{K}^T$ by

$$U = (\pi_{t_1,\dots,t_m})^{-1}(W_P)$$

for any $m \in \mathbb{N} \setminus \{0\}$, $t_1, \dots, t_m \in T$, $t_1 \prec \cdots \prec t_m$ and $P \in \mathbb{Q}[\Lambda_{t_1} \dots, \Lambda_{t_m}] \setminus \{0\}$. From Lemma 2.11 and Proposition 2.10 (cf. [37, Proposition 4.3]) we obtain

Proposition 2.12. $\mathcal{W}_T^{\mathbb{K}}$ *is a c-filter.*

In Section 2.2 we observed that there exists a c-filter $\Omega_{\mathbf{x}_0}^{\mathbb{K}}$ in $\mathbb{K}^T$ provided $T \subset \mathbb{R}$. Proposition 2.12 generalizes this to any set T.

From the definition of the sets W_P in the real and complex cases and from [37, Corollary 4.5], Proposition 2.10 and Lemma 2.11 we have

Corollary 2.13. $\mathcal{W}_T^{\mathbb{R}} = \{U \cap \mathbb{R}^T : U \in \mathcal{W}_T^{\mathbb{C}}\}$.

Remark 2.14. It is easy to see that for $\mathbb{K} = \mathbb{R}$ and $P = \Lambda_2 - \Lambda_1 \in \mathbb{Q}[\Lambda_1, \Lambda_2]$ we have $W_P = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 : \lambda_2 > \lambda_1\}$, so $\Lambda_2 \succ_{\mathcal{W}} \Lambda_1$ for the ordering $\succ_{\mathcal{W}}$ in $\mathcal{N}_{\mathcal{W}}^{\mathbb{R}}$ determined by the c-filter $\mathcal{W}$ in $\mathbb{R}^2$ (see Proposition 2.10). So, for any linearly ordered set T with ordering $\succ$ we have $t_1 \succ t_2$ iff $\Lambda_{t_1} \succ_{\mathcal{W}_T^{\mathbb{R}}} \Lambda_{t_2}$.

Remark 2.15. By Proposition 2.10 for any $U \in \mathcal{W}_T^{\mathbb{C}}$ we have $\overline{U} = \mathbb{C}^T$, so $\partial\,\mathcal{W}_T^{\mathbb{C}} = \bigcap_{U \in \mathcal{W}_T^{\mathbb{C}}} \overline{U} = \mathbb{C}^T$. On the other hand, $\partial\,\mathcal{W}_T^{\mathbb{R}} = \emptyset$. Indeed, take any $t \in T$. Then $U_n = \{x \in \mathbb{R}^T : x(t) > n\} \in \mathcal{W}_T^{\mathbb{R}}$ for all $n \in \mathbb{N}$, and so $\partial\,\mathcal{W}_T^{\mathbb{R}} \subset \bigcap_{n \in \mathbb{N}} \overline{U_n} = \emptyset$.

We will denote by $\mathcal{N}_T^{\mathbb{K}}$ the field of Nash functions $\mathcal{N}_{\Omega}^{\mathbb{K}}$, where $\Omega = \mathcal{W}_T^{\mathbb{K}}$ is the c-filter defined above. A similar argument to that for Corollary 2.9 gives

Proposition 2.16. *The mapping*

$$\Psi : \mathcal{N}_T^{\mathbb{C}} \ni f \mapsto \operatorname{Re} f|_{\mathbb{R}^T} + i \operatorname{Im} f|_{\mathbb{R}^T} \in \mathcal{N}_T^{\mathbb{R}}(i)$$

is an isomorphism of fields, where $f|_{\mathbb{R}^T}$ *is the restriction* $f|_{U \cap \mathbb{R}^T} : U \cap \mathbb{R}^T \to \mathbb{C}$, *provided* $f \in \mathcal{N}_T^{\mathbb{C}}(U)$, $U \in \mathcal{W}_T^{\mathbb{C}}$ *and* $i^2 = -1$. *Consequently,* $\mathcal{N}_T^{\mathbb{C}}$ *is an algebraic extension of* $\mathcal{N}_T^{\mathbb{R}}$ *of degree* 2. *Moreover, the field* $\mathcal{N}_T^{\mathbb{C}}$ *is the algebraic closure of* $\mathcal{N}_T^{\mathbb{R}}$.

Remark 2.17. By the definition of the c-filter $\mathcal{W}_T^{\mathbb{C}}$, any function $f \in \mathcal{N}_T^{\mathbb{C}}$ is holomorphic in an open connected, simply connected and dense subset of $\mathbb{C}^T$.

2.6. **Extensions of c-filters.** Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. Let (T_1,T_2) be a pair of nonempty disjoint linearly ordered sets with orderings $\succ_1$ and $\succ_2$, respectively. Then $T_1\cup T_2$ is linearly ordered by: for any $t,t'\in T_1\cup T_2$, $t\succ_{1,2} t'$ iff either $t\in T_2$ and $t'\in T_1$, or $t,t'\in T_1$ and $t\succ_1 t'$, or $t,t'\in T_2$ and $t\succ_2 t'$. Then one can consider the space $\mathbb{K}^{T_1}\times\mathbb{K}^{T_2}$ as $\mathbb{K}^{T_1\cup T_2}$.

Under the above convention, the construction of the c-filter $\mathcal{W}_T^{\mathbb{K}}$ gives

Proposition 2.18. *The c-filter $\mathcal{W}_{T_1\cup T_2}^{\mathbb{K}}$ of subsets of $\mathbb{K}^{T_1}\times\mathbb{K}^{T_2}$ contains the families $\{U\times\mathbb{K}^{T_2}:U\in\mathcal{W}_{T_1}^{\mathbb{K}}\}$ and $\{\mathbb{K}^{T_1}\times U:U\in\mathcal{W}_{T_2}^{\mathbb{K}}\}$.*

On account of the above proposition, the c-filter $\mathcal{W}_{T_1\cup T_2}^{\mathbb{K}}$ will be called an *extension* of $\mathcal{W}_{T_1}^{\mathbb{K}}$ and of $\mathcal{W}_{T_2}^{\mathbb{K}}$.

It is easy to observe that the assertion of Proposition 2.18 also holds for c-filters centered at points $\Omega_{\mathbf{x_1}}^{\mathbb{K}}$, $\Omega_{\mathbf{x_2}}^{\mathbb{K}}$, respectively in $\mathbb{K}^{T_1}$, $\mathbb{K}^{T_2}$, provided $T_1,T_2\subset\mathbb{R}$ are disjoint and their union $T_1\cup T_2$ is algebraically independent over $\mathbb{Q}$. In the case of arbitrary c-filters, a similar construction cannot be made, because it leads to many filters in the Cartesian product of appropriate spaces. For instance $\Omega=\{(0,\varepsilon):\varepsilon\in\mathbb{Q}_+\}$ is a c-filter in $\mathbb{R}$ but there are infinitely many c-filters in $\mathbb{R}^2$ containing $\{U\times\mathbb{R}:U\in\Omega\}$ and $\{\mathbb{R}\times U:U\in\Omega\}$.

Let (T_1,T_2) be a pair of nonempty disjoint linearly ordered sets.

Proposition 2.19. *The field $\mathcal{N}_{T_1\cup T_2}^{\mathbb{K}}$ is an extension of $\mathcal{N}_{T_1}^{\mathbb{K}}$ and $\mathcal{N}_{T_2}^{\mathbb{K}}$.*

Proof. Indeed, any function $f\in\mathcal{N}_{T_1}^{\mathbb{K}}$ has a representative $f:U\to\mathbb{K}$, where $U\in\mathcal{W}_{T_1}^{\mathbb{K}}$, which we may consider as a function $f:U\times\mathbb{K}^{T_2}\to\mathbb{K}$. So, $f\in\mathcal{N}_{T_1\cup T_2}^{\mathbb{K}}$. Obviously addition and multiplication extend from $\mathcal{N}_{T_1}^{\mathbb{K}}$ to $\mathcal{N}_{T_1\cup T_2}^{\mathbb{K}}$. Analogously we consider the case of $f\in\mathcal{N}_{T_2}^{\mathbb{K}}$. $\square$

3. A geometric model of an arbitrary differentially closed field

3.1. **Derivations on a field of Nash functions.** Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. Consider a c-filter Ω in $\mathbb{K}^T$ and the $\mathbb{K}$-field $\mathcal{N}_\Omega^{\mathbb{K}}$ of Nash functions. Take any family

$$g=(g_t\in\mathcal{N}_\Omega^{\mathbb{K}}:t\in T),$$

and let $\delta_g:\mathcal{N}_\Omega^{\mathbb{K}}\to\mathcal{N}_\Omega^{\mathbb{K}}$ be the mapping defined by

$$\delta_g(f)=\sum_{t\in T}g_t\frac{\partial f}{\partial\Lambda_t}\quad\text{for } f\in\mathcal{N}_\Omega^{\mathbb{K}}. \tag{2}$$

The mapping δ_g is well defined, because any representative of $f\in\mathcal{N}_\Omega^{\mathbb{K}}$ depends only on a finite number of variables, so $\frac{\partial f}{\partial\Lambda_t}\in\mathcal{N}_\Omega^{\mathbb{K}}$ and the sum in (2) is finite. We have

Proposition 3.1. *The mapping δ_g defined by* (2) *is a derivation (more precisely, a $\mathbb{Q}$-derivation) on $\mathcal{N}_\Omega^{\mathbb{K}}$. Moreover, any derivation on $\mathcal{N}_\Omega^{\mathbb{K}}$ is of the form* (2).

Proposition 3.2. *Assume that (K, δ) is a differential field of characteristic zero, let $\varphi : K \to \mathcal{N}_\Omega^{\mathbb{K}}$ be a $\mathbb{Q}$-embedding, and let $\mathcal{K} = \varphi(K)$. Then the mapping $\delta_\varphi : \mathcal{K} \to \mathcal{K}$ defined by*

$$\delta_\varphi(f) = \varphi(\delta(\varphi^{-1}(f)))$$

is a derivation on $\mathcal{K}$, and φ is a $\mathbb{Q}$-differential isomorphism of the differential fields (K, δ), $(\mathcal{K}, \delta_\varphi)$.

Proof. Obviously δ_φ is a $\mathbb{Q}$-linear mapping, and for any $f, g \in \mathcal{K}$,

$$\begin{aligned}\delta_\varphi(fg) &= \varphi(\delta(\varphi^{-1}(fg))) = \varphi(\delta(\varphi^{-1}(f)\varphi^{-1}(g))) \\ &= \varphi(\delta(\varphi^{-1}(f)))g + f\varphi(\delta(\varphi^{-1}(g))) = \delta_\varphi(f)g + f\delta_\varphi(g).\end{aligned}$$

On the other hand, for any $a \in K$, $\varphi(\delta(a)) = \varphi(\delta(\varphi^{-1}(\varphi(a)))) = \delta_\varphi(\varphi(a))$, which completes the proof. □

Theorem 3.3. *Let (K, δ) be a differentially closed field of characteristic zero. Then there exists an infinite set T such that (K, δ) is $\mathbb{Q}$-differentially isomorphic to $(\mathcal{N}_\Omega^{\mathbb{C}}, \delta_g)$ for an arbitrary c-filter Ω in $\mathbb{C}^T$ and some family*

$$g = (g_t \in \mathcal{N}_\Omega^{\mathbb{C}} : t \in T). \tag{3}$$

Proof. Let T be a transcendence basis of K over $\mathbb{Q}$. By Proposition 1.1, T is an infinite set. Since K is algebraically closed, being differentially closed, Proposition 2.5(b) implies that K is $\mathbb{Q}$-isomorphic to $\mathcal{N}_\Omega^{\mathbb{C}}$ for an arbitrary c-filter Ω in $\mathbb{C}^T$. Then, by Propositions 3.1 and 3.2 we see that (K, δ) is $\mathbb{Q}$-differentially isomorphic to $(\mathcal{N}_\Omega^{\mathbb{C}}, \delta_g)$ for some family g of the form (3). □

3.2. **A derivation which makes the field of Nash functions differentially closed.** Let T be a linearly ordered infinite set with ordering $\succ$. Let Ω be a c-filter in $\mathbb{C}^T$ (e.g., the one defined in Section 2.5). Set

$$\mathcal{K} = \mathcal{N}_\Omega^{\mathbb{C}}.$$

Consider the ring of polynomials

$$\mathcal{K}[Y] = \mathcal{K}[Y_j : j \in \mathbb{N}].$$

For any $\mathcal{P} \in \mathcal{K}[Y]$ we set

$$D(\mathcal{P}) = \left\{ t \in T : \frac{\partial \mathcal{P}}{\partial \Lambda_t} = 0 \right\}.$$

Obviously $T \setminus D(\mathcal{P})$ is a finite set.

For $\mathcal{P} \in \mathcal{K}[Y]$ with $\deg \mathcal{P} > 0$, we set

$$\alpha(\mathcal{P}) = \max \left\{ j \in \mathbb{N} : \deg_{Y_j} \mathcal{P} > 0 \right\},$$

where $\deg_{Y_j} \mathcal{P}$ denotes the degree of $\mathcal{P}$ as a polynomial in Y_j. Additionally we set $\alpha(\mathcal{P}) = -1$ if $\mathcal{P} \in \mathcal{K} \setminus \{0\}$, and $\alpha(0) = -\infty$.

Define

$$\mathcal{A} = \{(\mathcal{P}, \mathcal{Q}) \in \mathcal{K}[Y]^2 : \alpha(\mathcal{P}) > \alpha(\mathcal{Q}) \geq -1\}.$$

Fact 3.4. *The sets T, $\mathcal{K}$, $\mathcal{K}[Y]$ and $\mathcal{A}$ have the same cardinality.*

Proof. Since T is infinite, it has the same cardinality as the set $\mathrm{Fin}(T)$ of all finite subsets of T. So, $\mathbb{Q}[\Lambda_T]$ has cardinality $\operatorname{card} T$, because it is the union of the countable sets $\mathbb{Q}[\Lambda_{t_1}, \ldots, \Lambda_{t_m}]$ for $\{t_1, \ldots, t_m\} \in \mathrm{Fin}(T)$. Consequently, $\operatorname{card} \mathbb{Q}[\Lambda_T, Z] = \operatorname{card} T$. Hence $\operatorname{card} \mathcal{K} = \operatorname{card} T$, because any polynomial $P \in \mathbb{Q}[\Lambda_T, Z]$ determines a finite subset of $\mathcal{K}$ and the set $\mathbb{Q}[\Lambda_T]$ of cardinality $\operatorname{card} T$ is contained in $\mathcal{K}$. Analogously, $\mathcal{K}[Y]$ is the union of the sets $\mathcal{K}[Y_0, \ldots, Y_m]$, $m \in \mathbb{N}$, so $\operatorname{card} \mathcal{K}[Y] = \operatorname{card} T$. Since $[Y_1(\mathcal{K}[Y] \setminus \{0\})] \times \{1\} \subset \mathcal{A} \subset \mathcal{K}[Y]^2$, we obtain $\operatorname{card} \mathcal{A} = \operatorname{card} T$. $\square$

Fact 3.5. *There exists a family of pairwise disjoint infinite and countable subsets $T_{\mathcal{P},\mathcal{Q}} \subset T$, $(\mathcal{P}, \mathcal{Q}) \in \mathcal{A}$, such that*

$$T = \bigcup_{(\mathcal{P},\mathcal{Q}) \in \mathcal{A}} T_{\mathcal{P},\mathcal{Q}}.$$

Proof. Since T is infinite, there exists a bijection $\tau : \mathbb{N} \times T \to T$. By Fact 3.4 there exists a bijection $\eta : \mathcal{A} \to T$. Thus for $T_{\mathcal{P},\mathcal{Q}} = \tau(\mathbb{N} \times \{\eta(\mathcal{P}, \mathcal{Q})\}) \subset T$, $(\mathcal{P}, \mathcal{Q}) \in \mathcal{A}$, we obtain the assertion. $\square$

Fact 3.6. *Let $(\mathcal{P}, \mathcal{Q}) \in \mathcal{A}$. For any $t_{\mathcal{P},\mathcal{Q},0}, \ldots, t_{\mathcal{P},\mathcal{Q},\alpha(\mathcal{P})-1} \in D(\mathcal{P}) \cap D(\mathcal{Q}) \cap T_{\mathcal{P},\mathcal{Q}}$ such that $t_{\mathcal{P},\mathcal{Q},0} \prec \cdots \prec t_{\mathcal{P},\mathcal{Q},\alpha(\mathcal{P})-1}$ we have*

$$\mathcal{Q}(\Lambda_{t_{\mathcal{P},\mathcal{Q},0}}, \ldots, \Lambda_{t_{\mathcal{P},\mathcal{Q},\alpha(\mathcal{P})-1}}) \neq 0$$

and

$$\deg_{Y_{\alpha(\mathcal{P})}} \mathcal{P}(\Lambda_{t_{\mathcal{P},\mathcal{Q},0}}, \ldots, \Lambda_{t_{\mathcal{P},\mathcal{Q},\alpha(\mathcal{P})-1}}, Y_{\alpha(\mathcal{P})}) > 0,$$

under the natural convention when $\alpha(\mathcal{P}) = 0$. Moreover, points $t_{\mathcal{P},\mathcal{Q},0}, \ldots, t_{\mathcal{P},\mathcal{Q},\alpha(P)-1} \in D(\mathcal{P}) \cap D(\mathcal{Q}) \cap T_{\mathcal{P},\mathcal{Q}}$ such that $t_{\mathcal{P},\mathcal{Q},0} \prec \cdots \prec t_{\mathcal{P},\mathcal{Q},\alpha(\mathcal{P})-1}$ always exist, provided $\alpha(\mathcal{P}) > 0$.

Proof. If $\alpha(\mathcal{P}) = 0$ then the assertion is trivial. Assume that $\alpha(\mathcal{P}) > 0$. By the definition of $\mathcal{A}$, the polynomial $\mathcal{Q}$ depends on at most $\alpha(\mathcal{P}) - 1$ first variables Y_j. Since $D(\mathcal{P}) \cap D(\mathcal{Q}) \cap T_{\mathcal{P},\mathcal{Q}}$ is an infinite set, there exist $t_0, \ldots, t_{\alpha(\mathcal{P})-1} \in D(\mathcal{P}) \cap D(\mathcal{Q}) \cap T_{\mathcal{P},\mathcal{Q}}$ such that $t_0 \prec \cdots \prec t_{\alpha(\mathcal{P})-1}$, and hence we immediately deduce the assertion. $\square$

Assume that we have fixed a choice of $t_{\mathcal{P},\mathcal{Q},i}$ for $(\mathcal{P},\mathcal{Q})\in\mathcal{A}$ as in Fact 3.6.

Let $(\mathcal{P},\mathcal{Q})\in\mathcal{A}$ and let $g_{\mathcal{P},\mathcal{Q}}\in\mathcal{K}$ be a solution of the equation

$$\mathcal{P}(\Lambda_{t_{\mathcal{P},\mathcal{Q},0}},\dots,\Lambda_{t_{\mathcal{P},\mathcal{Q},\alpha(\mathcal{P})-1}},Y_{\alpha(\mathcal{P})})=0$$

with respect to $Y_{\alpha(\mathcal{P})}$. Recall that $\mathcal{K}=\mathcal{N}_{\Omega}^{\mathbb{C}}$ is an algebraically closed field, so $g_{\mathcal{P},\mathcal{Q}}\in\mathcal{K}$ always exists.

Define a family g of points $g_t\in\mathcal{K}$, $t\in T$, by

$$g_t=\begin{cases}\Lambda_{t_{\mathcal{P},\mathcal{Q},i+1}} & \text{for } t=t_{\mathcal{P},\mathcal{Q},i},\ i=0,\dots,\alpha(\mathcal{P})-2,\\ g_{\mathcal{P},\mathcal{Q}} & \text{for } t=t_{\mathcal{P},\mathcal{Q},\alpha(\mathcal{P})-1},\\ f_t & \text{for } t\in T_{\mathcal{P},\mathcal{Q}}\setminus\{t_{\mathcal{P},\mathcal{Q},0},\dots,t_{\mathcal{P},\mathcal{Q},\alpha(\mathcal{P})-1}\},\end{cases}\tag{4}$$

under the natural convention when $\alpha(\mathcal{P})\in\{0,1\}$, where $f_t\in\mathcal{K}$ are arbitrary for $t\in T_{\mathcal{P},\mathcal{Q}}\setminus\{t_{\mathcal{P},\mathcal{Q},0},\dots,t_{\mathcal{P},\mathcal{Q},\alpha(\mathcal{P})-1}\}$, for each $(\mathcal{P},\mathcal{Q})\in\mathcal{A}$. Since $t_{\mathcal{P},\mathcal{Q},i}\in T_{\mathcal{P},\mathcal{Q}}$, by Fact 3.5 the family g is well defined. Consider the derivation

$$\delta_g(f)=\sum_{t\in T}g_t\frac{\partial f}{\partial\Lambda_t}\quad\text{for } f\in\mathcal{K}.$$

Theorem 3.7. *$(\mathcal{K},\delta_g)$ is a differentially closed field.*

Proof. Obviously $(\mathcal{K},\delta_g)$ is a differential field. It suffices to prove that for each pair $p,q\in\mathcal{K}\{y\}$ of differential polynomials such that $\operatorname{ord}q<\operatorname{ord}p$, $q\neq 0$, there is some $f\in\mathcal{K}$ with $p(f)=0$ and $q(f)\neq 0$. Since the field $\mathcal{K}$ is algebraically closed, this condition obviously holds in the case $\operatorname{ord}p=0$. Assume that $\operatorname{ord}p>0$.

Since $\mathcal{K}\{y\}=\mathcal{K}[y_{\delta_g^n}:n\in\mathbb{N}]$, there exists a one-to-one correspondence between $\mathcal{K}\{y\}$ and $\mathcal{K}[Y]$ determined by $Y_j\mapsto y_{\delta_g^j}$ for $j\in\mathbb{N}$. So, for any $p,q\in\mathcal{K}\{y\}$ with $n=\operatorname{ord}p>\operatorname{ord}q$, $q\neq 0$, there exist $\mathcal{P},\mathcal{Q}\in\mathcal{K}[Y]$ with $\mathcal{P}=p^*$ and $\mathcal{Q}=q^*$, i.e.,

$$p=\mathcal{P}(y_0,y_\delta,\dots,y_{\delta^n})\quad\text{and}\quad q=\mathcal{Q}(y_0,y_\delta,\dots,y_{\delta^n}),$$

$\alpha(\mathcal{P})=n\geq 0$ and $\alpha(\mathcal{Q})=\operatorname{ord}q$. Then by the definition of δ_g for $f=\Lambda_{t_{\mathcal{P},\mathcal{Q},0}}\in\mathcal{K}$ we have $\delta_g^m(f)=\Lambda_{t_{\mathcal{P},\mathcal{Q},m}}$ for $0\leq m\leq n-1$ and $\delta_g^n(f)=g_{\mathcal{P},\mathcal{Q}}$. So, by Facts 3.5 and 3.6, $p(f)=0$ and $q(f)\neq 0$, which completes the proof. □

From the choice of g in (4) and Theorem 3.7 we have

Corollary 3.8. *The set of all derivations δ on $\mathcal{N}_{\Omega}^{\mathbb{C}}$ such that $(\mathcal{N}_{\Omega}^{\mathbb{C}},\delta)$ is a differentially closed field has cardinality $2^{\operatorname{card}(T)}$.*

3.3. **A universal extension of a differential field.** Let $T\neq\emptyset$ be a linearly ordered set. Take the c-filter $\mathcal{W}_T^{\mathbb{C}}$ in $\mathbb{C}^T$ defined in Section 2.5. Consider a pair (T_1,T_2) of nonempty sets, where $T_1=T$ and T_2 is a linearly ordered infinite set such that $T_1 \cap T_2 = \emptyset$ and $\operatorname{card} T_2 = \max\{\operatorname{card} T, \operatorname{card} \mathbb{N}\}$. Let $\succ$ be the ordering of $T_1 \cup T_2$ defined in Section 2.6. Let

$$\mathcal{K} = \mathcal{N}^{\mathbb{C}}_{T_1 \cup T_2}$$

be the extension of $\mathcal{N}^{\mathbb{C}}_T$ (see Proposition 2.19).

Let $T_2 = T_3 \cup T_4$, where $T_3 \cap T_4 = \emptyset$ and $\operatorname{card} T_3 = \operatorname{card} T_2$.

We will use similar notation to that of Section 3.2. Consider the ring of polynomials $\mathcal{K}[Y] = \mathcal{K}[Y_j : j \in \mathbb{N}]$ and set

$$D_{T_3}(\mathcal{P}) = \left\{ t \in T_3 : \frac{\partial \mathcal{P}}{\partial \Lambda_t} = 0 \right\} \quad \text{for } \mathcal{P} \in \mathcal{K}[Y].$$

Then $T_3 \setminus D_{T_3}(\mathcal{P})$ is a finite set.

Consider the set $\mathcal{A} = \{(\mathcal{P}, \mathcal{Q}) \in \mathcal{K}[Y]^2 : \alpha(\mathcal{P}) > \alpha(\mathcal{Q}) \geq -1\}$.

We easily see that the sets T_2, $\mathcal{K}$, $\mathcal{K}[Y]$ and $\mathcal{A}$ have the same cardinality (cf. Fact 3.4). By Fact 3.5 there exists a family of pairwise disjoint countable subsets $T_{\mathcal{P},\mathcal{Q}} \subset T_3$, $(\mathcal{P}, \mathcal{Q}) \in \mathcal{A}$, such that

$$T_3 = \bigcup_{(\mathcal{P},\mathcal{Q}) \in \mathcal{A}} T_{\mathcal{P},\mathcal{Q}}.$$

For any $(\mathcal{P}, \mathcal{Q}) \in \mathcal{A}$ the set $D_{T_3}(\mathcal{P}) \cap D_{T_3}(\mathcal{Q}) \cap T_{\mathcal{P},\mathcal{Q}}$ is infinite. So there are $t_{\mathcal{P},\mathcal{Q},\ell,0}, \ldots, t_{\mathcal{P},\mathcal{Q},\ell,\alpha(\mathcal{P})-1} \in D_{T_3}(\mathcal{P}) \cap D_{T_3}(\mathcal{Q}) \cap T_{\mathcal{P},\mathcal{Q}}$, where $1 \leq \ell \leq k$, $k = \deg_{Y_{\alpha(\mathcal{P})}} \mathcal{P} \geq 1$, such that

$$t_{\mathcal{P},\mathcal{Q},1,0} \prec \cdots \prec t_{\mathcal{P},\mathcal{Q},1,\alpha(\mathcal{P})-1} \prec \cdots \prec t_{\mathcal{P},\mathcal{Q},k,0} \prec \cdots \prec t_{\mathcal{P},\mathcal{Q},k,\alpha(\mathcal{P})-1}\,, \tag{5}$$

$$\mathcal{Q}(\Lambda_{t_{\mathcal{P},\mathcal{Q},\ell,0}}, \ldots, \Lambda_{t_{\mathcal{P},\mathcal{Q},\ell,\alpha(\mathcal{P})-1}}) \neq 0, \quad 1 \leq \ell \leq k,$$

and

$$k = \deg_{Y_{\alpha(\mathcal{P})}} \mathcal{P}(\Lambda_{t_{\mathcal{P},\mathcal{Q},\ell,0}}, \ldots, \Lambda_{t_{\mathcal{P},\mathcal{Q},\ell,\alpha(\mathcal{P})-1}}, Y_{\alpha(\mathcal{P})}), \quad 1 \leq \ell \leq k. \tag{6}$$

Hence there exist $h_{\mathcal{P},\mathcal{Q},\ell} \in \mathcal{N}^{\mathbb{C}}_{T \cup \{t_{\mathcal{P},\mathcal{Q},\ell,0}, \ldots, t_{\mathcal{P},\mathcal{Q},\ell,\alpha(\mathcal{P})-1}\}} \subset \mathcal{K}$, $1 \leq \ell \leq k$, such that

$$\mathcal{P}(\Lambda_{t_{\mathcal{P},\mathcal{Q},\ell,0}}, \ldots, \Lambda_{t_{\mathcal{P},\mathcal{Q},\ell,\alpha(\mathcal{P})-1}}, h_{\mathcal{P},\mathcal{Q},\ell}) = 0, \quad 1 \leq \ell \leq k, \tag{7}$$

and for some $1 \leq j \leq k$ (equivalently, for each $1 \leq j \leq k$),

$$\begin{aligned} &h_{\mathcal{P},\mathcal{Q},\ell}(\Lambda_{t_{\mathcal{P},\mathcal{Q},j,0}}, \ldots, \Lambda_{t_{\mathcal{P},\mathcal{Q},j,\alpha(\mathcal{P})-1}}),\ 1 \leq \ell \leq k, \text{ are all solutions counted} \\ &\text{with multiplicity of the equation } \mathcal{P}(\Lambda_{t_{\mathcal{P},\mathcal{Q},j,0}}, \ldots, \Lambda_{t_{\mathcal{P},\mathcal{Q},j,\alpha(\mathcal{P})-1}}, Y_{\alpha(\mathcal{P})}) = 0. \end{aligned} \tag{8}$$

Obviously $h_{\mathcal{P},\mathcal{Q},\ell}(\Lambda_{t_{\mathcal{P},\mathcal{Q},j,0}}, \ldots, \Lambda_{t_{\mathcal{P},\mathcal{Q},j,\alpha(\mathcal{P})-1}}) \in \mathcal{N}^{\mathbb{C}}_{T \cup \{t_{\mathcal{P},\mathcal{Q},j,0}, \ldots, t_{\mathcal{P},\mathcal{Q},j,\alpha(\mathcal{P})-1}\}} \subset \mathcal{K}$ for any $1 \leq j \leq k$.

Define a family h of points $h_t \in \mathcal{K}$, $t \in T_3$, by

$$h_t = \begin{cases} \Lambda_{t_{\mathcal{P},\mathcal{Q},\ell,i+1}} & \text{for } t = t_{\mathcal{P},\mathcal{Q},\ell,i},\ 0 \leq i \leq \alpha(\mathcal{P}) - 2,\ 1 \leq \ell \leq k, \\ h_{\mathcal{P},\mathcal{Q},\ell} & \text{for } t = t_{\mathcal{P},\mathcal{Q},\ell,\alpha(\mathcal{P})-1},\ 1 \leq \ell \leq k, \\ f_t & \text{for } t \in T_{\mathcal{P},\mathcal{Q}} \setminus \{t_{\mathcal{P},\mathcal{Q},1,0}, \ldots, t_{\mathcal{P},\mathcal{Q},k,\alpha(\mathcal{P})-1}\} \end{cases}$$

under the natural convention when $\alpha(\mathcal{P}) \in \{0,1\}$, we take $k = \deg_{Y_{\alpha(\mathcal{P})}} \mathcal{P}$ and $f_t \in \mathcal{K}$ are arbitrary for $t \in T_{\mathcal{P},\mathcal{Q}} \setminus \{t_{\mathcal{P},\mathcal{Q},1,0}, \ldots, t_{\mathcal{P},\mathcal{Q},k,\alpha(\mathcal{P})-1}\}$, for each $(\mathcal{P},\mathcal{Q}) \in \mathcal{A}$.

Let δ_g be a derivation on $\mathcal{N}_T^{\mathbb{C}}$ of the form

$$\delta_g(f) = \sum_{t\in T} g_t \frac{\partial f}{\partial \Lambda_t} \quad \text{for } f \in \mathcal{N}_{T_1}^{\mathbb{C}}$$

for some family $g = (g_t \in \mathcal{N}_T^{\mathbb{C}} : t \in T)$. Take any family $w = (w_t \in \mathcal{K} : t \in T_4)$. Then the mapping $\delta : \mathcal{K} \to \mathcal{K}$ defined by

$$\delta(f) = \sum_{t\in T} g_t \frac{\partial f}{\partial \Lambda_t} + \sum_{t\in T_3} h_t \frac{\partial f}{\partial \Lambda_t} + \sum_{t\in T_4} w_t \frac{\partial f}{\partial \Lambda_t}$$

is a derivation extending δ_g. So, by an analogous argument to that for Theorem 3.7 we deduce that

Corollary 3.9. *$(\mathcal{K},\delta)$ is a differentially closed field differentially extending $(\mathcal{N}_T^{\mathbb{C}}, \delta_g)$.*

Proof. Obviously $(\mathcal{K},\delta)$ is a differential extension of $(\mathcal{N}_T^{\mathbb{C}}, \delta_g)$. Take any $p, q \in \mathcal{K}\{y\}$ such that $\operatorname{ord} p > \operatorname{ord} q$ and $q \neq 0$. Then $p^* = \mathcal{P}$ and $q^* = \mathcal{Q}$ for some $(\mathcal{P},\mathcal{Q}) \in \mathcal{A}$. So, for $f = \Lambda_{t_{\mathcal{P},\mathcal{Q},1,0}}$ we have $p(f) = 0$ and $q(f) \neq 0$. □

Assume that $\operatorname{card} T_4 = \operatorname{card} \mathbb{N}$. For simplicity one can assume that $T_4 = \mathbb{N}$. Take a bijection $\sigma : T_4 \times \mathbb{N} \to T_4$ and a family $v = (v_t : t \in T_4)$ defined by

$$v_t = \Lambda_{\sigma(s,k+1)} \quad \text{if } t = \sigma(s,k) \in T_4,\ (s,k) \in T_4 \times \mathbb{N}.$$

Let $T_{4,s} = \sigma(\{s\} \times \mathbb{N})$, $s \in T_4$. Then the sets $T_{4,s}$ are countable, pairwise disjoint, and

$$T_4 = \bigcup_{s\in T_4} T_{4,s}.$$

Further, the mapping $\delta^* : \mathcal{K} \to \mathcal{K}$ defined by

$$\delta^*(f) = \sum_{t\in T} g_t \frac{\partial f}{\partial \Lambda_t} + \sum_{t\in T_3} h_t \frac{\partial f}{\partial \Lambda_t} + \sum_{t\in T_4} v_t \frac{\partial f}{\partial \Lambda_t}$$

is a derivation extending δ_g, and by Corollary 3.9, $(\mathcal{K},\delta^*)$ is a differentially closed field. Moreover, we have

Lemma 3.10. (a) *For any $s \in T_4$ the mapping δ^* is a derivation in $\mathcal{N}_{T\cup T_{4,s}}^{\mathbb{C}}$.*

(b) *If $(\mathcal{P},\mathcal{Q}) \in \mathcal{A}$ and $\mathcal{P} \in \mathcal{N}_T^{\mathbb{C}}[Y]$ then for any $1 \leq \ell \leq \deg_{Y_{\alpha(\mathcal{P})}} \mathcal{P}$ the mapping δ^* is a derivation in $\mathcal{N}_{T\cup T'}^{\mathbb{C}}$, where $T' = \{t_{\mathcal{P},\mathcal{Q},\ell,0}, \ldots, t_{\mathcal{P},\mathcal{Q},\ell,\alpha(\mathcal{P})-1}\}$.*

Lemma 3.11. *Let $(\mathcal{N}_T^{\mathbb{C}}, \delta)$ be a differential extension of a differential field $(\mathcal{L},\delta)$ such that $\mathcal{N}_T^{\mathbb{C}}$ is an algebraic extension of $\mathcal{L}$. Let $\mathcal{F} \subset \mathcal{N}_{T\cup T^\circ}^{\mathbb{C}}$, with $(T\cup T_2)\cap T^\circ = \emptyset$, be a field such that $(\mathcal{F},\delta_1)$ is a simply generated differential extension of $(\mathcal{L},\delta)$ and let c be a generator of the extension. Assume that c is transcendental over $\mathcal{L}$.*

(a) *If the sequence* $(\delta_1^n(c) : n \in \mathbb{N})$ *is algebraically independent*[1] *over* $\mathcal{L}$*, then for any* $s \in T_4$ *the mapping* $\Phi : \mathcal{F} \to \mathcal{N}^{\mathbb{C}}_{T\cup T_{4,s}} \subset \mathcal{K}$ *defined by*

$$\Phi(f) = f \quad \text{for } f \in \mathcal{L} \qquad \text{and} \qquad \Phi(\delta_1^n(c)) = \Lambda_{\sigma(s,n)} \quad \text{for } n \in \mathbb{N}, \tag{9}$$

is a differential embedding over $\mathcal{L}$*. Moreover, the field* $\mathcal{N}^{\mathbb{C}}_{T\cup T_{4,s}}$ *is an algebraic extension of* $\Phi(\mathcal{F})$*.*

(b) *If* $\delta_1^0(c), \ldots, \delta_1^{m-1}(c)$ *is the longest sequence algebraically independent over* $\mathcal{L}$*, then there exists* $(\mathcal{P}, \mathcal{Q}) \in \mathcal{A}$ *such that* $\alpha(\mathcal{P}) = m$ *and for some open and connected subset* $V \subset \mathbb{C}^{T\cup T^\circ}$*,*

$$\mathcal{P}(\delta_1^0(c), \ldots, \delta_1^{m-1}(c), \delta_1^m(c)) = 0 \quad \text{in } V, \tag{10}$$

$$\delta_1^m(c) = h_{\mathcal{P},\mathcal{Q},\ell}(\delta_1^0(c), \ldots, \delta_1^{m-1}(c)) \quad \text{in } V \text{ for some } 1 \le \ell \le \deg_{Y_m} \mathcal{P}. \tag{11}$$

Moreover, for any such $(\mathcal{P}, \mathcal{Q})$*,* $V \subset \mathbb{C}^{T\cup T^\circ}$ *and* $1 \le \ell \le \deg_{Y_m} \mathcal{P}$*, the mapping* $\Phi_1 : \mathcal{F} \to \mathcal{N}^{\mathbb{C}}_{T\cup T'} \subset \mathcal{K}$ *defined by*

$$\begin{aligned} \Phi_1(f) = f \quad \text{for } f \in \mathcal{L}, \quad \Phi_1(\delta_1^n(c)) = \Lambda_{t_{\mathcal{P},\mathcal{Q},\ell,n}} \ \text{for } 0 \le n \le m-1,\\ \text{and } \Phi_1(\delta_1^n(c)) = (\delta^*)^{n-m}(h_{\mathcal{P},\mathcal{Q},\ell}) \in \mathcal{N}^{\mathbb{C}}_{T\cup T'} \ \text{for } n \ge m, \end{aligned} \tag{12}$$

is a differential embedding over $\mathcal{L}$*, where* $T' = \{t_{\mathcal{P},\mathcal{Q},\ell,0}, \ldots, t_{\mathcal{P},\mathcal{Q},\ell,m-1}\}$*. In particular the field* $\mathcal{N}^{\mathbb{C}}_{T\cup T'}$ *is an algebraic extension of* $\Phi_1(\mathcal{F})$*.*

Proof. Let the sequence $(\delta_1^n(c) : n \in \mathbb{N})$ be algebraically independent over $\mathcal{L}$, let $s \in T_4$ and let $\Phi_1 : \mathcal{F} \to \mathcal{K}$ be the mapping defined by (9). By Lemma 3.10(a), δ^* is a derivation in $\mathcal{N}^{\mathbb{C}}_{T\cup T_{4,s}}$ and obviously $\Phi(\mathcal{F}) \subset \mathcal{N}^{\mathbb{C}}_{T\cup T_{4,s}}$. Moreover,

$$\Phi(\delta_1^n(c)) = (\delta^*)^n(\Lambda_{\sigma(s,0)}) = (\delta^*)^n(\Phi(c)) \quad \text{for } n \in \mathbb{N},$$

so, Φ is a differential embedding over $\mathcal{L}$. Obviously $\mathcal{N}^{\mathbb{C}}_{T\cup T_{4,s}}$ is an algebraic extension of $\Phi(\mathcal{F})$, which gives (a).

Assume now that $\delta_1^0(c), \ldots, \delta_1^{m-1}(c)$ is the longest sequence algebraically independent over $\mathcal{L}$. Then $\delta_1^n(c) \in \mathcal{N}^{\mathbb{C}}(W)$, $0 \le n \le m$, for some $W \in \mathcal{W}^{\mathbb{C}}_{T\cup T^\circ}$. Let

$$F : W \ni (\lambda, x) \mapsto (\lambda, \delta_1^0(c)(\lambda, x), \ldots, \delta_1^{m-1}(c)(\lambda, x)) \in \mathbb{C}^T \times \mathbb{C}^m,$$

where $\lambda \in \mathbb{C}^T$ and $x \in C^{T^\circ}$.

We claim that there exists an open connected subset $V \subset W$ such that $F(V) \subset \mathbb{C}^T \times \mathbb{C}^m$ has nonempty connected interior. Indeed, since $\delta_1^n(c)$, $0 \le n \le m-1$, depends on a finite number of variables, it suffices to consider the case when $T \cup T^\circ$ is a finite set. Let $X \subset \mathbb{C}^T \times \mathbb{C}^{T^\circ} \times \mathbb{C}^m$ be the graph of F, let Y be the Zariski closure of X, and let $\pi : Y \ni (\lambda, x, y) \mapsto (\lambda, y) \in \mathbb{C}^T \times \mathbb{C}^m$. Take the ideal $\mathcal{I} \subset \mathbb{C}[(\Lambda_t; t \in T), (x_{t^\circ} : t^\circ \in T^\circ), (y_0, \ldots, y_{m-1})]$ of polynomials vanishing on Y. Since X is the graph of a mapping which components are $\mathbb{Q}$-Nash functions

[1] i.e., $\delta_1^0(c), \ldots, \delta_1^m(c)$ are algebraically independent over $\mathcal{L}$, for any $m \in \mathbb{N}$.

on open connected set W, it is a connected complex analytic manifold and so, irreducible analytic subset of $W \times \mathbb{C}^m$ (see [15, Corollary 3, p.216]). So, by [15, Proposition 4, p. 217 and Corollary after Proposition 2, p. 408] the set Y is irreducible and consequently the ideal $\mathcal{I}$ is prime. Moreover, using Gröbner bases (see for instance [24, Theorem 2.4], [7, Section 9], see also [1], [8]), we obtain that $\mathcal{I}$ is generated by polynomials with rational coefficients, and that the ideal $\mathcal{J} \subset \mathbb{C}[(\Lambda_t; t \in T), (y_0, \dots, y_{m-1})]$ of the set $\overline{\pi(Y)} \subset \mathbb{C}^T \times \mathbb{C}^m$ is also generated by polynomials with rational coefficients.

By the assumption $\delta_1^0(c), \dots, \delta_1^{m-1}(c)$ are algebraically independent over $\mathcal{L}$ and $\mathcal{L} \subset \mathcal{N}_T^{\mathbb{C}}$ is an algebraic extension, so they are algebraically independent over $\mathcal{N}_T^{\mathbb{C}}$ and in particular – over $\mathbb{Q}(\Lambda_T)$. So Λ_t, $t \in T$, and $\delta_1^0(c), \dots, \delta_1^{m-1}(c)$ are algebraically independent over $\mathbb{Q}$. Since the ideal $\mathcal{J}$ is generated by polynomials with rational coefficients, [1, Theorem 1.22] gives that $\mathcal{J} = \{0\}$ and the set $\pi(Y)$ is a constructible (i.e., it is in the Boolean algebra generated by the closed algebraic sets, see [17]) and dense subset of $\mathbb{C}^T \times \mathbb{C}^m$. Since Y is an irreducible algebraic set, there exists a proper algebraic subset $Y_0 \subsetneq Y$ such that $\pi|_{Y \setminus Y_0} : Y \setminus Y_0 \to \mathbb{C}^T \times \mathbb{C}^m$ is an open mapping (see [17, Corollary 3.15] and the Riemann Open Mapping Theorem [15, Theorem V.6.2]). Consequently $F(W) = \pi(X)$ has nonempty interior because $X \subset Y$ and X contains a nonempty open subset of $Y \setminus Y_0$. This easily gives the announced claim.

By the assumption, $\delta_1^0(c), \dots, \delta_1^m(c)$ are algebraically dependent over $\mathcal{L}$. So, there exists $(\mathcal{P}, \mathcal{Q}) \in \mathcal{A}$ with $\alpha(\mathcal{P}) = m$ and $\mathcal{P} \in \mathcal{L}[Y] \subset \mathcal{N}_T^{\mathbb{C}}[Y]$ such that $\mathcal{P}(\delta_1^0(c), \dots, \delta_1^{m-1}(c), \delta_1^m(c)) = 0$. Take any such $(\mathcal{P}, \mathcal{Q})$, $t_{\mathcal{P},\mathcal{Q},\ell,0}, \dots, t_{\mathcal{P},\mathcal{Q},\ell,m-1} \in D_{T_3}(\mathcal{P}) \cap D_{T_3}(\mathcal{Q}) \cap T_{\mathcal{P},\mathcal{Q}}$, and $h_{\mathcal{P},\mathcal{Q},\ell} \in \mathcal{K}$, where $1 \le \ell \le k$, $k = \deg_{Y_m} \mathcal{P}$, for which (5) – (8) hold. Since $\mathcal{P} \in \mathcal{N}_T^{\mathbb{C}}[Y]$, we have $h_{\mathcal{P},\mathcal{Q},\ell} \in \mathcal{N}_{T \cup T'}^{\mathbb{C}}$, where $T' = \{t_{\mathcal{P},\mathcal{Q},\ell,0}, \dots, t_{\mathcal{P},\mathcal{Q},\ell,m-1}\}$. So, $\mathcal{P} \in \mathcal{N}^{\mathbb{C}}(U)[Y]$ and $h_{\mathcal{P},\mathcal{Q},\ell} \in \mathcal{N}^{\mathbb{C}}(U)$, $1 \le \ell \le k$, for some $U \in \mathcal{W}_{T \cup T'}^{\mathbb{C}}$. By Lemma 3.10(b) the mapping δ^* is a derivation in $\mathcal{N}_{T \cup T'}^{\mathbb{C}}$.

From Proposition 2.10, U is an open and dense subset of $\mathbb{C}^T \times \mathbb{C}^m$, so by the above claim for some nonempty open connected set $V \subset W$ we have that $F(V) \subset U$ is an open and connected set. Consequently, (10) holds in V and by (8), there exists $1 \le \ell_0 \le k$ such that

$$\delta_1^m(c)(\lambda, x) = h_{\mathcal{P},\mathcal{Q},\ell_0}(\lambda, \delta_1^0(c)(\lambda, x), \dots, \delta_1^{m-1}(c)(\lambda, x)) \quad \text{for } (\lambda, x) \in V$$

and (11) holds. So, it is easy to observe that the mapping $\Phi_1 : \mathcal{F} \to \mathcal{K}$ defined by (12) is an embedding over $\mathcal{L}$. By the definition of δ^* we conclude that Φ_1 is a differential embedding of $(\mathcal{F}, \delta_1)$ in $\mathcal{N}_{T \cup T'}^{\mathbb{C}}$ over $\mathcal{L}$. Furthermore, the homomorphism Φ_1 transforms the transcendence basis $\{\delta_1^0(c), \dots, \delta_1^{m-1}(c)\}$ of $\mathcal{F}$ over $\mathcal{L}$ onto the transcendence basis $\{\Lambda_t : t \in T'\}$ of $\mathcal{N}_{T \cup T'}^{\mathbb{C}}$ over $\mathcal{N}_T^{\mathbb{C}}$. Since $\mathcal{N}_{T \cup T'}^{\mathbb{C}}$ is an algebraically closed field and $\mathcal{N}_T^{\mathbb{C}}$ is an algebraic extension of $\mathcal{L}$, the field $\mathcal{N}_{T\cup T'}^{\mathbb{C}}$ is an algebraic extension of $\Phi_1(\mathcal{F})$, which gives (b) and completes the proof. □

Theorem 3.12. *$(\mathcal{K}, \delta^*)$ is a universal extension of $(\mathcal{N}_T^{\mathbb{C}}, \delta_g)$.*

Proof. We claim that $(\mathcal{K}, \delta^*)$ is a semiuniversal extension of $(\mathcal{N}_T^{\mathbb{C}}, \delta_g)$. Take any finitely generated differential extension $(\mathcal{F}, \delta_1)$ of $(\mathcal{N}_T^{\mathbb{C}}, \delta_g)$ and let $\{c_1, \ldots, c_N\}$ be the set of generators of the extension. Obviously $(\mathcal{F}, \delta_1)$ is equal to the quotient field of the differential domain $\mathcal{N}_T^{\mathbb{C}}[\delta_1^n(c_j) : n \in \mathbb{N},\ 1 \le j \le N]$ with derivation δ_1.

Let $\mathcal{F}_\nu$ be the quotient field of the domain $\mathcal{N}_T^{\mathbb{C}}[\delta_1^n(c_j) : n \in \mathbb{N},\ 1 \le j \le \nu]$, $0 \le \nu \le N$. Then $\mathcal{F}_0 = \mathcal{N}_T^{\mathbb{C}}$ and $(\mathcal{F}_{\nu+1}, \delta_1)$ is a simply generated extension of $(\mathcal{F}_\nu, \delta_1)$ for $1 \le \nu \le N-1$. By [12, Proposition II.2.3] one can assume that $c_{\nu+1}$ is transcendental over $\mathcal{F}_\nu$. If the sequence $(\delta_1^n(c_1) : n \in \mathbb{N})$ is algebraically independent over $\mathbb{N}_T^{\mathbb{C}}$ then by Lemma 3.11(a) for any $s \in T_4$ the mapping $\Phi : \mathcal{F}_1 \to \mathcal{N}_{T\cup T_{4,s}}^{\mathbb{C}} \subset \mathcal{K}$ defined by (9) with $c = c_1$ is a differential embedding over $\mathcal{N}_T^{\mathbb{C}}$ and $\mathcal{N}_{T\cup T_{4,s}}^{\mathbb{C}}$ is an algebraic extension of $\Phi(\mathcal{F}_1)$. Therefore, using several times Lemma 3.11(a), we may assume that for any $1 \le j \le N$ there exists $m_j \in \mathbb{N}$ such that $\delta_1^0(c_j), \ldots, \delta_1^{m_j-1}(c_j)$ is the longest sequence algebraically independent over $\mathcal{F}_{j-1}$. By Proposition 3.2 we may assume that $\mathcal{F}_j \subset \mathcal{N}_{T\cup T_j^\circ}^{\mathbb{C}} \subset \mathcal{N}_{T\cup T^\circ}^{\mathbb{C}}$ for some finite sets $T_1^\circ \subsetneq \ldots \subsetneq T_N^\circ = T^\circ$ with $(T \cup T_2) \cap T^\circ = \emptyset$, such that $\mathcal{N}_{T\cup T_j^\circ}^{\mathbb{C}}$ is an algebraic extension of $\mathcal{F}_j$. Then $\delta_1^n(c_j) \in \mathcal{N}^{\mathbb{C}}(W)$, $0 \le n \le m_j$, for some $W \in \mathcal{W}_{T\cup T^\circ}^{\mathbb{C}}$.

Since c_1 is transcendental over $\mathcal{N}_T^{\mathbb{C}}$, by Lemma 3.11(b) there exist $(\mathcal{P}, \mathcal{Q}) \in \mathcal{A}$ with $\alpha(\mathcal{P}) = m_1$, there exists $1 \le \ell \le \deg_{Y_{m_1}} \mathcal{P}$, $T_1' = \{t_{\mathcal{P},\mathcal{Q},\ell,0}, \ldots, t_{\mathcal{P},\mathcal{Q},\ell,m_1-1}\}$ and $h_{\mathcal{P},\mathcal{Q},\ell} \in \mathcal{N}_{T\cup T_1'}^{\mathbb{C}}$ such that $\Phi_1 : \mathcal{F}_1 \to \mathcal{K}$ defined by (12) with $c = c_1$ is a differential embedding over $\mathcal{N}_T^{\mathbb{C}}$ and $\mathcal{N}_{T\cup T_1'}^{\mathbb{C}}$ is an algebraic extension of $\Phi(\mathcal{F}_1)$. So, we may assume that $\mathcal{F}_1 \subset \mathcal{N}_{T\cup T_1'}^{\mathbb{C}}$ is an algebraic extension. Then c_2 is transcendental over $\mathcal{F}_1$ and we may repeat the above argument with c_2 and the extension $\mathcal{F}_1 \subset \mathcal{F}_2$. By applying Lemma 3.11(b) N times we find that $(\mathcal{F}, \delta_1) = (\mathcal{F}_N, \delta_1)$ differentially embeds over $\mathcal{N}_T^{\mathbb{C}}$ in $(\mathcal{N}_{T\cup T'}^{\mathbb{C}}, \delta^*)$ for some finite set $T' \subset T_3$. These iterations of Lemma 3.11 are possible, because for a fixed $\mathcal{P}_0 \in \mathcal{K}[Y]$, $\alpha(\mathcal{P}_0) \ge 0$, there are infinitely many $\mathcal{Q} \in \mathcal{K}[Y]$ such that $(\mathcal{P}_0, \mathcal{Q}) \in \mathcal{A}$, and so the family of sets $\{T_{\mathcal{P}_0,\mathcal{Q}} : (\mathcal{P}_0, \mathcal{Q}) \in \mathcal{A}\}$ is infinite and for any $\mathcal{P}$, $\mathcal{Q}$ we have defined all roots $h_{\mathcal{P},\mathcal{Q},\ell}$ of $\mathcal{P}(\Lambda_{t_{\mathcal{P},\mathcal{Q},1,0}}, \ldots, \Lambda_{t_{\mathcal{P},\mathcal{Q},1,\alpha(\mathcal{P})-1}}, Y_{\alpha(\mathcal{P})}) = 0$, so we can choose appropriate ℓ for which (11) holds. Summing up, $(\mathcal{K}, \delta^*)$ is a semiuniversal extension of $(\mathcal{N}_T^{\mathbb{C}}, \delta_g)$.

To complete the proof it suffices to prove that for any finitely generated differential extension $(\mathcal{F}, \delta^*)$ of $(\mathcal{N}_T^{\mathbb{C}}, \delta_g)$ in $(\mathcal{K}, \delta^*)$, the field $(\mathcal{K}, \delta^*)$ is a semiuniversal extension of $(\mathcal{F}, \delta^*)$. Indeed, by the above, there are a finite set $T' \subset T_3$ and a finite union $T'' = T_{4,s_1'} \cup \cdots \cup T_{4,s_\mu'} \subset T_4$ such that $\mathcal{N}_{T\cup T'\cup T''}^{\mathbb{C}}$ is an algebraic extension of $\mathcal{F}$. Let $(\mathcal{G}, \delta_1)$ be a finitely generated differential extension of $(\mathcal{F}, \delta^*)$.

Then, an analogous argument as in the first two paragraphs of the proof, but using $(\mathcal{G}, \delta_1)$ in place of $(\mathcal{F}, \delta^*)$ and $(\mathcal{F}, \delta^*)$ in place of $(\mathcal{N}_T^{\mathbb{C}}, \delta_g)$, gives that $(\mathcal{G}, \delta_1)$ can be differentially embedded in $(\mathcal{K}, \delta^*)$ over $(\mathcal{F}, \delta^*)$. This completes the proof. □

Remark 3.13. If $\mathcal{L} = \mathbb{Q}$ then $(\mathcal{K}, \delta^*)$ is the universal differential field.

4. An Archimedean ordered differentially closed field

4.1. **A geometric model of an arbitrary ordered differential field.** In [37] we proved that there exists a one-to-one correspondence between the family of orderings in $\mathbb{Q}(\Lambda_T)$ and the family of plain filters (see [37, Theorem 5.2, Proposition 2.4 and Corollary 2.5], cf. [6]). By a *plain filter* we mean a c-filter Ω of subsets of $\mathbb{R}^T$ defined by:

1) any $U \in \Omega$ is a connected component of the complement of a proper $\mathbb{Q}$-algebraic set $V \subset \mathbb{R}^T$,

2) for any proper $\mathbb{Q}$-algebraic set $V \subset \mathbb{R}^T$, some connected component U of the complement of V belongs to Ω.

The above mentioned correspondence is as follows:

Fact 4.1. *For any ordering $\succ$ of $\mathbb{Q}(\Lambda_T)$ there exists a unique plain filter Ω such that $f \succ 0$ iff $f > 0$ on some $U \in \Omega$. Conversely, any plain filter Ω determines a unique ordering $\succ$ of $\mathbb{Q}(\Lambda_T)$ in the above way.*

Since any ordering in $\mathcal{N}_\Omega^{\mathbb{R}}$ is uniquely determined by an ordering in $\mathbb{Q}(\Lambda_T)$, from the above fact we obtain (cf. Theorem 3.3 for differentially closed fields)

Corollary 4.2. *Let (K, δ) be an ordered differentially closed field. Then there exists an infinite set T such that (K, δ) is $\mathbb{Q}$-differentially order isomorphic to $(\mathcal{N}_\Omega^{\mathbb{R}}, \delta_g)$ for some c-filter Ω in $\mathbb{R}^T$ and some family*

$$g = (g_t \in \mathcal{N}_\Omega^{\mathbb{R}} : t \in T). \tag{13}$$

Proof. Let T be the transcendence basis of K over $\mathbb{Q}$. By Corollary 1.4, T is an infinite set. Since K is a real closed field, being ordered and differentially closed, Proposition 2.5(a) shows that K is $\mathbb{Q}$-order isomorphic to $\mathcal{N}_\Omega^{\mathbb{R}}$ for some plain filter Ω of subsets of $\mathbb{R}^T$. Then, by Propositions 3.1 and 3.2 we see that (K, δ) is $\mathbb{Q}$-differentially order isomorphic to $(\mathcal{N}_\Omega^{\mathbb{C}}, \delta_g)$ for some family g of the form (13). □

4.2. **A derivation which makes an Archimedean Nash field ordered differentially closed.** Let $T \subset \mathbb{R}$ be an infinite set algebraically independent over $\mathbb{Q}$

ordered by the usual ordering $>$ on $\mathbb{R}$. Let $\Omega = \Omega_{\mathbf{x}_0}^{\mathbb{R}}$ be the c-filter of subsets of $\mathbb{R}^T$ centered at $\mathbf{x}_0 \in \mathbb{R}^T$, defined by (1) in Section 2.4. Set

$$\mathcal{K} = \mathcal{N}_{\mathbf{x}_0}^{\mathbb{R}}.$$

By Theorem 2.6, the field $\mathcal{K}$ is Archimedean, where the ordering $\succ$ in $\mathcal{K}$ is described by $f \succ 0$ iff $f(\mathbf{x}_0) > 0$. Set

$$\mathcal{K}_{\mathbf{x}_0} = \{f(\mathbf{x}_0) : f \in \mathcal{K}\}.$$

Remark 4.3. By the definition of the c-filter $\Omega_{\mathbf{x}_0}^{\mathbb{R}}$, each $f \in \mathcal{K}$ is a real analytic function in a neighbourhood of $\mathbf{x}_0$, or more precisely, f is a germ of real analytic function at $\mathbf{x}_0$. Consequently, one can consider the elements f as sums of power series centered at $\mathbf{x}_0$ in a finite number of variables.

By Corollary 2.8 we have

Fact 4.4. *$\mathcal{K}_{\mathbf{x}_0}$ is a real closed field order isomorphic to $\mathcal{K}$.*

We will adopt the notation of Section 3.2. Consider the ring of polynomials

$$\mathcal{K}[Y] = \mathcal{K}[Y_j : j \in \mathbb{N}].$$

For a polynomial $\mathcal{R} \in \mathcal{K}[Y]$ of the form

$$\mathcal{R}(Y_0, \ldots, Y_k) = \sum_{j_0,\ldots,j_k \geq 0} f_{j_0,\ldots,j_k} Y_0^{j_0} \cdots Y_k^{j_k}$$

where $f_{i_0,\ldots,i_k} \in \mathcal{K}$ for all $i_0, \ldots i_k$, we denote by $\mathcal{R}_{\mathbf{x}_0}$ the polynomial in $\mathcal{K}_{\mathbf{x}_0}[Y]$ defined by

$$\mathcal{R}_{\mathbf{x}_0}(Y_0, \ldots, Y_k) = \sum_{j_0,\ldots,j_k \geq 0} f_{j_0,\ldots,j_k}(\mathbf{x}_0) Y_0^{j_0} \cdots Y_k^{j_k}.$$

Consider the sets

$$\mathcal{B}_{k,n} = \{(\mathcal{P}, \mathcal{Q}_1, \ldots, \mathcal{Q}_n) \in \mathcal{K}[Y]^{n+1} : k = \alpha(\mathcal{P}) \geq \alpha(\mathcal{Q}_s) \geq -1,\ s = 1, \ldots, n\}$$

for $k, n \in \mathbb{N}$, and let

$$(14)\quad \mathcal{Z} = \bigcup_{k,n=1}^{\infty} \Big\{(\mathcal{P}, \mathcal{Q}_1, \ldots, \mathcal{Q}_n, f_0, \ldots, f_k) \in \mathcal{B}_{k,n} \times \mathcal{K}^{k+1} : \mathcal{P}(f_0, \ldots, f_k) = 0,$$
$$\frac{\partial \mathcal{P}}{\partial x_k}(f_0, \ldots, f_k) \neq 0,\ \mathcal{Q}_s(f_0, \ldots, f_k) \succ 0,\ s = 1, \ldots, n\Big\}.$$

We immediately obtain the following fact (cf. Fact 3.4).

Fact 4.5. *The sets T, $\mathcal{K}$, $\mathcal{K}[Y]$ and $\mathcal{Z}$ have the same cardinality.*

Fact 4.6. *There exists a family of pairwise disjoint infinite and countable subsets $T_{\mathcal{P},\mathcal{Q}_1,\ldots,\mathcal{Q}_n,f_0,\ldots,f_k} \subset T$, $(\mathcal{P}, \mathcal{Q}_1, \ldots, \mathcal{Q}_n, f_0, \ldots, f_k) \in \mathcal{Z}$, such that*

$$T = \bigcup_{(\mathcal{P},\mathcal{Q}_1,\ldots,\mathcal{Q}_n,f_0,\ldots,f_k)\in\mathcal{Z}} T_{\mathcal{P},\mathcal{Q}_1,\ldots,\mathcal{Q}_n,f_0,\ldots,f_k}.$$

Proof. Since T is infinite, there exists a bijection $\tau : \mathbb{N} \times T \to T$. By Fact 4.5 there exists a bijection $\eta : \mathcal{Z} \to T$. Thus setting

$$T_{\mathcal{P},\mathcal{Q}_1,\ldots,\mathcal{Q}_n,f_0,\ldots,f_k} = \tau(\mathbb{N} \times \{\eta(\mathcal{P},\mathcal{Q}_1,\ldots,\mathcal{Q}_n,f_0,\ldots,f_k)\}) \subset T$$

for $(\mathcal{P},\mathcal{Q}_1,\ldots,\mathcal{Q}_n,f_0,\ldots,f_k) \in \mathcal{Z}$, we obtain the assertion. □

Proposition 4.7. *Let $z = (\mathcal{P},\mathcal{Q}_1,\ldots,\mathcal{Q}_n,f_0,\ldots,f_k) \in \mathcal{Z}$, $k = \alpha(\mathcal{P})$. For any*

$$t_{z,0},\ldots,t_{z,k-1} \in D(\mathcal{P}) \cap D(\mathcal{Q}_1) \cap \ldots \cap D(\mathcal{Q}_n) \cap T_z \tag{15}$$

such that $t_{z,0} < \cdots < t_{z,k-1}$ there are $r_{z,0},\ldots,r_{z,k-1} \in \mathbb{Q} \setminus \{0\}$, such that

$$\mathcal{P}(r_{z,0}\Lambda_{t_{z,0}},\ldots,r_{z,k-1}\Lambda_{t_{z,k-1}},f_z) = 0 \tag{16}$$

and

$$\mathcal{Q}_s(r_{z,0}\Lambda_{t_{z,0}},\ldots,r_{z,k-1}\Lambda_{t_{z,k-1}},f_z) \succ 0, \quad s = 1,\ldots,n, \tag{17}$$

for some $f_z \in \mathcal{K}$. Moreover, points in (15) *such that $t_{z,0} < \cdots < t_{z,k-1}$ always exist.*

Proof. By definition of $\mathcal{Z}$, the polynomial $\mathcal{Q}_s$ depends on at most the first $k+1$ variables Y_j. Since T_z is infinite, there exist $t_{z,0},\ldots,t_{z,k} \in D(\mathcal{P}) \cap D(\mathcal{Q}_1) \cap \cdots \cap D(\mathcal{Q}_n) \cap T_z$ such that $t_{z,0} < \cdots < t_{z,k}$. The set of coordinates of $\mathbf{x}_0$ is algebraically independent over $\mathbb{Q}$, so $\mathbf{x}_0(t_{z,j}) \neq 0$ for $j = 0,\ldots,k$.

Let $\xi_j = f_j(\mathbf{x}_0) \in \mathcal{K}_{\mathbf{x}_0}$, $j = 0,\ldots,k$. In view of the choice of the point z,

$$\mathcal{P}_{\mathbf{x}_0}(\xi_0,\ldots,\xi_k) = 0, \quad \frac{\partial \mathcal{P}_{\mathbf{x}_0}}{\partial x_k}(\xi_0,\ldots,\xi_k) \neq 0,$$
$$(\mathcal{Q}_s)_{\mathbf{x}_0}(\xi_0,\ldots,\xi_k) > 0, \quad s = 1,\ldots,n.$$

For any $r_j \in \mathcal{K}_{\mathbf{x}_0} \setminus \{0\}$ sufficiently close to $\frac{\xi_j}{\mathbf{x}_0(t_{z,j})}$ for $j = 0,\ldots,k$ we have

$$(\mathcal{Q}_s)_{\mathbf{x}_0}(r_0\mathbf{x}_0(t_{z,0}),\ldots,r_k\mathbf{x}_0(t_{z,k})) > 0, \quad s = 1,\ldots,n, \tag{18}$$

and moreover $r_j\mathbf{x}_0(t_{z,j}) = r_jF_j(\mathbf{x}_0) \in \mathcal{K}_{\mathbf{x}_0}$, where $F_j(\Lambda_T) = \Lambda_{t_{z,j}}$ for $j = 0,\ldots,k$. Then there exists $\varepsilon > 0$ such that any point of the set

$$\mathcal{U}_\varepsilon = \left\{ r = (r_0,\ldots,r_k) \in \mathcal{K}_{\mathbf{x}_0}^{k+1} : \left| r_j - \frac{\xi_j}{\mathbf{x}_0(t_{z,j})} \right| < \varepsilon,\ j = 0,\ldots,k \right\}$$

satisfies (18). Since $\frac{\partial \mathcal{P}_{\mathbf{x}_0}}{\partial x_k}(\xi_0,\ldots,\xi_k) \neq 0$ and $\mathcal{K}_{\mathbf{x}_0}$ is real closed, the function

$$\mathcal{K}_{\mathbf{x}_0} \ni \zeta \mapsto \mathcal{P}_{\mathbf{x}_0}(\xi_0,\ldots,\xi_{k-1},\zeta) \in \mathcal{K}_{\mathbf{x}_0}$$

changes sign at ξ_k. Thus there are $a, b \in \mathcal{K}_{\mathbf{x}_0}$ such that $a < b$ and

$$|a - \xi_k| < \varepsilon|\mathbf{x}_0(t_{z,k})|, \quad |b - \xi_k| < \varepsilon|\mathbf{x}_0(t_{z,k})| \tag{19}$$

and

$$\mathcal{P}_{\mathbf{x}_0}(\xi_0,\ldots,\xi_{k-1},a)\mathcal{P}_{\mathbf{x}_0}(\xi_0,\ldots,\xi_{k-1},b) < 0.$$

Since $\mathbb{Q}$ is a dense subset of $\mathcal{K}_{\mathbf{x}_0}$, there exists $r = (r_0, \dots, r_k) \in \mathcal{U}_\varepsilon \cap \mathbb{Q}^{k+1}$ such that $r_j \neq 0$ for $j = 0, \dots, k$, and

$$\mathcal{P}_{\mathbf{x}_0}(r_0\mathbf{x}_0(t_{z,0}), \dots, r_{k-1}\mathbf{x}_0(t_{z,k-1}), a)\mathcal{P}_{\mathbf{x}_0}(r_0\mathbf{x}_0(t_{z,0}), \dots, r_{k-1}\mathbf{x}_0(t_{z,k-1}), b) < 0.$$

As $\mathcal{K}_{\mathbf{x}_0}$ is real closed, this implies that there exists $\xi^* \in \mathcal{K}_{\mathbf{x}_0}$ such that $a < \xi^* < b$ and

$$\mathcal{P}_{\mathbf{x}_0}(r_0\mathbf{x_0}(t_{z,0}), \dots, r_{k-1}\mathbf{x_0}(t_{z,k-1}), \xi^*) = 0, \tag{20}$$

and by (19),

$$\left| \frac{\xi^*}{\mathbf{x}_0(t_{z,k})} - \frac{\xi_k}{\mathbf{x}_0(t_{z,k})} \right| < \varepsilon.$$

Hence, $\left(r_0, \dots, r_{k-1}, \frac{\xi^*}{\mathbf{x}_0(t_{z,k})}\right) \in \mathcal{U}_\varepsilon$, and consequently

$$(\mathcal{Q}_s)_{\mathbf{x}_0}(r_0\mathbf{x}_0(t_{z,0}), \dots, r_{k-1}\mathbf{x}_0(t_{z,k-1}), \xi^*) > 0, \quad s = 1, \dots, n. \tag{21}$$

By definition of $\mathcal{K}_{\mathbf{x}_0}$ there exists $f_z \in \mathcal{K}$ such that $f_z(\mathbf{x}_0) = \xi^*$. Moreover, $r_{z,j} := r_j \in \mathbb{Q}$ and so $r_{z,j}\Lambda_{t_{z,j}} \in \mathcal{K}$ for $j = 0, \dots, k-1$. Now, (20), (21) and Fact 4.4 immediately give the assertion. □

Assume that for any $z = (\mathcal{P}, \mathcal{Q}_1, \dots, \mathcal{Q}_n, f_0, \dots, f_k) \in \mathcal{Z}$, we have chosen points

$$t_{z,0}, \dots, t_{z,k-1} \in D(\mathcal{P}) \cap D(\mathcal{Q}_1) \cap \dots \cap D(\mathcal{Q}_n) \cap T_z,$$

where $k = \alpha(\mathcal{P})$, such that $t_{z,0} < \dots < t_{z,k-1}$ and $r_{z,0}, \dots, r_{z,k-1} \in \mathbb{Q} \setminus \{0\}$, and $f_z \in \mathcal{K}$ as in Proposition 4.7, i.e., (16) and (17) hold.

Define a family g of points $g_t \in \mathcal{K}$, $t \in T$, by

$$g_t = \begin{cases} \frac{r_{z,i+1}}{r_{z,i}}\Lambda_{t_{z,i+1}} & \text{for } t = t_{z,i},\ i = 0, \dots, \alpha(\mathcal{P}) - 2, \\ \frac{1}{r_{z,\alpha(\mathcal{P})-1}} f_z & \text{for } t = t_{z,\alpha(\mathcal{P})-1}, \\ h_t & \text{for } t \in T_z \setminus \{t_{z,0}, \dots, t_{z,\alpha(\mathcal{P})-1}\}, \end{cases} \tag{22}$$

where $h_t \in \mathcal{K}$ are arbitrary for $t \in T_z \setminus \{t_{z,0}, \dots, t_{z,\alpha(\mathcal{P})-1}\}$, for each $z = (\mathcal{P}, \mathcal{Q}_1, \dots, \mathcal{Q}_n, f_0, \dots, f_{\alpha(\mathcal{P})}) \in \mathcal{Z}$.

Consider the following derivation on $\mathcal{K}$:

$$\delta_g(f) = \sum_{t \in T} g_t \frac{\partial f}{\partial \Lambda_t} \quad \text{for } f \in \mathcal{K}. \tag{23}$$

Theorem 4.8. *$(\mathcal{K}, \delta_g)$ is an ordered differentially closed field.*

Proof. Obviously $(\mathcal{K}, \delta_g)$ is an ordered differential field and by Corollary 2.8, $\mathcal{K}$ is real closed. Take any $p, q_1, \dots, q_n \in \mathcal{K}\{y\}$ such that $k = \operatorname{ord} p \geq \operatorname{ord} q_j$, $1 \leq j \leq n$, and any $f_0, \dots, f_k \in \mathcal{K}$ such that $p^*(f_0, \dots, f_k) = 0$, $\frac{\partial p^*}{\partial x_k}(f_0, \dots, f_k) \neq 0$ and $q_j^*(f_0, \dots, f_k) \succ 0$, $1 \leq j \leq n$. Then $z = (p^*, q_1^*, \dots, q_n^*, f_0, \dots, f_k) \in \mathcal{Z}$ and $k = \alpha(p^*)$. Since $r_{z,j} \in \mathbb{Q}$, by (22) for $f = r_{z,0}\Lambda_{t_{z,0}}$ we have

$$\delta(f) = r_{z,0}\delta(\Lambda_{t_{z,0}}) = r_{z,1}\Lambda_{z,1}, \ \dots, \ \delta^{k-1}(f) = r_{z,k-1}\Lambda_{z,k-1}, \ \ \delta^k(f) = f_z.$$

So, Proposition 4.7, similarly to the proof of Theorem 3.7, shows that $p(f) = 0$ and $q_j(f) \succ 0$, $1 \leq j \leq n$, which gives the assertion. □

From the choice of g in (22) and Theorem 4.8 we have

Corollary 4.9. *The set of all derivations δ on $\mathcal{N}^{\mathbb{R}}_{\mathbf{x}_0}$ such that $(\mathcal{N}^{\mathbb{R}}_{\mathbf{x}_0}, \delta)$ is an ordered differentially closed field has cardinality $2^{\operatorname{card}(\mathcal{N}^{\mathbb{R}}_{\mathbf{x}_0})}$.*

Remark 4.10. By Corollary 1.6, to construct a derivation δ on $\mathcal{K}$ such that $(\mathcal{K}, \delta)$ becomes ordered differentially closed, it suffices to consider the set

$$\mathcal{Z} = \bigcup_{k,n=1}^{\infty} \Big\{ (\mathcal{P}, \varepsilon, f_0, \ldots, f_k) \in \mathcal{K}[Y] \times \mathbb{Q}_+ \times \mathcal{K}^{k+1} : k = \alpha(\mathcal{P}) \geq 0, \\ \mathcal{P}(f_0, \ldots, f_k) = 0,\ \frac{\partial \mathcal{P}}{\partial x_k}(f_0, \ldots, f_k) \neq 0 \Big\}$$

instead of the one defined in (14), and repeat the construction in Proposition 4.7 without taking into consideration the polynomials $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$.

Remark 4.11. Let $(\mathcal{N}^{\mathbb{R}}_{\mathbf{x}_0}, \delta_g)$ be the ordered differentially closed field with derivation δ_g defined by (22) and (23). By Proposition 1.3 (see also [36]), the field $\mathcal{N}^{\mathbb{R}}_{\mathbf{x}_0}(i)$ with the derivation

$$\delta(f_1 + i f_2) = \delta_g(f_1) + i\delta_g(f_2),$$

extending δ_g, is a differentially closed field.

Indeed, since i is algebraic over $\mathcal{N}^{\mathbb{R}}_{\mathbf{x}_0}$, it follows that δ is the unique derivation in $\mathcal{N}^{\mathbb{R}}_{\mathbf{x}_0}(i)$ extending δ_g. Thus Proposition 1.3 gives the assertion.

Remark 4.12. By Remarks 4.3, 4.11 and Corollary 2.9 we see that any function $f \in \mathcal{N}^{\mathbb{C}}_{\mathbf{x}_0} = \mathcal{N}^{\mathbb{R}}_{\mathbf{x}_0}(i)$ is holomorphic in a neighborhood of $\mathbf{x}_0$ in $\mathbb{C}^T$. Consequently, one can consider the elements f as sums of power series centered at $\mathbf{x}_0$ in a finite number of complex variables (or as germs of holomorphic functions).

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