

# ACM VECTOR BUNDLES ON PROJECTIVE SURFACES OF NONNEGATIVE KODAIRA DIMENSION

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ABSTRACT. In this paper we contribute to the construction of families of arithmetically Cohen-Macaulay (aCM) indecomposable vector bundles on a wide range of polarized surfaces  $(X, \mathcal{O}_X(1))$  for  $\mathcal{O}_X(1)$  an ample line bundle. In many cases, we show that for every positive integer  $r$  there exists a family of indecomposable aCM vector bundles of rank  $r$ , depending roughly on  $r$  parameters, and in particular they are of *wild representation type*. We also introduce a general setting to study the complexity of a polarized variety  $(X, \mathcal{O}_X(1))$  with respect to its category of aCM vector bundles. In many cases we construct indecomposable vector bundles on  $X$  which are aCM for all ample line bundles on  $X$ .

## 1. INTRODUCTION

In many areas of mathematics it plays a central role to understand the *complexity* of the objects one is interested in. This complexity can be measured in many different ways. For instance, in representation theory of quivers, Gabriel's theorem states that a connected quiver supports only finitely many irreducible representations, i.e., of indecomposable modules over the associated path algebra, if and only if it is of type  $A, D, E$ . The classification of *tame* quivers as *Euclidean graphs*, or *extended Dynkin diagrams*, of type  $\tilde{A}, \tilde{D}, \tilde{E}$  was obtained right after. Remarkably, any other quiver supports arbitrarily large families of indecomposable representations, i.e., they turn out to be of *wild representation type*.

Motivated by the results, similar questions were raised to understand the category of Cohen-Macaulay modules over an arbitrary  $\mathbf{k}$ -algebra  $R$ . When  $R := \mathbf{k}[x_0, \dots, x_n]/I$  is a graded algebra finitely generated in degree one over a field  $\mathbf{k}$ , Cohen-Macaulay modules correspond naturally to arithmetically Cohen-Macaulay sheaves over the closed subscheme  $\text{Proj}(R) \subset \mathbb{P}^n$ ; see [18].

**Definition 1.1.** A coherent sheaf  $\mathcal{E}$  on a projective scheme  $(X, \mathcal{O}_X(1))$  is called *arithmetically Cohen-Macaulay* (for short, aCM) if the following conditions hold:

- (i)  $\mathcal{E}$  is locally Cohen-Macaulay, i.e. the stalk  $\mathcal{E}_x$  has depth equal to  $\dim \mathcal{O}_{X,x}$  for any point  $x$  on  $X$ ;
- (ii)  $H^i(\mathcal{E}(t)) = 0$  for all  $t \in \mathbb{Z}$  and  $i = 1, \dots, \dim X - 1$ .

The forementioned correspondence allowed to use a geometrical approach to this kind of questions. A milestone on this area was due to Horrocks, stating that the only indecomposable aCM sheaf on  $\mathbb{P}^n$ , up to twist, is  $\mathcal{O}_{\mathbb{P}^n}$ ; see [15]. A similar classification was obtained for a smooth quadric hypersurface  $Q \subset \mathbb{P}^n$ : there exist, besides the structural sheaf  $\mathcal{O}_Q$ , only one (for  $n$  even) or two (for  $n$  odd) irreducible aCM sheaves, the well-studied Spinor bundles; see [19]. The combined work of many mathematicians allowed to complete the list of projective schemes -of positive dimension- supporting a finite number of aCM sheaves, called the varieties of *finite aCM-representation type*: they are either a projective space  $\mathbb{P}^n$ , a smooth quadric hypersurface  $X \subset \mathbb{P}^n$ , a cubic scroll in  $\mathbb{P}^4$ , the Veronese surface in  $\mathbb{P}^5$  or a rational normal curve; see [8].

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The next degree of complexity is offered by the elliptic curves: in this case, vector bundles of a given rank and degree on an elliptic curve  $C$  are in bijection with the points of  $C$ ; see [1]. They are called varieties of *tame aCM-representation type*. In [9] it was shown that smooth quartic surface scrolls in  $\mathbb{P}^5$  are also tame. Notice that all the projective schemes  $X \subset \mathbb{P}^n$  mentioned until now are arithmetically Cohen-Macaulay, namely the coordinate ring  $R := \mathbf{k}[x_0, \dots, x_n]/I_X$  is a Cohen-Macaulay ring. Indeed, the representation type of the remaining aCM projective schemes  $X \subset \mathbb{P}^n$  was set in [10]: they support arbitrarily large families of indecomposable non-isomorphic aCM sheaves. They are, therefore, of *wild aCM-representation type*.

On the other hand, up to our knowledge, a broader problem has been much less studied: which are the possible dimensions of families of aCM irreducible sheaves on polarized schemes  $(X, \mathcal{O}_X(1))$ , where the only requirement for the line bundle  $\mathcal{O}_X(1)$  is to be ample. With this setting it is proved in [6] and [7] that polarized surfaces  $(S, \mathcal{O}_S(1))$  such that  $p_g = 0$ ,  $q = 0$  or  $1$ , and  $\mathcal{O}_S(1)$  is very ample with  $h^1(\mathcal{O}_S(1)) = 0$  are of wild representation type. Indeed, the aCM vector bundles witnessing wilderness own a special property: they have the maximal permitted number of global sections, namely they are the so-called *Ulrich vector bundles*. Again for  $\mathcal{O}_X(1)$  very ample, it is proved in [22] that for polarized varieties  $(X, \mathcal{O}_X(1))$  of dimension at least two, the embedding given by  $\mathcal{O}_X(l)$  with  $l \geq 3$  is of wild representation type under some mild assumptions on  $\mathcal{O}_X(1)$ .

The goal of the present paper is to contribute to this set of problems: we are constructing families of aCM vector bundles on a large range of polarized integral surfaces  $(X, \mathcal{O}_X(1))$ . In the following Theorem we summarize the results obtained:

**Theorem 1.2.** *Let  $X$  be an integral projective surface with a fixed ample line bundle  $\mathcal{O}_X(1)$  listed below. Then for each integer  $r \geq 2$  there exists an  $b_X(r)$ -dimensional irreducible family  $\{\mathcal{E}_\alpha\}_{\alpha \in \Gamma}$  of indecomposable aCM vector bundles of rank  $r$  on  $X$  such that for each  $\alpha \in \Gamma$  there are only finitely many  $\beta \in \Gamma$  with  $\mathcal{E}_\alpha \cong \mathcal{E}_\beta$ .*

no.	$X$	$b_X(r)$
1	$\pi : X \rightarrow Y$ a birational morphism with $\omega_Y \cong \mathcal{O}_Y$ and $q(Y) = 0$ such that $\pi^{-1}(Y_{\text{sing}}) \cong Y_{\text{sing}}$	$2r$
2	$\omega_X \not\cong \mathcal{O}_X$ locally free with $h^0(\omega_X) = 0$ and $h^0(\omega_X^{\otimes 2}) = 1$ , and $q(X) = 0$	$2\lceil \frac{r}{2} \rceil$
3	smooth and $q(X) = 1$ with $\omega_X^\vee \otimes \mathcal{O}_X(1)$ trivial or ample	1
4	$\pi : X \rightarrow Y$ a birational morphism with an abelian surface $Y$ and $\omega_X^\vee \otimes \mathcal{O}_X(1)$ trivial or ample	$r + 1$
5	$\pi : X \rightarrow Y$ a birational morphism with a hyperelliptic surface $Y$	1
6	$\omega_X \cong \mathcal{O}_X(1)$ with $h^1(\omega_X^{\otimes n}) = 0$ for all $n \in \mathbb{Z}$ and $p_g \geq 3$	$r$

Theorem 1.2 shows that the projective surfaces of Kodaira dimension zero, possibly with singularity, are of wild representation type, except the case of hyperelliptic surfaces. G. Casnati proved in [7] that hyperelliptic surfaces are of wild representation type with respect to a very ample polarization. Note that we do not assume in Theorem 1.2 that  $X$  is minimal or  $\mathcal{O}_X(1)$  is very ample, while the result in [7] is more powerful in a sense that it gives wildness with respect to Ulrich vector bundles.

The strategy for Theorem 1.2 is two-fold. One is to consider zero-dimensional subschemes of length equal to the second Chern class of the aCM vector bundles in consideration, from which we construct aCM vector bundles of arbitrary rank by a series of extensions. The cases no. 1, 2 and 6 are handled by this method respectively in Theorem 2.4, Theorem 3.5 and Theorem 5.4; in case no. 6, for the construction of a family of aCM vector bundles of rank  $r$  even, it is enough to suppose that  $p_g \geq 2$ . The second strategy is to consider a family of aCM line bundles, parametrized by a non-empty open Zariski subset of  $\text{Pic}^0(X)$ , from which we construct aCM vector bundles of arbitrary rank by iterated extensions. the cases no. 3, 4 and 5 are handled by this method respectively in Proposition 4.1, Theorem 1.3 and Proposition 4.5.

Based on the results in Theorem 1.2 we introduce a set-up to measure the complexity of a polarized variety  $(X, \mathcal{O}_X(1))$ . Define

$$a_{X, \mathcal{O}_X(1)}(r) := \sup_{\Gamma} \left\{ \dim \Gamma \mid \begin{array}{l} \Gamma \text{ runs over the parameter spaces of indecomposable} \\ \text{aCM vector bundles of rank } r \text{ on } X \end{array} \right\}$$

with the convention that  $a_{X, \mathcal{O}_X(1)}(r) = -\infty$  if there is no indecomposable aCM vector bundle of rank  $r$ . Then we have  $a_{X, \mathcal{O}_X(1)}(r) \geq b_X(r)$  for the surfaces listed in Theorem 1.2. We also define

$$a_X(r) := \sup \{ a_{X, \mathcal{O}_X(1)}(r) \mid \mathcal{O}_X(1) \text{ ample} \}, \quad a'_X(r) := \inf \{ a_{X, \mathcal{O}_X(1)}(r) \mid \mathcal{O}_X(1) \text{ ample} \}.$$

In many construction of aCM vector bundles, the polarization is assumed to be very ample, in which case we give similar definitions for  $a_X(r)$  and  $a'_X(r)$ , if we consider only very ample polarizations in their definitions. Then we may raise several questions.

- For a given  $X$ , what can be said about the following limits?

$$\limsup_{r \rightarrow \infty} a_X(r), \quad \limsup_{r \rightarrow \infty} a'_X(r), \quad \liminf_{r \rightarrow \infty} a_X(r) \quad \text{and} \quad \liminf_{r \rightarrow \infty} a'_X(r)$$

- What can be said about following suprema

$$\sup_X \{ a_X(r) \} \quad \text{and} \quad \sup_X \{ a'_X(r) \},$$

where  $X$  runs over all smooth projective varieties, all varieties with a prescribed Kodaira dimension or all varieties in a prescribed interesting class, e.g. K3 surfaces?

In those questions with  $(X, \mathcal{O}_X(1))$  polarized surfaces, we may allow singular surfaces, but locally CM, e.g. normal or singularity with embedded dimension at most three, so that we may consider non-locally free aCM sheaves. We do not know if we may obtain bigger dimensional families of indecomposable aCM sheaves by considering non-locally free aCM sheaves.

For higher dimensional smooth varieties we prove the following result.

**Theorem 1.3.** *Let  $X$  be a smooth projective variety of dimension  $n \geq 2$ , birational to an abelian variety and fix an ample line bundle  $\mathcal{O}_X(1)$  with  $\omega_X^\vee \otimes \mathcal{O}_X(1)$  ample. Then  $X$  is wild with respect to  $\mathcal{O}_X(1)$  and*

$$a_{X, \mathcal{O}_X(1)}(r) \geq (n-1)r + 1.$$

For the proof of Theorem 1.3 we use in an essential way a construction by S. Mukai of vector bundles on abelian varieties in [21], a generic vanishing for smooth varieties with maximal Albanese dimension in [12, 13] and results on the local Hilbert schemes in [5, 11].

**Remark 1.4.** In cases no. 1, 2 and 6 of Theorem 1.2 the indecomposable vector bundles that we construct are aCM for any ample line bundles on  $X$ . On the other hand, in cases no. 3, 4 and 5 of Theorem 1.2 and Theorem 1.3 the indecomposable vector bundles that we construct are aCM for every ample line bundles  $\mathcal{O}_X(1)$  with  $\omega_X^\vee \otimes \mathcal{O}_X(1)$  ample.

Recall from Theorem 1.2 that we obtain irreducible families of indecomposable aCM vector bundles of rank  $r$  on several projective surfaces, whose dimensions are at most linear polynomials in  $r$ . Nonetheless, we may not expect that  $a_{X, \mathcal{O}_X(1)}(r)$  is linear in  $r$  for any projective surface. Indeed, Remark 1.5 shows that for  $X$  as in Theorem 1.5 with  $n \geq 3$  we get a lower bound for  $a_{X, \mathcal{O}_X(1)}(r)$  greater than linear, but less than quadratic, in  $r$ .

**Remark 1.5.** Let  $X$  be as in Theorem 1.3. For  $n \geq 3$  and  $r \gg 0$ ,  $\cup_r$  is reducible and there are positive constants  $\alpha_n$  and  $\beta_n$  such that

$$\alpha_n r^{2-2/n} \leq \dim B_f[r] \leq \beta_n r^{2-2/n}$$

by [5] and [11, page 6]. Then from  $\dim B_f[r] \leq a_{X, \mathcal{O}_X(1)}(r)$  we get

$$\liminf_{r \rightarrow \infty} \frac{a_r(X, \mathcal{O}_X(1))}{r^{2-2/n}} > 0.$$

On the other hand, in Section 6 we suggest examples of smooth surfaces of general type with at least a quadratic lower bound for  $a_{X, \mathcal{O}_X(1)}(r)$ .

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## 2. K3-LIKE SURFACES

In this section we assume that  $X$  is integral with  $\omega_X \cong \mathcal{O}_X$  and  $q(X) = 0$ . Set  $\tilde{g} := h^0(\mathcal{O}_X(1))$ ; if  $X$  is a K3 surface, then we have  $2\tilde{g} - 4 = d$  and  $g := \tilde{g} - 1$  is called the genus of  $X$ .

**Proposition 2.1.** *For each  $r \in \mathbb{Z}$  with  $2 \leq r \leq \tilde{g}$ , there exists an indecomposable aCM vector bundle  $\mathcal{E}$  of rank  $r$  on  $X$  with  $\det(\mathcal{E}) \cong \mathcal{O}_X$  and  $c_2(\mathcal{E}) = r$*

*Proof.* Take a general set of points  $S \subset X_{\text{reg}}$  with  $|S| = r$  and let  $\mathcal{E}$  be a general sheaf fitting into the following exact sequence

$$(1) \quad 0 \rightarrow \mathcal{O}_X^{\oplus(r-1)} \xrightarrow{j} \mathcal{E} \rightarrow \mathcal{I}_{S,X} \rightarrow 0.$$

Note that  $\text{ext}_X^1(\mathcal{I}_{S,X}, \mathcal{O}_X) = h^1(\mathcal{I}_{S,X}) = r - 1$  and the sheaf  $\text{Im}(j)$  is the image of the evaluation map  $H^0(\mathcal{E}) \otimes \mathcal{O}_X \rightarrow \mathcal{E}$ . By generality of the extension (1) we may choose a basis  $\{\varepsilon_1, \dots, \varepsilon_{r-1}\}$  of  $\text{Ext}_X^1(\mathcal{I}_{S,X}, \mathcal{O}_X)$  inducing (1). In particular,  $\mathcal{E}$  has no trivial factor. In case  $r = 2$ , the sheaf  $\mathcal{E}$  is locally free from the Cayley-Bacharach condition. For  $r > 2$ , the vector bundle  $\mathcal{F} \oplus \mathcal{O}_X^{\oplus(r-2)}$  with a vector bundle  $\mathcal{F}$  of rank two fitting into (1) as the middle term for  $r = 2$ , also fits into (1) as the middle term. Since local freeness is an open condition, the sheaf  $\mathcal{E}$  is locally free.

Assume  $\mathcal{E} \cong \mathcal{F}_1 \oplus \mathcal{F}_2$  with  $\text{rank}(\mathcal{F}_1) = s$  and  $0 < s < r$ . For each  $i \in \{1, 2\}$ , let  $\mathcal{G}_i \subseteq \mathcal{F}_i$  be the image of the evaluation map  $H^0(\mathcal{F}_i) \otimes \mathcal{O}_X \rightarrow \mathcal{F}_i$  with  $s_i := \text{rank}(\mathcal{G}_i)$ . Then we get  $\mathcal{G}_1 \oplus \mathcal{G}_2 \cong \mathcal{O}_X^{\oplus(r-1)}$ . In particular, each  $\mathcal{G}_i$  is trivial and  $s_1 \in \{s, s-1\}$ . Note that  $(\mathcal{F}_1/\mathcal{G}_1) \oplus (\mathcal{F}_2/\mathcal{G}_2) \cong \mathcal{I}_{S,X}$  has no torsion. If  $s_1 = s$ , then we get  $\mathcal{F}_1/\mathcal{G}_1 \cong 0$ , i.e.  $\mathcal{F}_1 \cong \mathcal{O}_X^{\oplus s}$ , which is impossible since  $\mathcal{E}$  has no trivial factor. If  $s_1 = s-1$ , then we would get a contradiction similarly from  $\mathcal{F}_2 \cong \mathcal{O}_X^{\oplus(r-s)}$ . Thus  $\mathcal{E}$  is indecomposable.

Then it remains to show that  $\mathcal{E}$  is aCM. Since  $h^0(\mathcal{O}_S) \leq h^0(\mathcal{O}_X(1))$  and  $S$  is general, we have  $h^1(\mathcal{I}_{S,X}(t)) = 0$  for all  $t > 0$ . Now  $\{\varepsilon_1, \dots, \varepsilon_{r-1}\}$  is a basis for  $\text{Ext}_X^1(\mathcal{I}_{S,X}, \mathcal{O}_X)$  and so it induces an isomorphism  $H^1(\mathcal{I}_{S,X}) \rightarrow H^2(\mathcal{O}_X^{\oplus(r-1)})$ . Thus we have  $h^0(\mathcal{E}(t)) = 0$  for all  $t \geq 0$ . For any  $\lambda \in \mathbf{k}$ , let  $\mathcal{E}_\lambda$  denote the middle term of the extension corresponding to  $(\varepsilon_1, \lambda\varepsilon_2, \dots, \lambda\varepsilon_{r-1})$ ; we have  $\mathcal{E}_\lambda \cong \mathcal{E}$  for  $\lambda \neq 0$  and  $\mathcal{E}_0 \cong \mathcal{G} \oplus \mathcal{O}_X^{\oplus(r-2)}$  with  $\mathcal{G}$  induced by the extension  $\varepsilon_1$ . As above we see that  $h^1(\mathcal{G}(t)) = 0$  for all  $t \geq 0$ . Since  $\mathcal{G}$  is locally free from the Cayley-Bacharach condition and generality of  $\varepsilon_1$ , we use Serre's duality to obtain  $h^1(\mathcal{G}(t)) = h^1(\mathcal{G}(-t)) = 0$  for  $t < 0$ . Thus  $\mathcal{E}_0$  is aCM. Now using the semicontinuity theorem for cohomology, we obtain  $h^1(\mathcal{E}(t)) = 0$  because  $\mathcal{E}_\lambda \cong \mathcal{E}$ .  $\square$

**Remark 2.2.** Consider the exact sequence (1) with  $r = 2$ . Since  $\text{ext}_X^1(\mathcal{I}_{S,X}, \mathcal{O}_X) = h^1(\mathcal{I}_{S,X}) = 1$ , there exists a unique nontrivial extension of  $\mathcal{I}_{S,X}$  by  $\mathcal{O}_X$ ; denote its middle term by  $\mathcal{G}_S$ . Since the Cayley-Bacharach condition is satisfied, the sheaf  $\mathcal{G}_S$  is an aCM vector bundle of rank two on  $X$ .

**Theorem 2.3.** *For each integer  $2 \leq r \leq \tilde{g}$ , there exists a  $2r$ -dimensional family  $\{\mathcal{E}_\alpha\}_{\alpha \in \Gamma}$  of indecomposable aCM vector bundles of rank  $r$  on  $X$  with  $\det(\mathcal{E}_\alpha) \cong \mathcal{O}_X$  and  $c_2(\mathcal{E}_\alpha) = r$  such that for each  $\alpha \in \Gamma$  there are only finitely many  $\beta \in \Gamma$  with  $\mathcal{E}_\beta \cong \mathcal{E}_\alpha$ .*

*Proof.* For any subset  $S \subset X_{\text{reg}}$  with  $|S| = r$ , define  $\mathbb{E}'(S)$  to be the subset of  $\mathbb{E}(S) := \text{Ext}_X^1(\mathcal{I}_{S,X}, \mathcal{O}_X^{\oplus(r-1)})$ , consisting of all extensions whose corresponding middle terms are aCM and indecomposable vector bundles. By Proposition 2.1,  $\mathbb{E}'(S)$  is a non-empty open subset of  $\mathbb{E}(S)$  and each  $[\mathcal{E}] \in \mathbb{E}'(S)$  has trivial determinant with  $c_2(\mathcal{E}) = r$ .

Letting  $\mathbb{U} := \{S \subset X_{\text{reg}} \mid |S| = r\}$ , there is a vector bundle  $\mathcal{V}$  of rank  $(r-1)^2$  on  $\mathbb{U}$  with  $\mathbb{E}(S)$  as its fibre over  $S \in \mathbb{U}$ , since  $\text{ext}_X^1(\mathcal{I}_{S,X}, \mathcal{O}_X^{\oplus(r-1)}) = (r-1)^2$  for all  $S \in \mathbb{U}$ . Then there is a non-empty open subset  $\mathcal{V}' \subset \mathcal{V}$  with  $\mathcal{V}'_S = \mathbb{E}'(S)$  for a general  $S \in \mathbb{U}$ . Thus there exists an irreducible variety  $\Gamma \subset \mathcal{V}'$  such that the restriction of the map  $\mathcal{V} \rightarrow \mathbb{U}$  to  $\Gamma$  is quasi-finite and dominant. In particular, we have  $\dim \Gamma = \dim \mathbb{U} = 2r$ .

For  $[\mathcal{E}] \in \mathbb{E}'(S)$  we have  $h^0(\mathcal{E}) = r - 1$  and the cokernel of the evaluation map  $H^0(\mathcal{E}) \otimes \mathcal{O}_X \rightarrow \mathcal{E}$  is isomorphic to  $\mathcal{I}_{S,X}$ . Thus for  $[\mathcal{E}] \in \mathbb{E}'(S)$  and  $[\mathcal{F}] \in \mathbb{E}'(S')$  with  $S \neq S' \in \mathbb{U}$ , we have  $\mathcal{E} \not\cong \mathcal{E}'$ . Since the map  $\Gamma \rightarrow \mathbb{U}$  is quasi-finite, the variety  $\Gamma$  satisfies the requirements for the assertion.  $\square$

**Theorem 2.4.** *For each integer  $r \geq 2$ , there exists an  $2r$ -dimensional family  $\{\mathcal{E}_\alpha\}_{\alpha \in \Gamma}$  of indecomposable aCM vector bundles of rank  $r$  on  $X$  with  $\det(\mathcal{E}_\alpha) \cong \mathcal{O}_X$  and  $c_2(\mathcal{E}_\alpha) = r$  such that for each  $\alpha \in \Gamma$  there are only finitely many  $\beta \in \Gamma$  with  $\mathcal{E}_\beta \cong \mathcal{E}_\alpha$ .*

For the proof of Theorem 2.4 we collect numerous technical results below. We fix subsets  $S_0, \dots, S_m \subset X_{\text{reg}}$  with  $|S_0| = 3$  and  $|S_i| = 2$  for all  $1 \leq i \leq m$  such that  $S_i \cap S_j = \emptyset$  for any  $i \neq j$ .

Set  $\mathbb{I}(S_1) := \{\mathcal{I}_{S_1,X}\}$  and define  $\mathbb{I}(S_1, \dots, S_i)$  for  $i \geq 2$  inductively to be the set of all sheaves admitting an extension of  $\mathcal{I}_{S_i,X}$  by an element in  $\mathbb{I}(S_1, \dots, S_{i-1})$ . Thus for each  $i \geq 2$  each sheaf  $\mathcal{J} \in \mathbb{I}(S_1, \dots, S_i)$  admits the following exact sequence for some  $\mathcal{J}' \in \mathbb{I}(S_1, \dots, S_{i-1})$

$$(2) \quad 0 \rightarrow \mathcal{J}' \rightarrow \mathcal{J} \rightarrow \mathcal{I}_{S_i,X} \rightarrow 0.$$

For a subset  $N = \{i_1, \dots, i_k\} \subset \{1, \dots, i\}$  with  $i_1 < \dots < i_k$ , we denote  $\mathbb{I}(S_{i_1}, \dots, S_{i_k})$  by  $\mathbb{I}(S_j; j \in N)$ .

Set  $\mathbb{J}(\emptyset; S_0) := \{\mathcal{I}_{S_0,X}\}$  and define  $\mathbb{J}(S_1, \dots, S_i; S_0)$  to be the set of all isomorphism classes of extensions of  $\mathcal{I}_{S_0,X}$  by an element in  $\mathbb{I}(S_1, \dots, S_i)$ . Similarly we define  $\mathbb{J}(S_j; j \in N; S_0)$ .

**Lemma 2.5.** *Each sheaf  $\mathcal{J} \in \mathbb{I}(S_1, \dots, S_i)$  admits an exact sequence*

$$(3) \quad 0 \rightarrow \mathcal{J} \xrightarrow{\iota} \mathcal{J}^{\vee\vee} \cong \mathcal{O}_X^{\oplus i} \rightarrow \mathcal{O}_{S_1 \cup \dots \cup S_i} \rightarrow 0,$$

where the map  $\iota$  is the double dual map. In particular, we have  $h^0(\mathcal{J}) = 0$  and  $h^1(\mathcal{J}) = h^2(\mathcal{J}) = i$ .

*Proof.* The assertion is clear for  $i = 1$ , i.e.  $\mathcal{J} = \mathcal{I}_{S_1,X}$ . Assume  $i \geq 2$  and consider an exact sequence (2) with  $\mathcal{J}' \in \mathbb{I}(S_1, \dots, S_{i-1})$ . By inductive hypothesis, the assertion holds for  $\mathcal{J}'$  and  $\mathcal{I}_{S_i,X}$  and we get the following commutative diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 \rightarrow & \mathcal{J}' & \rightarrow & \mathcal{J} & \rightarrow & \mathcal{I}_{S_i,X} & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & \mathcal{O}_X^{\oplus(i-1)} & \rightarrow & \mathcal{J}^{\vee\vee} & \rightarrow & \mathcal{O}_X & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & \mathcal{O}_{S_1 \cup \dots \cup S_{i-1}} & \rightarrow & \mathcal{J}^{\vee\vee} / \mathcal{J} & \rightarrow & \mathcal{O}_{S_i} & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array}$$

Since  $\text{ext}_X^1(\mathcal{O}_X, \mathcal{O}_X) = h^1(\mathcal{O}_X) = 0$ , we get  $\mathcal{J}^{\vee\vee} \cong \mathcal{O}_X^{\oplus i}$  from the second horizontal sequence. From the third horizontal sequence, we get  $\mathcal{J}^{\vee\vee} / \mathcal{J} \cong \mathcal{O}_{S_1 \cup \dots \cup S_i}$ , because  $S_i$ 's are disjoint to each other. Then we get the exact sequence (3). The vanishing  $H^0(\mathcal{J}) = 0$  can be obtained by induction on  $i$  and  $h^1(\mathcal{J}) = h^2(\mathcal{J}) = i$  can be obtained from (3).  $\square$

**Remark 2.6.** By the same argument in the proof of Lemma 2.5, we have an exact sequence

$$0 \rightarrow \tilde{\mathcal{J}} \rightarrow \tilde{\mathcal{J}}^{\vee\vee} \cong \mathcal{O}_X^{\oplus(i+1)} \rightarrow \mathcal{O}_{S_0 \cup S_1 \cup \dots \cup S_i} \rightarrow 0,$$

for  $\tilde{\mathcal{J}} \in \mathbb{J}(S_1, \dots, S_i; S_0)$ . This gives  $h^0(\tilde{\mathcal{J}}) = 0$ ,  $h^1(\tilde{\mathcal{J}}) = i + 2$  and  $h^2(\tilde{\mathcal{J}}) = i + 1$ .

**Lemma 2.7.** *For a sheaf  $\mathcal{J} \in \mathbb{I}(S_1, \dots, S_i)$  and any finite subset  $A \subset X$ ,*

- (i) *if  $A \not\subseteq S_j$  for all  $1 \leq j \leq i$ , then we have  $\text{Hom}_X(\mathcal{J}, \mathcal{I}_{A,X}) = 0$ ;*
- (ii) *if  $A \not\subseteq S_j$  for some  $1 \leq j \leq i$ , then we have  $\text{Hom}_X(\mathcal{I}_{A,X}, \mathcal{J}) = 0$ .*

*Proof.* We only prove part (i), because part (ii) can be obtained similarly. Let us use induction on  $i$ ; the case  $i = 1$  is true, because  $A \not\subseteq S_1$  is equivalent to  $\text{Hom}_X(\mathcal{I}_{S_1,X}, \mathcal{I}_{A,X}) = 0$ . Now assume  $i \geq 2$  and consider the sequence (2) with  $\mathcal{J} \in \mathbb{I}(S_1, \dots, S_{i-1})$ . Since  $\text{Hom}_X(\mathcal{I}_{S_i,X}, \mathcal{I}_{A,X}) = 0$ , any map  $f \in \text{Hom}_X(\mathcal{J}, \mathcal{I}_{A,X})$  is

uniquely determined by  $f' \in \text{Hom}_X(\mathcal{F}', \mathcal{F}_{A,X})$ . The inductive assumption gives  $f' = 0$  and so we have  $f = 0$ .  $\square$

**Lemma 2.8.** *We have  $\text{ext}_X^1(\mathcal{I}_{S_{i+1},X}, \mathcal{F}) = 2i$  for  $\mathcal{F} \in \mathbb{L}(S_1, \dots, S_i)$ .*

*Proof.* Let  $S := S_1 \cup \dots \cup S_i$  and apply the functor  $\text{Hom}_X(\mathcal{I}_{S_{i+1},X}, -)$  to the sequence (3) to obtain

$$\begin{aligned} 0 \rightarrow \text{Hom}_X(\mathcal{I}_{S_{i+1},X}, \mathcal{F}) &\rightarrow \text{Hom}_X(\mathcal{I}_{S_{i+1},X}, \mathcal{O}_X^{\oplus i}) \rightarrow \text{Hom}_X(\mathcal{I}_{S_{i+1},X}, \mathcal{O}_S) \\ &\rightarrow \text{Ext}_X^1(\mathcal{I}_{S_{i+1},X}, \mathcal{F}) \rightarrow \text{Ext}_X^1(\mathcal{I}_{S_{i+1},X}, \mathcal{O}_X^{\oplus i}) \rightarrow \text{Ext}_X^1(\mathcal{I}_{S_{i+1},X}, \mathcal{O}_S). \end{aligned}$$

Here, we have  $\text{hom}_X(\mathcal{I}_{S_{i+1},X}, \mathcal{O}_X^{\oplus i}) = i = \text{ext}_X^1(\mathcal{I}_{S_{i+1},X}, \mathcal{O}_X^{\oplus i})$ . We also get  $\text{hom}_X(\mathcal{I}_{S_{i+1},X}, \mathcal{O}_S) = 2i$ , because  $S$  is disjoint from  $S_{i+1}$ . Now apply the functor  $\text{Hom}_X(-, \mathcal{O}_S)$  to the standard exact sequence for  $S_{i+1} \subset X$  to obtain

$$\text{Ext}_X^1(\mathcal{O}_X, \mathcal{O}_S) \rightarrow \text{Ext}_X^1(\mathcal{I}_{S_{i+1},X}, \mathcal{O}_S) \rightarrow \text{Ext}_X^2(\mathcal{O}_{S_{i+1}}, \mathcal{O}_S).$$

Here, we have  $\text{ext}_X^1(\mathcal{O}_X, \mathcal{O}_S) = h^1(\mathcal{O}_S) = 0$  and  $\text{ext}_X^2(\mathcal{O}_{S_{i+1}}, \mathcal{O}_S) = 0$ . In particular, we get  $\text{ext}_X^1(\mathcal{I}_{S_{i+1},X}, \mathcal{O}_S) = 0$ . Finally, apply the functor  $\text{Hom}_X(\mathcal{I}_{S_{i+1},X}, -)$  to the sequence (2) to have

$$\text{Hom}_X(\mathcal{I}_{S_{i+1},X}, \mathcal{F}') \rightarrow \text{Hom}_X(\mathcal{I}_{S_{i+1},X}, \mathcal{F}) \rightarrow \text{Hom}_X(\mathcal{I}_{S_{i+1},X}, \mathcal{I}_{S_i,X}).$$

Since  $S_i \cap S_{i+1} = \emptyset$ , we get  $\text{hom}_X(\mathcal{I}_{S_{i+1},X}, \mathcal{I}_{S_i,X}) = 0$ . By inductive hypothesis, we get  $\text{hom}_X(\mathcal{I}_{S_{i+1},X}, \mathcal{F}') = 0$ . Thus we have  $\text{hom}_X(\mathcal{I}_{S_{i+1},X}, \mathcal{F}) = 0$  and we get the assertion.  $\square$

**Remark 2.9.** Similarly as in the proof of Lemma 2.8, we see that  $\text{ext}_X^1(\mathcal{I}_{S_0,X}, \mathcal{F}) = 3i$  for any  $\mathcal{F} \in \mathbb{L}(S_1, \dots, S_i)$ . In particular, there exists a non-trivial extension

$$0 \rightarrow \mathcal{F} \rightarrow \tilde{\mathcal{F}} \rightarrow \mathcal{I}_{S_0,X} \rightarrow 0.$$

In this case, we have  $\text{ext}_X^1(\mathcal{I}_{S_0,X}, \mathcal{O}_X^{\oplus i}) = 2i$  and the other numeric data in the proof of Lemma 2.8 are all same.

**Lemma 2.10.** *For each  $i \geq 1$ , there exists an indecomposable sheaf  $\mathcal{F} \in \mathbb{L}(S_1, \dots, S_i)$ .*

*Proof.* Since  $\mathcal{I}_{S_1,X}$  has rank one and  $X$  is an integral variety,  $\mathcal{I}_{S_1,X}$  is indecomposable. Thus we may assume  $i \geq 2$ . Note that each  $\mathcal{I}_{S_j,X}$  has the same Hilbert polynomial with respect to any polarization  $\mathcal{O}_X(1)$ . Thus any sheaf in  $\mathbb{L}(S_1, \dots, S_i)$  is strictly semistable with  $\oplus_{j=1}^i \mathcal{I}_{S_j,X}$  as its Jordan-Hölder grading. Let  $\mathcal{F}$  be a general sheaf fitting into an exact sequence

$$(4) \quad 0 \rightarrow \oplus_{j=1}^{i-1} \mathcal{I}_{S_j,X} \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{I}_{S_i,X} \rightarrow 0$$

and assume that  $\mathcal{F}$  is decomposable, say  $\mathcal{F} \cong \mathcal{A}_1 \oplus \dots \oplus \mathcal{A}_h$  with  $h \geq 2$  and each  $\mathcal{A}_j$  indecomposable. Since  $\mathcal{F}$  is strictly semistable with  $gr(\mathcal{F}) \cong \oplus_{j=1}^i \mathcal{I}_{S_j,X}$ , there is a subset  $N_j \subset \{1, \dots, h\}$  for each  $j \in \{1, \dots, h\}$  such that  $gr(\mathcal{A}_j) \cong \oplus_{k \in N_j} \mathcal{I}_{S_k,X}$ . Note that  $\{N_j | 1 \leq j \leq h\}$  forms a partition of  $\{1, \dots, i\}$  with each  $N_j$  non-empty.

Assume first that  $|N_j| = 1$  for all  $j$ . Then we have  $\mathcal{F} \cong \oplus_{j=1}^i \mathcal{I}_{S_j,X}$ . Since we have  $\text{Hom}_X(\mathcal{I}_{S_i,X}, \mathcal{I}_{S_j,X}) = 0$  for all  $j < i$  and  $\text{Hom}_X(\mathcal{I}_{S_i,X}, \mathcal{I}_{S_i,X}) \cong \mathbf{k}$ , we get that the sequence (4) splits, contradicting Lemma 2.8.

Now without loss of generality, assume  $e := |N_1| \geq 2$ . If  $i \notin N_1$ , then by permuting the first  $i-1$  indices of  $S_j$ 's we may assume  $\mathcal{A}_1 \in \mathbb{L}(S_1, \dots, S_e)$ . Then by Lemma 2.7 we have  $\text{hom}_X(\mathcal{I}_{S_j,X}, \mathcal{A}_1) = \text{hom}_X(\mathcal{A}_1, \mathcal{I}_{S_i,X}) = 0$  for all  $j \geq e+1$ . Thus  $f$  induces an isomorphism  $f' : \mathcal{A}_1 \rightarrow \oplus_{j=1}^e \mathcal{I}_{S_j,X}$ , contradicting the assumption  $e \geq 2$  and the indecomposability of  $\mathcal{A}_1$ . If  $i \in N_1$ , then by permuting the first  $i-1$  indices of  $S_j$ 's we may assume  $\mathcal{A}_1 \in \mathbb{L}(S_{i-e+1}, \dots, S_i)$ . From the case when  $i \notin N_1$  we may also assume  $|N_j| = 1$  for all  $j > 1$ , and this implies  $\mathcal{F} \cong \mathcal{A}_1 \oplus (\oplus_{j=1}^{i-e} \mathcal{I}_{S_j,X})$ . Then by Lemma 2.7 we have  $\text{Hom}_X(\mathcal{I}_{S_j,X}, \mathcal{A}_1) = 0$  for all  $j \leq i-e$ . In particular, the extension class  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{i-1})$  corresponding to (4) with  $\varepsilon_j \in \text{Ext}_X^1(\mathcal{I}_{S_i,X}, \mathcal{I}_{S_j,X})$  satisfies  $\varepsilon_j = 0$  for all  $j \leq i-e$ , contradicting Lemma 2.8 and the generality of  $\varepsilon$ .  $\square$

**Remark 2.11.** As in the proof of Lemma 2.10, let us consider a general sheaf  $\tilde{\mathcal{F}}$  fitting into an exact sequence

$$(5) \quad 0 \rightarrow \bigoplus_{j=1}^i \mathcal{I}_{S_j, X} \rightarrow \tilde{\mathcal{F}} \rightarrow \mathcal{I}_{S_0, X} \rightarrow 0.$$

By Remark 2.9 the extension (5) is non-trivial. Here,  $\tilde{\mathcal{F}} \in \mathbb{J}(S_1, \dots, S_i; S_0)$  and the sequence (5) is the Harder-Narasimhan filtration of  $\tilde{\mathcal{F}}$ . Assume that  $\tilde{\mathcal{F}}$  is decomposable, say  $\tilde{\mathcal{F}} \cong \tilde{\mathcal{A}}_1 \oplus \dots \oplus \tilde{\mathcal{A}}_h$ . Note that the HN filtration of  $\tilde{\mathcal{F}}$  is obtained from the ones of each  $\tilde{\mathcal{A}}_i$ . In particular, as in the proof of Lemma 2.10, we have a partition  $\{N_j | 1 \leq j \leq h\}$  of  $\{0, 1, \dots, i\}$  such that  $\tilde{\mathcal{A}}_j \in \mathbb{J}(S_k; k \in N_j)$  if  $0 \notin N_j$ , and  $\tilde{\mathcal{A}}_j \in \mathbb{J}(S_k; k \in N_j \setminus \{0\}; S_0)$ . Then by the same argument in the proof of Lemma 2.10, we get a contradiction. Thus we get an indecomposable sheaf in  $\mathbb{J}(S_1, \dots, S_i; S_0)$ .

**Lemma 2.12.** *For each integer  $i \geq 1$ , the set  $\mathbb{J}(S_1, \dots, S_i)$  is parametrized by an affine space  $T(S_1, \dots, S_i)$ , not necessarily finite-to-one, equipped with the universal sheaf, i.e. a sheaf  $\mathcal{S}(S_1, \dots, S_i)$  on  $T(S_1, \dots, S_i) \times X$  such that the fiber of  $\mathcal{S}(S_1, \dots, S_i)$  over  $\{\mathcal{J}\} \times X$  with  $\mathcal{J} \in \mathbb{J}(S_1, \dots, S_i)$  is the sheaf  $\mathcal{J}$  on  $X$ .*

*Proof.* For  $i = 1$  we may take as  $T(S_1)$  just a single point set, because  $\mathbb{J}(S_1) = \{\mathcal{I}_{S_1, X}\}$ . Assume that there exists an affine space  $T(S_1, \dots, S_{i-1})$  and a sheaf  $\mathcal{S}(S_1, \dots, S_{i-1})$  with prescribed property for  $i \geq 2$ . We set

$$\begin{aligned} T(S_1, \dots, S_i) &:= \mathcal{E}xt_{p_1}^1(\mathcal{S}(S_1, \dots, S_{i-1}), p_2^* \mathcal{I}_{S_i, X}) \\ &= R^i(p_{1*} \mathcal{H}om_{T(S_1, \dots, S_{i-1}) \times X}(\mathcal{S}(S_1, \dots, S_{i-1}), -))(p_2^* \mathcal{I}_{S_i, X}) \end{aligned}$$

to be the relative  $\mathcal{E}xt_{p_1}^1$ -sheaf, where  $p_j$  is the projection from  $T(S_1, \dots, S_{i-1}) \times X$  to its  $j$ -th factor; see [20, Proposition 3.1]. By Lemma 2.8 we have  $\text{ext}_X^1(\mathcal{J}', \mathcal{I}_{S_i, X}) = 2i - 2$  for each  $\mathcal{J}' \in T(S_1, \dots, S_{i-1})$ . This implies that  $T(S_1, \dots, S_i)$  is a vector bundle of rank  $2i - 2$  over  $T(S_1, \dots, S_{i-1})$  and so it is an affine space parametrizing  $\mathbb{J}(S_1, \dots, S_i)$  as required. We may also take as  $\mathcal{S}(S_1, \dots, S_i)$  the universal extension on  $T(S_1, \dots, S_i) \times X$  as in [20, Corollary 3.4].  $\square$

**Remark 2.13.** Following the same argument in the proof of Lemma 2.12, we can obtain an affine space  $\tilde{T}(S_1, \dots, S_i; S_0)$  parametrizing  $\mathbb{J}(S_1, \dots, S_i)$  equipped with the universal sheaf  $\tilde{\mathcal{S}}(S_1, \dots, S_i; S_0)$ .

*Proof of Theorem 2.4:* Assume that  $r$  is even and set  $m := r/2$ . Fix subsets  $S_1, \dots, S_m \subset X_{\text{reg}}$  such that  $|S_i| = 2$  for all  $i$  and  $S_i \cap S_j = \emptyset$  for all  $i \neq j$ . By Lemma 2.10 there exists an indecomposable sheaf  $\mathcal{J} \in \mathbb{J}(S_1, \dots, S_m)$ , for which we consider a general sheaf  $\mathcal{E}$  fitting into the following exact sequence:

$$(6) \quad 0 \rightarrow \mathcal{O}_X^{\oplus m} \xrightarrow{f} \mathcal{E} \rightarrow \mathcal{J} \rightarrow 0.$$

Note that  $\mathcal{E}$  has rank  $r$  with  $\det(\mathcal{E}) \cong \mathcal{O}_X$  and  $c_2(\mathcal{E}) = r$ . Let  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in \text{Ext}_X^1(\mathcal{J}, \mathcal{O}_X^{\oplus m})$  be the extension class corresponding to (6) with  $\varepsilon_i \in \text{Ext}_X^1(\mathcal{J}, \mathcal{O}_X)$ . Note that  $h^0(\mathcal{E}) = m$  and  $f(\mathcal{O}_X^{\oplus m})$  is the image of the evaluation map  $\rho_{\mathcal{E}} : H^0(\mathcal{E}) \otimes \mathcal{O}_X \rightarrow \mathcal{E}$  with  $\mathcal{J} = \text{coker}(\rho_{\mathcal{E}})$ .

By Lemma 2.5 and Serre's duality, we have  $\text{ext}_X^1(\mathcal{J}, \mathcal{O}_X) = h^1(\mathcal{J}) = m$ . From the generality of  $\varepsilon$  we see that the extensions  $\varepsilon_1, \dots, \varepsilon_m$  are linearly independent. In particular, we have  $A \cdot \varepsilon \neq 0$  for all  $A \in \text{GL}(m)$ , and so  $\mathcal{E} \not\cong \mathcal{O}_X \oplus \mathcal{G}$  with  $\mathcal{G}$  an extension of  $\mathcal{J}$  by  $\mathcal{O}_X^{\oplus(m-1)}$ . Since  $f(\mathcal{O}_X^{\oplus m}) \subset \mathcal{E}$  is the image of  $\rho_{\mathcal{E}}$ , we get that  $\mathcal{E} \not\cong \mathcal{O}_X \oplus \mathcal{G}$  for any sheaf  $\mathcal{G}$ , i.e.  $\mathcal{E}$  has no trivial factor.

Assume that  $\mathcal{E}$  is decomposable, say  $\mathcal{E} \cong \mathcal{E}_1 \oplus \mathcal{E}_2$  with each  $\mathcal{E}_i \neq 0$ . Since the global section functor  $H^0(-)$  and the evaluation map commute with direct sums, we have  $\mathcal{J} \cong \text{coker}(\rho_{\mathcal{E}_1}) \oplus \text{coker}(\rho_{\mathcal{E}_2})$ . Since  $\mathcal{J}$  is indecomposable, we get  $\text{coker}(\rho_{\mathcal{E}_i}) = 0$  for some  $i \in \{1, 2\}$ . This implies that  $\mathcal{E}_i$  is trivial, which is impossible because  $\mathcal{E}$  has no trivial factor.

To conclude the case  $r$  even we need to find a sheaf  $\mathcal{E}$  that is locally free and aCM. Consider the variety  $T(S_1, \dots, S_m)$  together with the sheaf  $\mathcal{S}(S_1, \dots, S_m)$  in Lemma 2.12. Define

$$\mathcal{V}(S_1, \dots, S_m) := \mathcal{E}xt_{p_2}^1(\mathcal{S}(S_1, \dots, S_m), p_2^* \mathcal{O}_X^{\oplus m})$$

to be the relative  $\mathcal{E}xt_{p_2}^1$ -sheaf as in [20, Proposition 3.1]; the fibre of  $\mathcal{V}(S_1, \dots, S_m)$  over a point  $\mathcal{J} \in T(S_1, \dots, S_m)$  is the set of all extensions of  $\mathcal{J}$  by  $\mathcal{O}_X^{\oplus m}$ . By Lemma 2.5 the sheaf  $\mathcal{V}(S_1, \dots, S_m)$  is a vector bundle of rank  $m^2$  on  $T(S_1, \dots, S_m)$  and so it is an affine space. Since  $\mathcal{G}_{S_1} \oplus \dots \oplus \mathcal{G}_{S_m}$  is locally free and aCM, the sheaf associated to a general point in  $\mathcal{V}$  is locally free and aCM. Define

$$\mathbb{U} := \{(S_1, \dots, S_m) \mid S_i \subset X_{\text{reg}} \text{ with } |S_i| = 2 \text{ and } S_i \cap S_j = \emptyset \text{ for all } i \neq j\}$$

and consider a vector bundle  $\mathcal{V}$  on  $\mathbb{U}$ , whose fibre over  $(S_1, \dots, S_m)$  is  $\mathcal{V}(S_1, \dots, S_m)$ . Then there exists a non-empty open subset  $\mathcal{V}' \subset \mathcal{V}$  such that the middle term of each extension in  $\mathcal{V}'$  is aCM and locally free. As in the proof of Theorem 2.3 we can choose an irreducible subvariety  $\Gamma \subset \mathcal{V}'$  such that the restriction of the map  $\mathcal{V}' \rightarrow \mathbb{U}$  to  $\Gamma$  is quasi-finite and dominant. Hence we get the assertion for the case  $r$  even.

Now assume that  $r$  is odd, say  $r = 2m + 3$ . The case  $m = 0$  is true by Proposition 2.1 with  $r = 3$ , because we have  $g = h^0(\mathcal{O}_X(1)) \geq 3$ . Now assume  $r \geq 5$ , i.e.  $m \geq 1$ , and that Theorem 2.4 is true for all odd integers less than  $r$ . We fix subsets  $S_0, \dots, S_m \subset X_{\text{reg}}$  with  $|S_0| = 3$  and  $|S_i| = 2$  for all  $i \geq 1$  such that  $S_i \cap S_j = \emptyset$  for all  $i \neq j$ . Define

$$\mathcal{W}(S_1, \dots, S_m; S_0) := \mathcal{E}xt_{p_2}^1(\tilde{\mathcal{F}}(S_1, \dots, S_m; S_0), p_2^* \mathcal{O}_X^{\oplus(m+2)}),$$

where  $\tilde{\mathcal{F}}(S_1, \dots, S_m; S_0)$  is the universal sheaf in Remark 2.13. Then it parametrizes all the extensions of some sheaf  $\tilde{\mathcal{F}} \in \mathbb{J}(S_1, \dots, S_m; S_0)$  by  $\mathcal{O}_X^{\oplus(m+2)}$ . Note that for each extension in  $\mathcal{W}(S_1, \dots, S_m; S_0)$  the corresponding middle term  $\mathcal{E}$  is torsion-free and has rank  $r = 2m + 3$  with  $\det(\mathcal{E}) \cong \mathcal{O}_X$  and  $c_2(\mathcal{E}) = r$ .

Let us denote by  $\mathcal{G}_{S_0}$  an aCM and indecomposable vector bundle of rank three, admitting an extension of  $\mathcal{I}_{S_0, X}$  by  $\mathcal{O}_X^{\oplus 2}$  as in Proposition 2.1. Then  $\bigoplus_{i=1}^m \mathcal{G}_{S_i}$  is the middle term of an extension in  $\mathcal{W}(S_1, \dots, S_m; S_0)$ , which is locally free and aCM. So the general extension in  $\mathcal{W}(S_1, \dots, S_m; S_0)$  has the aCM and indecomposable middle term. Now fix an indecomposable sheaf  $\tilde{\mathcal{F}} \in \mathbb{J}(S_1, \dots, S_m; S_0)$  in Remark 2.11 and consider a general sheaf  $\mathcal{E}$  fitting into the following exact sequence:

$$(7) \quad 0 \rightarrow \mathcal{O}_X^{\oplus(m+2)} \xrightarrow{f} \mathcal{E} \xrightarrow{g} \tilde{\mathcal{F}} \rightarrow 0.$$

Assume that  $\mathcal{E}$  is decomposable, say  $\mathcal{E} \cong \mathcal{E}_1 \oplus \mathcal{E}_2$  with each  $\mathcal{E}_i \neq 0$ . As before,  $f(\mathcal{O}_X^{\oplus(m+2)})$  is the image of the evaluation map  $\rho_{\mathcal{E}} : H^0(\mathcal{E}) \otimes \mathcal{O}_X \rightarrow \mathcal{E}$  and  $\text{coker}(\rho_{\mathcal{E}}) = \tilde{\mathcal{F}}$ . Since the global section functor  $H^0(-)$  and the evaluation map commute with finite direct sums, we have  $\tilde{\mathcal{F}} \cong \text{coker}(\rho_{\mathcal{E}_1}) \oplus \text{coker}(\rho_{\mathcal{E}_2})$ . Since  $\tilde{\mathcal{F}}$  is indecomposable, we get that  $\mathcal{E}_i$  is trivial for some  $i$ , which contradicts to the generality of the extension (7), because we have  $\text{ext}_X^1(\tilde{\mathcal{F}}, \mathcal{O}_X) = h^1(\tilde{\mathcal{F}}) = m + 2$  by Remark 2.6. As in the case  $r$  even, we define

$$\begin{aligned} \tilde{\mathbb{U}} := \{(S_0, S_1, \dots, S_m) \mid S_i \subset X_{\text{reg}} \text{ with } |S_0| = 3, \\ |S_i| = 2 \text{ for all } 1 \leq i \leq m \text{ and } S_i \cap S_j = \emptyset \text{ for all } i \neq j\}. \end{aligned}$$

We consider a vector bundle  $\mathcal{W}$  on  $\tilde{\mathbb{U}}$ , whose fibre over  $(S_0, S_1, \dots, S_m)$  is  $\mathcal{W}(S_1, \dots, S_m; S_0)$ . Then we get the assertion, following the same argument in the case  $r$  even.  $\square$

**Remark 2.14.** Let  $\pi : Y \rightarrow X$  be a birational morphism between integral projective surfaces with  $\omega_X \cong \mathcal{O}_X$  and  $q(X) = 0$  such that  $\pi$  induces an isomorphism  $\pi^{-1}(X_{\text{sing}}) \cong X_{\text{sing}}$ . In particular, we have  $Y_{\text{reg}} = \pi^{-1}(X_{\text{reg}})$ . This implies that  $\pi_* \mathcal{O}_Y \cong \mathcal{O}_X$  and  $R^1 \pi_* \mathcal{O}_Y \cong 0$ . Since each fiber of  $\pi$  has dimension at most one, we also have  $R^2 \pi_* \mathcal{F} \cong 0$  for any coherent sheaf  $\mathcal{F}$  on  $X$ . Thus we have  $q(Y) = 0$  and  $h^2(\mathcal{O}_Y) = 1$ . Since  $\pi$  induces an isomorphism between  $\pi^{-1}(X_{\text{sing}})$  and  $X_{\text{sing}}$ , the canonical sheaf  $\omega_Y$  is locally free with  $h^0(\omega_Y) = 1$  and so there is an effective divisor  $\Delta$  such that  $|\omega_Y| = \{\Delta\}$ ; we have  $\Delta = \emptyset$  if and only if  $\pi$  is an isomorphism. By Serre's duality we have  $\text{ext}_Y^1(\mathcal{I}_{S, Y}, \mathcal{O}_Y) = h^1(\mathcal{I}_{S, Y} \otimes \omega_Y)$ . Since  $|\omega_Y| = \{\Delta\}$  and  $S \cap \Delta = \emptyset$ , we may use the long exact sequence of cohomology of the following exact sequence

$$0 \rightarrow \mathcal{I}_{S, Y} \otimes \omega_Y \rightarrow \omega_Y \rightarrow \mathcal{O}_S \rightarrow 0$$

to obtain  $\text{ext}_Y^1(\mathcal{I}_{S, Y}, \mathcal{O}_Y) = |S| - 1$  for any finite subset  $S \subset Y_{\text{reg}} \setminus \Delta$ . Then the same statement of Theorem 2.4 holds for  $Y$ , using the same argument in its proof with subsets  $S_i \subset Y_{\text{reg}} \setminus \Delta$  for  $i = 0, \dots, m$ .

## 3. ENRIQUES SURFACES

In this section we assume that  $X$  is an integral projective surface with  $q(X) = 0$  and  $\omega_X \not\cong \mathcal{O}_X$  locally free such that  $h^0(\omega_X) = 0$  and  $h^0(\omega_X^{\otimes 2}) = 1$ . Let  $\Delta \geq 0$  be the effective divisor such that  $\omega_X^{\otimes 2} \cong \mathcal{O}_X(\Delta)$ . When  $X$  is smooth, the minimal model of  $X$  is an Enriques surface. Note that  $h^2(\mathcal{O}_X) = h^0(\omega_X) = 0$  and so  $\chi(\mathcal{O}_X) = 1$ . Set  $X' := X_{\text{reg}} \cap (X \setminus \Delta)$ .

**Remark 3.1.** We fix an ample line bundle  $\mathcal{O}_X(1)$  on  $X$  such that  $h^1(\mathcal{O}_X(t)) = 0$  for all  $t \in \mathbb{Z}$ ; at least in characteristic zero Kodaira's vanishing theorem shows that we only need this assumption for  $t \geq 0$ . The case  $t = 0$  is a general assumption of the surfaces considered in this article. Serre's duality gives  $h^1(\omega_X(t)) = 0$  for all  $t \in \mathbb{Z}$ . Notice that using Riemann-Roch it is easy to see that under these hypothesis  $h^0(\omega_X(1)) \neq 0$ . In summary, we take a polarization  $\mathcal{O}_X(1)$  such that  $h^0(\omega_X(1)) \neq 0$  and  $h^1(\mathcal{O}_X(t)) = h^1(\omega_X(t)) = 0$  for all  $t \in \mathbb{Z}$ . If  $\Delta = \emptyset$ , e.g. minimal Enriques surfaces, then we always have  $h^1(\mathcal{O}_X(t)) = 0$  for  $t > 0$ , because  $\omega_X(t)$  with  $t > 0$  is ample; it is numerically equivalent to  $\mathcal{O}_X(t)$  and so we can use Kodaira's vanishing theorem.

For any point  $p \in X_{\text{reg}}$ , we have  $\text{ext}_X^1(\mathcal{I}_{p,X}, \mathcal{O}_X) = h^1(\mathcal{I}_{p,X} \otimes \omega_X) = 1$  by Serre's duality. Thus, up to isomorphisms, there is a unique sheaf  $\mathcal{E}_p$  that fits into the following non-trivial extension:

$$(8) \quad 0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E}_p \rightarrow \mathcal{I}_{p,X} \rightarrow 0.$$

Obviously  $\mathcal{E}_p$  has rank two and it is locally free outside  $p$  with  $\det(\mathcal{E}_p) \cong \mathcal{O}_X$ . Since  $p \in X_{\text{reg}}$  and  $h^0(\omega_X) = 0$ , the Cayley-Bacharach condition is satisfied. Thus  $\mathcal{E}_p$  is locally free. Note that the point  $p$  is uniquely determined by the isomorphism class of  $\mathcal{E}_p$ , because we have  $h^0(\mathcal{E}_p) = 1$  by the sequence (8) and any non-zero section of  $\mathcal{E}_p$  vanishes only at  $p$ .

**Lemma 3.2.** *For a general  $p \in X_{\text{reg}}$  the vector bundle  $\mathcal{E}_p$  is aCM and indecomposable.*

*Proof.* The exact sequence (8) twisted by  $\mathcal{O}_X(t)$  gives  $h^1(\mathcal{E}_p(t)) = 0$  for all  $t \geq 0$ . From  $\mathcal{E}_p^\vee \cong \mathcal{E}_p$  we see that  $h^1(\mathcal{E}_p \otimes \omega_X) = h^1(\mathcal{E}_p) = 0$  by Serre's duality. Now fix an integer  $t < 0$ . The twist of the sequence (8) by  $\omega_X(-t)$  gives

$$h^1(\mathcal{E}_p \otimes \omega_X(-t)) \leq h^1(\omega_X(-t)) + h^1(\mathcal{I}_{p,X} \otimes \omega_X(-t)) = h^1(\mathcal{I}_{p,X} \otimes \omega_X(-t)).$$

Here, we have  $h^1(\omega_X(-t)) = 0$  by our assumptions on the polarization  $\mathcal{O}_X$ . We also have  $h^0(\omega_X(-t)) > 0$  from the assumption that  $h^0(\omega_X(1)) > 0$ . Since  $p$  is general, we have  $h^1(\mathcal{I}_{p,X} \otimes \omega_X(-t)) = 0$ . By Serre's duality, this implies that  $h^1(\mathcal{E}_p(t)) = h^1(\mathcal{E}_p \otimes \omega_X(-t)) = 0$ . Thus  $\mathcal{E}_p$  is aCM.

Assume that  $\mathcal{E}_p$  is decomposable; say  $\mathcal{E}_p \cong \mathcal{A}_1 \oplus \mathcal{A}_2$  with each  $\mathcal{A}_i$  a line bundle. Since  $h^0(\mathcal{E}_p) = 1$ , we may assume that  $h^0(\mathcal{A}_1) = 1$  and  $h^0(\mathcal{A}_2) = 0$ . Since the evaluation map commutes with direct sums and  $\mathcal{I}_{p,X}$  is isomorphic to the cokernel of the evaluation map  $H^0(\mathcal{E}_p) \otimes \mathcal{O}_X \rightarrow \mathcal{E}_p$ , we get  $\mathcal{A}_2 \cong \mathcal{I}_{p,X}$ , a contradiction.  $\square$

**Lemma 3.3.** *For any two general points  $p, q \in X_{\text{reg}}$ , we have  $\text{ext}_X^1(\mathcal{E}_p, \mathcal{E}_q) = 1$ .*

*Proof.* Since  $\det(\mathcal{E}_q) \cong \mathcal{O}_X$ , we have  $\mathcal{E}_q^\vee \cong \mathcal{E}_q$  and so  $\text{Ext}_X^1(\mathcal{E}_p, \mathcal{E}_q) \cong H^1(\mathcal{E}_p \otimes \mathcal{E}_q)$ . Tensoring the exact sequence (8) with  $\mathcal{E}_q$ , we get the exact sequence

$$(9) \quad 0 \rightarrow \mathcal{E}_q \rightarrow \mathcal{E}_p \otimes \mathcal{E}_q \rightarrow \mathcal{I}_{p,X} \otimes \mathcal{E}_q \rightarrow 0.$$

Since  $\mathcal{E}_q$  is aCM, we have  $h^1(\mathcal{E}_q) = 0$ . On the other hand, tensoring the sequence (8) for  $\mathcal{E}_q$  with  $\omega_X$  gives  $h^0(\mathcal{E}_q \otimes \omega_X) = 0$ , because  $\omega_X \not\cong \mathcal{O}_X$ . Thus by Serre's duality we get  $h^2(\mathcal{E}_q) = h^0(\mathcal{E}_q \otimes \omega_X) = 0$ . Then the assertion follows from the exact sequence

$$0 \rightarrow \mathcal{I}_{p,X} \otimes \mathcal{E}_q \rightarrow \mathcal{E}_q \rightarrow (\mathcal{E}_q)_{|p} \rightarrow 0$$

together with the fact that  $\mathcal{E}_q$  is an aCM vector bundle of rank two and  $H^0(\mathcal{E}_q)$  is one-dimensional whose nontrivial section vanishes only at  $q$  so that  $h^0(\mathcal{I}_{p,X} \otimes \mathcal{E}_q) = 0$ .  $\square$

**Proposition 3.4.** *Setting  $\tilde{g} := h^0(\mathcal{O}_X(1))$ , there exists an indecomposable aCM vector bundle  $\mathcal{E}$  of rank  $r$  on  $X$  with  $\det(\mathcal{E}) \cong \mathcal{O}_X$  and  $c_2(\mathcal{E}) = r - 1$  for each integer  $2 \leq r \leq \tilde{g} - 1$ .*

*Proof.* As in the proof of Proposition 2.1, consider a general sheaf  $\mathcal{E}$  fitting into the sequence (1) for a general  $S \subset X_{\text{reg}}$  with  $|S| = r - 1$ . Then we get  $\text{ext}_X^1(\mathcal{I}_{S,X}, \mathcal{O}_X) = r - 1$  and the proof of Proposition 2.1 works verbatim.  $\square$

**Theorem 3.5.** *Let  $X$  be an integral projective surface with  $q(X) = 0$  and  $\omega_X \not\cong \mathcal{O}_X$  locally free such that  $h^0(\omega_X) = 0$  and  $h^0(\omega_X^{\otimes 2}) = 1$ . Then for any  $r \geq 2$  there exists a family  $\{\mathcal{E}_\alpha\}_{\alpha \in \Gamma}$  of dimension  $2\lceil \frac{r}{2} \rceil$  of indecomposable rank  $r$  aCM vector bundles with  $c_1(\mathcal{E}_\alpha) \cong \mathcal{O}_X$  such that for each  $\alpha \in \Gamma$  there are only finitely many  $\beta \in \Gamma$  with  $\mathcal{E}_\beta \cong \mathcal{E}_\alpha$ .*

*Proof.* The proof follows exactly the same structure as in the case of Theorem 2.4. In the present setting, however, in the case of even rank  $r = 2m$ , the family  $\Gamma$  of indecomposable aCM vector bundles of rank  $r$  will be mapped by a quasi-finite dominant morphism to

$$\mathbb{U} := \{(S_1, \dots, S_m) \mid S_i \subset X_{\text{reg}} \text{ with } |S_i| = 1 \text{ and } S_i \cap S_j = \emptyset \text{ for all } i \neq j\},$$

a variety of dimension  $r$ , while in the odd case  $r = 2m + 3$  it will be mapped to

$$\begin{aligned} \tilde{\mathbb{U}} := \{(S_0, S_1, \dots, S_m) \mid S_i \subset X_{\text{reg}} \text{ with } |S_0| = 2, \\ |S_i| = 1 \text{ for all } 1 \leq i \leq m \text{ and } S_i \cap S_j = \emptyset \text{ for all } i \neq j\}. \end{aligned}$$

a variety of dimension  $2m + 4 = 2\lceil \frac{r}{2} \rceil$ .  $\square$

#### 4. IRREGULAR SURFACES

In this section we deal with surfaces with  $q(X) \geq 1$ .

**Proposition 4.1.** *Let  $X$  be a smooth projective surface with  $q(X) = 1$  and a fixed ample line bundle  $\mathcal{O}_X(1)$ , satisfying one of the following conditions:*

- (i)  $\mathcal{O}_X(1) \cong \omega_X$ ;
- (ii)  $\mathcal{O}_X(1) \otimes \omega_X^\vee$  is ample.

*Then for each positive integer  $r$  there exists a one-dimensional family  $\{\mathcal{E}_\alpha\}_{\alpha \in \Gamma}$  of indecomposable aCM vector bundles of rank  $r$  on  $X$  such that  $\mathcal{E}_\alpha$  for each  $\alpha \in \Gamma$  is strictly semistable with  $\det(\mathcal{E}_\alpha) \in \text{Pic}^0(X)$  and  $c_2(\mathcal{E}_\alpha) = 0$  with respect to any polarization of  $X$ , and there are only finitely many  $\beta \in \Gamma$  with  $\mathcal{E}_\beta \cong \mathcal{E}_\alpha$ .*

*Proof.* Fix a general line bundle  $\mathcal{L} \in \text{Pic}^0(X)$ . Then we have  $h^1(\mathcal{L}) = 0$ ; see [3, Th. 0.1], [12, Theorem 1] or [13, Theorem 0.1]. We also have  $h^1(\mathcal{L}(-t)) = 0$  for all  $t > 0$  by Kodaira's vanishing. Note that Serre's duality gives  $h^1(\mathcal{L}(t)) = h^1(\mathcal{L}^\vee \otimes \omega_X(-t))$ . Then we have  $h^1(\mathcal{L}^\vee \otimes \omega_X(-t)) = 0$  for all  $t > 0$ . Indeed, in case (i) we may apply Kodaira's vanishing for  $t \geq 2$  and  $h^1(\mathcal{L}^\vee) = 0$  for  $t = 1$ . In case (ii)  $\omega_X^\vee(t)$  is ample and so we may apply Kodaira's vanishing. Thus  $\mathcal{L}$  is aCM.

Let  $\varphi : X \rightarrow C$  be the Albanese map of  $X$  onto an elliptic curve  $C$ . We have  $\varphi_* \mathcal{O}_X \cong \mathcal{O}_C$  and  $\text{Pic}^0(X) = \varphi^* \text{Pic}(C)$ . By the classification of vector bundles on an elliptic curve in [1], there is an indecomposable vector bundle  $\mathcal{F}$  of rank  $r$  on  $C$ , which is an iterated extension of  $\mathcal{O}_C$ . Define

$$\mathcal{E}_\mathcal{L} := \varphi^* \mathcal{F} \otimes \mathcal{L}.$$

Then  $\mathcal{E}_\mathcal{L}$  is a vector bundle of rank  $r$  on  $X$  with  $\det(\mathcal{E}_\mathcal{L}) \cong \mathcal{L}^{\otimes r} \in \text{Pic}^0(X)$  and  $c_2(\mathcal{E}_\mathcal{L}) = 0$ , which is an iterated extension of  $\mathcal{L}$ . Since  $\mathcal{L}$  is aCM, so is  $\mathcal{E}_\mathcal{L}$ .

Assume that  $\mathcal{E}_\mathcal{L}$  is decomposable and this would imply that  $\varphi^* \mathcal{F}$  is also decomposable, say  $\varphi^* \mathcal{F} \cong \mathcal{F}_1 \oplus \mathcal{F}_2$  with each  $\mathcal{F}_i$  an aCM vector bundle of rank  $r_i$  with  $0 < r_i < r$ . By the projection formula and  $\varphi_* \mathcal{O}_X \cong \mathcal{O}_C$ , we have  $\mathcal{F} \cong \varphi_* \mathcal{F}_1 \oplus \varphi_* \mathcal{F}_2$ . Now take a non-empty subset of  $C$  so that

- we have  $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus r}$ , and
- $\varphi^{-1}(q)$  is a smooth projective curve for each  $q \in U$ .

Since  $(\varphi^* \mathcal{F})|_{\varphi^{-1}(q)}$  is the trivial vector bundle of rank  $r$  on the integral projective curve  $\varphi^{-1}(q)$ , we get  $\mathcal{F}_i|_{\varphi^{-1}(q)} \cong \mathcal{O}_{\varphi^{-1}(q)}^{\oplus r_i}$  for each  $i$ . In particular, we have  $\varphi_* \mathcal{F}_i$  is not zero for each  $i$ , a contradiction to the indecomposability of  $\mathcal{F}$ .  $\square$

**Remark 4.2.** Let  $X$  be a smooth and connected projective variety of dimension  $n \geq 2$  and  $\varphi : X \rightarrow \text{Alb}(X)$  its Albanese map. Assume that  $X$  has *maximal Albanese dimension*, i.e.  $\dim \varphi(X) = n$ . Note that this implies  $q(X) = \dim \text{Alb}(X) = n \geq 2$ . In particular, an abelian variety has maximal Albanese dimension. Let  $\mathcal{O}_X(1)$  be an ample line bundle on  $X$  such that  $\omega_X^\vee \otimes \mathcal{O}_X(1)$  is ample; if  $X$  is an abelian variety, then  $\mathcal{O}_X(1)$  can be arbitrary.

Now choose a general line bundle  $\mathcal{L} \in \text{Pic}^0(X)$ . Since  $X$  has Albanese dimension  $n$ , we have  $h^i(\mathcal{L}) = 0$  for all  $1 \leq i \leq n-1$  by [12, Theorem 1] or [13, Theorem 0.1]. Fix a positive integer  $t$ . By Kleiman's numerical criterion of ampleness in [17], we get that  $\mathcal{L}^\vee(t)$  and  $\omega_X^\vee \otimes \mathcal{L}(t)$  are ample for  $t > 0$ . Then Kodaira's vanishing gives  $h^i(\mathcal{L}(-t)) = h^i(\omega_X \otimes \mathcal{L}^\vee(-t)) = 0$  for all  $1 \leq i \leq n-1$ . On the other hand, Serre's duality gives  $h^i(\mathcal{L}(t)) = h^{n-i}(\omega_X \otimes \mathcal{L}^\vee(-t)) = 0$  for  $1 \leq i \leq n-1$ . This implies that  $\mathcal{L}$  is aCM. Since  $\dim \text{Pic}^0(X) = q(X)$ , there exists a  $n$ -dimensional family of pairwise non-isomorphic aCM lines bundles.

Now we work on the proof of Theorem 1.3 and the key tool is Mukai's study of vector bundles on abelian varieties; see [21].

*Proof of Theorem 1.3:* Since  $X$  is smooth and birational to an abelian variety, there are an  $n$ -dimensional abelian variety  $Y$  and a proper birational morphism  $\nu : X \rightarrow Y$ ; see [23, Proposition 9.12]. In particular, we have  $\nu_* \mathcal{O}_X \cong \mathcal{O}_Y$  by the Zariski Main Theorem in [14, Corollary III.11.4]). Let  $\hat{Y} = \text{Pic}^0(Y)$  denote the abelian variety dual to  $Y$ . As in [21, Definitions 4.4, 4.5, 4.6] we consider the following set

$$\mathbb{U}'_r := \{ \text{the unipotent vector bundles of rank } r \text{ on } Y \},$$

i.e. the set of all vector bundles of rank  $r$  on  $Y$ , obtained by  $(r-1)$ -times of iteration of  $\mathcal{O}_Y$ ; we have  $\mathbb{U}'_1 = \{\mathcal{O}_Y\}$  and  $\mathbb{U}'_r$  is the set of all vector bundles which admit extensions of  $\mathcal{O}_Y$  by an element of  $\mathbb{U}'_{r-1}$ . If we let  $R$  be the completion of the local ring  $\mathcal{O}_{\hat{Y},0}$  and  $B_f$  the set of all  $R$ -modules with finite length, then by [21, Theorem 4.12] there is a bijection between  $\mathbb{U}'_r$  and the set  $B_f[r]$  of the  $R$ -modules of length  $r$ . Note that this bijection preserves finite direct sums. Thus to an indecomposable vector bundle in  $\mathbb{U}'_r$  it is enough to consider an indecomposable elements of  $B_f[r]$ . Define a subset

$$\mathbb{U}_r := \left\{ \mathcal{A} \in \mathbb{U}'_r \mid \begin{array}{l} \mathcal{A} \text{ corresponds to an indecomposable elements of } B_f[r] \\ \text{of the form } R/I \text{ with } I \subset R \text{ an ideal of colength } r \end{array} \right\},$$

consisting of elements of the local Hilbert scheme of  $R$  corresponding to connected zero-dimensional subschemes of  $\hat{Y}$  of degree  $r$  with  $0$  as their support. Then we get an algebraic family  $\mathbb{U}_r$  of indecomposable unipotent vector bundles of rank  $r$ . For the known results on the dimension of  $\mathbb{U}_r$ , refer to [11, page 6]. For  $n = 2$  and arbitrary  $r$ ,  $\mathbb{U}_r$  is irreducible of dimension  $r-1$  by [4, 16], while it can be reducible for  $n \geq 3$  by [11, 16]. In any case with  $n \geq 2$ ,  $\mathbb{U}_r$  has an irreducible family of dimension  $(n-1)(r-1)$ , whose general element is curvilinear, or collinear, by [11, pages 5–6].

For any line bundle  $\mathcal{L} \in \text{Pic}^0(X)$ , set

$$\Theta_{\mathcal{L}} := \{ \nu^*(\mathcal{F}) \otimes \mathcal{L} \mid \mathcal{F} \in \mathbb{U}_r \}.$$

Each element of  $\Theta_{\mathcal{L}}$  is a vector bundle of rank  $r$  on  $X$ , which is an iterated extension of  $\mathcal{L}$ . Thus each element of  $\Theta_{\mathcal{L}}$  is strictly semistable with respect to any polarization on  $X$  and all its Chern classes are zero. Assume that  $\nu^*(\mathcal{F}) \otimes \mathcal{L} \cong \nu^*(\mathcal{G}) \otimes \mathcal{L}$  for  $\mathcal{F}, \mathcal{G} \in \mathbb{U}_r$ . Then we get  $\nu^*(\mathcal{F}) \cong \nu^*(\mathcal{G})$  and so  $\mathcal{F} \cong \mathcal{G}$  by the projection formula and  $\nu_* \mathcal{O}_X \cong \mathcal{O}_Y$ . In particular,  $\Theta_{\mathcal{L}}$  parametrizes one-to-one vector bundles of rank  $r$  on  $X$  and  $\dim \Theta_{\mathcal{L}} = \dim \mathbb{U}_r$ . Note that for each  $\mathcal{A} \in \Theta_{\mathcal{L}}$  there are only finitely many  $\mathcal{L}' \in \text{Pic}^0(X)$  such that  $\mathcal{A} \cong \mathcal{A}'$  for some  $\mathcal{A}' \in \Theta_{\mathcal{L}'}$ ; indeed, we have at most  $(2n)^r$  vector bundles  $\mathcal{A}'$ , because  $\det(\mathcal{A}) \cong \mathcal{L}^{\otimes r}$  and so  $\mathcal{L}' \otimes \mathcal{L}^\vee$  is an element of  $r$ -torsion of  $\text{Pic}^0(X)$ . Now a general line bundle  $\mathcal{L} \in \text{Pic}^0(X)$  is aCM by

Remark 4.2. Define a non-empty open subset

$$\mathbb{V} := \{ \mathcal{L} \in \text{Pic}^0(X) \mid \mathcal{L} \text{ is aCM} \},$$

which is an algebraic variety of dimension  $q(X) = n$ . For each  $\mathcal{L} \in \mathbb{V}$ , every vector bundle  $\mathcal{A} \in \Theta_{\mathcal{L}}$  is aCM, because it is an iterated extension of aCM vector bundles. Define a parameter space  $\Gamma$  over  $\mathbb{V}$  whose fibre over  $\mathcal{L}$  is  $\Theta_{\mathcal{L}}$ . Then it is a parameter space, finite-to-one, for indecomposable aCM vector bundles of rank  $r$  on  $X$  with  $\dim \Gamma = n + \dim \mathbb{U}_r = (n-1)r + 1$ .  $\square$

**Proposition 4.3.** *Let  $X$  be a smooth projective surface with  $q(X) \geq 2$  and a fixed ample line bundle  $\mathcal{O}_X(1)$  satisfying one of the following conditions:*

- (i)  $\mathcal{O}_X(1) \cong \omega_X$ ;
- (ii)  $\mathcal{O}_X(1) \otimes \omega_X^\vee$  is ample.

Then for each integer  $r$  with  $1 \leq r \leq q(X)$  there exists a  $q(X)$ -dimensional family  $\{\mathcal{E}_\alpha\}_{\alpha \in \Gamma}$  of indecomposable aCM vector bundles of rank  $r$  on  $X$  such that  $\mathcal{E}_\alpha$  for each  $\alpha \in \Gamma$  is strictly semistable with  $\det(\mathcal{E}_\alpha) \in \text{Pic}^0(X)$  and  $c_2(\mathcal{E}_\alpha) = 0$  with respect to any polarization of  $X$ , and there are only finitely many  $\beta \in \Gamma$  with  $\mathcal{E}_\beta \cong \mathcal{E}_\alpha$ .

*Proof.* Fix a general line bundle  $\mathcal{L} \in \text{Pic}^0(X)$ . Then as in Remark 4.2 we see that  $\mathcal{L}$  is aCM. Set  $\mathcal{G}_0 = 0$  the zero sheaf and  $\mathcal{G}_1 := \mathcal{L}$ . For an integer  $r \geq 2$ , we define  $\mathcal{G}_r$  inductively as a general sheaf fitting into the following extension

$$(10) \quad 0 \rightarrow \mathcal{G}_{r-1} \xrightarrow{u} \mathcal{G}_r \xrightarrow{v} \mathcal{L} \rightarrow 0.$$

Note that  $\mathcal{G}_r$  is strictly semistable for any polarization and  $\mathcal{G}_r \otimes \mathcal{L}^\vee$  is an iterated extension of  $\mathcal{O}_X$  for each  $r \geq 1$ . Since  $\mathcal{G}_{r-1} \otimes \mathcal{L}^\vee$  is an iterated extension of  $\mathcal{O}_X$ , we have  $\det(\mathcal{G}_{r-1} \otimes \mathcal{L}^\vee) \cong \mathcal{O}_X$  and  $c_2(\mathcal{G}_{r-1} \otimes \mathcal{L}^\vee) = 0$ . Moreover, we may choose  $\mathcal{G}_r$  admitting a non-trivial extension (10), because we have  $\text{ext}_X^1(\mathcal{L}, \mathcal{G}_{r-1}) > 0$ ; indeed, we have  $h^1(\mathcal{G}_{r-1} \otimes \mathcal{L}^\vee) \geq q(X) - r + 2$ , which is clearly true for  $r = 2$ . In general, we get the following exact sequence from (10)

$$H^0(\mathcal{O}_X) \rightarrow H^1(\mathcal{G}_{r-1} \otimes \mathcal{L}^\vee) \rightarrow H^1(\mathcal{G}_r \otimes \mathcal{L}^\vee).$$

Then we may apply the inductive hypothesis and  $h^0(\mathcal{O}_X) = 1$ .

Note that the coboundary map  $H^0(\mathcal{O}_X) \rightarrow H^1(\mathcal{G}_{r-1} \otimes \mathcal{L}^\vee)$  is zero if and only if (10) is the trivial extension. Since we take a non-trivial extension at each step, we have  $h^0(\mathcal{G}_r \otimes \mathcal{L}^\vee) = h^0(\mathcal{G}_{r-1} \otimes \mathcal{L}^\vee)$ . By induction on  $r$  we get  $h^0(\mathcal{G}_r \otimes \mathcal{L}^\vee) = 1$  for all  $r \leq q(X)$ . Assume now that  $\mathcal{G}_r$  is decomposable, say  $\mathcal{G}_r \cong \mathcal{F}_1 \oplus \mathcal{F}_2$  with each  $\mathcal{F}_i$  nonzero. Then each  $\mathcal{F}_i \otimes \mathcal{L}^\vee$  is a strictly semistable vector bundle with numerically trivial determinant. Since  $gr(\mathcal{G}_{r-1} \otimes \mathcal{L}^\vee) = \mathcal{O}_X^{\oplus(r-1)}$ , we that  $gr(\mathcal{F}_i \otimes \mathcal{L}^\vee)$  is trivial and so each  $\mathcal{F}_i \otimes \mathcal{L}^\vee$  has a subsheaf isomorphic to  $\mathcal{O}_X$ . In particular, we have  $h^0(\mathcal{G}_r \otimes \mathcal{L}^\vee) \geq 2$ , a contradiction.

Note that  $\det(\mathcal{G}_r) \cong \mathcal{L}^{\otimes r}$  and so there are only finitely many line bundles  $\mathcal{L}' \in \text{Pic}^0(X)$  such that  $\mathcal{G}_r$  is also an iterated extension of  $\mathcal{L}'$ . Hence we get the assertion from  $\dim \text{Pic}^0(X) = q(X)$ .  $\square$

**Remark 4.4.** Let  $Y$  be a hyperelliptic surface, i.e. a smooth projective surface with  $\omega_Y \not\cong \mathcal{O}_Y$ ,  $q(Y) = 1$  and  $\omega_Y^{\otimes 12} \cong \mathcal{O}_Y$ . In particular, we have  $h^2(\mathcal{O}_Y) = h^0(\omega_Y) = 0$  and so  $\chi(\mathcal{O}_Y) = 0$ . Let  $X$  be a smooth projective surface birational to  $Y$ . Then we have  $h^i(\mathcal{O}_X) = h^i(\mathcal{O}_Y)$  for each  $i$  and  $\omega_X \not\cong \mathcal{O}_X$  with  $h^0(\omega_X^{\otimes 12}) = 1$ . Fix an ample line bundle  $\mathcal{O}_X(1)$  on  $X$  and take a line bundle  $\mathcal{L} \in \text{Pic}^0(X) \setminus \{\mathcal{O}_X, \omega_X^\vee\}$ . Then we have  $h^0(\mathcal{L}) = h^2(\mathcal{L}) = 0$ . Since  $\mathcal{L}$  is numerically equivalent to  $\mathcal{O}_X$  and  $\chi(\mathcal{O}_X) = 0$ , we have  $\chi(\mathcal{L}) = 0$  and so  $h^1(\mathcal{L}) = 0$ . Note that  $\mathcal{L}(t)$  and  $\mathcal{L}^\vee \otimes \omega_X(t)$  are ample for  $t > 0$ , because they are numerically equivalent to the ample line bundle  $\mathcal{O}_X(t)$ . So we get  $h^1(\mathcal{L}(t)) = 0$  for all  $t \neq 0$  by Kodaira's vanishing and Serre's duality. Thus  $\mathcal{L}$  is aCM. Now we may construct indecomposable aCM vector bundles  $\mathcal{G}_r$  of rank  $r$  as in the case of abelian surfaces. Indeed, we have  $\text{ext}_X^1(\mathcal{L}, \mathcal{L}) = h^1(\mathcal{O}_X) = 1$  and  $\text{ext}_X^1(\mathcal{L}, \mathcal{G}_{r-1}) > 0$ . We have  $\det(\mathcal{G}_r) \cong \mathcal{L}^{\otimes r}$ . In particular, there are only finitely many line bundles  $\mathcal{L}' \in \text{Pic}^0(X)$  such that  $\mathcal{G}_r$  is an iterated extension of  $\mathcal{L}'$ . We get the following result from  $q(X) = 1$ .

**Proposition 4.5.** *Let  $X$  be a smooth projective surface, birational to a hyperelliptic surface, with any polarization. For any positive integer  $r$ , there exists a one-dimensional family  $\{\mathcal{E}_\alpha\}_{\alpha \in \Gamma}$  of indecomposable aCM vector bundles of rank  $r$  on  $X$  such that for each  $\alpha \in \Gamma$  there are only finitely many  $\beta \in \Gamma$  with  $\mathcal{E}_\beta \cong \mathcal{E}_\alpha$ .*

## 5. SURFACES OF GENERAL TYPE WITH AMPLE CANONICAL LINE BUNDLE

Let  $X$  be an integral projective surface, possibly singular, with ample  $\omega_X$  satisfying the following conditions:

- (i)  $h^1(\omega_X^{\otimes n}) = 0$  for all  $n \in \mathbb{Z}$ ;
- (ii- $\varepsilon$ )  $p_g := h^0(\omega_X) \geq 2 + \varepsilon$  with  $\varepsilon \in \{0, 1\}$ .

We set  $\mathcal{O}_X(1) := \omega_X$  with respect to which we consider aCM vector bundles on  $X$ .

**Remark 5.1.** Assume that  $X$  is smooth. The canonical line bundle  $\omega_X$  is ample if and only if  $X$  is a minimal surface of general type without  $(-2)$ -curves, i.e. a smooth surface of general type without smooth rational curves  $D \subset X$  with either  $D^2 = -1$  or  $D^2 = -2$ ; see [2]. There are surfaces  $X$  of general type with  $p_g = h^0(\omega_X) \leq 1$ , but most surfaces have  $p_g \geq 2$ . The condition (i) for  $n = 0$  is  $h^1(\mathcal{O}_X) = 0$ , i.e. the irregularity of  $X$  is  $q(X) = 0$ . This is a non-trivial requirement, but it is satisfied in many important cases. By Serre's duality this would imply that  $h^1(\omega_X) = q(X) = 0$ . In characteristic 0 the condition (i) for  $n < 0$  comes from Kodaira's vanishing theorem by the ampleness of  $\omega_X$ . Assume  $h^1(\omega_X^{\otimes n}) = 0$  for all  $n < 0$ . By Serre's duality we have  $h^1(\omega_X^{\otimes n}) = h^1(\omega_X^{\otimes(1-n)}) = 0$  for  $n \geq 2$ . Thus in characteristic 0 we have the condition (i) satisfied if and only if  $h^1(\mathcal{O}_X) = 0$ .

By the condition (ii- $\varepsilon$ ), the set

$$\Sigma := \text{Sing}(X) \cap (\text{the base locus of } |\omega_X|)$$

is a proper closed subset of  $X$ . By the same argument in Remark 2.14 using Serre's duality we get the following lemma.

**Lemma 5.2.** *For a finite subset  $S \subset X \setminus \Sigma$ , we have  $\text{ext}_X^1(\mathcal{I}_{S,X}, \omega_X) = |S| - 1$  and a general extension of  $\mathcal{I}_{S,X}$  by  $\omega_X$  is locally free.*

*Proof.* For the first assertion, we may apply the same argument in Remark 2.14 using Serre's duality. The second assertion is clear, because the Cayley-Bacharach condition for  $S$  and the linear system  $|\mathcal{O}_X|$  is satisfied.  $\square$

**Proposition 5.3.** *For a fixed integer  $2 \leq r \leq p_g$  and a general subset  $S \subset X \setminus \Sigma$  with  $|S| = r$ , the general sheaf  $\mathcal{E}$  fitting into an exact sequence*

$$(11) \quad 0 \rightarrow \omega_X^{\oplus(r-1)} \rightarrow \mathcal{E} \rightarrow \mathcal{I}_{S,X} \rightarrow 0$$

*is an indecomposable and aCM vector bundle of rank  $r$ .*

*Proof.* We get that  $\mathcal{E}$  is locally free with rank  $r$  from Lemma 5.2. Indeed, if  $r \geq 3$ , then  $\mathcal{E} \cong \mathcal{G} \oplus \omega_X^{\oplus(r-2)}$  with a general extension  $\mathcal{G}$  of  $\mathcal{I}_{S,X}$  by  $\omega_X$  is locally free. Then we may use openness of being locally free. Now since we have  $\text{ext}_X^1(\mathcal{I}_{S,X}, \omega_X) = r - 1$  by Lemma 5.2, the extension (11) is induced by a choice of a basis  $\{e_1, \dots, e_{r-1}\}$  of  $\text{Ext}_X^1(\mathcal{I}_{S,X}, \omega_X)$ . Thus the map  $\varphi: H^1(\mathcal{I}_{S,X}) \rightarrow H^2(\omega_X^{\oplus(r-1)}) \cong \mathbf{k}^{\oplus(r-1)}$  is bijective, and in particular we have  $h^1(\mathcal{E}) = 0$ . Recall that we assume  $\omega_X \cong \mathcal{O}_X(1)$ . Then by the condition (i) we get  $h^1(\omega_X(n)) = 0$  for all  $n \in \mathbb{Z}$  and we get

$$0 \rightarrow H^1(\mathcal{E}(n)) \rightarrow H^1(\mathcal{I}_{S,X}(n)) \rightarrow H^2(\omega_X(n))^{\oplus(r-1)}.$$

Assume first that  $n$  is positive and this implies  $h^2(\omega_X(n)) = h^0(\mathcal{O}_X(-n)) = 0$ . Since  $S$  is general with  $|S| = r \leq h^0(\mathcal{O}_X(1)) \leq h^0(\mathcal{O}_X(n))$ , we get  $h^1(\mathcal{I}_{S,X}(n)) = 0$ . Thus we have  $h^1(\mathcal{E}(n)) = 0$ . It remains to show that  $h^1(\mathcal{E}(-n)) = 0$  for  $n \geq 1$ . In fact, it is sufficient to prove the existence of an extension  $\mathcal{F}$  of  $\mathcal{I}_{S,X}$  by  $\omega_X^{\oplus(r-1)}$  satisfying  $h^1(\mathcal{F}(-n)) = 0$  for all  $n \geq 1$ . Take  $\mathcal{F} \cong \mathcal{G} \oplus \omega_X^{\oplus(r-2)}$  with a general extension  $\mathcal{G}$  of  $\mathcal{I}_{S,X}$  by  $\omega_X$  given by  $e_1$ . By the previous argument, we have  $h^1(\mathcal{G}(n)) = 0$  for all  $n \geq 1$ . By Lemma 5.2,  $\mathcal{G}$  is locally

free with  $\det(\mathcal{G}) \cong \omega_X$ . Serre's duality gives  $h^1(\mathcal{G}(-n)) = h^1(\mathcal{G}(n)) = 0$  for all  $n \geq 1$ . Thus we get that  $\mathcal{E}$  is aCM. Note that if  $r \geq 3$ , then  $\mathcal{G}$  is not aCM since we have  $h^1(\mathcal{G}) = r - 2$ .

For the indecomposability, we may use the same argument in the proof of Proposition 2.1 to  $\mathcal{E} \otimes \omega_X^\vee$ , because  $\mathcal{I}_{S,X} \otimes \omega_X^\vee$  is indecomposable.  $\square$

Now for the statement in Theorem 5.4, set  $\varepsilon = r - 2 \lfloor \frac{r}{2} \rfloor$  for which the condition (ii- $\varepsilon$ ) for  $X$  is assumed to be satisfied.

**Theorem 5.4.** *For each integer  $r \geq 2$ , there exists an  $r$ -dimensional family  $\{\mathcal{E}_\alpha\}_{\alpha \in \Gamma}$  of indecomposable aCM vector bundles of rank  $r$  on  $X$  with  $\det(\mathcal{E}_\alpha) \cong \omega_X^{\otimes \lfloor r/2 \rfloor}$  and  $c_2(\mathcal{E}_\alpha) = r$  such that for each  $\alpha \in \Gamma$  there are only finitely many  $\beta \in \Gamma$  with  $\mathcal{E}_\beta \cong \mathcal{E}_\alpha$ .*

*Proof.* We use the same notations in the proof of Theorem 2.4 such as  $\mathbb{I}(S_1, \dots, S_i)$  and  $\mathbb{J}(S_1, \dots, S_i; S_0)$ . Then we get the same assertions from Lemma 2.5 till Remark 2.13; the only difference occurs in Lemma 2.8 and Remark 2.9, where we have

$$\mathrm{ext}_X^1(\mathcal{I}_{S_{i+1}, X}, \mathcal{I}) = \mathrm{ext}_X^1(\mathcal{I}_{S_0, X}, \mathcal{I}) = i$$

for  $\mathcal{I} \in \mathbb{I}(S_1, \dots, S_i)$  from  $\mathrm{ext}_X^1(\mathcal{I}_{S_{i+1}, X}, \mathcal{O}_X) = \mathrm{ext}_X^1(\mathcal{I}_{S_0, X}, \mathcal{O}_X) = 0$ . Then we may consider the exact sequences (6) and (7) with  $\mathcal{O}_X$  replaced by  $\omega_X$ .  $\square$

## 6. SURFACES MAPPED TO A CURVE OF GENUS $\geq 3$ NOT AS THEIR ALBANESE IMAGE

Throughout this section,  $X$  is a smooth projective surface admitting a surjective map  $\nu : X \rightarrow C$  with  $g = g(C) \geq 3$ . Assume that  $C$  is such a curve achieving maximum possible genus  $g$  and that  $q(X) > g$ . For example, we may take as  $X$  any smooth surface birational to  $C \times D$ , where  $D$  is a smooth curve with  $1 \leq g(D) \leq g$ ; in this case we have  $q(X) = g + g(D)$ .

**Proposition 6.1.** *For each positive integer  $r$  there exists a family  $\{\mathcal{E}_\alpha\}_{\alpha \in \Gamma}$  of indecomposable aCM vector bundles of rank  $r$  on  $X$  such that  $\Gamma$  is an integral variety with*

$$\dim \Gamma \geq q(X) + \frac{(r-1)(r-2)(g-1)}{2} - \frac{r(r-1)}{2}$$

and each  $\mathcal{E}_\alpha$  is strictly semistable with  $\det(\mathcal{E}_\alpha) \in \mathrm{Pic}^0(X)$  and  $c_2(\mathcal{E}_\alpha) = 0$  with respect to any polarization of  $X$  such that there are only finitely many  $\beta \in \Gamma$  with  $\mathcal{E}_\beta \cong \mathcal{E}_\alpha$ .

Set  $\mathcal{A}_1 := \mathcal{O}_C$  and define inductively a vector bundle  $\mathcal{A}_{i+1}$  of rank  $i+1$  on  $C$  to be the middle term of the following extension:

$$(12) \quad 0 \rightarrow \mathcal{A}_i \rightarrow \mathcal{A}_{i+1} \rightarrow \mathcal{O}_C \rightarrow 0,$$

where  $\mathcal{A}_{i+1} = \mathcal{A}_{i+1}(e)$  corresponds to the extension class  $e \in \mathrm{Ext}_C^1(\mathcal{O}_C, \mathcal{A}_i) \cong H^1(\mathcal{A}_i)$ . The long exact sequence of cohomology of (12) gives  $g-1 \leq h^1(\mathcal{A}_{i+1}) - h^1(\mathcal{A}_i) \leq g$ . Since we have  $g \geq 3$  from the assumption, we get  $h^1(\mathcal{A}_{i+1}) \neq 0$ . In particular, we may assume that the extension (12) is non-trivial. The image of the coboundary map  $H^0(\mathcal{O}_C) \rightarrow H^1(\mathcal{A}_i)$  corresponds to the extension (12), up to a sign, and so we get  $h^0(\mathcal{A}_{i+1}) = h^0(\mathcal{A}_i)$  and  $h^1(\mathcal{A}_{i+1}) = h^1(\mathcal{A}_i) + g - 1$  for each  $i$ . By induction, we get

$$h^0(\mathcal{A}_i) = 1 \text{ and } h^1(\mathcal{A}_i) = i(g-1) + 1.$$

Note that each  $\mathcal{A}_i$  is an iterated extension of  $\mathcal{O}_C$ , and in particular it is strictly semistable with  $gr(\mathcal{A}_i) \cong \mathcal{O}_C^{\oplus i}$ . Assume  $\mathcal{A}_i \cong \mathcal{B}_1 \oplus \mathcal{B}_2$  with each  $\mathcal{B}_i \neq 0$ . Since each  $\mathcal{B}_i$  has a HN-filtration with  $\mathcal{O}_C$  as its first step, we have  $h^0(\mathcal{B}_i) > 0$  and so  $h^0(\mathcal{A}_i) \geq 2$ , a contradiction. Thus each  $\mathcal{A}_i$  is indecomposable.

**Remark 6.2.** Let  $u : \mathcal{A} \rightarrow \mathcal{B}$  be a surjection of sheaves on  $C$ . Since  $\dim C = 1$ , we have  $h^2(C, \ker(u)) = 0$ . Thus the surjection  $u$  induces a surjective map  $H^1(C, \mathcal{A}) \rightarrow H^1(C, \mathcal{B})$ .

**Lemma 6.3.** *Let  $\mathcal{M}, \mathcal{D}_1, \mathcal{D}_2$  be vector bundles on  $C$  fitting into exact sequences*

$$(13) \quad 0 \rightarrow \mathcal{M} \xrightarrow{u_i} \mathcal{D}_i \rightarrow \mathcal{O}_C \rightarrow 0,$$

corresponding to an extension class  $e_i \in \text{Ext}_C^1(\mathcal{O}_C, \mathcal{M}) \cong H^1(\mathcal{M})$  for each  $i$ . If there exists an isomorphism  $h: \mathcal{D}_2 \rightarrow \mathcal{D}_1$  such that  $h(u_2(\mathcal{M})) = u_1(\mathcal{M})$ , then  $e_1$  and  $e_2$  are in the same orbit of  $H^1(\mathcal{M})$  for the action of the group  $\text{Aut}(\mathcal{M})$ .

*Proof.* Note that  $h^0(\mathcal{M}) \leq h^0(\mathcal{D}_i) \leq h^0(\mathcal{M}) + 1$ , and  $h^0(\mathcal{M}) = h^0(\mathcal{D}_i)$  if and only if  $e_i \neq 0$ . Since  $h$  is an isomorphism,  $e_1 = 0$  if and only if  $e_2 = 0$ . Since the assertion is obvious when  $e_1 = e_2 = 0$ , we may assume  $e_1 \neq 0$  and  $e_2 \neq 0$ . Since  $h(u_2(\mathcal{M})) = u_1(\mathcal{M})$ ,  $h$  induces isomorphisms  $h': \mathcal{D}_2/u_2(\mathcal{M}) \rightarrow \mathcal{D}_1/u_1(\mathcal{M})$  and  $f: \mathcal{M} \rightarrow \mathcal{M}$ . Since  $\mathcal{D}_i/u_i(\mathcal{M}) \cong \mathcal{O}_C$ ,  $i = 1, 2$ ,  $h'$  is induced by the multiplication by a constant,  $c$ . Note that  $e_i$  is determined by the image of 1 by the coboundary map  $H^0(\mathcal{O}_C) \rightarrow H^1(\mathcal{M})$  in (13). Since  $e_1 \neq 0$  and  $e_2 \neq 0$ , we have  $c \neq 0$ . Taking  $(\frac{1}{c})h$  instead of  $h$  we reduce to the case in which  $h': \mathcal{O}_C \rightarrow \mathcal{O}_C$  is the identity map. Thus we get a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathcal{M} & \rightarrow & \mathcal{D}_2 & \rightarrow & \mathcal{O}_C & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \mathcal{M} & \rightarrow & \mathcal{D}_1 & \rightarrow & \mathcal{O}_C & \rightarrow & 0, \end{array}$$

in which the three vertical arrows are respectively  $f$ ,  $h$  and  $\text{Id}_{\mathcal{O}_C}$ . By the definition of  $\text{Ext}_C^1(\mathcal{O}_C, \mathcal{M})$  as short exact sequences modulo an equivalence relation, we get  $e_1 = f_*(e_2)$ , i.e.  $e_1 \in H^1(\mathcal{M})$  is contained in the orbit of  $e_2$  for the action of the group  $\text{Aut}(\mathcal{M})$ .  $\square$

We set  $\mathbf{T}_2 := H^1(\mathcal{O}_C) \setminus \{0\}$  and consider it as a parameter space, not finite-to-one, for non-trivial extensions of  $\mathcal{O}_C$  by  $\mathcal{O}_C$ . Then we get a family  $\{\mathcal{A}_2(e)\}_{e \in \mathbf{T}_2}$  of aCM vector bundles of rank two. Since we have  $h^1(\mathcal{A}_2(e)) = 2g - 3$  for each  $e \in \mathbf{T}_2$ , there is a vector bundle  $\pi_2: \mathbf{T}'_3 \rightarrow \mathbf{T}_2$  of rank  $2g - 3$  whose fibre over  $\mathcal{A}_2(e)$  is  $H^1(\mathcal{A}_2(e)) \cong \text{Ext}_C^1(\mathcal{O}_C, \mathcal{A}_2(e))$ . Then we get a family  $\{\mathcal{A}_3(e)\}_{e \in \mathbf{T}'_3}$  of aCM vector bundles of rank three on  $C$  such that for each  $e \in \mathbf{T}'_2$ ,  $\mathcal{A}_3(e)$  is an extension of  $\mathcal{O}_C$  by  $\mathcal{A}_2(\pi(e))$ . Let  $\mathbf{T}_3$  be the non-empty Zariski open subset of  $\mathbf{T}'_3$  parametrizing the non-trivial extensions of  $\mathcal{O}_C$  by  $\mathcal{A}_2(\pi(e))$ . Thus we have a family  $\{\mathcal{A}_3(e)\}_{e \in \mathbf{T}_3}$  of indecomposable aCM vector bundles of rank three, parametrized by  $\mathbf{T}_3$ .

Now we define a parameter space  $\mathbf{T}_i$  inductively: fix an integer  $i \geq 2$  and assume that  $\mathbf{T}_i$  is defined, together with a family  $\{\mathcal{A}_i(e)\}_{e \in \mathbf{T}_i}$  of indecomposable aCM vector bundles of rank  $i$ , parametrized by  $\mathbf{T}_i$ . Since we have  $h^1(\mathcal{A}_i(e)) = i(g - 1) + 1$ , there exists a vector bundle  $\pi_i: \mathbf{T}'_{i+1} \rightarrow \mathbf{T}_i$  of rank  $i(g - 1) + 1$  and a family  $\{\mathcal{A}_{i+1}(e)\}_{e \in \mathbf{T}'_{i+1}}$  of aCM vector bundles of rank  $i + 1$  on  $C$  such that for each  $e \in \mathbf{T}'_{i+1}$ ,  $\mathcal{A}_{i+1}(e)$  is an extension of  $\mathcal{O}_C$  by  $\mathcal{A}_i(\pi(e))$ . Let  $\mathbf{T}_{i+1}$  be the non-empty Zariski open subset of  $\mathbf{T}'_{i+1}$  parametrizing the non-trivial extensions of  $\mathcal{O}_C$  by  $\mathcal{A}_i(\pi(e))$ .

If a vector bundle  $\mathcal{A} = \mathcal{A}_r$  of rank  $r$  on  $C$  corresponding to  $e \in \mathbf{T}_r$  is obtained as a successive extension of  $\mathcal{O}_C$  by  $\mathcal{A}_i(e_{i-1})$  corresponding to  $e_i \in H^1(\mathcal{A}_i(e_{i-1})) \setminus \{0\}$  for each  $i \leq r$ , then we simply denote it by  $\mathcal{A}(e_1, \dots, e_{r-1}) := \mathcal{A}$  and it has a filtration

$$0 \subset \mathcal{A}_1 = \mathcal{O}_C \subset \mathcal{A}_2 = \mathcal{A}(e_1) \subset \mathcal{A}_3 = \mathcal{A}(e_1, e_2) \subset \dots \subset \mathcal{A}_r = \mathcal{A}(e_1, \dots, e_{r-1}).$$

Fix a general  $\mathcal{A} = \mathcal{A}(e_1, \dots, e_{r-1})$  that is a non-trivial extension of  $\mathcal{O}_C$  by  $\mathcal{A}' := \mathcal{A}(e_1, \dots, e_{r-2})$ . Letting  $u_{i,r}: \mathcal{A}_i \rightarrow \mathcal{A}$  with  $1 \leq i \leq r - 1$  be the inclusion arising by the extensions reaching  $\mathcal{A}$ , we have the

following commutative diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & \mathcal{A}_1 & = & \mathcal{A}_1 & & \\
& & \downarrow & & \downarrow & & \\
0 \rightarrow & & \mathcal{A}' & \rightarrow & \mathcal{A} & \rightarrow & \mathcal{O}_C \rightarrow 0 \\
& & \downarrow & & \downarrow & & \parallel \\
0 \rightarrow & & \mathcal{A}'/u_{1,r-1}(\mathcal{A}_1) & \rightarrow & \mathcal{A}/u_{1,r}(\mathcal{A}_1) & \rightarrow & \mathcal{O}_C \rightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 
\end{array}$$

so that  $\mathcal{A}/u_{1,r}(\mathcal{A}_1)$  is an extension of  $\mathcal{O}_C$  by  $\mathcal{A}'/u_{1,r}(\mathcal{A}_1)$ . Iterating the process, we see that  $\mathcal{A}/u_{1,r}(\mathcal{A}_1)$  is an iterated extension of  $\mathcal{O}_C$ .

**Lemma 6.4.** *Fix a general  $\mathcal{A}_r = \mathcal{A}(e_1, \dots, e_{r-1}) \in \mathbf{T}_r$  with a filtration  $\mathcal{A}_1 \subset \dots \subset \mathcal{A}_{r-1} \subset \mathcal{A}_r$ . Then we have*

- (i)  $h^0(\mathcal{A}_i/\mathcal{A}_j) = 1$  for all  $1 \leq j < i \leq r$ ;
- (ii)  $f(\mathcal{A}_i) \subset \mathcal{A}_i$  for any  $f \in \text{End}(\mathcal{A}_r)$  and each  $i$ ;
- (iii)  $\dim \text{End}(\mathcal{A}_r) \leq r$  and  $\dim \text{End}(\mathcal{A}_r) - \dim(\mathcal{A}_{r-1}) \leq 1$ .
- (iv)  $h(\mathcal{A}_i) = \mathcal{B}_i$  for all  $i$  and any isomorphism  $h : \mathcal{B}_r \rightarrow \mathcal{A}_r$ , where  $\mathcal{B}_r \in \mathbf{T}_r$  general with a filtration  $\mathcal{B}_1 \subset \dots \subset \mathcal{B}_{r-1} \subset \mathcal{B}_r$ .

*Proof.* For (i) consider the following sequence, obtained from (12):

$$(14) \quad 0 \rightarrow \mathcal{A}_i/\mathcal{A}_j \rightarrow \mathcal{A}_{i+1}/\mathcal{A}_j \rightarrow \mathcal{O}_C \rightarrow 0.$$

Since  $e_i \in H^1(\mathcal{A}_i)$  is general by the generality of  $\mathcal{A}_r$ , we get that (14) is a general extension and  $h^0(\mathcal{A}_{i+1}/\mathcal{A}_j) = h^0(\mathcal{A}_i/\mathcal{A}_j)$ . Thus to prove the assertion for  $j = 1$  it is enough to show it for the case  $i = 2$ , which is obvious from  $\mathcal{A}_2/\mathcal{A}_1 \cong \mathcal{O}_C$ . For  $j \geq 2$  we use (14) starting from the case  $i = j + 1$ , when we have  $\mathcal{A}_{j+1}/\mathcal{A}_j \cong \mathcal{O}_C$ .

For (ii) note first that  $\mathcal{A}_1 = \mathcal{O}_C$  and  $h^0(\mathcal{A}_r) = 1$ . This implies that  $\mathcal{A}_1$  is the image of the evaluation map  $H^0(\mathcal{A}_r) \otimes \mathcal{O}_C \rightarrow \mathcal{A}_r$  and so  $f(\mathcal{A}_1) \subset \mathcal{A}_1$ , concluding the case  $r = 2$ . Now  $f$  induces a map  $f' : \mathcal{A}_r/\mathcal{A}_1 \rightarrow \mathcal{A}_r/\mathcal{A}_1$ . Since  $h^0(\mathcal{A}_r/\mathcal{A}_1) = 1$  by (i) and  $\mathcal{A}_2/\mathcal{A}_1 \cong \mathcal{O}_C$ , we get  $f'(\mathcal{A}_2/\mathcal{A}_1) \subset \mathcal{A}_2/\mathcal{A}_1$  and so  $f(\mathcal{A}_2) \subset \mathcal{A}_2$ . Thus we get the assertion by continuing this process together with (i).

For (iii) since the case  $r = 1$  is trivial, we may assume  $r \geq 2$  and use induction on  $r$ . For  $f \in \text{End}(\mathcal{A}_r)$ , we have  $\mathcal{A}_1 = \mathcal{O}_C$  and  $f(\mathcal{A}_1) \subset \mathcal{A}_1$  by (ii). Thus there is  $c \in \mathbf{k}$  such that  $(f - c \cdot \text{Id}_{\mathcal{A}_r})(\mathcal{A}_1) = 0$ , and  $f - c \cdot \text{Id}_{\mathcal{A}_r}$  is uniquely determined by  $f' \in \text{End}(\mathcal{A}_r/\mathcal{A}_1)$ . Since we may apply (i) and (ii) to  $\mathcal{A}_r/\mathcal{A}_1$ , we conclude by induction on  $r$ .

For (iv) note that  $\mathcal{A}_1$  (resp.  $\mathcal{B}_1$ ) is the image of the evaluation map of  $\mathcal{A}_r$  (resp.  $\mathcal{B}_r$ ) and  $h$  is an isomorphism. In particular, we have  $h(\mathcal{A}_1) = \mathcal{B}_1$  and so  $h$  induces an isomorphism  $h' : \mathcal{A}_r/\mathcal{A}_1 \rightarrow \mathcal{B}_r/\mathcal{B}_1$ . Since  $h^0(\mathcal{A}_i/\mathcal{A}_j) = h^0(\mathcal{B}_i/\mathcal{B}_j) = 1$  for all  $i > j$  by (i), we iterate the previous argument.  $\square$

Define a subset  $\mathbf{J}_r$  to be

$$\mathbf{J}_r = \left\{ e \in \mathbf{T}_r \mid \begin{array}{l} \mathcal{A}_r(e) \text{ admits a filtration } \mathcal{A}_1 \subset \dots \subset \mathcal{A}_{r-1} \subset \mathcal{A}_r \\ \text{such that } h^0(\mathcal{A}_i/\mathcal{A}_j) = 1 \text{ for all } 1 \leq j < i \leq r \end{array} \right\},$$

i.e. the non-empty open subset of  $\mathbf{T}_r$  parametrizing the vector bundles  $\mathcal{A}_r$  satisfying (i) of Lemma 6.4; thus  $\mathcal{A}_r$  satisfies (ii), (iii) and (iv) of Lemma 6.4.

**Lemma 6.5.** *For a general  $\mathcal{A}_r \in \mathbf{J}_r$  there exists an algebraic subset of  $\mathbf{J}_r$ , parametrizing the vector bundles isomorphic to  $\mathcal{A}_r$ , with dimension at most  $\frac{r(r-1)}{2}$ .*

*Proof.* We use induction on  $r$ ; the case  $r = 1$  is trivial, because  $\mathbf{J}_1 = \mathbf{T}_1 = \{\mathcal{O}_C\}$ . We assume that  $r \geq 2$  and fix  $\mathcal{B}_r \in \mathbf{J}_r$ , isomorphic to  $\mathcal{A}_r$ , with a filtration  $\mathcal{B}_1 \subset \dots \subset \mathcal{B}_r$ . For any isomorphism  $h : \mathcal{B}_r \rightarrow \mathcal{A}_r$ , we have  $h(\mathcal{B}_{r-1}) = \mathcal{A}_{r-1}$  by (iv) of Lemma 6.4. Since  $\mathcal{A}_{r-1}$  is also general in  $\mathbf{J}_{r-1}$ , by inductive assumption there is an algebraic subset  $\mathbf{J}'$  of  $\mathbf{J}_{r-1}$  parametrizing the vector bundles isomorphic to  $\mathcal{A}_{r-1}$ . Fix  $\mathcal{M} \in \mathbf{J}'$  and

consider the subset  $\mathbf{T}' \subset \mathbf{T}_r$  of all extensions of  $\mathcal{O}_C$  by  $\mathcal{M}$  which are isomorphic to  $\mathcal{A}_r$ . By Lemma 6.3 and (iii) of Lemma 6.4, we have  $\dim \mathbf{T}' \leq r - 1$  and we get the assertion.  $\square$

*Proof of Proposition 6.1:* Note that

$$g - 1 + \sum_{i=2}^{r-1} (i(g-1) - 1) - \sum_{i=1}^{r-1} i = \frac{(r-1)(r-2)(g-1)}{2} - \frac{r(r-1)}{2}.$$

Set  $\Delta := \{v^*(\mathcal{A}) \mid \mathcal{A} \in \mathbf{J}_r\}$  and then each element of  $\Delta$  is indecomposable, because each  $\mathcal{A} \in \mathbf{J}_r$  is indecomposable. Since we have  $v_*v^*\mathcal{F} \cong \mathcal{F}$  for any vector bundle  $\mathcal{F}$  on  $C$  by the projection formula and  $v_*\mathcal{O}_X \cong \mathcal{O}_C$ , we have  $v^*\mathcal{A} \cong v^*\mathcal{B}$  if and only if  $\mathcal{A} \cong \mathcal{B}$  for any  $v^*\mathcal{A}, v^*\mathcal{B} \in \Delta$ .

Fix a general  $\mathcal{L} \in \text{Pic}^0(X)$  and set  $\Theta_{\mathcal{L}} := \{\mathcal{G} \otimes \mathcal{L} \mid \mathcal{G} \in \Delta\}$ . Each element of  $\Theta_{\mathcal{L}}$  is an indecomposable vector bundle of rank  $r$  on  $X$  and the isomorphism classes of elements in  $\Theta_{\mathcal{L}}$  are also parametrized by  $\mathbf{J}_r$ . We have  $h^1(\mathcal{L}) = 0$  by [3, Th. 0.1], because  $q(X) > g$  and by our definition of  $g$  there is no non-constant morphism from  $X$  to a curve of genus  $q(X)$ . Fix a positive interger  $t$ . By Kleiman's numerical criterion of ampleness in [17],  $\mathcal{L}^\vee(t)$  and  $\omega_X^\vee \otimes \mathcal{L}(t)$  are ample. So Kodaira' vanishing gives  $h^1(\mathcal{L}(-t)) = h^1(\omega_X \otimes \mathcal{L}^\vee(-t)) = 0$ . On the other hand, by Serre's duality we get  $h^1(\mathcal{L}(t)) = h^1(\omega_X \otimes \mathcal{L}^\vee(-t)) = 0$ . Thus  $\mathcal{L}$  is aCM.

Since each element of  $\Theta_{\mathcal{L}}$  is an iterated extension of  $\mathcal{L}$ , each element of  $\Theta_{\mathcal{L}}$  is also aCM. Note that each element of  $\Theta_{\mathcal{L}}$  is strictly semistable with  $gr(\mathcal{A}_r) \cong \mathcal{L}^{\oplus r}$  and so no element of  $\Theta_{\mathcal{L}}$  is isomorphic to an element of  $\Theta_{\mathcal{L}'}$  with  $\mathcal{L} \not\cong \mathcal{L}'$ . Now we may vary the general  $\mathcal{L} \in \text{Pic}^0(X)$  to obtain a family  $\Gamma$  whose fibre over  $\mathcal{L}$  is  $\Theta_{\mathcal{L}}$ . Then we get the inequality in the assertion and all the requirements for  $\Gamma$  are satisfied.  $\square$

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