GENUS 2 CURVES AND GENERALIZED THETA DIVISORS

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Abstract. In this paper we investigate generalized theta divisors Θ_r in the moduli spaces $\mathcal{U}_C(r,r)$ of semistable vector bundles on a curve C of genus 2. We provide a desingularization Φ of Θ_r in terms of a projective bundle $\pi : \mathbb{P}(\mathcal{V}) \to \mathcal{U}_C(r-1,r)$ which parametrizes extensions of stable vector bundles on the base by $\mathcal{O}_C$. Then, we study the composition of Φ with the well known theta map θ. We prove that, when it is restricted to the general fiber of π, we obtain a linear embedding.

Introduction

Theta divisors play a fundamental role in the study of moduli spaces of semistable vector bundles on curves. First of all, the classical notion of theta divisor of the Jacobian variety of a curve can be generalized to higher rank. Let C be a smooth, irreducible, complex, projective curve of genus $g \geq 2$. The study of isomorphism classes of stable vector bundles of fixed rank r and degree n goes back to Mumford. The compactification of this moduli space is denoted by $\mathcal{U}_C(r,n)$ and has been introduced by Seshadri. In the particular case when the degree is equal to $r(g-1)$ it admits a natural Brill-Noether locus Θ_r, which is called the *theta divisor* of $\mathcal{U}_C(r, r(g-1))$. Riemann's singularity Theorem extends to Θ_r, see [Las91].
When we restrict our attention to semistable vector bundles of rank r and fixed determinant $L \in \mathrm{Pic}^{r(g-1)}(C)$, we have the moduli space $\mathcal{SU}_C(r,L)$ and a Brill-Noether locus $\Theta_{r,L}$ which is called the *theta divisor* of $\mathcal{S}U_C(r,L)$. The line bundle associated to $\Theta_{r,L}$ is the ample generator $\mathcal{L}$ of the Picard variety of $\mathcal{SU}_C(r,L)$, which is called the *determinant line bundle*, see [DN89].

For semistable vector bundles with integer slope, one can also introduce the notion of *associated theta divisor*. In particular for a stable $E \in \mathcal{SU}_C(r,L)$ with $L \in \mathrm{Pic}^{r(g-1)}(C)$ we have that the set

$$\{N \in \mathrm{Pic}^0(C) \,|\, h^0(E \otimes N) \geq 1\}$$

is either all $\mathrm{Pic}^0(C)$ or an effective divisor Θ_E which is called the theta divisor of E. Moreover the map which associates to each bundle E its theta divisor Θ_E defines a rational map

$$\theta \colon \mathcal{SU}_C(r,L) \dashrightarrow |r\Theta_M|,$$

where Θ_M is a translate of the canonical theta divisor of $\mathrm{Pic}^{g-1}(C)$ and M is a line bundle such that $M^{\otimes r} = L$. Note that the indeterminacy locus of θ is given by set the vector bundles which does not admit a theta divisor.

Actually, this map is defined by the determinant line bundle $\mathcal{L}$, see [BNR89] and it has been studied by many authors. It has been completely described for $r = 2$ with the contributions of many authors. On the other hand, when $r \geq 3$, very little is known. In particular, the genus 2 case seems to be interesting. First of all, in this case we have that $\dim \mathcal{SU}_C(r,L) = \dim |r\Theta_M|$. For $r = 2$ it is proved in [NR69] that θ is an isomorphism, whereas, for $r = 3$ it is a double covering ramified along a sextic hypersurface (see [Ort05]). For $r \geq 4$ this is no longer a morphism, and it is generically finite and dominant, see [Bea06] and [BV07].

In this paper, we will consider a smooth curve C of genus 2. In this case, the theory of extensions of vector bundles allows us to give a birational description of the Theta divisor Θ_r as a projective bundle over the moduli space $\mathcal{U}_C(r-1,r)$. Our first result is Theorem 2.5 which can be stated as follows

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Theorem. *There exists a vector bundle $\mathcal{V}$ on $\mathcal{U}_C(r-1,r)$ of rank $2r-1$ whose fiber at the point $[F] \in \mathcal{U}_C(r-1,r)$ is $\mathrm{Ext}^1(F,\mathcal{O}_C)$. Let $\mathbb{P}(\mathcal{V})$ be the associated projective bundle and $\pi\colon \mathbb{P}(\mathcal{V}) \to \mathcal{U}_C(r-1,r)$ the natural projection. Then the map*

$$\Phi\colon \mathbb{P}(\mathcal{V}) \to \Theta_r$$

sending $[v]$ to the vector bundle which is extension of $\pi([v])$ by $\mathcal{O}_C$, is a birational morphism.

In particular, notice that this theorem gives a desingularization of Θ_r as $\mathbb{P}(\mathcal{V})$ is smooth. As a corollary of the above Theorem we have, (see 2.7), that $\Theta_{r,L}$ is birational to a projective bundle over the moduli space $\mathcal{SU}_C(r-1,L)$ for any $r \geq 3$. This has an interesting consequence (see Corollary 2.8):

Corollary. *$\Theta_{r,L}$ is a rational subvariety of $\mathcal{SU}_C(r,L)$.*

The proof of the Theorem and its corollaries can be found in Section 2.

The second result of this paper is contained in Section 3 and it involves the study of the restriction of Φ to the general fiber $\mathbb{P}_F = \pi^{-1}([F])$ of π and its composition with the theta map. The main result of this section is Theorem 3.4 which can be stated as follows:

Theorem. *For a general stable bundle $F \in \mathcal{SU}_C(r-1,L)$ the map*

$$\theta \circ \Phi|_{\mathbb{P}_F}\colon \mathbb{P}_F \to |r\Theta_M|$$

is a linear embedding.

In the proof we are actually more precise about the generality of F: we describe explicitely a open subset of the moduli space $\mathcal{SU}_C(r-1,L)$ where the above theorem holds. Let us stress that one of the key argument in the proof involves the very recent result about the stability of secant bundles $\mathcal{F}_2(E)$ over the two-symmetric product of a curve, see [BD18].

It would be interesting to extend the above results to a curve of genus $g \geq 3$, but the generalization is not straightforward as one can think. First of all, in order to have a projective bundle over the moduli space $\mathcal{U}_C(r-1, r(g-1))$, as in theorem 2.5, we need to assume that $r-1$ and $r(g-1)$ are coprime. Nevertheless, also in these hypothesis Φ is no more a morphism (see Remark 2.5.1 for more details). Finally, in order to generalize the second result, one need to consider secant bundles over g-symmetric product of a curve. Unfortunately, in this case, it is not known whether the secant bundle $\mathcal{F}_g(E)$ is stable when E is so, and this is one of the key argument of our proof in the case $g = 2$.

1. Background and known results

In this section we recall some definitions and useful results about generalized Theta divisors, secant bundles and 2-symmetric product of curves that we will use in the following sections.

1.1. **Theta divisors.**

Let C be a smooth, irreducible, complex, projective curve of genus $g = 2$. For any $r \geq 2$ and for any $n \in \mathbb{Z}$, let $\mathcal{U}_C(r,n)$ denote the moduli space of semistable vector bundles on the curve C with rank r and degree n. It is a normal, irreducible, projective variety of dimension r^2+1, whose points are S-equivalence classes of semistable vector bundles of rank r and degree n; we recall that two vector bundles are called to be S-equivalent if they have isomorphic graduates, where the graduate $gr(E)$ of E is the polystable bundle defined by a Jordan-Holder filtration of E, see [Ses82] and [LeP97]. We denote by $\mathcal{U}_C(r,n)^s$ the open subset correponding to isomorphism classes of stable bundles. For $r = 2$ one has that $\mathcal{U}_C(r,n)$ is smooth, whereas, for $r \geq 3$ one has

$$\mathrm{Sing}(\mathcal{U}_C(r,n)) = \mathcal{U}_C(r,n) \setminus \mathcal{U}_C(r,n)^s.$$

Moreover, $\mathcal{U}_C(r,n) \simeq \mathcal{U}_C(r,n')$ whenever $n'-n = kr$, with $k \in \mathbb{Z}$, and $\mathcal{U}_C(r,n)$ is a fine moduli space if and only if r and n are coprime.

For any line bundle $L \in \mathrm{Pic}^n(C)$, let $SU_C(r,L)$ denote the moduli space of semistable vector bundles on C with rank r and fixed determinant L. These moduli spaces are the fibres of the natural map $\mathcal{U}_C(r,n) \to \mathrm{Pic}^n(C)$ which associates to each vector bundle its determinant.

When $n = r$, we consider the following Brill-Noether loci:

$$\Theta_r = \{[E] \in \mathcal{U}_C(r,r) \mid h^0(gr(E)) \geq 1 \},$$

$$\Theta_{r,L} = \{[E] \in \mathcal{S}U_C(r,L) \mid h^0(gr(E)) \geq 1 \},$$

where $[E]$ denotes $S-$equivalence class of E. Actually, Θ_r (resp. $\Theta_{r,L}$) is an integral Cartier divisor which is called the *theta divisor* of $\mathcal{U}_C(r,r)$ (resp. $\mathcal{S}U(r,L)$), see [DN89]. The line bundle $\mathcal{L}$ associated to $\Theta_{r,L}$ is called the *determinant bundle* of $\mathcal{S}U_C(r,L)$ and it is the generator of its Picard variety. We denote by ${\Theta_r}^s \subset \Theta_r$ the open subset of stable points. Let $[E] \in {\Theta_r}^s$, then the multiplicity of Θ_r at the point $[E]$ is $h^0(E)$, see [Las91]. This implies:

$$\mathrm{Sing}(\Theta_r^s) = \{[E] \in \Theta_r^s \,|\, h^0(E) \geq 2\}.$$

For semistable vector bundles with integer slope we can introduce the notion of theta divisors as follows. Let E be a semistable vector bundle on C with integer slope $m = \frac{\deg E}{r}$.

The tensor product defines a morphism

$$\mu\colon \mathcal{U}_C(r,rm) \times \mathrm{Pic}^{1-m}(C) \to \mathcal{U}_C(r,r)$$

sending $([E],N) \to [E \otimes N]$.

The intersection $\mu^*\Theta_r \cdot ([E] \times \mathrm{Pic}^{1-m}(C))$ is either an effective divisor Θ_E on $Pic^{1-m}(C)$ which is called the *theta divisor* of E, or all $([E] \times \mathrm{Pic}^{1-m}(C))$, and in this case we will say that E *does not admit theta divisor*. For more details see [Bea03].

Set theoretically we have

$$\Theta_E = \{N \in \mathrm{Pic}^{1-m}(C) \mid h^0(gr(E) \otimes N) \geq 1 \}.$$

For all $L \in \mathrm{Pic}^{rm}(C)$ fixed we can choose a line bundle $M \in \mathrm{Pic}^m(C)$ such that $L = M^{\otimes r}$. If $[E] \in \mathcal{SU}_C(r,L)$, then $\Theta_E \in |r\Theta_M|$ where

$$\Theta_M = \{N \in \mathrm{Pic}^{1-m}(C) \,|h^0(M \otimes N) \geq 1 \}$$

is a translate of the canonical theta divisor $\Theta \subset \mathrm{Pic}^{g-1}(C)$. This defines a rational map, which is called the *theta map* of $\mathcal{SU}_C(r,L)$

$$\mathcal{SU}_C(r,L) \overset{\theta}{\dashrightarrow} |r\Theta_M|. \tag{1}$$

As previously recalled θ is the map induced by the determinant bundle $\mathcal{L}$ and the points $[E]$ which do not admit theta divisor give the indeterminacy locus of θ. Moreover θ is an isomsorphism for $r = 2$, it is a double covering ramified along a sextic hypersurface for $r = 3$. For $r \geq 4$ it is no longer a morphism: it is generically finite and dominant.

1.2. 2-symmetric product of curves.

Let $C^{(2)}$ denote the 2-symmetric product of C, parametrizing effective divisors d of degree 2 on the curve C. It is well known that $C^{(2)}$ is a smooth projective surface, see [ACGH85]. It is the quotient of the product $C \times C$ by the action of the symmetric group $\mathcal{S}_2$; we denote by

$$\pi\colon C \times C \to C^{(2)}, \quad \pi(x,y) = x + y,$$

the quotient map, which is a double covering of $C^{(2)}$, ramified along the diagonal $\Delta \subset C \times C$.

Let $N^1(C^{(2)})_{\mathbb{Z}}$ be the Neron-Severi group of $C^{(2)}$, i.e. the quotient group of numerical equivalence classes of divisors on $C^{(2)}$. For any $p \in C$, let 's consider the embedding

$$i_p\colon C \to C^{(2)}$$

sending $q \to q + p$, we denote the image by $C + p$ and we denote by x its numerical class in $N^1(C^{(2)})_{\mathbb{Z}}$. Let d_2 be the diagonal map

$$d_2\colon C \to C^{(2)}$$

sending $q \to 2q$. Then $d_2(C) = \pi(\Delta) \simeq C$, we denote by δ its numerical class in $N^1(C^{(2)})_{\mathbb{Z}}$. Finally, let's consider the Abel map

$$A\colon C^{(2)} \to \mathrm{Pic}^2(C) \simeq J(C)$$

sending $p + q \to O_C(p + q)$. Since $g(C) = 2$, it is well known that actually $C^{(2)}$ is the blow up of $\mathrm{Pic}^2(C)$ at ω_C with exeptional divisor

$$\mathfrak{E} = \{d \in C^{(2)} |\ \mathcal{O}_C(d) \simeq \omega_C\} \simeq \mathbb{P}^1.$$

This implies that:

$$K_{C^{(2)}} = A^*(K_{\mathrm{Pic}^2(C)}) + \mathfrak{E} = \mathfrak{E},$$

since $K_{\mathrm{Pic}^2(C)}$ is trivial.

Let $\Theta \subset J(C)$ be the theta divisor, its pull back $A^*(\Theta)$ is an effective divisor on $C^{(2)}$, we denote by θ its numerical class in $N^1(C^{(2)})_{\mathbb{Z}}$. It is well known that $\delta = 2(3x - \theta)$, or, equivalently,

$$\theta = 3x - \frac{\delta}{2}. \tag{2}$$

If C is a general curve of genus 2 then $N^1(C^{(2)})_{\mathbb{Z}}$ is generated by the classes x and $\frac{\delta}{2}$ (see [ACGH85]). The Neron-Severi lattice is identified by the relations

$$x \cdot x = 1, \quad x \cdot \frac{\delta}{2} = 1, \quad \frac{\delta}{2} \cdot \frac{\delta}{2} = -1.$$

1.3. **Secant bundles on 2-symmetric product of curves.**

Let's consider the *universal effective divisor* of degree 2 of C:

$$\mathcal{I}_2 = \{(d, y) \in C^{(2)} \times C \mid y \in \mathrm{Supp}(d)\},$$

it is a smooth irreducible divisor on $C^{(2)} \times C$. Let ι be the embedding of $\mathcal{I}_2$ in $C^{(2)} \times C$, r_1 and r_2 be the natural projections of $C^{(2)} \times C$ onto factors and $q_i = r_i \circ \iota$ the restriction to $\mathcal{I}_2$ of r_i. Then q_1 is a surjective map of degree 2. Denote also with p_1 and p_2 the natural projections of $C \times C$ onto factors.

We have a natural isomorphism

$$\nu\colon C \times C \to \mathcal{I}_2, \quad (x, y) \to (x + y, y)$$

and, under this isomorphism, the map $q_1\colon \mathcal{I}_2 \to C^{(2)}$ can be identified with the map $\pi\colon C \times C \to C^{(2)}$. It is also easy to see that the map q_2, under the isomorphism ν, can be identified with the projection p_2. We have then a commutative diagram

$$\begin{tikzcd}
 & C^{(2)} \times C \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\
C^{(2)} & \mathcal{I}_2 \arrow[l, "q_1"] \arrow[u, "\iota"] \arrow[r, "q_2"] & C \\
 & C \times C \arrow[ul, "\pi"] \arrow[u, "\nu"] \arrow[ur, "p_2"'] &
\end{tikzcd}$$

Now we will introduce the secant bundle $\mathcal{F}_2(E)$ associated to a vector bundle E on C as well as some properties which will be useful in the sequel. For an introduction on these topics one can refer to [Sch64] or the Ph.D. thesis of E. Mistretta, whereas some interesting recent results can be found in [BN12] and [BD18].

Let E be a vector bundle of rank r on C, we can associate to E a sheaf on $C^{(2)}$ which is defined as

$$\mathcal{F}_2(E) = q_{1*}(q_2^*(E)). \tag{3}$$

$\mathcal{F}_2(E)$ is a vector bundles of rank $2r$ which is called the *secant bundle associated to* E on $C^{(2)}$.

Let's consider the pull back of the secant bundle on $C \times C$: $\pi^*\mathcal{F}_2(E)$. Outside the diagonal $\Delta \subset C \times C$ we have:

$$\pi^*\mathcal{F}_2(E) \simeq p_1^* E \oplus p_2^*(E).$$

Actually, these bundles are related by the following exact sequence:

$$0 \to \pi^*\mathcal{F}_2(E) \to p_1^* E \oplus p_2^*(E) \to p_1^*(E)_{|\Delta} = p_2^*(E)_{|\Delta} \simeq E \to 0, \tag{4}$$

where the last map sends $(u, v) \to u_{|\Delta} - v_{|\Delta}$.

Finally, from the exact sequence on $C^{(2)} \times C$:

$$0 \to \mathcal{O}_{C^{(2)}\times C}(-\mathcal{I}_2) \to \mathcal{O}_{C^{(2)}\times C} \to \iota_* \mathcal{O}_{\mathcal{I}_2} \to 0,$$

tensoring with $r_2^*(E)$ we get:

$$0 \to r_2^*(E)(-\mathcal{I}_2) \to r_2^*(E) \to \iota_*(q_2^* E) \to 0,$$

where, to simplify notations, we set $r_2^*(E) \otimes \mathcal{O}_{C^{(2)}\times C}(-\mathcal{I}_2) = r_2^*(E)(-\mathcal{I}_2)$ and we have used the projection formula

$$r_2^*(E) \otimes \iota_* \mathcal{O}_{\mathcal{I}_2} = \iota_*(\iota^*(r_2^* E) \otimes \mathcal{O}_{\mathcal{I}_2}) = \iota_*(q_2^* E).$$

By applying r_{1*} we get

$$\begin{aligned} 0 \to r_{1*}(r_2^*(E)(-\mathcal{I}_2)) \to H^0(E) \otimes O_{C^{(2)}} \to \mathcal{F}_2(E) \to \\ \to R^1 r_{1*}(r_2^*(E)(-\mathcal{I}_2)) \to H^1(E) \otimes O_{C^{(2)}} \to \cdots \end{aligned} \tag{5}$$

since we have: $r_{1*}(\iota_*(q_2^* E)) = q_{1*} q_2^* E = \mathcal{F}_2(E)$ and

$$R^p r_{1*} r_2^* E = H^p(E) \otimes \mathcal{O}_{\mathbb{C}^{(2)}}.$$

Moreover, by projection formula $H^0(C^{(2)}, \mathcal{F}_2(E)) \simeq H^0(C, E)$ and the map

$$H^0(E) \otimes O_{C^{(2)}} \to \mathcal{F}_2(E)$$

appearing in (5) is actually the evaluation map of global sections of the secant bundle; we will denoted it by ev. Notice that, if we have $h^1(E) = 0$, the exact sequence (5) becomes

$$0 \to r_{1*}(r_2^*(E)(-\mathcal{I}_2)) \to H^0(E) \otimes \mathcal{O}_{C^{(2)}} \xrightarrow{ev} \mathcal{F}_2(E) \to R^1 r_{1*}(r_2^*(E)(-\mathcal{I}_2)) \to 0 \tag{6}$$

We will call the exact sequence (5) (and its particular case (6)) the *exact sequence induced by the evaluation map of the secant bundle.* If $degE = n$, then the Chern character of $\mathcal{F}_2(E)$ is given by the following formula:

$$ch(\mathcal{F}_2(E)) = n(1 - e^{-x}) - r + r(3 + \theta)e^{-x},$$

where x and θ are the numerical classes defined above. From this we can deduce the Chern classes of $\mathcal{F}_2(E)$:

$$c_1(\mathcal{F}_2(E)) = (n - 3r)x + r\theta, \tag{7}$$

$$c_2(\mathcal{F}_2(E)) = \frac{1}{2}(n - 3r)(n + r + 1) + r^2 + 2r. \tag{8}$$

We recall the following definition:

Definition 1.1. *Let X be a smooth, irreducible, complex projective surface and let H be an ample divisor on X. For a torsion free sheaf E on X we define the slope of E with respect to H:*

$$\mu_H(E) = \frac{c_1(E) \cdot H}{rk(E)}.$$

E is said semistable with respect to H if for any non zero proper subsheaf F of E we have $\mu_H(F) \leq \mu_H(E)$, it is said stable with respect to H if for any proper subsheaf F with $0 < rk(F) < rk(E)$ we have $\mu_H(F) < \mu_H(E)$.

One of the key arguments of the proof of our main theorems will use the following interesting result which can be found in [BD18]:

Proposition 1.1. *Let E be a semistable vector bundle on C with rank r and deg$(E) \geq r$, then $\mathcal{F}_2(E)$ is semistable with respect to the ample class x; if deg$(E) > r$ and E is stable, then $\mathcal{F}_2(E)$ is stable too with respect to the ample class x.*

2. Description of Θ_r and $\Theta_{r,L}$.

In this section we will give a description of Θ_r (resp. $\Theta_{r,L}$) which gives a natural desingularization. Fix $r \geq 3$.

Lemma 2.1. *Let E be a stable vector bundle with $[E] \in \Theta_r$, then there exists a vector bundle F such that E fit into the following exact sequence:*

$$0 \to \mathcal{O}_C \to E \to F \to 0,$$

with $[F] \in \mathcal{U}_C(r-1, r)$.

Proof. Since E is stable, $E \simeq gr(E)$ and, as $[E] \in \Theta_r$, $h^0(E) \geq 1$. Let $s \in H^0(E)$ be a non zero global section, since E is stable of slope 1, s cannot be zero in any point of C, so it defines an injective map of sheaves

$$i_s \colon \mathcal{O}_C \to E$$

which induces the following exact sequence of vector bundles:

$$0 \to \mathcal{O}_C \to E \to F \to 0,$$

where the quotient F is a vector bundle of rank $r-1$ and degree r. We will prove that F is semistable, hence $[F] \in \mathcal{U}_C(r-1, r)$, which implies that it is also stable.

Let G be a non trivial destabilizing quotient of F of degree k and rank s with $1 \leq s \leq r-2$. Since G is also a quotient of E, by stability of E we have

$$1 = \mu(E) < \mu(G) \leq \mu(F) = \frac{r}{r-1},$$

i.e.

$$1 < \frac{k}{s} \leq 1 + \frac{1}{r-1}.$$

Hence we have

$$s < k \leq s + \frac{s}{r-1}$$

which is impossible since $s < r-1$. □

A short exact sequence of vector bundles

$$0 \to G \to E \to F \to 0,$$

is said to be *an extension of F by G*, see [Ati57]. Recall that equivalence classes of extensions of F by G are parametrized by

$$H^1(\mathcal{H}om(F, G)) \simeq \mathrm{Ext}^1(F, G);$$

where the extension corresponding to $0 \in \mathrm{Ext}^1(F, G)$ is $G \oplus F$ and it is called the trivial extension. Given $v \in \mathrm{Ext}^1(F, G)$ we will denote by E_v the vector bundle which is the extension of F by G in the exact sequence corresponding to v. Moreover, if $v_2 = \lambda v_1$ for some $\lambda \in \mathbb{C}^*$, we have $E_{v_1} \simeq E_{v_2}$. Lastly, recall that Ext^1 is a functorial construction so are well defined on isomorphism classes of vector bundles.

Lemma 2.2. *Let $[F] \in \mathcal{U}_C(r-1, r)$, then* $\dim \mathrm{Ext}^1(F, \mathcal{O}_C) = 2r - 1$.

Proof. We have: $\mathrm{Ext}^1(F, \mathcal{O}_C) \simeq H^1(F^\vee) \simeq H^0(F \otimes \omega_C)^\vee$, so by Riemann-Roch theorem:

$$\chi_C(F \otimes \omega_C) = \deg(F \otimes \omega_C) + \mathrm{rk}(F \otimes \omega_C)(1 - g(C)) = 2r - 1.$$

Finally, since $\mu(F \otimes \omega_C) = 3 + \frac{1}{r-1} \geq 2g - 1 = 3$, then $h^1(F \otimes \omega_C) = 0$. □

Let F be a stable bundle, with $[F] \in \mathcal{U}_C(r-1, r)$. The trivial extension $E_0 = \mathcal{O}_C \oplus F$ gives an unstable vector bundle. However, this is the only unstable extension of F by $\mathcal{O}_C$ as it is proved in the following Lemma.

Lemma 2.3. *Let $[F] \in \mathcal{U}_C(r-1,r)$ and $v \in \operatorname{Ext}^1(F, O_C)$ be a non zero vector. Then E_v is a semistable vector bundle of rank r and degree r, moreover $[E_v] \in \Theta_r$.*

Proof. By lemma 2.2 $\dim \operatorname{Ext}^1(F, \mathcal{O}_C) = 2r - 1 > 0$, let $v \in \operatorname{Ext}^1(F, \mathcal{O}_C)$ be a non zero vector and denote by E_v the corresponding vector bundle. By construction we have an exact sequence of vector bundles

$$0 \to \mathcal{O}_C \to E_v \to F \to 0$$

from which we deduce that E_v has rank r and degree r.

Assume that E_v is not semistable. Then there exists a proper subbundle G of E_v with $\mu(G) > \mu(E_v) = 1$. Denote with s and k respectively the rank and the degree of G. Hence we have

$$1 \leq s \leq r-1 \qquad k > s.$$

Let α be the composition of the inclusion $G \hookrightarrow E_v$ with the surjection $\varphi : E_v \to F$, let $K = \ker \alpha$. Then we have a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r, hook] \arrow[d] & G \arrow[r] \arrow[d] \arrow[dr, "\alpha"] & Im(\alpha) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_C \arrow[r, hook] & E_v \arrow[r, "\varphi"'] & F \arrow[r] & 0
\end{tikzcd}$$

If $K = 0$ then G is a subsheaf of F, which is stable, so

$$\mu(G) = \frac{k}{s} < \mu(F) = 1 + \frac{1}{r-1}$$

and

$$s < k < s + \frac{s}{r-1},$$

which is impossible as $1 \leq s \leq r-1$. Hence we have that α has non trivial kernel K, which is a subsheaf of $\mathcal{O}_C$, so $K = \mathcal{O}_C(-A)$ for some divisor $A \geq 0$ with degree $a \geq 0$. Then $Im(\alpha)$ is a subsheaf of F, which is stable so:

$$\frac{k+a}{s-1} < 1 + \frac{1}{r-1},$$

hence we have

$$s + a < k + a < s - 1 + \frac{s-1}{r-1}$$

and

$$a < -1 + \frac{s-1}{r-1}$$

which is impossible as $a \geq 0$. This proves that E_v is semistable. Finally, note that we have $h^0(E_v) \geq h^0(\mathcal{O}_C) = 1$, so $[E] \in \Theta_r$. □

We would like to study extensions of $F \in \mathcal{U}_C(r-1,r)$ by $\mathcal{O}_C$ which give vector bundles of $\Theta_r \setminus \Theta_r^s$. Note that if E_v is not stable, then there exists a proper subbundle S of E_v with slope 1. We will prove that any such S actually comes from a subsheaf of F of slope 1.

Let $[F] \in \mathcal{U}_C(r-1,r)$, observe that any proper subbsheaf S of F has slope $\mu(S) \leq 1$. Indeed, let $s = \operatorname{rk}(S) \leq r-1$, by stability of F we have

$$\frac{\deg(S)}{s} < 1 + \frac{1}{r-1},$$

which implies $\deg(S) < s + \frac{s}{r-1}$, hence $\deg(S) \leq s$. Assume that S is a subsheaf of slope 1. Then we are in one of the following cases:

- A subsheaf S of F with slope 1 and rank $s \leq r-2$ is a subbundle of F and it is called *a maximal subbundle* of F of rank s. Note that any maximal subbundle S is semistable and thus $[S] \in \mathcal{U}_C(s,s)$. Moreover, the set $\mathcal{M}_s(F)$ of maximal subbundles of F of rank s has a natural scheme structure given by identifying it with a Quot-scheme (see [LN83], [LN02] for details).
- A subsheaf S of F of slope 1 and rank $r-1$ is obtained by *an elementary transformation* of F at a point $p \in C$, i.e. it fits into an exact sequence as follows:

$$0 \to S \to F \to \mathbb{C}_p \to 0.$$

More precisely, let's denote with F_p the fiber of F at p, all the elementary transformations of F at p are parametrized by $\mathbb{P}(\mathrm{Hom}(F_p, \mathbb{C}))$. In fact, for any non zero form $\gamma \in \mathrm{Hom}(F_p, \mathbb{C})$, by composing it with the restriction map $F \to F_p$, we obtain a surjective morphism $F \to \mathbb{C}_p$ and then an exact sequence

$$0 \to G_\gamma \to F \to \mathbb{C}_p \to 0,$$

where G_γ is actually a vector bundle which is obtained by the elementary tranformation of F at p defined by γ. Finally, $G_{\gamma_1} \simeq G_{\gamma_2}$ if and only if $[\gamma_1] = [\gamma_2]$ in $\mathbb{P}(\mathrm{Hom}(F_p, \mathbb{C}))$, see [Mar82] and [Bri17].

We have the following result:

Proposition 2.1. *Let $[F] \in \mathcal{U}_C(r-1, r)$, $v \in \mathrm{Ext}^1(F, \mathcal{O}_C)$ a non zero vector and E_v the extension of F defined by v. If G is a proper subbundle of E_v of slope 1, then G is semistable and satisfies one of the following conditions:*

- *G is a maximal subbundle of F and $1 \leq \mathrm{rk}(G) \leq r-2$;*
- *G has rank $r-1$ and it is obtained by an elementary transformation of F.*

Proof. Let $s = \mathrm{rk}(G) = \deg(G)$. As in the proof of Lemma 2.1 we can construct a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r, hook] \arrow[d] & G \arrow[r] \arrow[d] \arrow[dr, "\alpha"] & Im(\alpha) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_C \arrow[r, hook] & E_v \arrow[r, "\varphi"'] & F \arrow[r] & 0
\end{tikzcd}$$

form which we obtain that either $K = 0$ of $K = \mathcal{O}_C(-A)$ with $A \geq 0$. In the second case, let a be the degree of A. As in the proof of Lemma 2.1, we have that the slope of $Im(\alpha)$ satisfies

$$\mu(Im(\alpha)) = \frac{s+a}{s-1} < 1 + \frac{1}{r-1}$$

which gives a contradiction

$$0 \leq a < -1 + \frac{s-1}{r-1}.$$

So can assume that $K = 0$, so $\alpha \colon G \to F$ is an injective map of sheaves, we denote by Q the quotient.

If $s = r-1$ we have that Q is a torsion sheaf of degree 1, i.e. a skyscraper sheaf over a point with the only non trivial fiber of dimension 1. Hence G is obtained by an elementary transformation of F at a point $p \in C$.

If $s \leq r-2$, we claim that α is an injective map of vector bundles. On the contrary, if G is not a subbundle, then Q is not locally free, so there exists a subbundle $G_f \subset F$ containing $\alpha(G)$, with

$\mathrm{rk}(G_f) = \mathrm{rk}(G)$ and $deg(G_f) = degG + b$, $b \geq 0$:

$$\begin{tikzcd}
 & 0 \arrow[dr] & 0 \arrow[d] & 0 \arrow[d] & & \\
 & & G \arrow[r, hook] \arrow[d] \arrow[dr, "\alpha"] & G_f \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}_C \arrow[r, hook] & E_v \arrow[r, "\phi"'] & F \arrow[r] \arrow[d] \arrow[dr] & 0 & \\
 & & & Q_f \arrow[d] & Q \arrow[l, two heads] \arrow[dr] & \\
 & & & 0 & & 0
\end{tikzcd}$$

Then, as F is stable, we have:

$$\mu(G_f) = \frac{s+b}{s} = 1 + \frac{b}{s} < 1 + \frac{1}{r-1} = \mu(F),$$

hence

$$0 \leq b < \frac{s}{r-1}$$

which implies $b = 0$.

Finally, note that G is semistable. In fact, since $\mu(G) = \mu(E_v)$, a subsheaf of G destabilizing G would be a subsheaf destabilizing E_v. $\square$

Let S be a subsheaf of F with slope 1, we ask when S is a subbundle of the extension E_v of F by $\mathcal{O}_C$ defined by v.

Definition 2.1. *Let $\varphi : E \to F$ and $f : S \to F$ be morphisms of sheaves. We say that f can be lifted to $\tilde{f} : S \to E$ if we have a commutative diagram*

$$\begin{tikzcd}
 & S \arrow[dl, "\tilde{f}"'] \arrow[d, "f"] \\
E \arrow[r, "\varphi"'] & F
\end{tikzcd}$$

we say that $\tilde{f}$ is a lift of f.

Lemma 2.4. *Let $F \in \mathcal{U}_C(r-1, r)$, $v \in \mathrm{Ext}^1(F, \mathcal{O}_C)$ be a non zero vector and E_v the extension of F defined by v. Let S be a vector bundle of slope 1 and $\iota \colon S \to F$ be an injective map of sheaves. Then i can be lifted to E_v if and only if $v \in \ker H^1(\iota^*)$ where*

$$H^1(i^*) \colon H^1(C, \mathcal{H}om(F, \mathcal{O}_C)) \to H^1(C, \mathcal{H}om(S, \mathcal{O}_C))$$

is the map induced by i. If $v \in \ker H^1(\iota^*)$ we will say that v extends ι.

For the proof see [NR69]. The above lemma allows us to prove the following result:

Proposition 2.2. *Let $[F] \in \mathcal{U}_C(r-1, r)$. Then:*

- *Let G_γ be the elementary transformation of F at $p \in C$ defined by $[\gamma] \in \mathbb{P}(F_p^\vee)$, there exists a unique $[v] \in \mathbb{P}(\mathrm{Ext}^1(F, \mathcal{O}_C))$ such that the inclusion $G_\gamma \hookrightarrow F$ can be lifted to E_v.*
- *Let S be a maximal subbundle of F of rank s and $\iota : S \hookrightarrow F$ the inclusion, then the set of classes $[v]$ which extend ι is a linear subspace of $\mathbb{P}(\mathrm{Ext}^1(F, \mathcal{O}_C))$ of dimension $2r - 2s - 2$.*

In particular, for any maximal subbundle of F and for any elementary transformation, we obtain at least an extension of F which is in $\Theta_r \setminus \Theta_r^s$.

Proof. Let's start with the case of elementary transformation. We are looking for the extensions of F by $\mathcal{O}_C$ such that there exists a lift $\tilde{\iota}\colon G_\gamma \to E_v$ such that the diagram

$$\begin{tikzcd}
 & & & G_\gamma \arrow[dl, hook', bend right] \arrow[d, hook, "\iota"] & \\
0 \arrow[r] & \mathcal{O}_C \arrow[r, hook] \arrow[d] & E_v \arrow[r, "\phi_v"'] \arrow[d] & F \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \mathcal{O}_C \arrow[r] \arrow[d] & \mathcal{O}_C(p) \arrow[r] \arrow[d] & \mathbb{C}_p \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

commutes. By Lemma 2.4, there exists $\tilde{\iota}$ if and only if the class of the extension E_v lives in the kernel of $H^1(\iota^*)$ in the diagram

(9)
$$\begin{tikzcd}
\operatorname{Hom}(F,\mathcal{O}_C) \arrow[r, hook] \arrow[d, "\iota^*"] & \operatorname{Hom}(F,E_v) \arrow[r, "\phi_v^*"] \arrow[d, "\iota^*"] & \operatorname{Hom}(F,F) \arrow[r, "\delta_v"] \arrow[d, "\iota^*"] & \operatorname{Ext}^1(F,\mathcal{O}_C) \arrow[d, "H^1(\iota^*)"] \\
\operatorname{Hom}(G_\gamma,\mathcal{O}_C) \arrow[r, hook] & \operatorname{Hom}(G_\gamma,E_v) \arrow[r, "\phi_v^*"'] & \operatorname{Hom}(G_\gamma,F) \arrow[r, "\delta"'] & \operatorname{Ext}^1(G_\gamma,\mathcal{O}_C)
\end{tikzcd}$$

If we apply the functor $\mathcal{H}om(-,\mathcal{O}_C)$ to the vertical exact sequence we obtain the exact sequence

$$0 \to F^\vee \to G_\gamma^\vee \to \mathbb{C}_p \to 0$$

from which we obtain

$$\cdots \to H^1(F^\vee) \to H^1(G_\gamma^\vee) \to 0.$$

In particular, the map $H^1(\iota^*)$ is surjective so its kernel has dimension

$$\begin{aligned}(10)\quad \dim(\ker(H^1(\iota^*))) &= \operatorname{ext}^1(F,\mathcal{O}_C) - \operatorname{ext}^1(G_\gamma,\mathcal{O}_C) = \\ &= h^0(F\otimes\omega_C) - h^0(G_\gamma\otimes\omega_C) = 2r-1-2(r-1) = 1.\end{aligned}$$

Hence there exist only one possible extension which extend ι.

Let S be a maximal subbundle of F of rank s, $1 \le s \le r-2$, and let $\iota : S \to F$ the inclusion. By Lemma 2.4, we have that the set of $[v]$ which extends ι lifts is $\mathbb{P}(\ker(H^1(\iota^*))$. As in the previous case, one can verify that $H^1(\iota^*)$ is surjective and

$$\begin{aligned}(11)\quad \dim(\ker(H^1(\iota^*))) &= \operatorname{ext}^1(F,\mathcal{O}_C) - \operatorname{ext}^1(S,\mathcal{O}_C) = \\ &= h^0(F\otimes\omega_C) - h^0(S\otimes\omega_C) = 2r-1-2(s) = 2r-2s-1.\end{aligned}$$

$\square$

The above properties of extensions allow us to give the following description of theta divisor of $\mathcal{U}_C(r,r)$:

Theorem 2.5. *There exists a vector bundle $\mathcal{V}$ on $\mathcal{U}_C(r-1,r)$ of rank $2r-1$ whose fiber at the point $[F] \in \mathcal{U}_C(r-1,r)$ is $\operatorname{Ext}^1(F,\mathcal{O}_C)$. Let $\mathbb{P}(\mathcal{V})$ be the associated projective bundle and $\pi\colon \mathbb{P}(\mathcal{V}) \to \mathcal{U}_C(r-1,r)$ the natural projection. Then, the map*

$$\Phi\colon \mathbb{P}(\mathcal{V}) \to \Theta_r$$

sending $[v]$ to $[E_v]$, where E_v is the extension of $\pi([v])$ by $\mathcal{O}_C$ defined by v, is a birational morphism.

Proof. As r and $r-1$ are coprime, there exists a Poincaré bundle $\mathcal{P}$ on $\mathcal{U}_C(r-1,r)$, i.e. $\mathcal{P}$ is a vector bundle on $C \times \mathcal{U}_C(r-1,r)$ such that $\mathcal{P}|_{C\times[F]} \simeq F$ for any $[F] \in \mathcal{U}_C(r-1,r)$, see [Ram73]. Let p_1 and p_2 denote the projections of $C \times \mathcal{U}_C(r-1,r)$ onto factors. Consider on $C \times \mathcal{U}_C(r-1,r)$

the vector bundle $p_1^*(\mathcal{O}_C)$, note that $p_1^*(\mathcal{O}_C)|_{C\times[F]} \simeq \mathcal{O}_C$, for any $[F] \in \mathcal{U}_C(r-1,r)$. Let consider on $\mathcal{U}_C(r-1,r)$ the first direct image of the sheaf $\mathcal{H}om(\mathcal{P}, p_1^*(\mathcal{O}_C))$, i.e. the sheaf

$$\mathcal{V} = R^1 p_{2*}\mathcal{H}om(\mathcal{P}, p_1^*(\mathcal{O}_C)). \tag{12}$$

For any $[F] \in \mathcal{U}_C(r-1,r)$ we have

$$\mathcal{V}_{[F]} = H^1(C, \mathcal{H}om(\mathcal{P}, p_1^*(\mathcal{O}_C))|_{C\times[F]}) = H^1(C, \mathcal{H}om(F, \mathcal{O}_C)) = \mathrm{Ext}^1(F, \mathcal{O}_C)$$

which, by lemma 2.2, has dimension $2r-1$. Hence we can conclude that $\mathcal{V}$ is a vector bundle on $\mathcal{U}_C(r-1,r)$ of rank $2r-1$ whose fibre at $[F]$ is actually $\mathrm{Ext}^1(F,\mathcal{O}_C)$. Let's consider the projective bundle associated to $\mathcal{V}$ and the natural projection map

$$\pi : \mathbb{P}(\mathcal{V}) \to \mathcal{U}_C(r-1,r).$$

Note that for any $[F] \in \mathcal{U}_C(r-1,r)$ we have:

$$H^0(C, \mathcal{H}om(\mathcal{P}, p_1^*(\mathcal{O}_C))|_{C\times[F]}) = H^0(C, \mathcal{H}om(F, \mathcal{O}_C)) = H^0(C, F^*) = 0,$$

since F is stable with positive slope. Then by [NR69, Proposition 3.1], there exists a vector bundle $\mathcal{E}$ on $C \times \mathcal{V}$ such that for any point $v \in \mathcal{V}$ the restriction $\mathcal{E}|_{C\times v}$ is naturally identified with the extension E_v of F by $\mathcal{O}_C$ defined by $v \in \mathrm{Ext}^1(F,\mathcal{O}_C)$ which, by lemma 2.3 is semistable and has sections, unless $v = 0$. Denote by $\mathcal{V}_0$ the zero section of the vector bundle $\mathcal{V}$, i.e. the locus parametrizing trivial extensions by $\mathcal{O}_C$. Then $\mathcal{V}\setminus\mathcal{V}_0$ parametrize a family of semistable extensions of elements in $\mathcal{U}_C(r-1,r)$ by $\mathcal{O}_C$. This implies that the map sending $v \in \mathcal{V}\setminus\mathcal{V}_0$ to $[E_v]$ is a morphism. Moreover this induces a morphism

$$\Phi : \mathbb{P}(\mathcal{V}) \to \Theta_r$$

sending $[v] \in \mathbb{P}(\mathrm{Ext}^1(F,\mathcal{O}_C))$ to $[E_v]$.

Note that we have:

$$\dim \mathbb{P}(\mathcal{V}) = \dim \mathcal{U}_C(r-1,r) + 2r - 2 = (r-1)^2 + 1 + 2r - 2 = r^2 = \dim \Theta_r.$$

Moreover, by lemma 2.1, Φ is dominant so we can conclude that Φ is a generically finite morphism onto Θ_r.

In order to conclude the proof it is enough to produce an open subset $U \subset \Theta_r$ such that the restriction

$$\Phi|_{\Phi^{-1}(U)} \colon \Phi^{-1}(U) \to U$$

has degree 1. Let U be the open subset of Θ_r given by the stable classes $[E]$ with $h^0(E) = 1$. Now, consider $[v_1],[v_2] \in \Phi^{-1}(U)$ and assume that $\Phi([v_1]) = \Phi([v_2]) = [E]$. As $h^0(E) = 1$ we have that $\pi([v_1]) = \pi([v_2]) = [F]$ and we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_C \arrow[r, "s_1"] \arrow[d, "id"'] & E \arrow[r] \arrow[d, "\lambda id"'] & F \arrow[r] \arrow[d, "\lambda id"'] & 0 \\
0 \arrow[r] & \mathcal{O}_C \arrow[r, "\lambda s_1"'] & E \arrow[r] & F \arrow[r] & 0
\end{tikzcd}$$

with $\lambda \in \mathbb{C}^*$. But this implies that the class of the extensions are multiples so we have $[v_1] = [v_2]$ and the degree is 1. □

Remark 2.5.1. We want to stress the importance of the assumption on the genus of C in the Theorem. Assume that C is a curve of genus $g \geq 3$. Then one can also study extensions of a stable vector bundle $F \in \mathcal{U}_C(r-1, r(g-1))$ by O_C. In order to get a projective bundle $\mathbb{P}(\mathcal{V})$ parametrizing all extensions, as in theorem 2.5, we need the existence of a Poincaré vector bundle $\mathcal{P}$ on the moduli space $\mathcal{U}_C(r-1, r(g-1))$. This actually exists if and only if $r-1$ and $r(g-1)$ are coprime, see [Ram73] (notice that this is always true if $g = 2$ and $r \geq 3$). Nevertheless, also under this further assumption, we can find extensions of F by $\mathcal{O}_C$ which are unstable, hence the map Φ fails to be a morphism.

In the proof of Theorem 2.5 we have seen that the fiber of Φ over a stable point $[E]$ with $h^0(E) = 1$ is a single point. For stable points it is possible to say something similar:

Lemma 2.6. *Let $[E] \in \Theta_r^s$, there is a bijective morphism*

$$\nu\colon \mathbb{P}(H^0(E)) \to \Phi^{-1}(E).$$

Proof. Let $s \in H^0(E)$ be a non zero global section of E. As in the proof of lemma 2.1, s induces an exact sequence of vector bundles:

$$0 \to \mathcal{O}_C \to E \to F_s \to 0, \tag{13}$$

where F_s is stable, $[F_s] \in \mathcal{U}_C(r-1, r)$ and E is the extension of F_s by a non zero vector $v_s \in \mathrm{Ext}^1(F_s, \mathcal{O}_C)$. By tensoring 13 with F_s^* and taking cohomology, since $h^0(F_s^*) = h^0(F_s^* \otimes E) = 0$, we get:

$$0 \to H^0(F_s^* \otimes F_s) \xrightarrow{\delta} H^1(F_s^*) \xrightarrow{\lambda_s} H^1(F_s^* \otimes E) \to H^1(F_s^* \otimes F_s) \to 0, \tag{14}$$

from which we see that $\langle v_s \rangle$ is the kernel of λ_s.

So we have a natural map:

$$H^0(E) \setminus \{0\} \to \mathbb{P}(\mathcal{V})$$

sending a non zero global section $s \in H^0(E)$ to $[v_s]$. Let s and s' be non zero global sections such that $s' = \lambda s$, with $\lambda \in C^*$. As in the proof of Theorem 2.5, it turns out that $v_{s'} = \lambda v_s$ in $\mathrm{Ext}^1(F, \mathcal{O}_C)$. So we have a map:

$$\nu\colon \mathbb{P}(H^0(E)) \to \mathbb{P}(\mathcal{V})$$

sending $[s] \to [v_s]$, whose image is actually $\Phi^{-1}(E)$.

We claim that this map is a morphism. Let $\mathbb{P}^n = \mathbb{P}(H^0(E))$, with $n \geq 1$, one can prove that there exists a vector bundle $\mathcal{Q}$ on $\mathbb{P}^n \times C$ of rank $r-1$ such that $\mathcal{Q}|_{[s]\times C} \simeq F_s$. Hence we have a morphism $\sigma\colon \mathbb{P}^n \to \mathcal{U}_C(r-1, r)$, sending $[s] \to [F_s]$, and a vector bundle $\sigma^*\mathcal{V}$ on $\mathbb{P}^n$. Finally, there exists a vector bundle $\mathcal{G}$ on $\mathbb{P}^n$ with $\mathcal{G}_{[s]} = H^1(F_s^* \otimes E)$ and a map of vector bundles:

$$\lambda\colon \sigma^*(\mathcal{V}) \to \mathcal{G},$$

where $\lambda_{[s]}$ is the map appearing in 14. Since $\langle v_s \rangle = \ker \lambda_s$, this implies the claim.

To conclude the proof, we show that ν is injective. Let s_1 and s_2 be global sections and assume that $[v_{s_1}] = [v_{s_2}]$. Then s_1 and s_2 defines two exact sequences which give two extensions which are multiples of each other. Then, there exists an isomorphism σ of E such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_C \arrow[r, "s_1"] \arrow[d, "id"'] & E \arrow[r] \arrow[d, "\sigma"'] & F \arrow[r] \arrow[d, "\lambda id"'] & 0 \\
0 \arrow[r] & \mathcal{O}_C \arrow[r, "s_2"'] & E \arrow[r] & F \arrow[r] & 0
\end{tikzcd}$$

is commutative. But E is stable, so $\sigma = \lambda id$. Then, clearly, $\sigma_1 = \lambda\sigma_2$.

□

Let $L \in \mathrm{Pic}^r(C)$ and $\mathcal{SU}_C(r-1, L)$ be the moduli space of stable vector bundles with determinant L. As we have seen, $\mathcal{SU}_C(r-1, L)$ can be seen as a subvariety of $\mathcal{U}_C(r-1, r)$. Let $\mathcal{V}$ be the vector bundle on $\mathcal{U}_C(r-1, r)$ defined in the proof of Theorem 2.5. Let $\mathcal{V}_L$ denote the restriction of $\mathcal{V}$ to $\mathcal{SU}_C(r-1, L)$. We will denote with $\pi : \mathbb{P}(\mathcal{V}_L) \to \mathcal{SU}_C(r-1, L)$ the projection map. Then, with the same arguments of the proof of Theorem 2.5 we have the following:

Corollary 2.7. *Fix $L \in \mathrm{Pic}^r(C)$. The map*

$$\Phi_L : \mathbb{P}(\mathcal{V}_L) \to \Theta_{r,L}$$

sending $[v]$ to the extension $[E_v]$ of $\pi([v])$ by $\mathcal{O}_C$ defined by v, is a birational morphism.

As r and $r-1$ are coprime, we have that $\mathcal{SU}_C(r-1, L)$ is a rational variety, see [New75, KS99]. Hence, as a consequence of our theorem we have also this interesting corollary:

Corollary 2.8. *For any $L \in \mathrm{Pic}^r(C)$, $\Theta_{r,L}$ is a rational subvariety of $\mathcal{SU}_C(r,L)$.*

3. General fibers of π and θ map

In this section, we would restrict the morphism Φ to extensions of a general vector bundle $[F] \in \mathcal{U}_C(r-1,r)$. First of all we will deduce some properties of general elements of $\mathcal{U}_C(r-1,r)$.

For any vector bundle F, let $\mathcal{M}_1(F^*)$ be the scheme of maximal line subbundles of F^*. Note that, if $[F] \in \mathcal{U}_C(r-1,r)$, then maximal line subbundles of F^* are exactly the line subbundles of degree -2.

Proposition 3.1. *Let $r \geq 3$, a general $[F] \in \mathcal{U}_C(r-1,r)$ satisfies the following properties:*

1. *if $r \geq 4$, F does not admit maximal subbundles of rank $s \leq r-3$;*
2. *F admits finitely many maximal subbundles of rank $r-2$;*
3. *we have $\mathcal{M}_{r-2}(F) \simeq \mathcal{M}_1(F^*)$.*

Proof. For any $1 \leq s \leq r-2$ let's consider the following locus:

$$T_s = \{[F] \in \mathcal{U}_C(r-1,r) \quad | \quad \exists S \hookrightarrow F \ \text{ with } \deg(S) = \mathrm{rk}(S) = s\}.$$

The set T_s is locally closed, irreducible of dimension

$$\dim T_s = (r-1)^2 + 1 + s(s-r+2),$$

see [LN02], [RT99]. If $r \geq 4$ and $s \leq r-3$, then $\dim T_s < \dim \mathcal{U}_C(r-1,r)$, which proves (1).
(2) Let $r \geq 3$ and $s = r-2$. Then actually $T_{r-2} = \mathcal{U}_C(r-1,r)$ and a general $[F] \in \mathcal{U}_C(r-1,r)$ has finitely many maximal subbundles of rank $r-2$. See [LN02], [RT99] for a proof in the general case and [LN83] for $r = 3$, where actually the property actually holds for any $[F] \in \mathcal{U}_C(2,3)$.
(3) Let $[F] \in \mathcal{U}_C(r-1,r)$ be a general element and $[S] \in \mathcal{M}_{r-2}(F)$, then S is semistable and we have an exact sequence

$$0 \to S \to F \to Q \to 0$$

with $Q \in \mathrm{Pic}^2(C)$. Moreover S and Q are general in their moduli spaces as in [LN02]. This implies that $\mathrm{Hom}(F,Q) \simeq \mathbb{C}$. In fact, by taking the dual of the above sequence and tensoring with Q we obtain

$$0 \to Q^* \otimes Q \to F^* \otimes Q \to S^* \otimes Q \to 0$$

and, passing to cohomology we get

$$0 \to H^0(O_C) \to H^0(F^* \otimes Q) \to H^0(S^* \otimes Q) \to \cdots .$$

Since S and Q are general $h^0(S^* \otimes Q) = 0$ and we can conclude

$$\mathrm{Hom}(F,Q) \simeq H^0(F^* \otimes Q) \simeq H^0(O_C) = \mathbb{C}.$$

We have a natural map $q\colon \mathcal{M}_{r-2}(F) \to \mathcal{M}_1(F^*)$ sending S to Q^*. The map q is surjective as any maximal line subbundle $Q^* \hookrightarrow F^*$ gives a surjective map $\phi\colon F \to Q$ whose kernel is a maximal subbundle S of F. The map is also injective. Indeed, assume that $[S_1]$ and $[S_2]$ are maximal subbundles such that $q(S_1) = q(S_2) = Q^*$. Then $S_1 = \ker \phi_1$ and $S_2 = \ker \phi_2$, with $\phi_i \in \mathrm{Hom}(F,Q) \simeq \mathbb{C}$. This implies that $\phi_2 = \rho\phi_1$, $\rho \in \mathbb{C}^*$, hence $S_1 \simeq S_2$. Moreover, note that the above construction works for any flat family of semistable maximal subbundles of F, hence q is a morphism. Finally, the same construction gives a morphism $q' : \mathcal{M}_1(F^*) \to \mathcal{M}_{r-2}(F)$ which turns out to be the inverse of q. This concludes the proof of (3). □

Lemma 3.2. *For any $r \geq 3$ and $[F] \in \mathcal{U}_C(r-1,r)$, let ev be the evaluation map of the secant bundle $\mathcal{F}_2(F \otimes \omega_C)$. If $\mathcal{M}_1(F^*)$ is finite, then ev is generically surjective and its degeneracy locus Z is the following:*

$$Z = \{d \in C^{(2)} \,|\, \mathcal{O}_C(-d) \in \mathcal{M}_1(F^*)\}.$$

Moreover, $Z \simeq \mathcal{M}_1(F^*)$ if and only if $h^0(F) = 1$; if $h^0(F) \geq 2$ then $Z = \mathfrak{E} \cup Z'$, where $\mathfrak{E} = |\omega_C|$ (see Section 1) and Z' is a finite set.

Proof. As we have seen in section 1, $\mathcal{F}_2(F\otimes\omega_C)$ is a vector bundle of rank $2r-2$ on $C^{(2)}$ and $H^0(C^{(2)},\mathcal{F}_2(F\otimes\omega_C))\simeq H^0(C,F\otimes\omega_C)$. Recall that the evaluation map of the secant bundle of $F\otimes\omega_C$ is the map

$$ev\colon H^0(F\otimes\omega_C)\otimes O_{C^{(2)}}\to\mathcal{F}_2(F\otimes\omega_C)$$

and is such that, for any $d\in C^{(2)}$, ev_d can be identified with the restriction map

$$H^0(F\otimes\omega_C)\to(F\otimes\omega_C)_d,\quad s\to s|_d.$$

Observe that

$$H^1(F\otimes\omega_C(-d))\simeq H^0(F^*\otimes O_C(d))^*\simeq\mathrm{Hom}(F,O_C(d))^*. \tag{15}$$

Note that for any $d\in C^{(2)}$ we have:

$$\mathrm{coker}(ev_d)\simeq H^1(F\otimes\omega_C(-d)),$$

hence ev_d is not surjective if and only if $\mathrm{Hom}(F,O_C(d))\neq 0$, that is $O_C(-d)$ is a maximal line subbundle of F^*. If F has finitely many maximal line subbundles we can conclude that ev is generically surjective and its degeneracy locus is the following:

$$Z=\{d\in C^{(2)}\mid \mathrm{rk}(ev_d)<2r-2\}=\{d\in C^{(2)}\mid O_C(-d)\in\mathcal{M}_1(F^*)\}.$$

Let $a\colon C^{(2)}\to\mathrm{Pic}^{-2}(C)$ be the map sending $d\to O_C(-d)$, a is the composition of $A\colon C^{(2)}\to\mathrm{Pic}^2(C)$ sending d to $\mathcal{O}_C(d)$ with the isomorphism $\sigma\colon\mathrm{Pic}^2(C)\to\mathrm{Pic}^{-2}(C)$ sending $Q\to Q^*$. Then $Z=a^{-1}(\mathcal{M}_1(F^*))$. Note that

$$Z\simeq\mathcal{M}_1(F^*)\iff\omega_C^{-1}\notin\mathcal{M}_1(F^*)\iff h^0(F)=1. \tag{16}$$

If $h^0(F)\geq 2$, then $\mathfrak{E}=|\omega_C|\subset Z$ and this concludes the proof. $\square$

Remark 3.2.1. *Under the hypothesis of Lemma 3.2, the evaluation map fit into an exact sequence*

$$0\to M\to H^0(F\otimes\omega_C)\otimes O_{C^{(2)}}\to\mathcal{F}_2(F\otimes\omega_C)\to T\to 0, \tag{17}$$

where M is a line bundle and $\mathrm{Supp}(T)=Z$.

Remark 3.2.2. Let $[F]\in\mathcal{U}_C(r-1,r)$ be a general vector bundle, by proposition 3.1, $\mathcal{M}_1(F^*)\simeq\mathcal{M}_{r-2}(F)$ is a finite set, moreover $\mathrm{Hom}(F,O_C(d))\simeq\mathbb{C}$ when $\mathcal{O}_C(-d)\in\mathcal{M}_1(F^*)$. Finally, $[F]$ being general, we have $h^0(F)=1$ and this implies

$$Z\simeq\mathcal{M}_1(F^*).$$

Taking the dual sequence of 17 we have:

$$0\to\mathcal{F}_2(F\otimes\omega_C)^*\to H^0(F\otimes\omega_C)^*\otimes O_{C^{(2)}}\to M^*\otimes J_Z\to 0,$$

and computing Chern classes we obtain:

$$c_1(M^*)=c_1(\mathcal{F}_2(F\otimes\omega_C)),$$

$$c_1(\mathcal{F}_2(F\otimes\omega_C)^*)c_1(M^*)+c_2(\mathcal{F}_2(F\otimes\omega_C)^*)+l(Z)=0,$$

from which we deduce:

$$l(Z)=c_1(\mathcal{F}_2(F\otimes\omega_C))^2-c_2(\mathcal{F}_2(F\otimes\omega_C)).$$

We have:

$$c_1(\mathcal{F}_2(F\otimes\omega_C))=x+(r-1)\theta,\quad c_2(\mathcal{F}_2(F\otimes\omega_C))=r^2+2r-2,$$

so we obtain:

$$l(Z)=(r-1)^2.$$

This gives the cardinality of $\mathcal{M}_{r-2}(F)$ and of $\mathcal{M}_1(F^*)$. This formula actually holds also for $F\in\mathcal{U}_C(r,d)$, see [Ghi81, Lan85] for $r=3$ and [OT02, Oxb00] for $r\geq 4$.

The stability properties of the secant bundles, on the two-symmetric product of a curve, allow us to prove the following.

Proposition 3.3. *Let $r\geq 3$ and $[F]\in\mathcal{U}_C(r-1,r)$ with $h^0(F)\leq 2$. If $\mathcal{M}_1(F^*)$ is finite, then every non trivial extension of F by $\mathcal{O}_C$ gives a vector bundle which admits theta divisor.*

Proof. Let E be an extension of F by $\mathcal{O}_C$ which does not admit a theta divisor. Hence

$$0 \to \mathcal{O}_C \to E \to F \to 0,$$

and, by tensoring with ω_C we obtain

$$0 \to \omega_C \to \tilde{E} \xrightarrow{\psi} \tilde{F} \to 0, \tag{18}$$

where, to simplify the notations, we have set $\tilde{E} = E \otimes \omega_C$ and $\tilde{F} = F \otimes \omega_C$. Note that $\tilde{E}$ does not admit theta divisor too, hence

$$\{l \in \mathrm{Pic}^{-2}(C) | h^0(\tilde{E} \otimes l) \geq 1\} = \mathrm{Pic}^{-2}(C).$$

This implies that $\forall d \in C^{(2)}$ we have $h^0(\tilde{E} \otimes O_C(-d)) \geq 1$ too. Let's consider the cohomology exact sequence induced by the exact sequence (18)

$$0 \to H^0(\omega_C) \to H^0(\tilde{E}) \xrightarrow{\psi_0} H^0(\tilde{F}) \to H^1(\omega_C) \to 0,$$

where we have used $h^1(\tilde{E}) = 0$ as $\mu(\tilde{E}) = 3 \geq 2$. Let's consider the subspace of $H^0(\tilde{F})$ given by the image of ψ_0, i.e.

$$V = \psi_0(H^0(\tilde{E})).$$

In particular $\dim V = h^0(\tilde{F}) - 1 = 2r - 2$ so V is an hyperplane.

Claim: For any $d \in C^{(2)} \setminus \mathfrak{C}$ we have $V \cap H^0(\tilde{F} \otimes O_C(-d)) \neq 0$.

In fact, by tensoring the exact sequence (18) with $O_C(-d)$ we have:

$$0 \to \omega_C \otimes O_C(-d) \to \tilde{E} \otimes O_C(-d) \to \tilde{F} \otimes O_C(-d) \to 0,$$

for a general $d \in C^{(2)}$, then passing to cohomology we obtain the inclusion:

$$0 \to H^0(\tilde{E} \otimes O_C(-d)) \to H^0(\tilde{F} \otimes O_C(-d)),$$

which implies the claim since $h^0(\tilde{E} \otimes O_C(-d)) \neq 0$.
Let $ev \colon H^0(\tilde{F}) \otimes O_{C^{(2)}} \to \mathcal{F}_2(\tilde{F})$ be the evaluation map of the secant bundle associated to $\tilde{F}$ and consider its restriction to $V \otimes O_{C^{(2)}}$. We have a diagramm as follows:

$$\begin{tikzcd}
0 \arrow[r] & \ker(ev_V) \arrow[r, hook] \arrow[d, hook] & V \otimes \mathcal{O}_{C^{(2)}} \arrow[r, two heads, "ev_V"] \arrow[d, hook] & im(ev_V) \arrow[r] \arrow[d, hook] & 0 & \\
0 \arrow[r] & M \arrow[r, hook] \arrow[d] & H^0(\tilde{F}) \otimes \mathcal{O}_{C^{(2)}} \arrow[r, "ev"'] & \mathcal{F}_2(\tilde{F}) \arrow[r] & T \arrow[r, two heads] & 0 \\
& Q & & & &
\end{tikzcd} \tag{19}$$

where M is a line bundle, T has support on Z as in Lemma 3.2. For any $d \in C^{(2)}$ we have that the stalk of $\ker(ev_V)$ at d is

$$\ker(ev_V)_d = \ker\Big((ev_V)_d : V \otimes \mathcal{O}_d \to \mathcal{F}_2(\tilde{F})_d\Big) = H^0(\tilde{F} \otimes \mathcal{O}_C(-d)) \cap V.$$

Notice that, as a consequence of the claim,

$$\dim\Big(H^0(\tilde{F} \otimes \mathcal{O}_C(-d)) \cap V\Big) \geq 1$$

for any non canonical divisor d. Hence $\ker(ev_V)$ is a torsion free sheaf of rank 1. For all $d \in C^{(2)} \setminus Z$ we have $h^0(\tilde{F} \otimes \mathcal{O}_C(-d)) = 1$, hence, for these points, we have

$$\ker(ev_V)_d = H^0(\tilde{F} \otimes \mathcal{O}_C(-d)).$$

In particular, as M and $\ker(ev_V)$ coincide outside Z, we have that the support of Q is cointained in Z.

In order to conclude the proof we will use the stability property of the secant bundle. With this aim, recall that, as seen in 3.2, $c_1(\mathcal{F}(\tilde{F})) = x + (r-1)\theta$ and thus, $c_1(\mathcal{F}(\tilde{F})) \cdot x = 2r-1$. In particular, if H is an ample divisor with numerical class x we have

$$\mu_H(\mathcal{F}(\tilde{F})) = \frac{2r-1}{2r-2}. \tag{20}$$

We will distinguish two cases depending on the value of $h^1(F)$.

Assume that $h^0(F) = 1$. In this case $Z \simeq \mathcal{M}_1(F^*)$ is a finite set (see Lemma 3.2). The support of T is finite too so we have

$$c_1(im(ev_V)) = -c_1(\ker(ev_V)) = -c_1(M) = c_1(\mathcal{F}_2(\tilde{F})).$$

Hence, we can conclude that $im(ev_V)$ is a proper subsheaf of the secant bundle with rank $2r-3$ and with the same first Chern class. Hence

$$\mu_H(im(ev_V)) = \frac{c_1(im(ev_V)) \cdot x}{2r-3} = \frac{x \cdot (x + (r-1)\theta)}{2r-3} = \frac{2r-1}{2r-3} \tag{21}$$

but this contraddicts Proposition 1.1. This conclude this case.

Assume that $h^0(F) = 2$. In this case $Z = \mathfrak{E} \cup Z'$ with Z' of dimension 0 by Lemma 3.2. Recall that the numerical class of $\mathfrak{E}$ in $C^{(2)}$ is $\theta - x$ (see Section 1). Observe that $Supp(T) = \mathfrak{E} \cup Z'$ and for any $d \in \mathfrak{E}$ we have: $\dim T_d = 1$. From the exact sequence of the evaluation map of the secant bundle we obtain:

$$c_1(M) = \mathfrak{E} - c_1(\mathcal{F}_2(\tilde{F})).$$

Since $Supp(Q) \subset Z$, we distinguish two cases depending on its dimension.

(a) If $\dim Supp(Q) = 0$, then we have

$$c_1(im(ev_V)) = -c_1(\ker(ev_V)) = -c_1(M),$$

hence $c_1(im(ev_V)) = c_1(\mathcal{F}_2(\tilde{F})) - \mathfrak{E}$. Then

$$\mu_H(im(ev_V)) = \frac{x \cdot (2x + (r-2)\theta)}{2r-3} = \frac{2r-2}{2r-3} \tag{22}$$

But this is impossible since the secant bundle is semistable by Proposition 1.1.

(b) If $\dim Supp(Q) = 1$, since $Supp(Q) \subset Z$ and $\mathfrak{E}$ is irreducible, then $Supp(Q) = \mathfrak{E} \cup Z'$, with Z' finite or empty. Observe that for any $d \in \mathfrak{E}$ we have: $\dim Q_d = 1$. So we have

$$c_1(im(ev_V)) = -c_1(\ker(ev_V)) = -c_1(M) + \mathfrak{E},$$

hence $c_1(im(ev_V)) = c_1(\mathcal{F}_2(\tilde{F}))$ and we can conclude as above. □

Fix a line bundle $L = M^{\otimes r}$, with $M \in \mathrm{Pic}^1(C)$. Let $[F] \in \mathcal{SU}_C(r-1, L)$, we consider the fibre of the projective bundle $\pi \colon \mathbb{P}(\mathcal{V}) \to \mathcal{U}_C(r-1, r)$ at $[F]$:

$$\mathbb{P}_F = \mathbb{P}(\mathrm{Ext}^1(F, \mathcal{O}_C)) = \pi^{-1}([F]) \simeq \mathbb{P}^{2r-2},$$

and the restriction of the morphism Φ to $\mathbb{P}_F$:

$$\Phi_F = \Phi|_{\mathbb{P}_F} \colon \mathbb{P}_F \to \Theta_{r,L}. \tag{23}$$

By Corollary 2.7 the map

$$\Phi_L \colon \mathbb{P}(\mathcal{V}_L) \to \Theta_{r,L}$$

is a birational morphism. Then, there exists a non empty open subset $U \subset \Theta_{r,L}$ such that

$$\Phi_{L|{\Phi_L}^{-1}(U)} \colon {\Phi_L}^{-1}(U) \to U$$

is an isomorphism. Hence, for general $F \in \mathcal{SU}_C(r-1, L)$ the intersection $\Phi^{-1}(U) \cap \mathbb{P}_F$ is a non empty open subset of $\mathbb{P}_F$ and

$$\Phi_F \colon \mathbb{P}_F \to \Theta_{r,L}$$

is a birational morphism onto its image.

Recall that

$$\mathcal{SU}_C(r,L) \overset{\theta}{\dashrightarrow} |r\Theta_M|. \tag{24}$$

is the rational map which sends $[E]$ to Θ_E. Note that if F is generic then, by Proposition 3.3, we have that θ is defined in each element of $im(\Phi_F)$ so it makes sense to study the composition of Φ_F with θ which is then a morphism:

$$\begin{tikzcd} \mathbb{P}_F \arrow[r, "\Phi_F"] \arrow[dr, bend right, "\theta\circ\Phi_F"'] & \Theta_{r,L} \arrow[d, dashed, "\theta"] \\ & |r\Theta_M| \end{tikzcd}$$

We have the following result:

Theorem 3.4. *For a general stable bundle $F \in \mathcal{SU}_C(r-1,L)$ the map*

$$\theta\circ\Phi_F \colon \mathbb{P}_F \to |r\Theta_M|$$

is a linear embedding.

Proof. As previously noted, as F is generic we have that

$$\Phi_F \colon \mathbb{P}_F \to \Theta_{r,L}$$

is a birational morphism onto its image and that the composition $\theta\circ\phi_F$ is a morphism by proposition 3.3. We recall that θ is defined by the determinat line bundle $\mathcal{L} \in \mathrm{Pic}^0(\mathcal{SU}_C(r,L))$. For simplicity, we set $\mathbb{P}^N = |r\Theta_M|$.

In order to prove that, for F general, $\theta\circ\Phi_F$ is a linear embedding, first of all we will prove that $(\theta\circ\Phi_F)^*(\mathcal{O}_{\mathbb{P}^N}(1)) \simeq \mathcal{O}_{\mathbb{P}_F}(1)$.

For any $\xi \in \mathrm{Pic}^0(C)$ the locus

$$D_\xi = \overline{\{[E] \in \mathcal{SU}_C(r,L)^s \colon h^0(E\otimes\xi) \geq 1\}}$$

is an effective divisor in $\mathcal{SU}_C(r,L)$ and $\mathcal{O}_{\mathcal{SU}_C(r,L)}(D_\xi) \simeq \mathcal{L}$, see [DN89].

Note that

$$(\theta\circ\Phi_F)^*(\mathcal{O}_{\mathbb{P}^N}(1)) = \Phi_F^*(\theta^*(\mathcal{O}_{\mathbb{P}^N}(1))) = \Phi_F^*(\mathcal{L}|_{\Theta_{r,L}}) = \Phi_F^*(\mathcal{O}_{\Theta_{r,L}}(D_\xi)). \tag{25}$$

Moreover, one can verify that for general $E \in \Theta^s_{r,L}$ there exists an irreducible reduced divisor D_ξ passing through E such that E is a smooth point of the intersection $D_\xi \cap \Theta_{r,L}$. This implies that for general F the pull back ${\Phi_F}^*(D_\xi)$ is a reduced divisor.

Observe that if ξ is such that if $h^1(F\otimes\xi) \geq 1$ (this happens, for example, if $\xi = 0$), then any extension E_v of F has sections:

$$h^0(E_v\otimes\xi) = h^1(E_v\otimes\xi) \geq 1.$$

In particular this implies that $\Phi_F(\mathbb{P}_F) \subset D_\xi$. On the other hand this does not happen for ξ general and we are also able to be more precise about this. Indeed, let $\xi \in \mathrm{Pic}^0(C)$, then there exists an effective divisor $d \in C^{(2)}$ such that $\xi = \omega_C(-d)$. We have that $h^1(F\otimes\xi) \geq 1$ if and only $d \in Z$, where Z is defined in Lemma 3.2. Moreover, we can assume that Z is finite by Proposition 3.2 as F is generic. From now on we will assume that $d \notin |\omega_C|$ and $d \notin Z$. We can consider the locus

$$H_\xi = \{[v] \in \mathbb{P}_F | \quad h^0(E_v\otimes\xi) \geq 1\}.$$

We will prove that H_ξ is an hyperplane in $\mathbb{P}_F$ and $\Phi_F^*(D_\xi) = H_\xi$.

From the exact sequence

$$0 \to \xi \to E_v\otimes\xi \to F\otimes\xi \to 0,$$

passing to cohomology, since $h^0(\xi) = 0$ we have

$$0 \to H^0(E_v\otimes\xi) \to H^0(F\otimes\xi) \to \cdots$$

from which we deduce that $[v] \in H_\xi$ if and only if there exists a non zero global section of $H^0(F\otimes\xi)$ which is in the image of $H^0(E_v\otimes\xi)$. Since $d \notin Z$, then $h^0(F\otimes\xi)=1$, let's denote by s a generator of $H^0(F\otimes\xi)$.

Claim: if ξ is general, we can assume that the zero locus $Z(s)$ of s is actually empty. This can be seen as follows. By stability of $F\otimes\xi$ we have that $Z(s)$ has degree at most 1. Suppose that $Z(s)=x$, with $x\in C$. Then we would have an injective map $\mathcal{O}_C(x)\hookrightarrow F\otimes\xi$ of vector bundles which gives us $\xi^{-1}(x)\in\mathcal{M}_1(F)$. Since F is general, if $r\geq 4$ then $\mathcal{M}_1(F)$ is empty by Proposition 3.1 so the zero locus of s is indeed empty. If $r=3$, then

$$\mathcal{M}_1(F)=\{T_1,\ldots,T_m\}$$

is finite. For each $i\in\{1,\ldots,m\}$ consider the locus

$$T_{F,i}=\{\xi\in \mathrm{Pic}^0(C)\,|\,\exists x\in C\,:\,\xi^{-1}(x)=T_i\}.$$

This is a closed subset of $\mathrm{Pic}^0(C)$ of dimension 1. Indeed, $T_{F,i}$ is the image, under the embedding $\mu_i : C\to \mathrm{Pic}^0(C)$ which send x to $T_i(-x)$. Hence the claim follows by choosing ξ outside the divisor $\bigcup_{i=1}^m T_{F,i}$.

As consequence of the claim, we have that s induces an exact sequence of vector bundles

$$0\longrightarrow \mathcal{O}_C\xrightarrow{\iota_s} F\otimes\xi\longrightarrow Q\longrightarrow 0.$$

Observe that $[v]\in H_\xi$ if and only if ι_s can be lifted to a map $\tilde{\iota_s}:\mathcal{O}_C\to E\otimes\xi$. Then, by Lemma 2.4, we have that H_ξ is actually the projectivization of the kernel of the following map:

$$H^1(\iota_s^*) : H^1(\mathcal{H}om(F\otimes\xi,\xi))\to H^1(\mathcal{H}om(\mathcal{O}_C,\xi))$$

which proves that H_ξ is an hyperplane as $H^1(\iota_s^*)$ is surjective and

$$H^1(\mathcal{H}om(\mathcal{O}_C,\xi))\simeq H^1(\xi)\simeq\mathbb{C}.$$

Note that we have the inclusion ${\Phi_F}^*(D_\xi)\subseteq H_\xi$. Since both are effective divisors and H_ξ is irreducible we can conlude that they have the same support. Finally, since ${\Phi_F}^*(D_\xi)$ is reduced, then they are the same divisor. In particular, as claimed, we have

$$\Phi_F^*(\mathcal{O}_{\Theta_{r,L}}(D_\xi))=\mathcal{O}_{\mathbb{P}(F)}(1).$$

In order to conclude we simply need to observe that the map is induced by the full linear system $|\mathcal{O}_{\mathbb{P}_F}(1)|$. But this easily follows from the fact that $\theta\circ\Phi_F$ is a morphism. Hence $\theta\circ\Phi_F$ is a linear embedding and the Theorem is proved. □

Remark 3.4.1. *The above Theorem implies that $\Phi_L^*(\mathcal{L})$ is a unisecant line bundle on the projective bundle $\mathbb{P}(\mathcal{V}_L)$.*

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