INVARIANT RINGS OF SUMS OF FUNDAMENTAL REPRESENTATIONS OF SL_n AND COLORED HYPERGRAPHS

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ABSTRACT. The fundamental representations of the special linear group SL_n over the complex numbers are the exterior powers of $\mathbb{C}^n$. We consider the invariant rings of sums of arbitrary many copies of these SL_n-modules. The symbolic method for antisymmetric tensors developed by Grosshans, Rota and Stein is used, but instead of brackets, we associate colored hypergraphs to the invariants. This approach allows us to use results and insights from graph theory. In particular, we determine (minimal) generating sets of the invariant rings in the case of SL_4 and SL_5, as well as syzygies for SL_4. Since the invariants constitute incidence geometry of linear subspaces of the projective space $\mathbb{P}_{n-1}$, the generating invariants provide (minimal) sets of geometric relations that are able to describe all others.

1. INTRODUCTION

Classical invariant theory deals with invariants of linear reductive groups and their syzygies. From its beginnings in the nineteenth century, it was not only a forerunner for modern invariant theory but, for example through *Hilberts Nullstellensatz* and *Basissatz*, for modern algebraic geometry and algebra in general. The literature on the subject is vast, we refer to [9, 20, 24, 35] for an overview.

One of the most important fields of research in the nineteenth century was the theory of invariants of *binary forms*, so to say of $\mathrm{SL}_2(\mathbb{C})$-invariants of symmetric tensors. This is still an active area of research, see [2, 3, 8, 21]. Besides the case of binary forms, among others, invariants and syzygies have been found for *systems of vectors and covectors* of the classical groups, see [26, §9.3, 9.4]. At least the theory of binary forms relies heavily on the *symbolic method*, using *brackets* to denote complete contractions of tensors made up of the relevant symmetric ones and the covariant tensor det.

Weitzenböck applied the symbolic method to *antisymmetric* tensors in [33, 34], but it was not until 1987, when Grosshans, Rota, and Stein in [10], see also [28], formulated a rigorous symbolic method for both symmetric and antisymmetric tensors using *superalgebras*. Nevertheless, besides findings on invariants of $(\wedge^2(\mathbb{C}^n))^1$ and $(\wedge^2(\mathbb{C}^4))^{n_2}$, see [10, §5.4], [11, Th. 34.9] and [5, 22, 29, 31, 32, 36] as well as invariants of up to four linear subspaces of projective space [16, 17], and some statements on invariants of $\wedge^3(\mathbb{C}^n)$ for small n, see [11, §35] and [4], the only progress in finding generators for the ring of invariants has been made by Rosa Huang, a student of Rota, in the case of $(\wedge^2(\mathbb{C}^4))^{n_2}$, see [18]. But the system of generators she found was by no means minimal. As vanishing of invariants describes the geometry of linear subspaces of projective space, see [30] and [7, §11] for a discussion, determination of a *minimal* generating set in this context means nothing less than finding a minimal set of geometric relations that are able to describe all the others.

Gurevich [11, §35] as well as Sturmfels [30, p. 173] and Procesi [27, §6.8] consider this a complicated and involved problem.

The aim of the present paper is first to develop a method for approaching this problem and second, to demonstrate the power of this method by finding generators of the ring of invariants for SL_4 and SL_5 as well as relations for SL_4.

We fix some notation. Let $n \geq 2$ be a fixed integer. We work over the field $\mathbb{C}$ of complex numbers and denote by SL_n the special linear group of degree n over $\mathbb{C}$. This group acts on $V = \mathbb{C}^n$ by multiplication from the left, which is the standard representation. This induces an action of SL_n on $\wedge^i V$ for any $i \in \{1, \ldots, n-1\}$, the fundamental representations. Now set $V_{i,j} := \wedge^i V$ and

$$W := \bigoplus_{i=1}^{n-1} \bigoplus_{j=1}^{n_i} V_{i,j}$$

for fixed $n_1, \ldots, n_{n-1} \geq 0$. We call the induced action of SL_n on W the *action on sums of fundamental representations*. The special case $n_i = 0$ for $i \neq 1, n-1$ is equivalent to the action on vectors and covectors.

We informally describe the symbolic method from [10] now, ignoring signs. Let m, $n_{i,j}$ be nonnegative integers. A bracket monomial is a product of m brackets, where every bracket contains n of the following letters: to every $V_{i,j}$ associate letters $a_{i,j,k}$, so that for every $1 \leq k \leq n_{i,j}$, the letter $a_{i,j,k}$ turns up i times in the bracket monomial. Now consider the mapping of an element $\sum t_{i,j}$ of W to

$$\left(\bigotimes_{i,j} t_{i,j}^{\otimes n_{i,j}} \right) \otimes \det^{\otimes m},$$

followed by the complete contraction, where two indices of the k-th appearance of $t_{i,j}$ and the l-th appearance of det in the tensor are contracted if and only if the letter $a_{i,j,k}$ turns up in the l-th bracket. This is the invariant associated to the bracket monomial.

We have two fundamental statements: first, all *invariants* come from bracket polynomials and second, all *relations* between bracket polynomials come from the *Plücker relation*

$$\sum_{(u_1,u_2) \vdash (u)} [u_1][u_2w] = 0,$$

where u is a word of length $n+1$, w a word of length $n-1$, and we sum over all partitions of u in two subwords u_1 of length n and u_2 of length one. In the case $n_i = 0$ for $i \neq 1$, this gives the standard Plücker relations (without sign, which is due to the nature of the involved superalgebras, see Section 2).
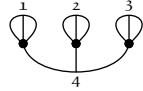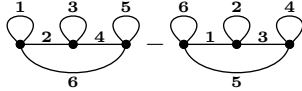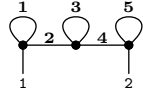
This sets the starting point for our method. We associate to each bracket monomial a *colored hypergraph* defined as follows: for each bracket, we have a vertex, and for each letter $a_{i,j,k}$, we have an *i*-edge of *color j* and *shading k*. We ignore the shading for a moment, as it just affects sign. Now if the letter $a_{i,j,k}$ turns up in a bracket, the respective *i*-edge has a connection to the vertex associated to the bracket. For a similar (somewhat dual) approach involving directed graphs in the case of binary forms see [25] and also [23, 24, 26]. This approach has just recently been applied to determine the ideal of relations of several points on the projective line, see [12, 13, 14, 15, 19].
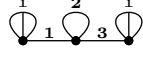
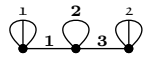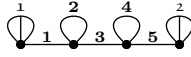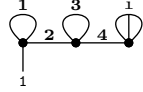At first sight, the problem is much more involved in our case: excluding the already settled cases $n = 2, 3$, we deal with vertices of degree $n \geq 4$ and *i*-edges with $i \leq n-1$. But it turns out that the Plücker relations from above can be used to substantially simplify the involved hypergraphs, which in turn allows us to effectively use combinatorial and graph theoretical results. We develop suitable

techniques in Sections 3, 4, and 5. One of the greatest advantages of the approach is the self-containedness, making it comprehensible for everyone with basic knowledge on combinatorics.

In the case of SL_4, we explicitly list a minimal set of generators of $\mathbb{C}[W]^{\mathrm{SL}_4}$ in the following theorem. Here and throughout the paper, for the colors of 1-edges we use typewriter font, for those of 2-edges bold letters and for 3-edges Fraktur letters. We denote the standard coordinate functions on $V_{1,j}$ by $x_{\mathtt{j}1}, \ldots, x_{\mathtt{j}4}$, those on $V_{2,j}$ by $y_{\boldsymbol{j}12}, \ldots, y_{\boldsymbol{j}34}$, and on $V_{3,j}$ by $z_{\mathfrak{j}123}, \ldots, z_{\mathfrak{j}234}$. Moreover, to ease notation, colors $1, 2, \ldots, N$ of i-edges stand representatively for arbitrary but ascending colors $1 \le k_1, k_2, \ldots, k_N \le n_i$.

Theorem 1.1. *Let* SL_4 *act on an arbitrary sum of fundamental representations* W. *Then a minimal generating set of* $\mathbb{C}[W]^{\mathrm{SL}_4}$ *is the following, where if the respective invariant is too big, only the number of monomials is given. In these cases, consult the appendix* [1] *for the actual invariant.*

Graph	Invariant	Symbol
	$\begin{vmatrix} x_{11} \cdots x_{41} \\ \vdots \ddots \vdots \\ x_{14} \cdots x_{44} \end{vmatrix}$	$\big|\mathtt{1234}\big|$
	$\begin{vmatrix} z_{\mathfrak{1}234} \cdots z_{\mathfrak{4}234} \\ \vdots \ddots \vdots \\ z_{\mathfrak{1}123} \cdots z_{\mathfrak{4}123} \end{vmatrix}$	$\big|\mathfrak{1234}\big|$
	$2(y_{\boldsymbol{1}12}y_{\boldsymbol{1}34} - y_{\boldsymbol{1}13}y_{\boldsymbol{1}24} + y_{\boldsymbol{1}14}y_{\boldsymbol{1}23})$	$\big|\boldsymbol{11}\big|$
	$y_{\boldsymbol{1}12}y_{\boldsymbol{2}34} - y_{\boldsymbol{1}13}y_{\boldsymbol{2}24} + y_{\boldsymbol{1}14}y_{\boldsymbol{2}23}$ $+ y_{\boldsymbol{1}34}y_{\boldsymbol{2}12} - y_{\boldsymbol{1}24}y_{\boldsymbol{2}13} + y_{\boldsymbol{1}23}y_{\boldsymbol{2}14}$	$\big|\boldsymbol{12}\big|$
	$\begin{vmatrix} y_{\boldsymbol{1}12} \cdots y_{\boldsymbol{6}12} \\ \vdots \ddots \vdots \\ y_{\boldsymbol{1}34} \cdots y_{\boldsymbol{6}34} \end{vmatrix}$	$\big|\boldsymbol{123456}\big|$
	$x_{11}z_{\mathfrak{1}234} - x_{12}z_{\mathfrak{1}134} + x_{13}z_{\mathfrak{1}124} - x_{14}z_{\mathfrak{1}123}$	$\big|\mathtt{1}^{\mathfrak{1}}\big|$
	$y_{\boldsymbol{1}12}(x_{13}x_{24} - x_{14}x_{23}) + y_{\boldsymbol{1}13}(x_{14}x_{22} - x_{12}x_{24})$ $+ y_{\boldsymbol{1}14}(x_{12}x_{23} - x_{13}x_{22}) + y_{\boldsymbol{1}23}(x_{11}x_{24} - x_{14}x_{21})$ $+ y_{\boldsymbol{1}24}(x_{13}x_{21} - x_{11}x_{23}) + y_{\boldsymbol{1}34}(x_{11}x_{22} - x_{12}x_{21})$	$\big|\mathtt{1}\boldsymbol{1}\mathtt{2}\big|$
	96	$\big|\mathtt{1}\boldsymbol{123}\mathtt{1}\big|$
	108	$\big|\mathtt{1}\boldsymbol{123}\mathtt{2}\big|$
	972	$\big|\mathtt{1}\boldsymbol{12345}\mathtt{2}\big|$

	12	$\left|{}^{1}\mathbf{1}^{2}\right|$
	96	$\left|{}^{1}\mathbf{123}^{1}\right|$
	108	$\left|{}^{1}\mathbf{123}^{2}\right|$
	972	$\left|{}^{1}\mathbf{12345}^{2}\right|$
	36	$\left|{}_{1}\mathbf{12}^{1}\right|$
	324	$\left|{}_{1}\mathbf{1234}^{1}\right|$

Corollary 1.2. *All geometric incidence relations between n_1 points, n_2 lines, and n_3 planes in $\mathbb{P}_3(\mathbb{C})$ can be expressed by means of the invariants from Theorem 1.1.*

It follows a generating set for the respective invariants of SL_5. Here the number of graphs coming into consideration turns out to be considerably larger as in the case of SL_4. Thus in the following theorem, we do not give all different colorings of each graph. We did this exemplarily in the special case $n_i = 0$ for $i \neq 2$ (for graphs only with 2-edges in other words), see Proposition 5.1 at the beginning of Section 5.

Moreover, by the duality of $\wedge^i V$ and $\wedge^{n-i} V$, we have *mirror invariants* and also *mirror graphs*, where i-edges of the one correspond to $(n-i)$-edges of the other. For example, the first two graphs from Theorem 1.1 are mirrors, while graphs three to six are their own mirror each. On the invariant side, it is straightforward to get the mirror by replacing x_{ja} by z_{jbcd} and vice versa, as well as $y_{\boldsymbol{j}ab}$ by $y_{\boldsymbol{j}cd}$ - where $\{a, b, c, d\} = \{1, 2, 3, 4\}$ - in the case of SL_4, and analogously for greater n. Thus we consider only one graph of each mirror pair in the following theorem. Lastly, there are the following types of 'building blocks' that can be attached to some of the graphs:

Theorem 1.3. *The following graphs with all possible combinations of building blocks attached and all possible colorings with respect to the conditions:*

- *the number of vertices with at least one looping 2-edge plus the number of non-looping two edges is less than or equal to nine,*
- *the number of blocks $\textcircled{a}$ plus the number of 3-edges that are not part of a block $\textcircled{a}$ is less than or equal to nine,*
- *the number of 3-edges is less than or equal to the number of 2-edges,*

together with their mirrors constitute a generating set of $\mathbb{C}[W]^{\mathrm{SL}_5}$.
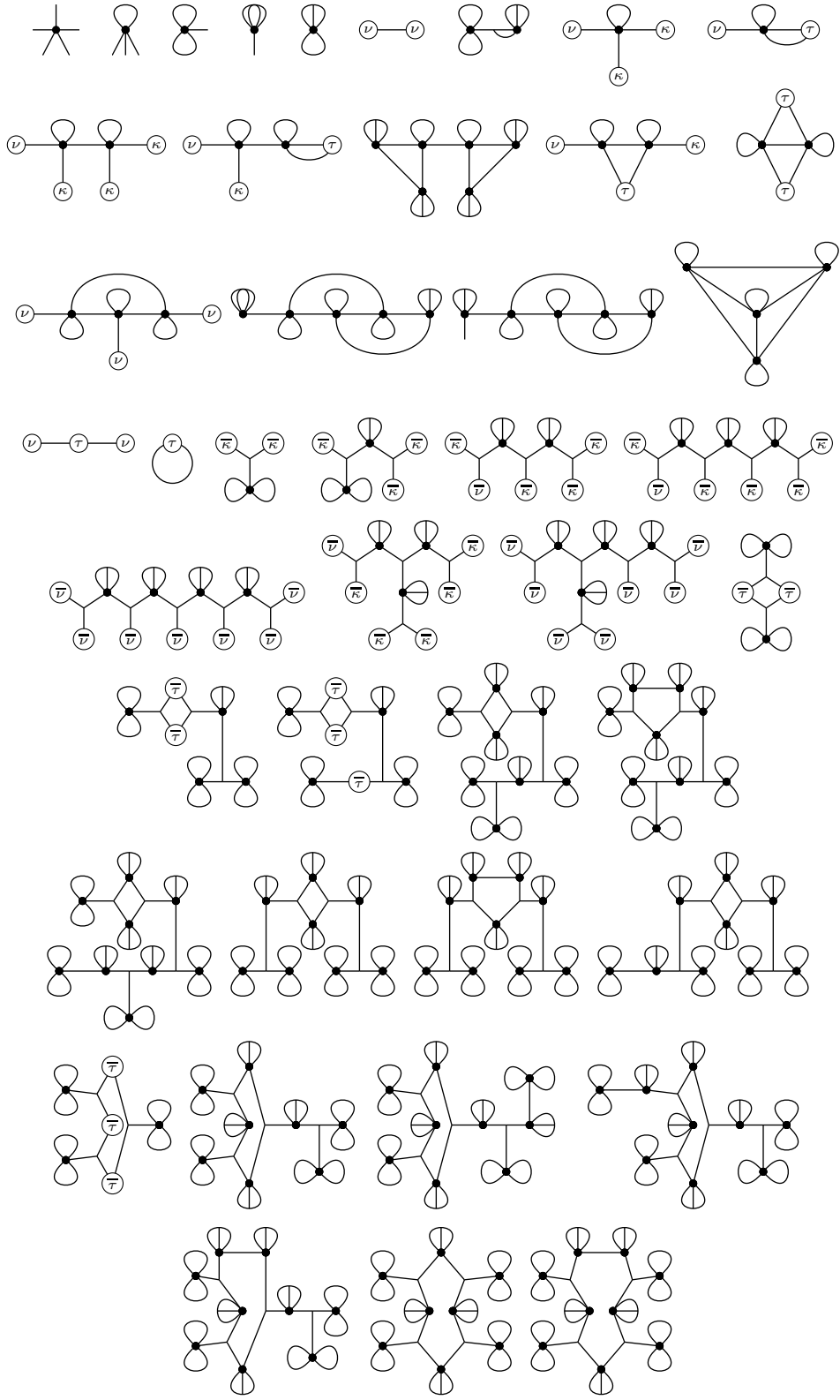
Corollary 1.4. *All geometric incidence relations between n_1 points, n_2 lines, n_3 planes, and n_4 hyperplanes in $\mathbb{P}_4(\mathbb{C})$ can be expressed by means of the invariants from Theorem 1.3.*

Finally, we present relations that hold between the generators of $\mathbb{C}[W]^{\mathrm{SL}_4}$. Most of them can be generalized to $n \geq 5$. The notation

$$(u_1, \ldots, u_M) \vdash (u)$$

means that we sum over all partitions of the word u in M subwords $u_k = l_{k,1} \cdots l_{k,N_k}$ of the lengths N_k. If we have an ordering on the letters of u, then we require u and all subwords to be ordered and for every summand define $\mathrm{sgn}(\vdash)$ to be the sign of the underlying permutation of letters $u \mapsto u_1 \cdots u_M$.

Theorem 1.5. *The following sums of graphs correspond to polynomials in the ideal of relations of $\mathbb{C}[W]^{\mathrm{SL}_4}$, where the not necessarily connected (sub-)graphs Γ and Γ_i have to be chosen such that all involved graphs are out of the minimal generating set from Theorem 1.1.*

$$\Upsilon_1 = \sum_{(\mathtt{ijkl,m}) \vdash (\mathtt{12345})} \quad , \qquad \Upsilon_2 = \quad - \sum_{(\mathtt{ij,kl}) \vdash (\mathtt{1234})} \quad ,$$

$$\Upsilon_3 = \quad - \sum_{(\mathtt{ij,k,l}) \vdash (\mathtt{1234})} \quad ,$$

$$\Upsilon_4 = \quad + \sum_{(\mathtt{ij,k,l}) \vdash (\mathtt{1234})} \quad ,$$

$$\Upsilon_5 = \quad - \sum_{(\mathtt{i,jk,l}) \vdash (\mathtt{1234})} \quad ,$$

$$\Upsilon_6 = \sum_{(\mathtt{i,j}) \vdash (\mathtt{12})} \quad$$

$$- \sum_{(\mathtt{i,j}) \vdash (\mathtt{12})} \mathrm{sgn}(\vdash) \left(\quad + \quad + \quad \right) ,$$

$$\Upsilon_7 = \sum_{(\mathtt{i,j,k}) \vdash (\mathtt{123})} \quad$$

$$+ \sum_{(\mathtt{i,jk}) \vdash (\mathtt{123})} \left(\quad + \quad + \quad \right) ,$$

$$\Upsilon_8 = 2 \quad \left(\quad - \quad \right)$$

$$+ \sum_{(\boldsymbol{g,h}) \vdash (\mathbf{12})} \sum_{(\boldsymbol{i,jklmn}) \vdash (\mathbf{345678})} (-1)^i \quad \left(\quad - \quad \right)$$

$$- \sum_{(\boldsymbol{i,jk,l}) \vdash (\mathbf{3456})} \mathrm{sgn}(\vdash) \left(\quad + \quad \right) ,$$

$$\Upsilon_9 = \underset{i_6}{\overset{\overset{i_1 \quad i_3 \quad i_5}{i_2 \quad i_4}}{\bigcirc\!\bigcirc\!\bigcirc}} \quad \underset{j_6}{\overset{\overset{j_1 \quad j_3 \quad j_5}{j_2 \quad j_4}}{\bigcirc\!\bigcirc\!\bigcirc}} + \det\left(\underset{j_s}{\overset{i_r}{\bigcirc\!\bigcirc}}\right)_{1 \le r, s \le 6}$$

$$\Upsilon_{10} = \sum_{(i_1 \cdots i_6, j_1) \vdash (k_1 \ldots k_7)} \underset{i_6}{\overset{\overset{i_1 \quad i_3 \quad i_5}{i_2 \quad i_4}}{\bigcirc\!\bigcirc\!\bigcirc}} \quad \underset{j_6}{\overset{\overset{j_1 \quad j_3 \quad j_5}{j_2 \quad j_4}}{\bigcirc\!\bigcirc\!\bigcirc}} \ .$$

These relations emerge from somewhat natural principles that are discussed in the proof of Theorem 1.5 in Section 6. This fact together with calculations of the Hilbert series for small values of n_i using [6, §4.6] and Xin's algorithm [37] for MacMahon partition analysis leads us to our final conjecture:

Conjecture 1.6. The graphsums from Theorem 1.5 generate the ideal of relations of $\mathbb{C}[W]^{\mathrm{SL}_4}$.

The paper is organized as follows: in Section 2, we give the necessary background on the symbolic method from [10], then in Section 3 we introduce (edge-)colored hypergraphs and give some general statements on the behaviour of these under relations, and restrictions concerning colorings. In Sections 4 and 5, we develop techniques in order to find the generating sets of $\mathbb{C}[W]^{\mathrm{SL}_4}$ and $\mathbb{C}[W]^{\mathrm{SL}_5}$ of Theorems 1.1 and 1.3 respectively. Most of these techniques are applicable for general n. In Section 6, we prove Theorem 1.5. The relation-generating principles are applicable for $n \ge 5$ as well. Finally, Section 7 gives an outlook on possible further applications of the hypergraph method.

Contents

2. Invariants and brackets

This section is merely a summary of the parts of [10] that are relevant for anti-symmetric tensors. All notation is as close as possible to the one from [10]. Fix a natural number n, a complex vector space V of dimension n and a basis $e_1, \ldots, e_n$ of V. Now let the special linear group SL_n act on V by multiplication from the left. This induces an action of SL_n on $\wedge^i V$ for every $1 \le i \le n-1$. Fix some integer $n_i \ge 0$ for every such i and set $V_{i,j} := \wedge^i V$ for every $1 \le i \le n-1$ and $1 \le j \le n_i$. Then the action of SL_n on V finally induces an action on

$$W := W_{(n_1, \ldots, n_{n-1})} = \bigoplus_{i=1}^{n-1} \bigoplus_{j=1}^{n_i} V_{i,j} = \bigoplus_{i=1}^{n-1} \left(\bigwedge\nolimits^i V\right)^{n_i}.$$

We can identify the ring of polynomial functions on W with $\mathbb{C}\left[T_{i,j,\iota_1 \cdots \iota_i}\right]$, where $1 \le j \le n_i$, $\{\iota_1 < \ldots < \iota_i\} \subseteq \{1, \ldots, n\}$. We do this by linearly mapping $T_{i,j,\iota_1 \cdots \iota_i}$

onto a function f so that for an element

$$t = \sum_{i=1}^{n-1} \sum_{j=1}^{n_i} \sum_{\iota_1 < \ldots < \iota_{\nu_i}} t_{i,j,\iota_1 \cdots \iota_i} e_{\iota_1} \wedge \ldots \wedge e_{\iota_i}$$

of W we have $f(t) = t_{i,j\cdots\iota_i}$.

Now following [10] we introduce an ordered alphabet $P = \{1, \ldots, n\}$ with $1 < \ldots < n$ of so called *places* and the algebra $\mathrm{Ext}(P)$, which is the exterior algebra generated by the places. We denote multiplication in $\mathrm{Ext}(P)$ by juxtaposition. Moreover, for every $V_{i,j}$, we introduce an infinite number of so called *letters* $a_{i,j,k}$ for all $k \in \mathbb{N}$ forming the alphabet L. We set $a_{i_1,j_1,k_1} < a_{i_2,j_2,k_2}$ if either $i_1 < i_2$ or $i_1 = i_2$ and $j_1 < j_2$ or $i_1 = i_2$ and $j_1 = j_2$ and $k_1 < k_2$.

Definition 2.1. Let A be an alphabet, then the *divided powers algebra* $\mathrm{Div}(A)$ is the commutative algebra generated by symbols $a^{(i)}$, where $a \in A$ and $i \in \mathbb{N}$. We denote multiplication in $\mathrm{Div}(A)$ by juxtaposition. Moreover, we set $a^{(0)} = 1$ and $a^{(1)} = a$ and impose the identity

$$a^{(i)} a^{(j)} = \binom{i+j}{j} a^{(i+j)}.$$

We define the length of the word $a^{(i)}$ to be $|a^{(i)}| = i$.

Now we proceed with the divided powers algebra $\mathrm{Div}(L)$ generated by the alphabet L of letters and define a third alphabet $[L|P]$, the *letterplace alphabet* having as elements pairs $(x|\alpha)$, where $x \in L$, $\alpha \in P$. The algebra $\mathrm{Ext}([L|P])$ is called the *fourfold algebra*.

Definition 2.2. We define a bilinear form

$$(*|*) : \mathrm{Div}(L) \times \mathrm{Ext}(P) \to \mathrm{Ext}([L|P]),$$

called the *biproduct*, by the following:

(1) $(w|v) = 0$ if w and v are words of different length.
(2) $(w|v) = (x|\alpha)$ if $w = x$ is a letter and $v = \alpha$ is a place, thus the image is a single letterplace.
(3) $(1|1) = 1$.
(4) $(w|vu) = \sum_{w_1 w_2 = w} (w_1|v)(w_2|u)$, where the sum ranges over all pairs w_1, w_2 of subwords of w such that $w_1 w_2 = w$.
(5) $(vu|w) = \sum_{w_1 w_2 = \pm w} (-1)^{\delta(w_1,w_2)} (v|w_1)(u|w_2)$, where $\delta(w_1, w_2)$ is the number of transpositions needed to obtain the word w from the word $w_1 w_2$.

We give some examples to clarify these rules.

Example 2.3. Let $a \in L$. We have a look at the image of $(a^{(2)}, 12)$ under the biproduct. First we want to use Rule (iv) with $v = 1$, $u = 2$. Due to Rule (i) we only have to consider pairs of subwords of length one. We have $a^{(2)} = \frac{1}{2}aa$ and we have *two* pairs of possible subwords of length one, since we have to distinguish the two a's. Thus we get

$$(a^{(2)}|12) = \frac{1}{2}(aa|12) = \frac{1}{2}((a|1)(a|2) + (a|1)(a|2)) = (a|1)(a|2).$$

Using Rule (v) instead, we compute

$$(a^{(2)}|12) = \frac{1}{2}(aa|12) = \frac{1}{2}((a|1)(a|2) - (a|2)(a|1)) = (a|1)(a|2).$$

More generally, for arbitrary i, j, k, l, we get

$$\left(a_{i,j,k}^{(l)}|i_1 \ldots i_l\right) = (a_{i,j,k}|i_1) \cdots (a_{i,j,k}|i_l).$$

For different letters $a, b \in L$, we compute
$$(ab|12) = (a|1)(b|2) + (b|1)(a|2) = (ba|12).$$

Now for letters $a_1, \ldots, a_n$, we define the *bracket* in $a_1, \ldots, a_n$ to be the element
$$[a_1 \ldots a_n] := (a_1 \cdots a_n | 1 \cdots n)$$
of $\mathrm{Ext}([L|P])$. A *bracket monomial* is a product of brackets and a *bracket polynomial* is a linear combination of bracket monomials. We denote the subalgebra of all bracket polynomials of $\mathrm{Ext}([L|P])$ by $\mathrm{Br}(L)$.

Lemma 2.4 ([10], 'Exchange Lemma', p. 60). *Let u, v, w be words in $\mathrm{Div}(L)$. Then*
$$\sum_{u_1 u_2 = u} [u_1 v]\,[u_2 w] \;=\; (-1)^{n - |w|} \sum_{v_1 v_2 = v} [v_1 u]\,[v_2 w].$$

Proposition 2.5. *All identities among bracket polynomials can be deduced from the identity of Lemma 2.4 with $|u| = 2$, $|w| = n - 1$.*

Proof. The fact that all identities can be deduced from Lemma 2.4 follows directly from Theorem 8 of [10]. Thus - as was stated in [10] on page xv - it can be used for an abstract definition of (skew) brackets. The fact that all those identities stem from the ones with $\mathrm{Length}(u) = 2$ is clear. $\qquad\square$

Remark 2.6. In Proposition 2.5, one can replace '$|u| = 2$, $|w| = n - 1$' with '$|u| = n + 1$, $|w| = 0$'.

Finally, we bring together brackets and invariants of the action of SL_n on W by the following linear map.

Definition 2.7. Let the linear *umbral operator*
$$U : \mathrm{Ext}([L|P]) \to \mathbb{C}[W]$$
$$f \mapsto \langle U, f \rangle$$
be defined by the following:

(i) $\left\langle U, \left(a_{i,j,k}^{(i)} | \iota_1 \ldots \iota_i \right) \right\rangle = T_{i,j,\iota_1,\ldots,\iota_i}$,

(ii) $\left\langle U, \left(a_{i,j,k}^{(l)} | \iota_1 \ldots \iota_l \right) \right\rangle = 0$ $\qquad\qquad\qquad\qquad$ if $l \neq i$,

(iii) $\left\langle U, \prod_{i,j,k} \left(a_{i,j,k}^{(l_{i,j,k})} | \iota_1 \ldots \iota_{l_{i,j,k}} \right) \right\rangle = \prod_{i,j,k} \left\langle U, \left(a_{i,j,k}^{(l_{i,j,k})} | \iota_1 \ldots \iota_{l_{i,j,k}} \right) \right\rangle$,

where in (iii), the order of the letterplaces in the word $\prod_{i,j,k} (a_{i,j,k}^{(l_{i,j,k})} | \iota_1 \ldots \iota_{l_{i,j,k}})$ must be according to the order of the letters $a_{i,j,k}$.

Example 2.8. For arbitrary n and any permutation $\sigma \in S_n$, we have
$$\langle U, [a_{1,1,1} \cdots a_{1,n,1}] \rangle = \left\langle U, \left[a_{1,\sigma(1),1} \cdots a_{1,\sigma(n),1} \right] \right\rangle = \begin{vmatrix} T_{1,1,1} & \cdots & T_{1,n,1} \\ \vdots & \ddots & \vdots \\ T_{1,1,n} & \cdots & T_{1,n,n} \end{vmatrix}$$
as $a_{1,1,1} < \ldots < a_{1,n,1}$.

Theorem 2.9 ([10], Thm. 18). *The umbral operator $U : \mathrm{Ext}([L|P]) \to \mathbb{C}[W]$ is surjective and its restriction to the bracket polynomials $\mathrm{Br}(L)$ is onto $\mathbb{C}[W]^{\mathrm{SL}_n}$.*

We are only interested in bracket polynomials that are not in the kernel of U. Thus in the following, we consider the subalgebra Bra(L) of *appropriate* bracket polynomials, where if a letter $a_{i,j,k}$ turns up in an appropriate bracket monomial, it does so exactly i times. Of course, the restriction of U to Bra(L) is still onto $\mathbb{C}[W]^{\mathrm{SL}_n}$.

3. Brackets and graphs

In this section, we develop the basis of our method: bracket polynomials are associated with formal sums of colored hypergraphs.

Definition 3.1. Let X be a set. Then we denote by $\mathcal{M}(X)$ the set of nonempty multisets composed of elements of X.

Definition 3.2. Let m be a positive integer. An undirected n-regular *colored hypergraph* Γ with m vertices is a pair $\Gamma = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1 < \ldots < v_m\}$ is the ordered set of *vertices* and $\mathcal{E} \in \mathcal{P}(\mathcal{M}(\mathcal{V}) \times \mathbb{N} \times \mathbb{N})$ is the set of colored *hyperedges* $e = (e_1, e_2, e_3)$, and for all vertices v, we have

$$\sum_{e \in E} \#_v(e_1) = n,$$

where $\#_v(e_1)$ is the *number of connections* of e to v, i.e. the number of occurences of the element v in the multiset e_1. By the *virtual degree* of a vertex v, we mean the number

$$\mathrm{vdeg}(v) = n - \sum_{(\{v,\ldots,v\}, e_2, e_3) \in E} |\{v, \ldots, v\}|.$$

By the *effective graph* Γ_{eff} of $\Gamma = (\mathcal{V}, \mathcal{E})$, we denote the subgraph

$$\Gamma_{\mathrm{eff}} = (\mathcal{V}, \mathcal{E} \setminus \{e | e_1 = \{v, \ldots, v\}, v \in \mathcal{V}\}).$$

If $e = (e_1, e_2, e_3)$ is a hyperedge, then we call $k = |e_1|$ the *size*, $e_2 \in \mathbb{N}$ the *color* and $e_3 \in \mathbb{N}$ the *shading* of e. We call e a k-edge. We say that e is connected to v, if $v \in e_1$. If e is connected to only one v, then we call it a *looping edge*.

Observe that multiple edges and loops are allowed in this definition of a hypergraph, so it is truly a *pseudo-hypergraph.*

Definition 3.3. Now let G be the $\mathbb{C}$-vector space of formal sums of $\mathbb{C}$-multiples of n-regular colored hypergraphs. On G, we define a (non-abelian) multiplication as follows. For $\Gamma_1 = (\{v_1 < \ldots < v_m\}, \mathcal{E}_1)$ and $\Gamma_2 = (\{w_1 < \ldots < w_m\}, \mathcal{E}_2)$ in G, we set

$$\Gamma_1 \Gamma_2 := \Gamma_1 \cdot \Gamma_2 := (\{v_1 < \ldots < v_m < w_1 < \ldots < w_m\}, \mathcal{E}_1 \cup \mathcal{E}_2)$$

and extend to formal sums of graphs in the obvious way. This makes G a $\mathbb{C}$-algebra. We call elements $\Upsilon = \sum a_i \Gamma_i \in G$ *graphsums.*

To a hypergraph $\Gamma = (\{v_1 < \ldots < v_m\}, \mathcal{E}_1)$, we associate a bracket monomial $p_\Gamma = b_1 \cdots b_m$ with brackets $b_1, \ldots, b_m$ defined by:

$$b_i := \left[\prod_{e \in \mathcal{E}} a_{|e_1|, e_2, e_3}^{(\#_{v_i}(e_1))} \right].$$

This gives a linear surjective map $\gamma : G \to \mathrm{Bra}(\mathrm{L})$ by setting

$$\gamma : \sum a_i \Gamma_i \mapsto \sum a_i p_{\Gamma_i}.$$

Now we set $\mathcal{G} := G/\ker(\gamma)$ and by $\gamma' : \mathcal{G} \to \mathrm{Bra}(L)$ denote the induced isomorphism.

Example 3.4. We associate the bracket monomial

$$\left[a_{3,1,1}^{(3)} a_{3,4,1} \right] \left[a_{2,2,3}^{(2)} a_{3,4,1} a_{2,1,1} \right] \left[a_{3,4,1} a_{2,1,1} a_{1,7,1} a_{1,2,5} \right]$$

to the color-and-shading-labeled hypergraph

.
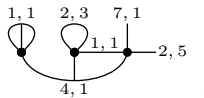
Convention 3.5. We will often speak only of *graphs*, when we mean colored hypergraphs. Moreover, we will not number vertices of graphs, but will assume that they are ordered ascending from left to right. We also ignore shading of edges, assuming that all k-edges of the same color have different shading.

Definition 3.6. We say that two graphsums Υ_1 and Υ_2 are *equivalent*, writing $\Upsilon_1 \simeq \Upsilon_2$, if $\Upsilon_1 - \Upsilon_2 \in \ker(U \circ \gamma)$. We call a graphsum Υ *reducible*, if it is equivalent to zero or to some $\sum a_i \Gamma_i$ with all Γ_i disconnected. A graphsum is irreducible if it is not reducible.

Several graphsums $\Upsilon_1, \ldots, \Upsilon_N$ are called *reducibly independent*, if the only reducible linear combination $\sum a_i \Upsilon_i$ is the trivial one. If for two graphsums Υ_1, Υ_2 the linear combination $\Upsilon_1 - \Upsilon_2$ is reducible, we call them *reducibly equivalent* and write $\Upsilon_1 \simeq_r \Upsilon_2$. We say that a set of reducibly independent irreducible graphs has property (RI).

Remark 3.7. The Exchange Lemma 2.4 leads to equivalencies between graphsums. If two graphsums are equivalent in this way, for any pair of graphs occuring in the two graphsums, there is a color-preserving one-to-one-correspondence between the edges.

Theorem 3.8. *Let M be a maximal set with property (RI). Then M is in one-to-one-correspondence to a minimal generating set of $\mathbb{C}[W]^{\mathrm{SL}_n}$ by $\Gamma \mapsto U \circ \gamma(\Gamma)$.*

Proof. Let M be a maximal set of reducibly independent irreducible graphs in G. Let $F := U \circ \gamma(M)$. Since $U \circ \gamma$ is surjective, for any element f of $\mathbb{C}[W]^{\mathrm{SL}_n}$, we have a graphsum Γ with $U \circ \gamma(\Gamma) = f$. If Γ is irreducible, then either $\Gamma \in M$ and thus $f \in F$, or if $\Gamma \notin M$, due to maximality of M, there is a reducible nontrivial linear combination

$$\Gamma + \sum_{\Gamma' \in M} a_{\Gamma'} \Gamma'.$$

Due to linearity of $U \circ \gamma$, we can proceed with reducible Γ. Either $\Gamma = 0$, then $f = 0$, or Γ is equivalent to a graphsum of disconnected graphs. We can assume all connected subgraphs are irreducible and thus proceed with an irreducible graph, where the number of vertices is strictly less than that of Γ. Since the number of vertices of graphs is bounded from below, this procedure comes to an end. So F generates $\mathbb{C}[W]^{\mathrm{SL}_n}$. The minimality of F follows immediately from M being reducibly independent. $\square$

Lemma 3.9. *Let the graph $\Gamma = (\mathcal{V}, \mathcal{E})$ have a hyperedge $e' = (e'_1, e'_2, e'_3)$ with $v \in e'_1$. Then Γ is equivalent to a graphsum $\sum_i (\mathcal{V}, \mathcal{E}_i)$ where $\mathcal{E}_i \setminus \{e | v \in e_1\} \subseteq \mathcal{E} \setminus \{e | v \in e_1\}$ and $(\{v, ..., v\}, e'_2, e'_3) \in \mathcal{E}_i$ for all i.*

Proof. We can assume that v is the smallest element of $\mathcal{V}$. If $\#_v(e'_1) = |e'_1|$, we are done. Thus take $\#_v(e'_1) = \kappa < |e'_1|$. We can assume that for the second smallest element v' of $\mathcal{V}$ we have $\#'_v(e'_1) = \nu \geq 1$. So

$$\gamma(\Gamma) = \left[a^{(\kappa)}_{|e'_1|, e'_2, e'_3} w_1 \right] \left[a^{(\nu)}_{|e'_1|, e'_2, e'_3} w_2 \right] \mathfrak{m}$$

for some $w_1, w_2 \in L$ and $\mathfrak{m}$ a bracket monomial. Now applying Lemma 2.4 with $u = a^{(\kappa + \nu)}_{|e'_1|, e'_2, e'_3}$, we see that Γ is equivalent to a graphsum $\sum_i (\mathcal{V}, \mathcal{E}_i)$ where $\mathcal{E}_i \setminus \{e | v \in e_1\} \subseteq \mathcal{E} \setminus \{e | v \in e_1\}$ and $(e''_1, e'_2, e'_3) \in \mathcal{E}_i$ with $\#_v(e''_1) = \kappa + \nu$. Iterating this gives the desired result. $\square$

Remark 3.10. When searching for a maximal set of reducibly independent irreducible graphs, Lemma 3.9 can be used to simplify the respective effective graphs.

Proposition 3.11. *Let Γ be an irreducible graph with arbitrary coloring which is not reducibly equivalent to $c\Upsilon_{\det(J)}$ for any $J \subseteq \{1, \ldots, n_k\}$ and $c \in \mathbb{C}$. Let M be the set of graphs of the same form as Γ but with k-edges of only $\binom{n}{k} - 1$ different colors. Then at least one element of M is irreducible. Moreover, if $n_k \geq \binom{n}{k}$ and $J \subseteq \{1, \ldots, n_k\}$ has cardinality $\binom{n}{k}$, then $\det : \oplus_{j \in J} V_{k,j} \to \mathbb{C}$ equals $U \circ \gamma(\Upsilon_{\det(J)})$ for some irreducible graphsum $\Upsilon_{\det(J)}$.*

Proof. This follows directly from Theorem 9.2 of [26]. $\qquad\square$

4. Invariants of SL_4

In the case of SL_4, it turns out that elementary graph theoretical and combinatorial considerations suffice to determine a minimal generating set of $\mathbb{C}[W]^{\mathrm{SL}_4}$. This means that the present section is almost totally self-contained.

**Proposition 4.1.** *The following graphs constitute a maximal set of reducibly independent irreducible graphs for the action of SL_4 on W.*
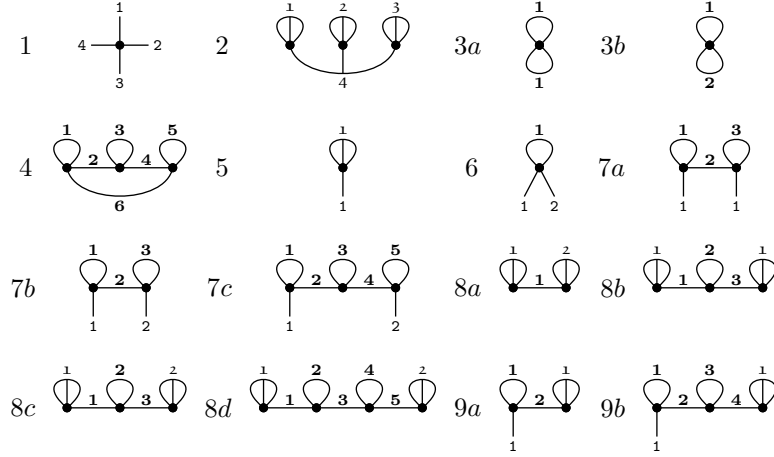
Remark 4.2. According to Proposition 3.11, some multiple of graph no. 4 must be reducibly equivalent to some graphsum Υ with $U \circ \gamma(\Upsilon) = \det$. In fact,

has this property, as it is alternating in the colors, i.e. Υ is not only reducibly, but truly equivalent to $\mathrm{sgn}(\sigma)\Upsilon_\sigma$. The more pleasant display has of course graph no. 4, while for some purposes like for example finding relations, Υ will do better. The corresponding bracket polynomial, as well as those of graphs no. 3a and 3b, turns also up in [22, 29, 36], while no attempt is made there to show that these give a minimal generating set of $\mathbb{C}[W_{(0,n_2,0,0)}]^{\mathrm{SL}_4}$.

Lemma 4.3. *Every irreducible graph for the action of SL_4 on W is reducibly equivalent to a graphsum $\sum a_i \Gamma_i$, where all Γ_i are of the same form. This form is one of the following:*
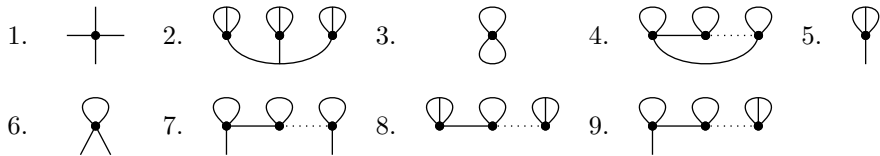
Proof. Let Γ be an irreducible graph. First assume Γ has only 1-edges. Then Γ is connected and thus irreducible only if it has one vertex. We are in Case 1.

Now assume Γ has only 3-edges. Then with Lemma 3.9, it is reducibly equivalent to a sum of graphs Γ_i having only vertices of virtual degree one. Now take one vertex v_1 of an arbitrary Γ_i. There must be a non-looping 3-edge e with one connection to v_1. Since all vertices have virtual degree one, there must be two vertices v_2 and v_3 with $e = \{v_1, v_2, v_3\}$. Thus v_1, v_2 and v_3 together with e and their respective looping 3-edge are a connected component of Γ_i and since Γ_i is connected, we are in Case 2.

If now Γ has only 2-edges and it has only one vertex, we are in Case 3. Assume it has more than one vertex. Again by Lemma 3.9, Γ is reducibly equivalent to a sum of connected graphs Γ_i where for each of them every vertex has one (and only one) looping edge and is thus of virtual degree two. The effective graph of an arbitrary Γ_i must be a *connected simple 2-regular graph*, i.e. a cycle, and we are in Case 4.

We come to the cases where Γ has hyperedges of two different sizes. Let us begin assuming there are 1- and 3-edges. Again using Lemma 3.9, we can move on to some Γ_i with vertices $v_1, \ldots, v_N$ each with a looping 3-edge. If there is an additional vertex v_{N+1} with a connection to a 3-edge, we can move on to a graph having $N+1$ vertices with a looping 3-edge. If there is an additional vertex with no connection to a 3-edge, it constitutes a connected component like in Case 1. Thus we can assume all vertices of Γ_i have a looping 3-edge. Now if any vertex despite for its looping 3-edge has a connection to another 3-edge, we have a connected component like in Case 2. Thus each vertex must have a connection to a 1-edge. Since Γ_i is connected we are in Case 5.

Now let Γ have 1- and 2-edges. We use Lemma 3.9, move on to some Γ_i and can with the same argumentation as in the previous case assume that every vertex has a looping 2-edge. If there is a vertex connected to two 1-edges, we are in Case 6. If not, there are vertices of virtual degree one and two. Thus the effective graph of Γ_i must be a *chain* and we are in Case 7.

The same argumentation goes through if Γ has 2- and 3-edges. By Lemma 3.9, we have vertices of virtual degree one and two and the effective graph of Γ_i must be a chain, giving Case 8.

Finally let Γ have 1-, 2-, and 3-edges. Once more, we have vertices of virtual degree one and two by Lemma 3.9 and the effective graph of Γ_i must be a chain, leading to Case 9. $\qquad\square$

When we say 'Γ is of the form n' in the following, we mean that the graph Γ falls under Case n of Lemma 4.3, while we use the term 'graph no. n' for the colored graphs from Proposition 4.1.

Lemma 4.4. *Let Γ be one of the graphs from Lemma 4.3 with two or more vertices and arbitrary coloring, let σ be a permutation of the colors of the 2-edges and Γ_σ be the graph with permuted colors. Then $\Gamma \simeq_r \operatorname{sgn}(\sigma)\Gamma_\sigma$.*

Proof. We show that $\Gamma \simeq_r -\Gamma_\sigma$, where σ swaps the colors of a looping and a non-looping 2-edge connected with the same vertex. From this the assertion follows immediately.

So let $v \neq v'$ be vertices of Γ and $(\{v, v\}, j_1, k_1)$, $(\{v, v'\}, j_2, k_2)$ the respective two edges. We use Lemma 2.4 with $u = a^{(2)}_{2, j_1, k_1} a^{(2)}_{2, j_2, k_2}$ and get $\Gamma + \Gamma_\sigma \simeq -\Gamma'$, where Γ' has the looping 2-edges $(\{v, v\}, j_1, k_1)$, $(\{v, v\}, j_2, k_2)$ and is thus reducible. So $\Gamma \simeq_r -\Gamma_\sigma$. $\qquad\square$

Proof of Proposition 4.1. For graphs Γ of the forms 1, 2 and 5, the corresponding invariants are 'determinants' and 'dot products', in classical terms 'invariants of systems of vectors and linear forms', see for example [26, p. 254]. Graphs of the form 3 either give the 'Pfaffian' if both edges have the same color, or a variation of it. Graphs Γ of the form 6 neither are disconnected nor $U \circ \gamma(\Gamma) = 0$ unless both 1-edges have the same color.

The remaining types to check for irreducibility are 4, 7, 8, 9. By the duality of V and $\wedge^{n-1} V$, we can reduce form 8 to form 7. Thus 4, 7, 9 remain. In all these cases we require all 2-edges to be of pairwise different color, otherwise Lemma 4.4 with σ swapping two edges of the same color would result in $\Gamma \simeq_r (\Gamma + \Gamma_\sigma)/2$, which is reducible.

Now first we show that if a graph of one of these types has four or more vertices, it is reducible. We observe graphs Γ_i of the forms:

$$\Gamma_4 = \ \cdots \qquad \Gamma_7 = \ \cdots \qquad \Gamma_9 = \ \cdots$$
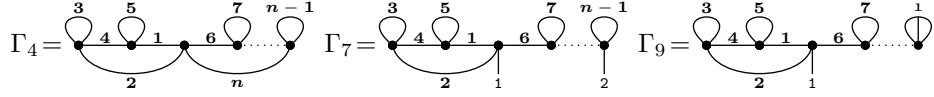
We proceed exemplarily with Γ_4. Applying Lemma 2.4 with u the word corresponding to the 2-edge of color **1**, we get that

$$\Gamma_4 + \ \cdots \ + \ \cdots$$
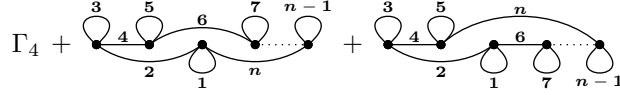
is reducible. Let us call the second and third graph in this sum $\Gamma_{4,1}$ and $\Gamma_{4,2}$ respectively. Since graphs with permuted vertices are equivalent, by swapping colors 1 and 5 as well as 2 and 4, we get $\Gamma_{4,2} \simeq_r \Gamma_{4,1}$, so $\Gamma_4 \simeq_r -2\Gamma_{4,1}$. Applying Lemma 2.4 again, now with u the word corresponding to the 2-edge of color 6, we get that

$$\Gamma_4 + \ \cdots \ + \ \cdots$$
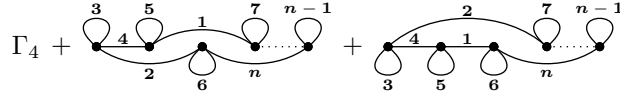
is reducible. Calling the second and third graph in this sum $\Gamma_{4,3}$ and $\Gamma_{4,4}$ respectively, by symmetry reasons, we have $\Gamma_{4,3} \simeq_r \Gamma_{4,4}$ and by swapping the edges of colors 1 and 6 in $\Gamma_{4,3}$ we get $\Gamma_{4,3} \simeq_r -\Gamma_{4,1}$, thus $2\Gamma_{4,1} \simeq_r -2\Gamma_{4,1}$ and $\Gamma_{4,1}$ must be reducible. Exactly the same procedure, namely two times applying Lemma 2.4, one time on the edge of color 1, one time on that of color 6, leads to reducibility of graphs of the forms 4, 7, and 9 with four or more vertices. We observe these with three or less vertices in the following.
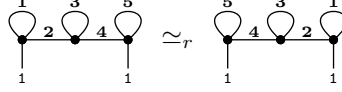
Case 1: Γ is of the form 7. Here Γ has either two or three vertices. We first check these with two vertices. All of the graphs with 2-edges of pairwise different colors **1**, **2**, **3**, and 1-edges of possibly non-different colors 1 and 2 are either reducible or of the form Γ_σ, where σ permutes colors of 2-edges and
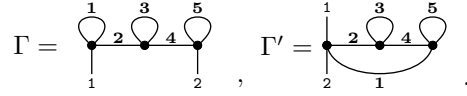
$$\Gamma = \ \cdots \ .$$
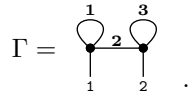
We define a map $\phi : G \to \mathbb{C}$ by setting $\phi(c\Gamma_\sigma) = \mathrm{sgn}(\sigma)c$ and $\phi(\Gamma') = 0$ for graphs Γ' of other forms. Since $\Gamma \notin \ker(U \circ \gamma)$ and all relations from Lemma 2.4 with $|u| = 2$ involving Γ are compatible with ϕ in the sense that $\phi(\Upsilon_1) = \phi(\Upsilon_2)$ if

$\gamma(\Upsilon_1) = \gamma(\Upsilon_2)$ for two graphsums $\Upsilon_1 = \Gamma + \Upsilon_1'$ and Υ_2, ϕ induces a well-defined map $\phi' : G/\ker(U \circ \gamma) \to \mathbb{C}$ and Γ is irreducible.

Now if Γ is of type 7 with three vertices and the two 1-edges are of the same color, we have $U \circ \gamma(\Gamma) = -U \circ \gamma(\Gamma_\tau)$, where τ interchanges the two 1-edges. On the other hand, we have

$$\overset{1\ \ \ 3\ \ \ 5}{\underset{1\ \ \ \ \ \ \ \ \ 1}{\bullet\ {}_{2}\ \bullet\ {}_{4}\ \bullet}} \quad \simeq_r \quad \overset{5\ \ \ 3\ \ \ 1}{\underset{1\ \ \ \ \ \ \ \ \ 1}{\bullet\ {}_{4}\ \bullet\ {}_{2}\ \bullet}}$$

by swapping colors **1** and **5** as well as **2** and **4** of 2-edges and see that Γ is reducible. Now all graphs with three vertices and with 2-edges of pairwise different colors **1**-**5** and 1-edges of different colors **1** and **2** are either reducible or of the form Γ_σ, Γ_σ', where σ permutes 2-edges and

$$\Gamma = \overset{1\ \ \ 3\ \ \ 5}{\underset{1\ \ \ \ \ \ \ \ \ 2}{\bullet\ {}_{2}\ \bullet\ {}_{4}\ \bullet}}\ , \quad \Gamma' = \overset{1\ \ \ 3\ \ \ 5}{\underset{2\ \ \ \ \ \ \ \ \ 1}{\bullet\ {}_{2}\ \bullet\ {}_{4}\ \bullet}}\ .$$
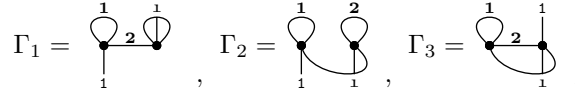
By applying Lemma 2.4 on the 2-edge of Γ' of color **1**, we see that $\Gamma' \simeq_r -2\Gamma$. Similar as we did before, we define a map $\phi : G \to \mathbb{C}$ by setting $\phi(c\Gamma_\sigma) = \mathrm{sgn}(\sigma)c$, $\phi(c\Gamma_\sigma') = -\mathrm{sgn}(\sigma)2c$ and $\phi(\Gamma^*) = 0$ for graphs Γ^* of other forms. As before, ϕ induces a well-defined map $\phi' : G/\ker(U \circ \gamma) \to \mathbb{C}$ and Γ thus is irreducible.

Case 2: Γ is of the form 4. If such a graph has two vertices, it has two edges $(\{v_1, v_2\}, j_1, k_1)$ and $(\{v_1, v_2\}, j_2, k_2)$. Then by Lemma 2.4 with $u = a_{2,j_1,k_1}^{(2)} a_{2,j_2,k_2}^{(2)}$, it is reducible. The case of three vertices remains. All graphs with three vertices and six 2-edges of different colors **1**-**6** are either reducible or of the form Γ_σ, where
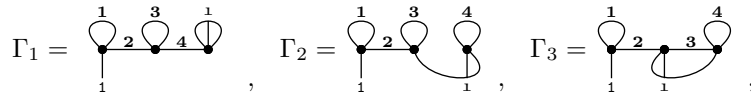
$$\Gamma = \overset{1\ \ \ 3\ \ \ 5}{\underset{6}{\bullet\ {}_{2}\ \bullet\ {}_{4}\ \bullet}}\ .$$
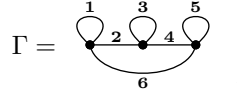
Again the map $\phi : G \to \mathbb{C}$ defined by $\phi(c\Gamma_\sigma) = \mathrm{sgn}(\sigma)c$ and $\phi(\Gamma^*) = 0$ for graphs Γ^* of other forms induces a well-defined map $\phi' : G/\ker(U \circ \gamma) \to \mathbb{C}$ and Γ thus is irreducible.
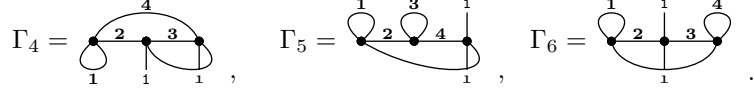
Case 3: Γ is of the form 9. Here Γ can have two or three vertices. We begin with the case of two. All graphs with one 1-edge of color **1**, two 2-edges of colors **1** and **2**, and one 3-edge of color **1** are either reducible or of the form $(\Gamma_i)_\sigma$ with

$$\Gamma_1 = \overset{1\ \ \ 1}{\underset{1}{\bullet\ {}_{2}\ \bullet}}\ , \quad \Gamma_2 = \overset{1\ \ \ \ 2}{\underset{1\ \ \ \ 1}{\bullet\ \ \ \bullet}}\ , \quad \Gamma_3 = \overset{1\ \ \ \ 1}{\underset{1}{\bullet\ {}_{2}\ \bullet}}\ .$$

Applying Lemma 2.4 with u the word corresponding to the 3-edge, we see $\Gamma_3 \simeq_r \Gamma_2 \simeq_r \Gamma_1$. The map $\phi : G \to \mathbb{C}$ defined by $\phi(c(\Gamma_i)_\sigma) = \mathrm{sgn}(\sigma)c$ and $\phi(\Gamma^*) = 0$ for graphs Γ' of other forms induces a well-defined map $\phi' : G/\ker(U \circ \gamma) \to \mathbb{C}$ and Γ_1 thus is irreducible.

We come to those graphs with three vertices. All graphs with one 1-edge of color **1**, four 2-edges of pairwise different colors **1**-**4**, and one 3-edge of color **1** are either reducible or of the form $(\Gamma_i)_\sigma$ with

$$\Gamma_1 = \overset{1\ \ \ 3\ \ \ 1}{\underset{1}{\bullet\ {}_{2}\ \bullet\ {}_{4}\ \bullet}}\ , \quad \Gamma_2 = \overset{1\ \ \ 3\ \ \ 4}{\underset{1\ \ \ \ \ \ \ \ 1}{\bullet\ {}_{2}\ \bullet\ \ \bullet}}\ , \quad \Gamma_3 = \overset{1\ \ \ \ \ \ \ \ 4}{\underset{1\ \ \ \ \ \ 1}{\bullet\ {}_{2}\ \bullet\ {}_{3}\ \bullet}}\ ,$$

$$\Gamma_4 = \qquad , \qquad \Gamma_5 = \qquad , \qquad \Gamma_6 = \qquad .$$

We apply Lemma 2.4 with u the word corresponding to the 3-edge (to two ends of the 3-edge in the case of Γ_6) and get $\Gamma_1 \simeq_r \Gamma_3 \simeq_r -\Gamma_2 \simeq_r -\Gamma_5$ and $\Gamma_6 \simeq_r \Gamma_4 \simeq_r -2\Gamma_1$. The map $\phi : G \to \mathbb{C}$ defined in the usual form induces a well-defined map $\phi' : G/\ker(U \circ \gamma) \to \mathbb{C}$ and Γ_1 thus is irreducible. The proof is complete.

$\square$

Remark 4.5. To show that the graphs with four (five in the case of those of form 8 respectively) or more vertices are reducible, we also could use Proposition 3.11 together with the fact that graphs with two 2-edges of the same color are reducible. We preferred the more self contained version here, because it provides more insight *why* this is so from our combinatorial viewpoint. We want to stress that in our opinion, this reducibility is almost impossible to see without the graph notation, which might explain why Huang in [18] could not reduce her generating set of cycles to a minimal one.

Moreover, to show the irreducibility of the remaining graphs, one can also compute the Hilbert series for small but sufficiently large values of respective n_i's using [6, §4.6] and Xin's algorithm [37] for MacMahon partition analysis. In fact, Xin's algorithm performs very good for such small values in our case.

Proof of Theorem 1.1. Proposition 4.1 provides us with a maximal set of reducibly independent irreducible graphs. The corresponding invariants can be computed according to the rules from Definition 2.7 or by computing the complete contractions given in the introduction. The author used the *DifferentialGeometry* package of MAPLE for these computations.

$\square$

5. INVARIANTS OF SL_5

In order to prove Theorem 1.3, we need some more techniques than the ones we developed for SL_4. The duality of $\wedge^k V$ and $\wedge^{n-k} V$ becomes very important and we introduce a new reducibility notion that is essential (and will be even more in higher dimensions). As Theorem 1.3 does not provide any colorings, we give these exemplarily for $W_{(0,n_2,0,0)}$ in the following.

Proposition 5.1. *The following graphs constitute a maximal set of reducibly independent irreducible graphs for the action of* SL_5 *on* $W_{(0,n_2,0,0)}$*, where in each case* $1 \le iiE1 < \mathbf{2} < \ldots \le n_2$ *are pairwise different colors of the 2-edges. Moreover,*
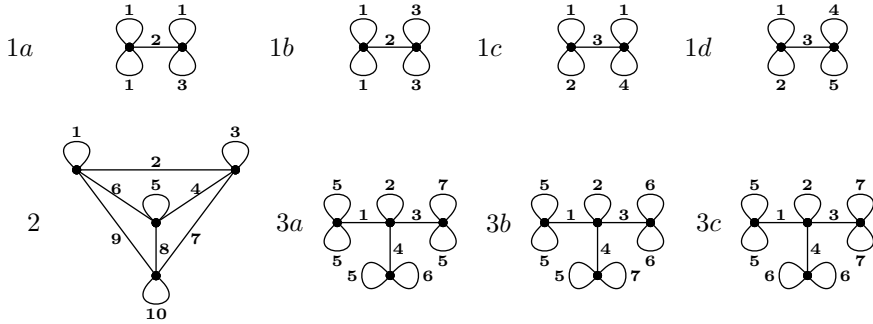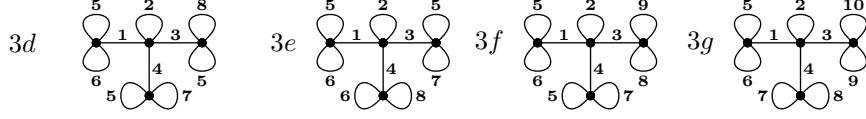
Definition 5.2. We say that a vertex is of type $\mathcal{V}_{k_1 \cdots k_r}^{l_1 \cdots l_s}$, if it has a looping k_i-edge for every $1 \leq i \leq r$ and has one connection to a l_j-edge for every $1 \leq j \leq s$.

Definition 5.3. The *virtual degree type* $d(\Gamma)$ of a graph Γ with k vertices is the descending sequence $(d_1, \ldots, d_k)$ of virtual degrees of vertices of Γ. We define a partial order on the set of graphs for the action of SL$_n$ with k vertices by setting

$$\Gamma < \Gamma' :\Leftrightarrow (d_1, \ldots, d_k) = d(\Gamma) < d(\Gamma') = (d_1', \ldots, d_k')$$
$$:\Leftrightarrow d_1 \leq d_1', \ldots, d_{k-1} \leq d_{k-1}', d_k < d_k'.$$

We call a graphsum $\sum \Gamma_i$ *degree-reducible*, if it is reducibly equivalent either to 0 or to a graphsum $\sum \Gamma_j'$ with $d(\Gamma_j') < d(\Gamma_i)$ for all i, j. Moreover, in analogy to Definition 3.6, we say that graphsums $\Upsilon_1, \ldots, \Upsilon_N$ are *degree-reducibly independent*, if a linear combination $\sum a_i \Upsilon_i$ is degree-reducible only if all a_i are equal to zero. If for two graphsums Υ_1, Υ_2 the linear combination $\Upsilon_1 - \Upsilon_2$ is degree-reducible, we call them *degree-reducibly equivalent* and write $\Upsilon_1 \simeq_d \Upsilon_2$. We say that a set of degree-reducibly independent degree-irreducible graphs has property (*DI*).

Lemma 5.4. *A maximal set with property* (*DI*) *is also a maximal set with property* (*RI*), *i.e. a set of reducibly independent irreducible graphs.*

Proof. Let M be a maximal set with property (*DI*). Of course, M has property (*RI*). Assume M is not maximal with that property. Then there is a graph Γ, so that $M' = M \cup \{\Gamma\}$ still has property (*RI*), but not (*DI*). Thus $\Gamma + \sum a_i \Gamma_i$ is degree-reducible for some $\Gamma_i \in M$. So $\Gamma + \sum a_i \Gamma_i \simeq_r \sum b_j \Gamma_j'$ with reducibly independent irreducible Γ_j' so that $d(\Gamma_j') < d(\Gamma)$ for all j. Since M' has property (*RI*), not all Γ_j' can be elements of M'. Take $\Gamma_k' \notin M'$ and assume $M \cup \{\Gamma_k'\}$ does not have property (*RI*). Since Γ_k' is irreducible, there must be a reducible sum $\sum c_i \Gamma_i + \Gamma_k'$ with not all c_i equal to zero. So we find

$$\Gamma + \sum (a_i + b_i c_i) \Gamma_i \simeq_r \sum_{j \neq k} b_j \Gamma_j'$$

Thus there must be some Γ_l' so that $M'' = M \cup \{\Gamma_k'\}$ has property (*RI*) and $d(\Gamma_k') < d(\Gamma)$. Since $(0, \ldots, 0)$ is a lower bound for the virtual degree type, iterating this procedure gives a contradiction. $\square$

Proof of Proposition 5.1. Due to Lemma 5.4, we only have to consider degree-irreducible graphs. Due to Lemma 3.9, such a graph can be assumed to have two types of vertices: such with one - type $\mathcal{V}_2$ - and such with two looping 2-edges - type $\mathcal{V}_{22}$ - , being of virtual degree three and one respectively. A graph with a multiple edge is not necessarily reducible but degree-reducible, due to
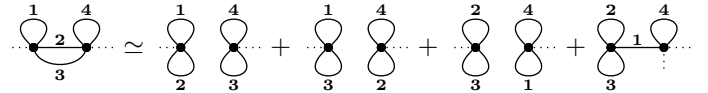
So we can exclude such graphs as well. We call vertices with two looping edges *black holes*, because they 'absorb colors' in the sense that we can not interchange the colors of the two looping edges with other edges' colors in the way we are used to from the SL$_4$-case. Colors can only be extracted if two of the adjacent edges have the same color:

$$\Gamma_1 \;+\; 2\;\Gamma_2 \;\simeq 0$$

On the other hand, if all three adjacent edges have the same color, the graph evaluates to zero under $U \circ \gamma$. So we exclude this case as well and first let Γ have two vertices, then it clearly is of the form 1 from the proposition and we get the relevant colorings by evaluating all other non-equivalent colorings to zero.

Let now Γ have four or more vertices and Γ_2 be the subgraph consisting of vertices of type $\mathcal{V}_2$ and all edges with a connection to one of these vertices. Let $\Gamma_{\sigma,2}$ be the graph Γ with the colors inside Γ_2 permuted by σ. Then similar as in the SL$_4$-case, but now with degree-reducibly equivalence, we get $\Gamma \simeq_d \operatorname{sgn}(\sigma)\Gamma_{\sigma,2}$, due to

$$\cdots + \cdots + \sum \cdots \simeq 0 \;.$$

Thus we can use Proposition 3.11 to conclude that either Γ has four vertices and Γ_2 ten edges or Γ_2 has at most $9 = \binom{5}{2} - 1$ edges. In the first case, Γ_{eff} is the simple cubic connected graph K_4 and we find Graph 2 from the proposition. In the second case, we distinguish between the number of vertices of Γ_2:

Case 1: Γ_2 has one vertex. Here Γ must be of the form:
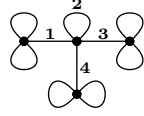
If this graph is reducible for any coloring, it must be reducible for a coloring where the remaining looping edges are of colors $\mathbf{1}, \ldots, \mathbf{9}$. If the two looping edges of one black hole are $\mathbf{1}, \ldots, \mathbf{4}$-colored, the graph is reducible. If one looping edge of a black hole is $\mathbf{1}, \ldots, \mathbf{4}$-colored, say $\mathbf{1}$, and the other $\mathbf{5}, \ldots, \mathbf{9}$-colored, by moving the $\mathbf{1}$-colored edge of Γ_2 to the black hole, this graph is reducibly equivalent to the respective one with two looping edges of color $\mathbf{1}$ at the black hole and one edge of color $\mathbf{5}$ in Γ_2. So by swapping colors $\mathbf{1}$ and $\mathbf{5}$, we can assume that the looping edges of black holes are $\mathbf{5}, \ldots, \mathbf{9}$-colored. If the four looping edges of two black holes are colored with only one color, the graph is reducible by moving one of the colored edges of the first to the second black hole. If two black holes each have colored their looping edges with the same two colors, say $\mathbf{5}$ and $\mathbf{6}$, then we get

$$\cdots \simeq_r -\cdots - \cdots \simeq_r \tfrac{1}{2}\cdots + \tfrac{1}{2}\cdots$$

$$\simeq_r \tfrac{1}{4}\cdots + \tfrac{1}{4}\cdots\;,$$

and any such graph is reducible. The only remaining possible form for a $\mathbf{1}, \ldots, \mathbf{6}$-colored graph is reducible as well:

$$\cdots \simeq -\cdots - \cdots$$

For graphs with more than six colors, we get the reducibly independent possibilities 3*a*-3*g*.

Case 2: Γ_2 has two vertices. Then Γ must have six and is of the form
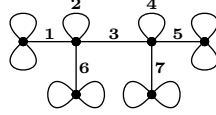
Due to the considerations from the previous case, this graph must be irreducible for some coloring of the remaining edges with colors $\mathbf{8}, \mathbf{9}$. But any such graph is reducible.

Case 3: Γ_2 has three vertices. It either contains more than nine edges, or Γ must be of the form
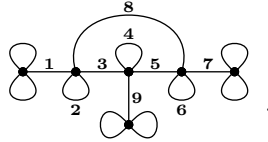
But this graph is reducible for all the remaining edges colored with colors $\mathbf{1}, \ldots, \mathbf{9}$. So it is reducible for any coloring.

Case 4: Γ_2 has four or more vertices. Then it contains more than nine edges, which is a contradiction. The proof is complete. $\square$

Proof of Theorem 1.3. We only consider degree-irreducible graphs and do not list explicit colorings. We also can assume that the number of 2-edges is greater or equal to the number of 3-edges due to the duality of $\wedge^k V$ and $\wedge^{n-k} V$. First of all, we have vertices of the type $\mathcal{V}_{11111}$, these are a connected component. Besides, there are vertices of virtual degree one of the types $\mathcal{V}_4$, $\mathcal{V}_{22}$, $\mathcal{V}_{13}$, and $\mathcal{V}_{112}$. We have vertices of virtual degree two of the types $\mathcal{V}_3$ and $\mathcal{V}_{21}$. Lastly, we have vertices of virtual degree three of the type $\mathcal{V}_2$. We have no multiple 2-edges due to degree-irreducibility.

All possible graphs with one vertex are irreducible for a suitable choice of colors. Those with two either have a non-looping 2-edge and any combination of the types $\mathcal{V}_4$, $\mathcal{V}_{22}$, $\mathcal{V}_{13}$, and $\mathcal{V}_{112}$, or they have two 2-edges and two 3-edges and due to degree-irreducibility, the form of such graph is unique.

So let Γ have three or more vertices. We can assume that there is no non-looping 4-edge, since if there is one and it is only connected to vertices with looping 4-edges, this constitues a connected component with mirror the graph with one vertex of the type $\mathcal{V}_{11111}$, on the other hand, if it is connected to a vertex without a looping 4-edge, we can pull it over to this vertex.

If there is a vertex of type $\mathcal{V}_2$, $\mathcal{V}_{12}$ or $\mathcal{V}_{112}$, then there is no non-looping 3-edge due to degree-irreducibility. All 2-edges but the looping ones of black holes $\mathcal{V}_{22}$ can be permuted and we have $\Gamma_\sigma \simeq_d \text{sgn}(\sigma)\Gamma$ as usual.

Case 1: Γ has a vertex of type $\mathcal{V}_2$. Here in principle, all graphs stem from those from the proof of Proposition 5.1, with three possible modifications. Firstly, vertices of type $\mathcal{V}_{22}$ can be replaced by such of types $\mathcal{V}_4$, $\mathcal{V}_{13}$ or $\mathcal{V}_{112}$. Secondly, arms or cycles can be prolonged by inserting vertices of type $\mathcal{V}_3$ and $\mathcal{V}_{12}$, and lastly, two arms can be connected to a cycle by replacing the two 'end-vertices' with one vertex of type $\mathcal{V}_3$ or $\mathcal{V}_{12}$. The number of 2-vertices here is always bounded by Proposition 3.11. We can assume that vertices of types $\mathcal{V}_3$ and $\mathcal{V}_{12}$ only are on two of three sides of a vertex of type $\mathcal{V}_2$ by the following:
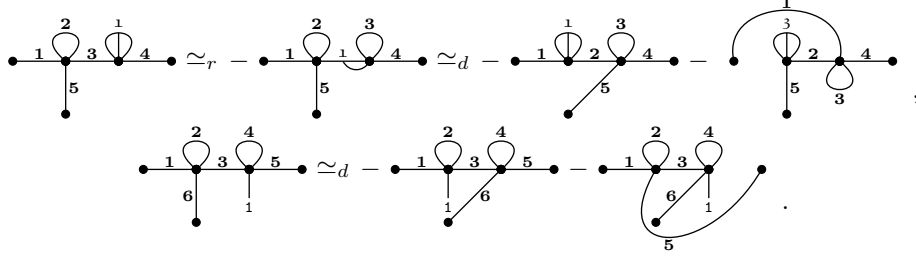
Moreover, a graph with two vertices of types $\mathcal{V}_3$ and $\mathcal{V}_{12}$ joined by a 2-edge and the graph with these two vertices swapped differ degree-reducibly by a graph with an additional vertex of type $\mathcal{V}_2$:
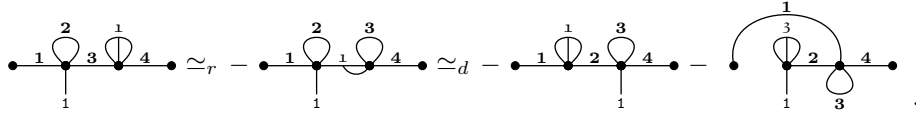
This graph is either reducible or is considered in the list of generators as well, so we can in fact swap two such vertices. This directly leads to the graphs from the Theorem.

Case 2: Γ has no vertex of type $\mathcal{V}_2$ and no non-looping 3-edge. In this case, the only non-looping edges are still those of size two. But now, we only have vertices of virtual degree one and two. Thus we have two types: chains and cycles.

Case 3: Γ has a non-looping 3-edge. We have no vertices of types $\mathcal{V}_2$, $\mathcal{V}_{12}$ or $\mathcal{V}_{112}$. Assume a non-looping 3-edge of Γ has two connections to one vertex, then at this vertex due to degree-irreducibility, there is a looping 3-edge. At the second vertex connected to the non-looping 3-edge, there must be a looping 3-edge as well. Thus this part of the graph must have the looks

If Γ has more than one cycle, the number of 3-edges exceeds the number of 2-edges. So first assume Γ has no cycle. By
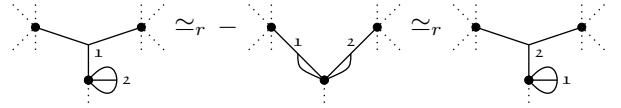
we can swap 3-edges adjacent to a vertex. In fact, we see that if both have the same color, the graph evaluates to zero since changing shadings of two k-edges for odd k results in reversed sign. Moreover, we can assume 2-edges to be only on two sides of a non-looping 3-edge due to
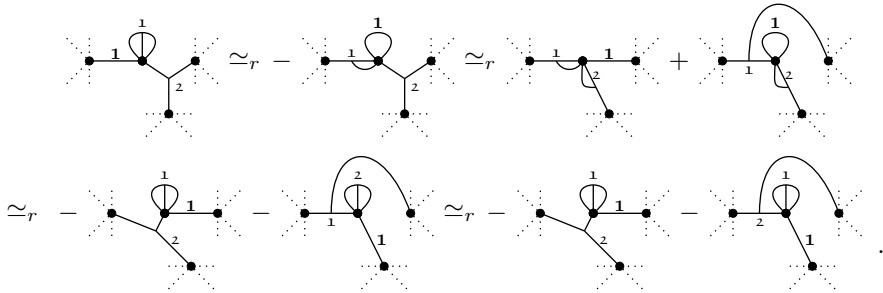
This is the mirrored version of the first equation from Case 1. The mirrored version of vertices $\mathcal{V}_{22}$ are the blocks ⊙ from Theorem 1.3. We conclude that the number of blocks ⊙ plus the number of 3-edges that are not part of a block ⊙ is less than or equal to nine. Only for one non-looping 3-edge of such graph there can be non-looping 3-edges on three sides, i.e. we have a 'star'. Here only three or four non-looping 3-edges are possible. If we have a 'chain', up to five non-looping 3-edges are possible.

Now assume Γ has a cycle (of non-looping 3-edges). Then all vertices of virtual degree one must be of type $\mathcal{V}_{22}$, otherwise the number of 3-edges would exceed the number of 2-edges. The cycle can be made up by two, three, or four 3-edges, where the total number of non-looping 3-edges is smaller or equal to four. We get the remaining graphs from the theorem.

Finally consider the restrictions on numbers of edges. The last one - number of 3-edges less than or equal number of 2-edges - can be made since we are considering the mirror graphs as well. For all graphs with no non-looping 3-edges the first two restrictions are due to the previous observations in Cases 1 and 2, and due to the last restriction as well as the form of the graphs. For graphs with looping 3-edges, the same arguments hold with 2- and 3-edges interchanged. □

6. Relations of SL$_4$

Example 6.1. Consider graphs no. 1 and 2 from Proposition 4.1. By applying the Plücker relation from Lemma 2.4 three times, we can *pull over* the 3-edge of color 4 and get the well known - see [26, p. 255] - relation:

Proof of Theorem 1.5. We identify four somewhat *natural principles* generating relations between invariants. All relations from Theorem 1.5 stem from these principles.

Principle 1a: permuting five 1-edges. This comes from applying the Plücker relation from Lemma 2.4 on five 1-edges *once*. Let Γ be an arbitrary graph, connected to a 1-edge, then

Principle 1b: pulling over a k-edge. If we have a product of graph no. 1 and a second graph with a 2- or 3-edge, we can *pull this k-edge over to graph no. 1 and distribute a total of k 1-edges to the second graph*. There are at most k applications of Lemma 2.4 necessary. In the case of a non-looping 2-edge, we get

where Γ must not be connected. In the case of a looping 2-edge, we have

$$4 \underset{3}{\overset{1}{\rule{0pt}{0pt}}}\!\!\!\! 2 \;\Gamma \;\simeq\; -\tfrac{1}{2} \sum_{(\mathtt{ijk,l})\vdash(\mathtt{1234})} k \,\overset{\mathtt{i}}{\underset{\mathtt{j}}{\rule{0pt}{0pt}}}\, \mathbf{1}\, \overset{\Gamma}{\underset{\mathtt{l}}{\rule{0pt}{0pt}}} \;\simeq\; \sum_{(\mathtt{ij,kl})\vdash(\mathtt{1234})} \overset{\mathbf{1}}{\underset{\mathtt{i\ j\ k\ l}}{\rule{0pt}{0pt}}}\Gamma ,$$

which is fine if Γ is a looping 2-edge. If not, then the vertex connected to the 2-edge of color **1** is connected to another, but non-looping, 2-edge - say of color **2** - and

$$4 \underset{3}{\overset{1}{\rule{0pt}{0pt}}}\!\!\!\! 2 \;\overset{\mathbf{1}}{\underset{\Gamma}{\mathbf{2}}} \;\simeq\; \sum_{(\mathtt{ij,kl})\vdash(\mathtt{1234})} \overset{\mathbf{1}}{\underset{\mathtt{i\ j\ k}}{\rule{0pt}{0pt}}}\, \mathbf{2}\, \Gamma \;\simeq\; - \sum_{(\mathtt{ij,k,l})\vdash(\mathtt{1234})} \overset{\mathbf{1}\ \mathbf{2}}{\underset{\mathtt{i\ j\ k}}{\rule{0pt}{0pt}}}\Gamma$$

holds. Now in the case of a 3-edge *pulled over* to graph no. 1, Example 6.1 shows what happens for the non-looping 3-edge of graph no. 2. Since we can interchange the edges of graph no. 2 as we want by a simple application of Lemma 2.4, we get the same relation for one of the looping 3-edges. Thus we can exclude graph no. 2 in the following, which means that we only have to consider graphs with a looping 3-edge on a vertex connected to a 2-edge (of color **1**). Here we get

$$4 \underset{3}{\overset{1}{\rule{0pt}{0pt}}}\!\!\!\! 2 \;\overset{\mathbf{1}}{\rule{0pt}{0pt}}\Gamma \;\simeq\; -\tfrac{1}{3} \sum_{(\mathtt{ijk,l})\vdash(\mathtt{1234})} k \,\overset{\mathtt{i}}{\underset{\mathtt{j}}{\rule{0pt}{0pt}}}\,\overset{\mathtt{1}}{\underset{\mathtt{l}}{\rule{0pt}{0pt}}}\,\mathbf{1}\,\Gamma$$

$$\simeq\; - \sum_{(\mathtt{i,jkl})\vdash(\mathtt{1234})} \overset{\mathtt{1}}{\underset{\mathtt{i}}{\rule{0pt}{0pt}}}\; j\,\overset{\mathtt{k}}{\underset{\mathtt{l}}{\rule{0pt}{0pt}}}\,\mathbf{1}\,\Gamma \;\simeq\; \sum_{(\mathtt{i,jk,l})\vdash(\mathtt{1234})} \overset{\mathtt{1}}{\underset{\mathtt{i}}{\rule{0pt}{0pt}}}\,\overset{\mathbf{1}}{\underset{\mathtt{j\ k}}{\rule{0pt}{0pt}}}\,\overset{\Gamma}{\underset{\mathtt{l}}{\rule{0pt}{0pt}}} .$$

Observe that Principle 1a can be seen as a special case of Principle 1b, namely pulling over a 1-edge.

Principle 2: bringing together two 1-edges (of two different graphs). This only works if one of the 1-edges is connected to a vertex with one looping 2-edge and one non-looping 2-edge. So any combination of graphs no. 1, 5, and 6 gives no or no new relation. If one of the graphs is graph no. 1, we can reduce to Principle 1 or 2. So we exclude this as well and get

$$\overset{\Gamma_1}{\underset{1}{\rule{0pt}{0pt}}}\,\overset{\mathbf{1}}{\underset{2}{\rule{0pt}{0pt}}}\,\mathbf{2}\,\Gamma_2 \;+\; \overset{\Gamma_1}{\underset{2}{\rule{0pt}{0pt}}}\,\overset{\mathbf{1}}{\underset{1}{\rule{0pt}{0pt}}}\,\mathbf{2}\,\Gamma_2 \;+\; \overset{\mathbf{1}}{\underset{1\ 2}{\rule{0pt}{0pt}}}\,\Gamma_1\,\mathbf{2}\,\Gamma_2 \;\simeq\; -\,\Gamma_1\,\overset{1}{\underset{2}{\rule{0pt}{0pt}}}\,\mathbf{2}\,\Gamma_2$$

$$\simeq\; \overset{\Gamma_2}{\underset{1}{\rule{0pt}{0pt}}}\,\overset{\mathbf{2}}{\underset{2}{\rule{0pt}{0pt}}}\,\mathbf{1}\,\Gamma_1 \;+\; \overset{\Gamma_2}{\underset{2}{\rule{0pt}{0pt}}}\,\overset{\mathbf{2}}{\underset{1}{\rule{0pt}{0pt}}}\,\mathbf{1}\,\Gamma_1 \;+\; \overset{\mathbf{2}}{\underset{1\ 2}{\rule{0pt}{0pt}}}\,\Gamma_1\,\mathbf{1}\,\Gamma_2$$

$$\simeq\; \overset{\Gamma_2}{\underset{1}{\rule{0pt}{0pt}}}\,\overset{\mathbf{1}}{\underset{2}{\rule{0pt}{0pt}}}\,\mathbf{2}\,\Gamma_1 \;+\; \overset{\Gamma_2}{\underset{2}{\rule{0pt}{0pt}}}\,\overset{\mathbf{1}}{\underset{1}{\rule{0pt}{0pt}}}\,\mathbf{2}\,\Gamma_1 \;+\; \overset{\mathbf{2}}{\underset{1\ 2}{\rule{0pt}{0pt}}}\,\Gamma_1\,\mathbf{1}\,\Gamma_2 \;+\; \overset{\mathbf{1}}{\underset{\mathbf{2}\ 2}{\rule{0pt}{0pt}}}\,\Gamma_1\,\Gamma_2 \;+\; \overset{\mathbf{1}}{\underset{\mathbf{2}\ 1\ 2}{\rule{0pt}{0pt}}}\,\Gamma_1\,\Gamma_2 .$$

Principle 3: bringing together three 1-edges. Here as well one of the 1-edges must be connected to a vertex with a looping and a non-looping 2-edge and none of the graphs must be no. 1. We have

$$\sum_{(\mathtt{i,j,k})\vdash(\mathtt{123})} \overset{\Gamma_1\ \Gamma_2}{\underset{\mathtt{i\ j}}{\rule{0pt}{0pt}}}\,\overset{\mathbf{1}}{\underset{\mathtt{k}}{\rule{0pt}{0pt}}}\,\mathbf{2}\,\Gamma_3 \;+\; \sum_{(\mathtt{i,jk})\vdash(\mathtt{123})} \overset{\Gamma_1}{\underset{\mathtt{i}}{\rule{0pt}{0pt}}}\; \overset{\mathbf{1}}{\underset{\mathtt{j\ k}}{\rule{0pt}{0pt}}}\,\Gamma_2\,\mathbf{2}\,\Gamma_3$$

$$\simeq\; - \sum_{(\mathtt{i,jk})\vdash(\mathtt{123})} \overset{\Gamma_1\ \Gamma_2}{\underset{\mathtt{i}}{\rule{0pt}{0pt}}}\; j\,\overset{\mathtt{j}}{\underset{\mathtt{k}}{\rule{0pt}{0pt}}}\,\mathbf{1}\,\mathbf{2}\,\Gamma_3 \;\simeq\; \left(j\,\overset{\mathtt{i}}{\underset{\mathtt{k}}{\rule{0pt}{0pt}}}\,\mathbf{2}\,\Gamma_3\,\Gamma_1\,\Gamma_2\,\mathbf{1} \;+\; j\,\overset{\mathtt{i}}{\underset{\mathtt{k}}{\rule{0pt}{0pt}}}\,\mathbf{1}\,\Gamma_2\,\Gamma_1\,\Gamma_3\,\mathbf{2} \right)$$

$$\simeq\ -\sum_{(\mathtt{i},\mathtt{jk})\vdash(\mathbf{123})}\left(\ \cdots\ +\ \cdots\ \right).$$

Principle 4a: going around circular graphs. We get the second identity in the following by going around the circular graph with four vertices first with the 2-edge of color **1** and then with the 2-edge of color **2**:

$$\cdots\ +\ \cdots\ \simeq\ -\ \cdots$$

$$\simeq\ \sum_{(i,jklmn)\vdash(\mathbf{345678})}\left((-1)^{i+1}\ \cdots\right)+\ \cdots$$

$$+\ \sum_{(i,jklmn)\vdash(\mathbf{345678})}\left((-1)^{i+1}\ \cdots\right)+\ \cdots\ .$$

Thus we get

$$2\ \cdots$$

$$\simeq\ \sum_{(i,jklmn)\vdash(\mathbf{345678})}(-1)^{i+1}\left(\ \cdots\ +\ \cdots\right.$$

$$\left.+\ \cdots\ +\ \cdots\right)$$

$$\simeq\ \sum_{(g,h)\vdash(\mathbf{12})}\ \sum_{(i,jklmn)\vdash(\mathbf{345678})}(-1)^{i+1}\ \cdots\left(\ \cdots\ -\ \cdots\right)$$

$$+\ \sum_{(i,jk,l)\vdash(\mathbf{3456})}\operatorname{sgn}(\vdash)\left(\ \cdots\ +\ \cdots\right).$$

Principle 4b: determinantal relations. Consider the matrices

$$B:=\begin{pmatrix} y_{j_1}34 & y_{j_2}34 & y_{j_3}34 & y_{j_4}34 & y_{j_5}34 & y_{j_6}34 \\ -y_{j_1}24 & -y_{j_2}24 & -y_{j_3}24 & -y_{j_4}24 & -y_{j_5}24 & -y_{j_6}24 \\ y_{j_1}23 & y_{j_2}23 & y_{j_3}23 & y_{j_4}23 & y_{j_5}23 & y_{j_6}23 \\ y_{j_1}14 & y_{j_2}14 & y_{j_3}14 & y_{j_4}14 & y_{j_5}14 & y_{j_6}14 \\ -y_{j_1}13 & -y_{j_2}13 & -y_{j_3}13 & -y_{j_4}13 & -y_{j_5}13 & -y_{j_6}13 \\ y_{j_1}12 & y_{j_2}12 & y_{j_3}12 & y_{j_4}12 & y_{j_5}12 & y_{j_6}12 \end{pmatrix},$$

$$A:=\begin{pmatrix} y_{i_1}12 & y_{i_2}12 & y_{i_3}12 & y_{i_4}12 & y_{i_5}12 & y_{i_6}12 \\ y_{i_1}13 & y_{i_2}13 & y_{i_3}13 & y_{i_4}13 & y_{i_5}13 & y_{i_6}13 \\ y_{i_1}14 & y_{i_2}14 & y_{i_3}14 & y_{i_4}14 & y_{i_5}14 & y_{i_6}14 \\ y_{i_1}23 & y_{i_2}23 & y_{i_3}23 & y_{i_4}23 & y_{i_5}23 & y_{i_6}23 \\ y_{i_1}24 & y_{i_2}24 & y_{i_3}24 & y_{i_4}24 & y_{i_5}24 & y_{i_6}24 \\ y_{i_1}34 & y_{i_2}34 & y_{i_3}34 & y_{i_4}34 & y_{i_5}34 & y_{i_6}34 \end{pmatrix},\quad C:=\begin{pmatrix} |i_1 j_1| & \cdots & |i_1 j_6| \\ \vdots & \ddots & \vdots \\ |i_6 j_1| & \cdots & |i_6 j_6| \end{pmatrix}.$$

We have $\det(A) = |\boldsymbol{i_1}\cdots\boldsymbol{i_6}|$, $\det(B) = -|\boldsymbol{j_1}\cdots\boldsymbol{j_6}|$ and $A^{\mathrm{T}}B = C$. Thus

$$\det(C) + |\boldsymbol{i_1}\cdots\boldsymbol{i_6}||\boldsymbol{i_1}\cdots\boldsymbol{i_6}| = 0$$

holds. Moreover, we have the standard Plücker identity for determinants of matrices of the form A:

$$\sum_{(i_1\cdots i_6, j_1) \vdash (k_1\ldots k_7)} |\boldsymbol{i_1}\cdots\boldsymbol{i_6}||\boldsymbol{j_1}\cdots\boldsymbol{j_6}| = 0.$$

Both of the above identities could also be achieved via 'going around circular graphs', which turns out to be a lot harder as this approach. Since we found all relations from Theorem 1.5, the proof is complete. $\qquad\square$

7. An outlook

In this section, we want to give a short compendium of possible further applications of our method. First of all, at least for small n, some $n_i = 0$ and aid of computers, determination of generating sets for $\mathbb{C}[W]^{\mathrm{SL}_n}$ seems to be possible.

On the other hand, as Section 6 shows, at least our method provides some intuitive processes to generate relations, while showing (in general) that these generate the ideal of relations requires more and possibly totally different considerations.

Of course our method is not restricted to antisymmetric tensors. As we mentioned in the introduction, related methods have been applied to binary forms. On the other hand, the symbolic method elaborated by Grosshans, Rota and Stein is able to deal with combinations of symmetric and antisymmetric tensors, so it seems likely to apply our graph method to such combinations. In this case, edges of different 'behaviour' would correspond to symmetric or antisymmetric tensors respectively. The presumably easiest nontrivial case would be that of SL_3 acting on symmetric and antisymmetric 2-tensors.

Another direction for generalization is that of changing the acting group. The classical groups SO_n and Sp_n for example have a *principal tensor* g besides det. This tensor is an (anti-)symmetric bilinear form, see [26, §9.5]. It could be represented by vertices of degree two that now behave differently than the vertices of degree n corresponding to det.

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