

RANDOM WALKS, WPD ACTIONS, AND THE CREMONA GROUP

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ABSTRACT. We study random walks on the Cremona group. We show that almost surely the dynamical degree of a sequence of random Cremona transformations grows exponentially fast, and a random walk produces infinitely many different normal subgroups with probability 1. Moreover, we study the structure of such random subgroups.

We prove these results in general for groups of isometries of (non-proper) hyperbolic spaces which possess at least one WPD element. As another application, we answer a question of D. Margalit showing that a random normal subgroup of the mapping class group is free.

1. INTRODUCTION

The *Cremona group* is the group $G = \text{Bir } \mathbb{P}^2(\mathbb{C})$ of birational transformations of the projective plane. Its study has been initiated by E. De Jonquières, L. Cremona, and M. Noether in the 1800's (see [Fav08] for a survey), and a great deal of progress has been obtained in the last decade. In particular, Cantat and Lamy [CL13] proved a conjecture of Mumford, showing that the Cremona group is not simple. In fact, they produced infinitely many different normal subgroups.

A technique to produce many examples of a mathematical structure is to use probability; indeed, even if it is hard to construct an explicit example, it may be simpler to show that *almost all* objects satisfy the desired property (a famous example is *expander graphs*, see e.g. [Lub94], Section 1.2).

In this paper, we prove the following strengthening of [CL13] by looking at random walks. To define a random walk, let us fix a probability measure μ on the Cremona group, with countable support. Let us denote as Γ_μ the semigroup generated by the support of μ . Then let us draw a sequence (g_n) of elements independently with distribution μ , and consider the random product

$$w_n := g_1 g_2 \dots g_n.$$

We prove the following.

Theorem 1.1. *Let μ be a probability measure on the Cremona group $G = \text{Bir } \mathbb{P}^2(\mathbb{C})$ so that Γ_μ is a primitive subgroup which contains a WPD element. For any sample path $\omega = (w_n)$, consider the normal closure $N_n(\omega) := \langle\langle w_n \rangle\rangle$. Then we have:*

- (1) *for almost every sample path ω , the sequence*

$$(N_1(\omega), N_2(\omega), \dots, N_n(\omega), \dots)$$

contains infinitely many different normal subgroups of $\text{Bir } \mathbb{P}^2(\mathbb{C})$.

- (2) *Let the injectivity radius of a subgroup $H < G$ be defined as*

$$\text{inj}(H) := \inf_{f \in H \setminus \{1\}} \deg f.$$

Then, for any $R > 0$ the probability that $\text{inj}(N_n) \geq R$ tends to 1 as $n \rightarrow \infty$;
 (3) The probability that the normal closure $\langle\langle w_n \rangle\rangle$ of w_n in G is free satisfies

$$\mathbb{P}(\langle\langle w_n \rangle\rangle \text{ is free}) \rightarrow 1$$

as $n \rightarrow \infty$.

We will in fact provide estimates on the rate of convergence in (3) (see Theorem 1.5), and discuss the non-primitive case in detail. Let us now introduce some definitions.

1.1. The dynamical degree. Let $f : \mathbb{P}^2(\mathbb{C}) \rightarrow \mathbb{P}^2(\mathbb{C})$ be a birational map. Then f is given in homogeneous coordinates by

$$f([x : y : z]) := [P : Q : R]$$

where P, Q, R are polynomials of degree d without common factors. We call d the *degree* of f , and we denote it as $\deg f$.

Now, one notes that $\deg(f^{n+m}) \leq \deg(f^n) \cdot \deg(f^m)$, but the equality need not hold: the most famous example is the *Cremona involution*

$$g([x : y : z]) := [yz : xz : xy]$$

which has degree 2, but g^2 is the identity; the Cremona group is in fact generated by degree 1 transformations and the Cremona involution. Hence, following [Fri95], [RS97] we define the *dynamical degree* of f as

$$\lambda(f) := \lim_{n \rightarrow \infty} (\deg f^n)^{1/n}.$$

The dynamical degree is always an algebraic integer [DF01], and it is related to the topological entropy by $h_{\text{top}}(f) \leq \log \lambda(f)$. In fact, equality is conjectured [Fri95].

The Cremona group acts by isometries on an infinite dimensional hyperbolic space $\mathbb{H}_{\mathbb{P}^2}$ which is contained in the *Picard-Manin space* (see Section 3). Thus, Cremona transformations can be classified as *elliptic*, *parabolic*, or *loxodromic* ([DF01] [Can11]). In particular, a Cremona transformation is *loxodromic* if $\lambda(f) > 1$, and we say it is *WPD* if it is loxodromic and not conjugate to a monomial transformation. A subgroup $\Gamma < G$ is *primitive* if no non-trivial element of Γ fixes the limit set $\Lambda(\Gamma) \subseteq \partial\mathbb{H}_{\mathbb{P}^2}$ pointwise. There are many such subgroups (see Remark 1.7).

A measure μ on the Cremona group has *finite first moment* if $\int \log \deg f \, d\mu(f) < +\infty$, and is *bounded* if there exists $D < +\infty$ such that $\deg f \leq D$ for any $f \in \text{supp}(\mu)$. Moreover, it is *non-elementary* if Γ_μ contains two loxodromic elements with disjoint fixed sets.

We prove that the degree and dynamical degree of a random Cremona transformation grow exponentially fast.

Theorem 1.2. *Let μ be a countable non-elementary probability measure on the Cremona group with finite first moment. Then there exists $L > 0$ such that for almost every random product $w_n = g_1 \dots g_n$ of elements of the Cremona group we have the limit*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \deg(w_n) = L.$$

Moreover, if μ is bounded then for almost every sample path we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \lambda(w_n) = L.$$

Moreover, we obtain the following characterization of the Poisson boundary (see Section 1.4).

Theorem 1.3. *Let μ be a non-elementary probability measure on the Cremona group with finite entropy and finite logarithmic moment, and suppose that Γ_μ contains a WPD element. Then the Gromov boundary of the hyperboloid $\mathbb{H}_{\mathbb{P}^2}$ with the hitting measure is a model for the Poisson boundary of (G, μ) .*

Note that for simplicity we deal with the Cremona group over \mathbb{C} , but Theorems 1.2, 1.1, and 1.3 are still true (and with the same proofs) for the Cremona group over any algebraically closed field k .

1.2. General setup. WPD actions. We will actually prove our results on the Cremona group under the more general framework of groups of isometries of non-proper hyperbolic spaces.

Recall that a metric space (X, d) is δ -hyperbolic if geodesic triangles are δ -thin, and is *proper* if closed balls are compact. Let us consider a group G acting by isometries on X .

If the group G is not hyperbolic, then it cannot admit a proper, cocompact action on a hyperbolic metric space, but there are many interesting actions on *non-proper* hyperbolic metric spaces. Notable examples include *relatively hyperbolic groups* which act on the coned-off Cayley graph ([Far98], [Osi16]); *right-angled Artin groups*, acting on the extension graph [KK14]; the *mapping class group* of a surface, which acts on the *curve complex* ([MM99], [Bow08]); and the group $\text{Out}(F_n)$ of outer automorphisms of the free group ([BF14], [HM13]).

Recall that a δ -hyperbolic space X is equipped with the *Gromov boundary* ∂X given by asymptote classes of quasigeodesic rays. Under mild conditions on μ , we proved in [MT18] that almost every sample path $(w_n x)$ converges to a point on the boundary ∂X , and that the random walk has positive drift.

Since the spaces on which G acts are not proper, some weak notion of properness is still needed in order to be able to extract information on the group from the action, and several candidate notions have been proposed in the last two decades.

First of all, following [Sel97], [Bow08], [Osi16], the action of a group G on X is *acylindrical* if for any two points x, y in X which are sufficiently far apart, the set of group elements which coarsely fixes both x and y has bounded cardinality. More precisely, given a constant $K \geq 0$, we define the *joint coarse stabilizer* of x and y as

$$\text{Stab}_K(x, y) := \{g \in G : d(x, gx) \leq K \text{ and } d(y, gy) \leq K\}.$$

Then the action of G on X is acylindrical if for any $K \geq 0$, there are constants $R(K)$ and $N(K)$ such that for all points x and y in X with $d(x, y) \geq R(K)$, we have the following bound (where $\#|A|$ is the cardinality of A):

$$\#\text{Stab}_K(x, y) \leq N(K). \tag{1}$$

This condition is quite useful, and it is verified in certain important cases (e.g. the action of the mapping class group on the curve complex [Bow08], or the action of a RAAG on its extension graph [KK14]).

However, there are several interesting actions of groups on hyperbolic spaces which are not acylindrical; in particular, certain actions of $\text{Out}(F_n)$ and of the Cremona group. For this reason, in this paper we will consider group actions which satisfy the *weak proper discontinuity (WPD)* property, a weaker notion introduced

by Bestvina and Fujiwara [BF02] in the context of mapping class groups. Intuitively, an element is WPD if it acts properly on its axis. In formulas, an element $g \in G$ is *WPD* if for any $x \in X$ and any $K \geq 0$ there exists $N > 0$ such that

$$\#\text{Stab}_K(x, g^N x) < +\infty. \quad (2)$$

In other words, the finiteness condition is not required of all pairs of points in the space, but only of points along the axis of a given loxodromic element.

Let μ be a probability measure on the group G . We say that μ is *countable* if the support of μ is countable, and we denote as Γ_μ the semigroup generated by the support of μ . In this paper we show that as long as the semigroup Γ_μ contains *at least one* WPD element, then generic elements have all the properness properties one could wish for. In particular, one can identify the Poisson boundary and study the normal closure of random elements. As an application, we will use this condition to derive results on the Cremona group.

Note that the action of the Cremona group on the infinite-dimensional hyperbolic space is not acylindrical, but WPD elements actually exist: in particular, by Shepherd-Barron [SB13], a loxodromic map is WPD if and only if it is not conjugate to a monomial map (see also [Ure20]). Moreover, by ([Lon16], Proposition 4), for each $n \geq 2$, the transformation given in affine coordinates by $(x, y) \mapsto (y, y^n - x)$ is WPD.

A related notion to WPD is the notion of *tight* element from [CL13]. In fact, in order to produce new normal subgroups, Cantat and Lamy take the normal closure of tight elements. Let us note that in the Cremona group, centralizers of loxodromic elements are virtually cyclic; as a consequence, if an element is tight then it is also WPD.

1.3. Normal closure. Let us now formulate our theorem on the normal closure for general WPD actions on a hyperbolic space. In order to state the theorem, we need some assumptions. We call a measure μ *reversible* if the semigroup Γ_μ generated by the support of μ is indeed a group. This condition is satisfied e.g. when the support of μ is closed under taking inverses. A measure μ on G is *admissible* with respect to an action on X if it is countable, non-elementary, reversible, bounded, and WPD.

Given a subgroup $H < G$, we define its *injectivity radius* as

$$\text{inj}(H) := \inf_{\substack{g \in H \setminus \{1\} \\ x \in X}} d(x, gx).$$

We prove that the injectivity radius of the normal closure of a random element is almost surely unbounded, and taking the normal closure of random elements yields many different normal subgroups.

To be precise, let us denote as $\Lambda_\mu \subseteq \partial X$ the limit set of the group Γ_μ , and $E_\mu := \{g \in G : gx = x \text{ for all } x \in \Lambda_\mu\}$ the pointwise stabilizer of Λ_μ . Note that if $G = \Gamma_\mu$, then $E_\mu = E(G)$ is the maximal finite normal subgroup of G (i.e., the largest finite subgroup of G which is normal: that such a subgroup exists is a consequence of the WPD property).

Since E_μ is normal in Γ_μ , conjugacy yields a homomorphism

$$\Gamma_\mu \rightarrow \text{Aut } E_\mu.$$

Let us denote as H_μ the image of Γ_μ in $\text{Aut } E_\mu$. Then the *characteristic index* $k(\mu)$ of μ is the cardinality of H_μ .

Theorem 1.4. *(Abundance of normal subgroups.) Let G be a group acting on a Gromov hyperbolic space X , and let μ be an admissible probability measure on G . Let $k = k(\mu)$ be the characteristic index of μ . Then, if we consider the normal closure $N_n(\omega) := \langle\langle w_n^k \rangle\rangle$, we have:*

- (1) *for any $R > 0$, the probability that $\text{inj}(N_n) \geq R$ tends to 1 as $n \rightarrow \infty$;*
- (2) *for almost every sample path ω , the sequence*

$$(N_1(\omega), N_2(\omega), \dots, N_n(\omega), \dots)$$

contains infinitely many different normal subgroups of G .

The characteristic index also determines the structure of the normal closure of a random element, in particular whether it is free.

Theorem 1.5. *(Structure of the normal closure.) Let G be a group acting on a Gromov hyperbolic space X , and let μ be an admissible probability measure on G with characteristic index $k(\mu)$. Then:*

- (1) *the probability that the normal closure $\langle\langle w_n \rangle\rangle$ of w_n in G is free satisfies*

$$\mathbb{P}(\langle\langle w_n \rangle\rangle \text{ is free}) \rightarrow \frac{1}{k(\mu)}$$

as $n \rightarrow \infty$.

- (2) *Moreover, if $k = k(\mu)$, then*

$$\mathbb{P}(\langle\langle w_n^k \rangle\rangle \text{ is free}) \rightarrow 1$$

as $n \rightarrow \infty$, and indeed there exist constant $B > 0, c < 1$ such that

$$\mathbb{P}(\langle\langle w_n^k \rangle\rangle \text{ is free}) \geq 1 - Bc^n$$

for any n .

Moreover, as a corollary of Theorem 1.5, the probability that the normal closure of a random element is free detects the following algebraic property of the group:

Corollary 1.6. *Let G be a group acting on a Gromov hyperbolic space X , and let μ be an admissible probability measure on G . If $\Gamma_\mu = G$, then*

$$\mathbb{P}(\langle\langle w_n \rangle\rangle \text{ is free}) \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

if and only if the maximal finite normal subgroup $E(G)$ equals the center $Z(G)$.

In particular, we will show later that this is the case for mapping class groups.

Remark 1.7. Let us note that it is not hard (e.g. in the Cremona group) to choose a measure μ such that Γ_μ is primitive, i.e. $k(\mu) = 1$. Indeed, let f be a loxodromic WPD element. Let us now pick $g \notin E^+(f) = \text{Stab}_G(\text{Fix}(f))$. Then $E := E^+(f) \cap E^+(gfg^{-1})$ is a finite group. For each $g_i \in E$, the set $\text{Fix}(g_i)$ of fixed points of g_i on the boundary of $\mathbb{H}_{\mathbb{P}^2}$ has codimension at least 1 in $\partial\mathbb{H}_{\mathbb{P}^2}$. Now, pick a loxodromic h such that $\text{Fix}(h) \cap \cup_{i=1}^r \text{Fix}(g_i) = \emptyset$. Then the group $\Gamma := \langle f, g, h \rangle$ is primitive.

1.4. The Poisson boundary. The well-known *Poisson representation formula* expresses a duality between bounded harmonic functions on the unit disk and bounded functions on its boundary circle. Indeed, bounded harmonic functions admit radial limit values almost surely, while integrating a boundary function against the Poisson kernel gives a harmonic function on the interior of the disk.

This picture is intimately connected with the geometry of $SL_2(\mathbb{R})$; then in the 1960's Furstenberg and others extended this duality to more general groups. In particular, let G be a countable group of isometries of a Riemannian manifold X , and let us consider a probability measure μ on G . One defines μ -harmonic functions as functions on G which satisfy the mean value property with respect to averaging using μ ; in formulas $f : G \rightarrow \mathbb{R}$ is μ -harmonic if

$$f(g) = \sum_{h \in G} f(gh) \mu(h) \quad \forall g \in G.$$

Following Furstenberg [Fur63], a measure space (M, ν) on which G acts is then a boundary if there is a duality between bounded, μ -harmonic functions on G and L^∞ functions on M .

A related way to interpret this duality is by looking at random walks on G . In many situations, (e.g. when X is hyperbolic) the space X is equipped naturally with a topological boundary ∂X , and almost every sample path $(w_n x)$ converges to some point on the boundary of X . Hence, one can define the *hitting measure* of the random walk as the measure ν on ∂X given on a subset $A \subseteq \partial X$ by

$$\nu(A) := \mathbb{P} \left(\lim_{n \rightarrow \infty} w_n x \in A \right).$$

A fundamental question in the field is then whether the pair $(\partial X, \nu)$ equals indeed the Poisson boundary of the random walk (G, μ) , i.e. if all harmonic functions on G can be obtained by integrating a bounded, measurable function on ∂X .

In the proper case, the classical criteria in order to identify the Poisson boundary can be applied and one gets that the Gromov boundary $(\partial X, \nu)$ with the hitting measure is a model for the Poisson boundary. In the non-proper case, the classical entropy criterion is not expected to work, as there may be infinitely many group elements contained in a ball of fixed diameter.

We prove, however, that as long as Γ_μ contains a WPD element, the Poisson boundary indeed coincides with the Gromov boundary.

Theorem 1.8. (*Poisson boundary for WPD actions.*) *Let G be a countable group which acts by isometries on a δ -hyperbolic metric space (X, d) , and let μ be a non-elementary probability measure on G with finite logarithmic moment and finite entropy. Suppose that there exists at least one WPD element h in the semigroup generated by the support of μ . Then the Gromov boundary of X with the hitting measure is a model for the Poisson boundary of the random walk (G, μ) .*

The result extends our earlier result in [MT18] for acylindrical actions.

1.5. Mapping class groups. Let $S_{g,n}$ be a topological surface with genus g and n punctures, and let $Mod(S_{g,n})$ be its mapping class group, i.e. the group of homeomorphisms of $S_{g,n}$, up to isotopy. The mapping class group acts on a locally infinite, δ -hyperbolic graph, known as the *curve complex* [MM99]. Loxodromic elements for this action are the pseudo-Anosov mapping classes, and as they are all WPD elements, all results in our paper apply.

As an application of Theorem 1.5, we prove that the normal closure of random mapping classes is a free group, answering a question of Margalit [Mar19, Problem 10.11].

Theorem 1.9. *Let $G = \text{Mod}(S_{g,n})$ be the mapping class group of a surface of finite type, and suppose that G is infinite. Let μ be a probability measure on G with bounded support in the curve complex and such that $\Gamma_\mu = G$, and let w_n be the n^{th} step of the random walk generated by μ . Then the probability that the normal closure $\langle\langle w_n \rangle\rangle$ is free tends to 1 as $n \rightarrow \infty$, with exponential decay.*

The result follows from Theorem 1.5 and the fact that, by the Nielsen realization theorem, the maximal normal subgroup of $\text{Mod}(S_{g,n})$ always equals its center (which is trivial unless the mapping class group contains a central hyperelliptic involution). See Section 11.2 for details. Note that in fact the action is acylindrical [Bow08], hence some applications such as the Poisson boundary already follow from [MT18].

1.6. Outer automorphisms of the free group. Another application of our setup is to the group $\text{Out}(F_n)$ of outer automorphisms of a finitely generated free group F_n of rank $n \geq 2$.

There are several hyperbolic graphs on which $\text{Out}(F_n)$ acts: the main two are the *free factor complex* and the *free splitting complex*. In particular, the free factor complex $\mathcal{FF}(F_n)$ is hyperbolic by work of Bestvina and Feighn [BF14]. Moreover, an element is loxodromic on $\mathcal{FF}(F_n)$ if and only if it is *fully irreducible*, and all fully irreducible elements satisfy the WPD property. However, it is not known whether the action of $\text{Out}(F_n)$ on the free factor complex is acylindrical.

On the other hand, the free splitting complex is also hyperbolic, but the action on the free splitting complex $\mathcal{FS}(F_n)$ is known not to be acylindrical, by work of Handel and Mosher [HM13]. Moreover, an element is loxodromic if and only if it admits a filling lamination pair. This is a weaker condition than being fully irreducible, and the stabilizer of a quasixis of a loxodromic element need not be virtually cyclic.

Thus, this is an example of an action for which not every loxodromic element satisfies the WPD property. However, by Theorem 1.11 even for this action WPD elements are generic for the random walk. Note that genericity of fully irreducible elements was already known by [Riv10].

We have the following identification for the Poisson boundary of $\text{Out}(F_n)$.

Theorem 1.10. *Let μ be a measure on $\text{Out}(F_n)$ such that the semigroup generated by the support of μ contains at least two independent fully irreducible elements. Moreover, suppose that μ has finite entropy and finite logarithmic moment for the simplicial metric on the free factor complex. Then the Gromov boundary of the free factor complex is a model for the Poisson boundary of (G, μ) .*

Proof. By [BF14], the action of fully irreducible elements on the free factor complex is WPD. Hence, the claim follows by Theorem 1.8. \square

Note that the identification of the Poisson boundary for $\text{Out}(F_n)$ has been obtained by Horbez [Hor16] using the action of $\text{Out}(F_n)$ on the outer space CV_n . This gives an identification of the Poisson boundary with both ∂CV_n and $\partial \mathcal{FF}(F_n)$, as there is a coarsely defined Lipschitz map $CV_n \rightarrow \mathcal{FF}(F_n)$. In our theorem above, the moment condition required is a bit weaker, as we only need the logarithmic

moment condition to hold with respect to the metric on $\mathcal{FF}(F_n)$ instead of the metric on CV_n .

1.7. Tame automorphism groups. Other groups arising in algebraic geometry admit an action on a non-proper δ -hyperbolic space with WPD elements.

First of all, the group $\text{Aut}(\mathbb{C}^2)$ of polynomial automorphisms of \mathbb{C}^2 (see [FL10] and references therein, as well as [MO15]) can be written as an amalgamated product of two of its subgroups, hence it acts on the corresponding Bass-Serre tree, which is a Gromov hyperbolic space; in fact, for this action every loxodromic element is WPD, but the action is not acylindrical.

Remarkably, Lamy and Przytycki recently extended this work to three variables. They considered the *tame automorphism group* $\text{Tame}(\mathbb{C}^3)$, which is the group generated by affine and elementary automorphisms of \mathbb{C}^3 (see [LP19] for a precise definition), and showed that this group also acts on a Gromov hyperbolic complex and there are WPD elements, so the methods of the present paper apply.

Let us finally remark that much less is known about the structure of the Cremona group in three variables, and these methods do not easily apply since there is no immediate analog of the hyperboloid, as the Cremona group no longer preserves a quadratic form.

1.8. Genericity of WPD elements. Maher [Mah11] and Rivin [Riv08] considered random walks on the mapping class group acting on the curve complex, and showed that pseudo-Anosov mapping classes are typical for random walks. More generally, in [MT18], we showed that for a group G acting non-elementarily on a Gromov hyperbolic space X , loxodromic elements are typical for the random walk: i.e., the probability that the random product of n elements is loxodromic tends to one as n tends to infinity.

One of the ingredients in our proofs is that, as long as there is one WPD element in the support of the measure generating the random walk, then WPD elements are generic.

We say that a measure μ is *non-elementary* if Γ_μ contains at least two independent loxodromic elements, and is *bounded* if for some $x \in X$ the set $(gx)_{g \in \text{supp } \mu}$ is bounded in X . Finally, μ is *WPD* if Γ_μ contains an element h which is WPD in G .

We will show that generic elements are WPD with an explicit bound on the rate of convergence: we say that a sequence of numbers (p_n) tends to 1 with *exponential decay* if there are constants $B > 0$ and $c < 1$ such that $p_n \geq 1 - Bc^n$.

Theorem 1.11. (*Genericity of WPD elements.*) *Let G be a group acting on a Gromov hyperbolic space X , and let μ be a countable, non-elementary, bounded, WPD probability measure on G . Then*

$$\mathbb{P}(w_n \text{ is WPD}) \rightarrow 1$$

as $n \rightarrow \infty$, with exponential decay.

In fact, we obtain that most random elements have bounded coarse stabilizer, where the bound does not depend on the point chosen. We call this property *asymptotic acylindricity*. We prove the following estimate on the joint coarse stabilizer.

Theorem 1.12. (*Asymptotic acylindricity.*) *Let G be a group acting on a Gromov hyperbolic space X . Let μ be a countable, non-elementary, bounded, WPD*

probability measure on G , and let $x \in X$. Then for any $K \geq 0$ there is an $N > 0$ such that

$$\mathbb{P}(\#\text{Stab}_K(x, w_n x) \leq N) \rightarrow 1,$$

with exponential decay.

1.9. Matching estimates and rates. In order to obtain our results, we need to show that a random element has finite joint coarse stabilizer, and to do so we recur to what we call *matching estimates*.

Following [CM15], we say that two geodesics γ and γ' in X have a *match* if there is a subsegment of γ close to a G -translate of a subsegment of γ' (see Definition 7.1). Let $x \in X$ be a basepoint and (w_n) be a sample path. The two key estimates we will prove and use are the following.

- (1) *Matching estimate* (Proposition 8.2): given a loxodromic element g , we show that the probability that the geodesic $[x, w_n x]$ has a match with a translate of the axis of g is at least $1 - Bc^n$.
- (2) *Non-matching estimate* (Proposition 9.2): given a geodesic segment η in X of length s , the probability that there is a match between $[x, w_n x]$ and a G -translate of η is at most Bc^s .

1.10. Asymmetric elements. Another important tool in our proofs is the notion of asymmetric element, which was introduced in [MS19]. We call a loxodromic element $g \in G$ *asymmetric* if any element which coarsely stabilizes a segment of the axis of g actually coarsely stabilizes the set $\{g^i x\}_{i \in \mathbb{Z}}$ (see Definition 10.1 for the precise statement). In [MS19] it is proven that if the action of G is acylindrical, then asymmetric elements are generic. In this paper, we generalize this result to WPD actions, and use it to prove the other results.

Let G_{WPD} be the set of WPD elements in G . For a loxodromic $g \in G$, let us denote as $\Lambda(g) := \{\lambda_g^+, \lambda_g^-\}$ the two fixed points of g on ∂X . We denote as $E_G(g)$ the stabilizer of $\Lambda(g)$ as a set, and as $E_G^+(g)$ the pointwise stabilizer of $\Lambda(g)$. Moreover, for a subgroup $H < G$ we denote as

$$E_G(H) := \bigcap_{H \cap G_{WPD}} E_G(h)$$

the intersection of all $E_G(h)$ as h lies in $H \cap G_{WPD}$ (a priori, this set may be smaller than the set of WPD elements for the action of H on X). Note that $E_G(G)$ is the maximal finite normal subgroup of G .

We have the following characterization of $E_G(w_n)$ for generic elements w_n . Let $E_\mu := E_G^+(\Gamma_\mu)$.

Theorem 1.13. *Given $\delta \geq 0$ there are constants K and L with the following properties. Let G be a group acting by isometries on a δ -hyperbolic space X , and let μ be a countable, non-elementary, reversible, bounded, WPD probability distribution on G . Then there are constants $B > 0$ and $c < 1$ such that the probability that w_n is loxodromic, $(1, L, K)$ -asymmetric, and WPD with*

$$E_G(w_n) = E_G^+(w_n) = \langle w_n \rangle \times E_\mu$$

is at least $1 - Bc^n$.

Note that the action of E_μ on $E_G(w_n)$ is precisely responsible for the value of k in Theorems 1.4 and 1.5. Indeed, one obtains that the cyclic group $\langle w_n \rangle$ is normal in $E_G(w_n)$ if and only if the image of w_n in $\text{Aut } E_\mu$ is trivial. Now, the random

walk on Γ_μ pushes forward to a random walk on the finite group $\text{Aut } E_\mu$, and this random walk equidistributes on the image of Γ_μ inside $\text{Aut } E_\mu$, which we denote as H_μ . This explains the asymptotic probability of $\frac{1}{\#H_\mu}$ in Theorem 1.5.

1.11. Further questions. We conclude with a few questions for further exploration.

- (1) Can one drop “reversible” as an hypothesis in Theorem 1.5?
- (2) Do our results still hold for measures μ with finite exponential moment, rather than bounded measures?
- (3) Does the radius of injectivity $\text{inj}(N_n)$ typically goes to infinity as $n \rightarrow \infty$, and at what rate?

We believe that the answers to all these questions should be positive, but we do not attempt to solve them here.

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2. BACKGROUND MATERIAL

Let X be a δ -hyperbolic metric space, and let G be a group of isometries of X . Let μ be a probability measure on G . This defines a *random walk* by choosing for each n an element g_n of G with distribution μ independently of the previous ones, and considering the product

$$w_n := g_1 \dots g_n.$$

The sequence $(w_n)_{n \geq 0}$ is called a *sample path* of the random walk, and we are interested in the asymptotic behavior of typical sample paths.

2.1. Isometries of hyperbolic spaces. Recall that isometries of a δ -hyperbolic space (even if it is not proper) can be classified into three types (see [Gro87], [DSU17]). In particular, $g \in \text{Isom}(X)$ is:

- *elliptic* if g has bounded orbits;
- *parabolic* if it has unbounded orbits, but zero translation length;
- *loxodromic* (or *hyperbolic*) if it has positive translation length.

Here, the *translation length* of $g \in \text{Isom}(X)$ is the quantity

$$\tau(g) := \lim_{n \rightarrow \infty} \frac{d(g^n x, x)}{n}, \tag{3}$$

where the limit always exists and is independent of the choice of x . Moreover, a loxodromic element has two fixed points on the Gromov boundary of X , one attracting and one repelling.

A semigroup inside $\text{Isom}(X)$ is *non-elementary* if it contains two loxodromic elements which have disjoint fixed point sets on ∂X .

We will use the following elementary properties of δ -hyperbolic spaces, which we state without proof. A *quasiaxis* for a loxodromic isometry g of X is a quasigeodesic which is invariant under g . In fact, the constants may be chosen to depend only on δ , see for example Bonk and Schramm [BS00, Proposition 5.2] or Kapovich and Benakli [KB02, Remark 2.16].

Proposition 2.1. *Given a constant $\delta \geq 0$, there is a constant K_1 such that every loxodromic isometry of a δ -hyperbolic space has a $(1, K_1)$ -quasiaxis.*

To simplify notation, we will refer to a $(1, K_1)$ -quasiaxis as a *quasiaxis* for g .

We will use the following fellow travelling properties of quasigeodesics in Gromov hyperbolic spaces.

The Morse lemma states that a quasigeodesic in a δ -hyperbolic space is contained in an L -neighborhood of a geodesic connecting its endpoints, where L depends only on δ and the quasigeodesic constants. The following result is a mild generalization of the Morse lemma, and is exactly the Morse lemma if K equals zero. Given a finite quasigeodesic γ with endpoints x and y , let $\gamma_K^- := \gamma \setminus (B_K(x) \cup B_K(y))$.

Proposition 2.2 ([Del96, Proposition 1.3.3]). *Given $\delta \geq 0$ and $K_1 \geq 0$, there is a constant L such that for any $K \geq 0$ and for any two $(1, K_1)$ -quasigeodesics γ and η in a δ -hyperbolic space, with endpoints distance at most K apart, any point on γ_K^- lies within distance at most L from a point on η .*

We say a set γ is Q -*quasiconvex* if for any points x and y in γ , any geodesic $[x, y]$ is contained in a Q -neighbourhood of γ . Given a Q -quasiconvex set γ in a hyperbolic space X and a point $x \in X$, let $\pi_\gamma(x)$ be a nearest point on γ to x . In a δ -hyperbolic space, the nearest point projection is not unique, but any two projections have uniformly bounded distance, where the bound only depends on δ and K , hence we shall always pick one and denote it $\pi_\gamma(x)$.

If two points x and y have nearest point projections $\pi_\gamma(x)$ and $\pi_\gamma(y)$ which are sufficiently far apart, then the piecewise geodesic running through x , $\pi_\gamma(x)$, $\pi_\gamma(y)$ and then y , which we shall call a *nearest point projection path*, is a quasigeodesic:

Proposition 2.3 ([CDP90, Proposition 10.2.1]). *Given δ and Q , there are constants L and K such that for any δ -hyperbolic space X , for any Q -quasiconvex set γ in X , and any pair of points x and y in X , if $d(\pi_\gamma(x), \pi_\gamma(y)) \geq L$, then the nearest point projection path*

$$[x, \pi_\gamma(x)] \cup [\pi_\gamma(x), \pi_\gamma(y)] \cup [\pi_\gamma(y), y]$$

is a $(1, K)$ -quasigeodesic.

Let us recall that given $x, y \in X$ and $R \geq 0$, we define the *shadow* $S_x(y, R)$ as

$$S_x(y, R) := \{z \in X : (z \cdot y)_x \geq d(x, y) - R\}.$$

The number $r = d(x, y) - R$ is called the *distance parameter* of the shadow.

Proposition 2.4. *Given constants $\delta \geq 0, K_1 \geq 0$ and $R \geq 0$, there are constants D and L with the following properties. Let x and y be two points in a δ -hyperbolic space X with $d(x, y) \geq D$. Let $A = S_x(y, R)$ and $B = S_y(x, R)$ be the corresponding shadows, and let γ be a $(1, K_1)$ -quasigeodesic with one endpoint in A and the other endpoint in B . Then any geodesic $[x, y]$ is contained in an L -neighborhood of γ .*

Proof. Let p, q be the endpoints of γ , with $p \in A$ and $q \in B$, and let p', q' be, respectively, a nearest point projection of p to $[x, y]$ and of q to $[x, z]$. Then, by [MT18, Proposition 2.4], $d(p', y) \leq R + O(\delta)$ and $d(q', z) \leq R + O(\delta)$. We shall assume that we have chosen $D \geq 2R + L_1 + O(\delta)$, where L_1 is the constant from Proposition 2.3. Then by Proposition 2.3, the piecewise geodesic through p, p', q' and q is a quasigeodesic, with quasigeodesic constants depending only on δ . As quasigeodesics fellow travel, $[p, q]$ is contained in an L -neighborhood of γ , where L depends only on δ, K_1 and R , as the quasigeodesic fellow travelling constants depend only on δ and K_1 . \square

2.2. Random walks on weakly hyperbolic groups. In [MT18], we established many properties of typical sample paths for random walks on general groups of isometries of δ -hyperbolic spaces. Namely:

Theorem 2.5 ([MT18]). *Let μ be a countable, non-elementary measure on a group of isometries of a δ -hyperbolic metric space X , and let $x \in X$. Then*

- (1) *almost every sample path $(w_n x)$ converges to some point ξ in the Gromov boundary of X ;*
- (2) *if μ has finite first moment in X , there exists $L > 0$ such that for almost all sample paths we have*

$$\lim_{n \rightarrow \infty} \frac{d(w_n x, x)}{n} = L;$$

- (3) *moreover, if μ is bounded, there exists $L > 0, B \geq 0$ and $c < 1$ such that the translation length grows linearly with exponential decay:*

$$\mathbb{P}(\tau(w_n) \geq nL) \geq 1 - Bc^n.$$

Note that in [MT18] the previous result is proven under the assumption that X is *separable*, i.e. it contains a countable dense set. However, since the measure μ is countable one can drop the separability assumption, as remarked in [GST20, Remark 4]. In fact, the only point where separability is used is to prove convergence to the boundary, and one can prove it for general metric spaces from the separable case and the following fact.

Lemma 2.6 ([GST20, Remark 4]). *Let Γ be a countable group of isometries of a δ -hyperbolic metric space X . Then there exists a separable metric space X' (in fact, a simplicial graph with countably many vertices) and a Γ -equivariant quasi-isometric embedding $i: X' \rightarrow X$. As a consequence, i extends continuously to a Γ -equivariant inclusion $\partial X' \rightarrow \partial X$ between the Gromov boundaries.*

By the theorem in the separable case, given $x' \in X'$ almost every sample path $(w_n x')$ converges to a point $\xi' \in \partial X'$, hence if $x = i(x')$ then almost every sample path $(w_n x)$ converges to $i(\xi') \in \partial X$, hence the random walk on X converges almost surely to the boundary.

Another ingredient in the proof of the previous theorem is the following lemma about the measure of shadows [MT18, Proposition 5.4], which we will use in the later sections.

Proposition 2.7. *Let G be a non-elementary, countable group acting by isometries on a Gromov hyperbolic space X , and let μ be a non-elementary probability distribution on G . Then there is a number R_0 such that if $g, h \in G$ are group elements*

such that h and $h^{-1}g$ lie in the semigroup generated by the support of μ , then

$$\nu(\overline{S_{hx}(gx, R_0)}) > 0,$$

where \overline{A} denotes the closure in $X \cup \partial X$.

We will also use the well-known fact that in a Gromov hyperbolic space the complement of a shadow is approximately a shadow, as in the following proposition (see [MT18], Proposition 2.4 and Corollary 2.5).

Proposition 2.8. *Given non-negative constants δ, K and L , there are constants C and D , such that in any δ -hyperbolic space X we have:*

(1) *for any pair of points x, y in X and any $R \geq 0$ we have*

$$X \setminus S_x(y, R) \subseteq S_y(x, d(x, y) - R + C);$$

(2) *for any $R \geq 0$, and any bi-infinite (K, L) -quasigeodesic γ , parameterized such that $\gamma(0)$ is a nearest point on γ to the basepoint x , then for any shadow set $V = S_x(\gamma(t), R)$ which does not contain x , with $t \geq 0$, and for any point $y \in U = S_x(\gamma(t + D), R)$, we have the inclusion*

$$X \setminus V \subseteq S_y(x, d(x, \gamma(t)) - R + C).$$

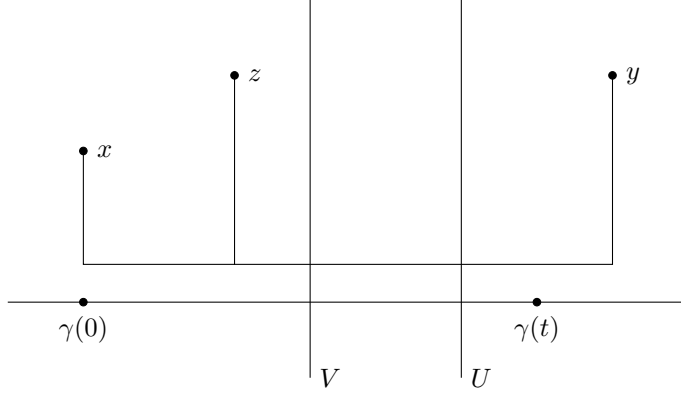


FIGURE 1. The complement of a shadow is contained in a shadow.

We will also use the following exponential decay estimates. For $Y \subset X$ let $H^+(Y)$ denote the probability that the random walk ever hits Y , i.e. that there is at least one index $n \in \mathbb{N}$ such that $w_n x \in Y$.

Lemma 2.9 (Exponential decay of shadows, [Mah12, Lemma 2.10]). *Let G be a group which acts by isometries on a Gromov hyperbolic space X , and let μ be a countable, non-elementary, bounded probability measure on G . Then there exist constants $B > 0$ and $c < 1$ such that for any shadow $S_x(y, R)$ with distance parameter $r = d(x, y) - R$, we have the estimates*

$$\nu(\overline{S_x(y, R)}) \leq Bc^r, \tag{4}$$

and

$$H^+(S_x(y, R)) \leq Bc^r. \tag{5}$$

In particular, for all n :

$$\mathbb{P}(w_n x \in S_x(y, R)) \leq Bc^r. \quad (6)$$

Indeed, equation (4) is [Mah10, Lemma 5.4], and equation (5) follows from (4) as in [MT18, Equation (5.3)]. Equation (6) is an immediate consequence of (5).

Finally, we will also use the following positive drift, or linear progress, result.

Proposition 2.10 (Exponential decay of linear progress, [Mah12]). *Let G be a group acting on a hyperbolic space X . Let μ be a countable, non-elementary measure on G which has bounded support in X . Then there exist constants $B > 0$, $L > 0$ and $0 < c < 1$ such that for all n :*

$$\mathbb{P}(d(x, w_n x) \leq Ln) \leq Bc^n.$$

2.3. The Poisson boundary. Given a countable group G and a probability measure μ on G , one defines the space of bounded μ -harmonic functions as

$$H^\infty(G, \mu) := \left\{ f : G \rightarrow \mathbb{R} \text{ bounded} \quad : \quad f(g) = \sum_{h \in G} f(gh)\mu(h) \quad \forall g \in G \right\}.$$

Suppose now that G acts by homeomorphisms on a measure space (M, ν) . Then the measure ν is μ -stationary if

$$\nu = \sum_{h \in G} \mu(h) h_* \nu.$$

A G -space M with a μ -stationary measure ν is called a μ -boundary if for almost every sample path (w_n) the measure $w_n \nu$ converges to a δ -measure. Given a μ -boundary, one has the *Poisson transform* $\Phi : L^\infty(M, \nu) \rightarrow H^\infty(G, \mu)$ defined as

$$\Phi(f)(g) := \int_M f(gx) d\nu(x).$$

Definition 2.11. The space (M, ν) is the *Furstenberg-Poisson boundary* of (G, μ) if the Poisson transform Φ is a bijection between $L^\infty(M, \nu)$ and $H^\infty(G, \mu)$.

It turns out that the Furstenberg-Poisson boundary is well-defined up to G -equivariant measurable isomorphisms. Moreover, it is the maximal μ -boundary in following sense: if (B_{FP}, ν_{FP}) is the Furstenberg-Poisson boundary and (B, ν) is another μ -boundary, then there exists a G -equivariant measurable map $(B_{FP}, \nu_{FP}) \rightarrow (B, \nu)$. Finally, such a boundary can be defined as the measurable quotient of the sample space of the random walk (G, μ) by identifying two sample paths if they eventually coincide (to be precise, one should cast this definition in the context of measurable partitions, as defined by Rokhlin [Roh52]).

2.4. The strip criterion. In order to obtain the Poisson boundary for WPD actions, we will use Kaimanovich's *strip criterion*. This basically says that if bi-infinite paths for the random walks can be approximated by subsets of G , called *strips*, then one can conclude that the relative entropies of the conditional random walks vanish, hence the proposed geometric boundary is indeed the Poisson boundary.

Given a measure μ on G , its *reflected measure* is $\check{\mu}(g) := \mu(g^{-1})$. Moreover, we denote as $\check{\nu}$ the hitting measure for the random walk associated to the reflected measure $\check{\mu}$. We say that the measure μ has *finite entropy* if

$$H(\mu) := - \sum_{g \in G} \mu(g) \log \mu(g) < \infty.$$

Let $x \in X$ be a basepoint. The measure μ has *finite logarithmic moment* if $\int_G \log^+ d(x, gx) d\mu(g) < \infty$. Let us denote as

$$B_G(g) := \{h \in G : d(x, hx) \leq d(x, gx)\}.$$

We shall use the following *strip criterion* by Kaimanovich.

Theorem 2.12 ([Kai00]). *Let μ be a probability measure with finite entropy on G , and let $(\partial X, \nu)$ and $(\partial X, \tilde{\nu})$ be μ - and $\tilde{\mu}$ -boundaries, respectively. If there exists a measurable G -equivariant map S assigning to almost every pair of points $(\alpha, \beta) \in \partial X \times \partial X$ a non-empty “strip” $S(\alpha, \beta) \subset G$, such that for all g*

$$\frac{1}{n} \log |S(\alpha, \beta)g \cap B_G(w_n)| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

for $\nu \times \tilde{\nu}$ -almost every $(\alpha, \beta) \in \partial X \times \partial X$, then $(\partial X, \nu)$ and $(\partial X, \tilde{\nu})$ are the Poisson boundaries of the random walks (G, μ) and $(G, \tilde{\mu})$, respectively.

3. BACKGROUND ON THE CREMONA GROUP

We will start by recalling some fundamental facts about the Cremona group, and especially its action on the Picard-Manin space. For more details, see [CL13], [DF01], [Fav08] and references therein.

3.1. The Picard-Manin space. If X is a smooth, projective, rational surface the group

$$N^1(X) := H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{R})$$

is called the *Néron-Severi group*. Its elements are Cartier divisors on X modulo numerical equivalence. The intersection form defines an integral quadratic form on $N^1(X)$. We denote $N^1(X)_{\mathbb{R}} := N^1(X) \otimes \mathbb{R}$.

If $f : X \rightarrow Y$ is a birational morphism, then the pullback map $f^* : N^1(Y) \rightarrow N^1(X)$ is injective and preserves the intersection form, so $N^1(Y)_{\mathbb{R}}$ can be thought of as a subspace of $N^1(X)_{\mathbb{R}}$.

A *model* for $\mathbb{P}^2(\mathbb{C})$ is a smooth projective surface X with a birational morphism $X \rightarrow \mathbb{P}^2(\mathbb{C})$. We say that a model $\pi' : X' \rightarrow \mathbb{P}^2(\mathbb{C})$ dominates the model $\pi : X \rightarrow \mathbb{P}^2(\mathbb{C})$ if the induced birational map $\pi^{-1} \circ \pi' : X' \dashrightarrow X$ is a morphism. By considering the set \mathcal{B}_X of all models which dominate X , one defines the space of *finite Picard-Manin classes* as the injective limit

$$\mathcal{Z}(X) := \lim_{X' \in \mathcal{B}_X} N^1(X')_{\mathbb{R}}.$$

In order to find a basis for $\mathcal{Z}(X)$, one defines an equivalence relation on the set of pairs (p, Y) where Y is a model of X and p a point in Y , as follows. One declares $(p, Y) \sim (p', Y')$ if the induced birational map $Y \dashrightarrow Y'$ maps p to p' and is an isomorphism in a neighbourhood of p . We denote the quotient space as \mathcal{V}_X . Finally, the *Picard-Manin space* of X is the L^2 -completion

$$\mathcal{Z}(X) := \left\{ [D] + \sum_{p \in \mathcal{V}_X} a_p [E_p] : [D] \in N^1(X)_{\mathbb{R}}, a_p \in \mathbb{R}, \sum_{p \in \mathcal{V}_X} a_p^2 < +\infty \right\}.$$

In this paper, we will only focus on the case $X = \mathbb{P}^2(\mathbb{C})$. Then the Néron-Severi group of $\mathbb{P}^2(\mathbb{C})$ is generated by the class $[H]$ of a line, with self-intersection $+1$.

Thus, the Picard-Manin space is

$$\overline{\mathcal{Z}}(\mathbb{P}^2) := \left\{ a_0[H] + \sum_{p \in \mathcal{V}_{\mathbb{P}^2(\mathbb{C})}} a_p[E_p], \sum_p a_p^2 < +\infty \right\}.$$

It is well-known that if one blows up a point in the plane, then the corresponding exceptional divisor has self-intersection -1 , and intersection zero with divisors on the original surface.

Thus, the classes $[E_p]$ have self-intersection -1 , are mutually orthogonal, and are orthogonal to $N^1(X)$. Hence, the space $\overline{\mathcal{Z}}(\mathbb{P}^2)$ is naturally equipped with a quadratic form of signature $(1, \infty)$, thus making it a Minkowski space of uncountably infinite dimension. Thus, just as classical hyperbolic space can be realized as one sheet of a hyperboloid inside a Minkowski space, inside the Picard-Manin space one defines

$$\mathbb{H}_{\mathbb{P}^2} := \{[D] \in \overline{\mathcal{Z}}(\mathbb{P}^2) : [D]^2 = 1, [H] \cdot [D] > 0\}$$

which is one sheet of a two-sheeted hyperboloid. The restriction of the quadratic intersection form to $\mathbb{H}_{\mathbb{P}^2}$ defines a Riemannian metric of constant curvature -1 , thus making $\mathbb{H}_{\mathbb{P}^2}$ into an infinite-dimensional hyperbolic space. More precisely, the induced distance dist satisfies the formula

$$\cosh \text{dist}([D_1], [D_2]) = [D_1] \cdot [D_2].$$

Each birational map f acts on $\overline{\mathcal{Z}}$ by orthogonal transformations. To define the action, recall that for any rational map $f : \mathbb{P}^2(\mathbb{C}) \dashrightarrow \mathbb{P}^2(\mathbb{C})$ there exist a surface X and morphisms $\pi, \sigma : X \rightarrow \mathbb{P}^2(\mathbb{C})$ such that $f = \sigma \circ \pi^{-1}$. Then we define $f^* = (\pi^*)^{-1} \circ \sigma^*$, and $f_* = (f^{-1})^*$. Moreover, f_* preserves the intersection form, hence it acts as an isometry of $\mathbb{H}_{\mathbb{P}^2}$: in other words, the map $f \mapsto f_*$ is a group homomorphism

$$\text{Bir } \mathbb{P}^2(\mathbb{C}) \rightarrow \text{Isom}(\mathbb{H}_{\mathbb{P}^2})$$

hence one can apply to the Cremona group the theory of random walks on groups acting on non-proper δ -hyperbolic spaces.

The space $\mathbb{H}_{\mathbb{P}^2}$ is not separable; however, any countable subgroup of the Cremona group preserves a closed, totally geodesic, separable, subset of $\mathbb{H}_{\mathbb{P}^2}$ (see also [DP12], Remark 1).

Definition 3.1. The *dynamical degree* of a birational transformation $f : X \dashrightarrow X$ is defined as

$$\lambda(f) := \lim_{n \rightarrow \infty} \|(f^n)^*\|^{1/n}$$

where $\|\cdot\|$ is any operator norm on the space of endomorphisms of $H^*(X, \mathbb{R})$.

Note that $\lambda(f) = \lambda(gfg^{-1})$ is invariant by conjugacy. Moreover, if f is represented by three homogeneous polynomials of degree d without common factors, then the action of f^* on the class $[H]$ of a line is $f^*([H]) = d[H]$, hence

$$\lambda(f) = \lim_{n \rightarrow \infty} \deg(f^n)^{1/n}.$$

Moreover, the degree is related to the displacement in the hyperbolic space $\mathbb{H}_{\mathbb{P}^2}$: in fact, (see [Fav08], page 17)

$$\deg(f) = f^*[H] \cdot [H] = [H] \cdot f_*[H] = \cosh d(x, fx)$$

if $x = [H] \in \mathbb{H}_{\mathbb{P}^2}$. As a consequence, the dynamical degree $\lambda(f)$ of a transformation f is related to its translation length $\tau(f)$ by the equation ([CL13], Remark 4.5):

$$\tau(f) = \lim_{n \rightarrow \infty} \frac{\text{dist}(x, f^n x)}{n} = \lim_{n \rightarrow \infty} \frac{\cosh^{-1} \deg(f^n)}{n} = \log \lambda(f).$$

Hence, a Cremona transformation f is loxodromic if and only if $\lambda(f) > 1$.

4. GROWTH OF TRANSLATION LENGTH

Let us now start by proving that for bounded probability measures translation length grows linearly along almost every sample path. This is a variation of [MT18, Theorem 1.2] and [DH18, Theorem 1.2].

Theorem 4.1. *Let G be a group acting on a Gromov hyperbolic space X . Let μ be a countable non-elementary measure on G whose support is bounded in X . Then for almost every sample path we have*

$$\lim_{n \rightarrow \infty} \frac{\tau(w_n)}{n} = L$$

where $L > 0$ is the drift of the random walk.

Proof of Theorem 4.1. Since the support is bounded in X , by Theorem 2.5 there exists $L > 0$ such that almost surely

$$\lim_{n \rightarrow \infty} \frac{d(x, w_n x)}{n} = L.$$

Moreover, proceeding as in [MT18, Section 5.8] and using the exponential decay of shadows [MT18, eq. (16)], (see also [DH18, proof of Prop. 2.6]), there exist $B > 0$ and $0 < c < 1$ such that for any $\epsilon > 0$ we have

$$\mathbb{P}((w_n x \cdot w_n^{-1} x)_x \geq \epsilon n) \leq Bc^{\epsilon n}. \quad (7)$$

Now, by Borel-Cantelli, we obtain almost surely

$$\lim_{n \rightarrow \infty} \frac{(w_n x \cdot w_n^{-1} x)_x}{n} = 0.$$

The claim then follows by using the well-known formula (see [MT18], Appendix A)

$$\tau(g) = d(x, gx) - 2(gx \cdot g^{-1}x)_x + O(\delta).$$

□

5. WPD ACTIONS

5.1. The WPD condition. Let G be a group acting by isometries on a metric space X . Recall that the action of G on X is *proper* if the map $G \times X \rightarrow X \times X$ given by $(g, x) \mapsto (x, gx)$ is proper, i.e. the preimages of compact sets are compact. A related notion is that the action is *properly discontinuous* if for every $x \in X$ there exists an open neighbourhood U of x such that $gU \cap U \neq \emptyset$ holds for at most finitely many elements g . If the space X is not proper, it is very restrictive to ask for the action to be proper (for instance, point stabilizers for a proper action must be finite). However, Bestvina-Fujiwara [BF02] defined the notion of *weak proper discontinuity*, or *WPD*; essentially, a loxodromic isometry g is a WPD element if its action is proper in the direction of its axis.

Definition 5.1. Let G be a group acting on a hyperbolic space X , and h a loxodromic element of G . One says that h satisfies the *weak proper discontinuity condition* (or h is a WPD element) if for every $K > 0$ and every $x \in X$, there exists $M \in \mathbb{N}$ such that

$$\#\{g \in G : d(x, gx) < K, d(h^M x, gh^M x) < K\} < \infty.$$

If we define the *joint coarse stabilizer* of two points $x, y \in X$ as

$$\text{Stab}_K(x, y) := \{g \in G : d(x, gx) \leq K \text{ and } d(y, gy) \leq K\}$$

then the WPD condition says that for any K and any x there exists an integer M such that $\text{Stab}_K(x, h^M x)$ is a finite set. A trivial consequence of the definition of WPD is the following.

Lemma 5.2. *Let G be a group acting on a Gromov hyperbolic space X , and let h be a WPD element in G . Then there are functions $M_W : \mathbb{R}_{\geq 0} \rightarrow \mathbb{N}$ and $N_W : \mathbb{R}_{\geq 0} \rightarrow \mathbb{N}$ such that for any $x \in X$, any $K \geq 0$, and for any $f \in G$ one has*

$$\#\text{Stab}_K(fx, fh^{M_W(K)}x) \leq N_W(K).$$

Proof. By definition, note that

$$\text{Stab}_K(fx, fy) = f\text{Stab}_K(x, y)f^{-1}$$

hence the cardinality

$$\#\text{Stab}_K(fx, fh^M x) = \#\{f(\text{Stab}_K(x, h^M x))f^{-1}\} = \#\text{Stab}_K(x, h^M x)$$

is finite and independent of f , proving the claim. \square

Given a loxodromic element g , its associated *maximal elementary subgroup* $E_G(g)$ is defined as the stabilizer of the two endpoints of a quasixis of g , i.e.

$$E_G(g) = \text{Stab}^G(\{\lambda_g^+, \lambda_g^-\})$$

(note that elements of $E_G(g)$ may permute the two fixed points). We will use the following result due to Bestvina and Fujiwara [BF02, Proposition 6].

Theorem 5.3. *Let G act on X with a WPD element h , with a quasixis α_h . Then $E_G(h)$ is the unique maximal virtually cyclic subgroup containing h . Furthermore, for any constant $K \geq 0$ there is a number L , depending on h, δ, K_1 and K , such that if $g \in G$ is an element which K -coarsely stabilizes a subsegment of α_h of length L , then g lies in $E_G(h)$.*

That is, if α_h is a quasixis of a WPD element h , then

$$E_G(h) = \{g \in G : d_{\text{Haus}}(g\alpha_h, \alpha_h) < \infty\}.$$

This is stated in [BF02] for a group action in which all loxodromic elements are WPD, but the proof works for any group acting non-elementarily on a Gromov hyperbolic space as long as h is a WPD element.

6. THE POISSON BOUNDARY

Let us now use the WPD property to prove that the Poisson boundary coincides with the Gromov boundary, proving Theorem 1.8 in the Introduction.

Similarly to [MT18, Section 6], the idea is to define appropriately the strips for Kaimanovich’s criterion using “elements of bounded geometry” as below, and using the WPD condition to show that the number of elements in such strips grows at most linearly.

The main difference is that we do not obtain a bound on the growth of all strips, but only on strips between *almost all* pairs of boundary points. In fact, if h is a WPD element, then one can use the WPD condition to obtain a bound on the number of bounded geometry elements in a ball (see Lemma 6.2 below). Moreover, by ergodicity, for almost every pair of boundary points, any $(1, K_1)$ -quasigeodesic between them will follow a translate of a quasisixis of h , hence we can use the previous claim to bound the number of elements in any strip between almost every pair of boundary points.

6.1. Elements of bounded geometry. Let $R \geq 0$ and $v \in G$. Then for any pair $(\alpha, \beta) \in \partial X \times \partial X$, with $\alpha \neq \beta$, define the set of *bounded geometry elements* as

$$\mathcal{O}_{R,v}(\alpha, \beta) := \{g \in G : \alpha \in \overline{S_{gvx}(gx, R)} \text{ and } \beta \in \overline{S_{gx}(gvx, R)}\}.$$

An example of a bounded geometry element is illustrated below in Figure 2. Note that for any $g \in G$ we have $\mathcal{O}_{R,v}(g\alpha, g\beta) = g\mathcal{O}_{R,v}(\alpha, \beta)$. Moreover, we define the ball in the group with respect to the metric on X as

$$B_G(y, r) := \{g \in G : d(y, gx) \leq r\}$$

where $y \in X$ and $r \geq 0$.

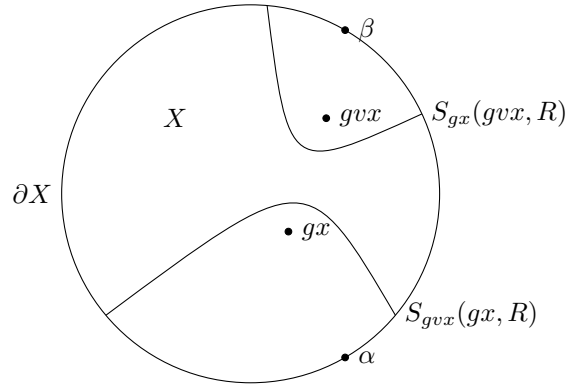


FIGURE 2. A bounded geometry element g in $\mathcal{O}_{R,v}(\alpha, \beta)$.

The most crucial property of bounded geometry elements is that their number in a ball grows linearly with the radius of the ball.

Proposition 6.1. *Let G be a group acting on a hyperbolic space X , let $x \in X$, and let h be a WPD element. Then for any $R > 0$, there is a positive power $v = h^M$ of*

the WPD element and a constant C such that for any radius $r > 0$ and any pair of distinct boundary points $\alpha, \beta \in \partial X$ one has

$$\#|B_G(x, r) \cap \mathcal{O}_{R,v}(\alpha, \beta)| \leq Cr.$$

This fact follows from the next lemma, which uses the WPD property in a crucial way.

Lemma 6.2. *Let G be a group acting on a hyperbolic space X , let $x \in X$, and let h be a WPD element. Then for any $R > 0$, there are positive constants L , M and N , such that if $v = h^M$ then*

$$\#|B_G(z, L) \cap \mathcal{O}_{R,v}(\alpha, \beta)| \leq N$$

for any $z \in X$ and any pair of distinct boundary points α, β .

Proof. We choose L to be the maximum of the two fellow travelling constants (both called L) in Propositions 2.2 and 2.4.

We choose M to be sufficiently large such that $d(x, h^M x)$ is at least the separation distance D from Proposition 2.4 and M is also at least $M_W(10L)$, where M_W is the function arising from the WPD condition in Lemma 5.2. Finally, we choose N to be $N_W(10L)$, where again N_W is the function described in Lemma 5.2.

Let us consider two elements g, g' which belong to $\mathcal{O}(\alpha, \beta) \cap B_G(z, 4K)$. Then if we let $f = g'g^{-1}$, then

$$d(gx, fgx) \leq 2L. \tag{8}$$

Let γ be a $(1, K_1)$ -quasigeodesic which joins α and β , and denote $S_1 := \overline{S_{gvx}(gx, R)}$, $S_2 := \overline{S_{gx}(gvx, R)}$. By construction, α belongs to both S_1 and fS_1 hence both α and $f\alpha$ belong to fS_1 ; similarly, β and $f\beta$ belong to fS_2 . Hence, the two quasigeodesics γ and $f\gamma$ have endpoints in fS_1 and fS_2 , hence they must fellow travel in their middle: more precisely, by Proposition 2.4 they must pass within distance L from both fgx and $y := fgvx$. Hence, if we call q a nearest point to fgx on $f\gamma$, we have $d(fgx, q) \leq L$. Moreover, if we call p a nearest point on γ to y , and p' a nearest point on $f\gamma$ to y , we have

$$d(p, p') \leq d(p, y) + d(y, p') \leq 2L.$$

Combining this with eq. (8) we get

$$|d(gx, p) - d(fgx, p')| \leq 4L.$$

Moreover, since f is an isometry we have $d(fgx, fp) = d(gx, p)$, hence

$$|d(fgx, fp) - d(fgx, p')| \leq 4L. \tag{9}$$

Now, the points q, p' and fp both lie on the quasigeodesic $f\gamma$; let us assume that fp lies in between q and p' , and draw a geodesic segment γ' between q and p' , and let p'' be a nearest point projection of fp to γ' (the case where p' lies between q and fp is completely analogous). By fellow travelling (Proposition 2.2), we have $d(fp, p'') \leq L$. Then, since p', p'' and q lie on a geodesic, we have

$$d(p', p'') = |d(q, p') - d(q, p'')| \leq$$

and by using eq. (9)

$$\leq |d(fgx, p') - d(fgx, fp)| + d(fgx, q) + d(fgx, q) + d(fp, p'') \leq 4L + L + L + L$$

hence

$$d(fp, p') \leq d(fp, p'') + d(p', p'') \leq 7L + L = 8L$$

and finally

$$d(y, fy) \leq d(y, p') + d(p', fp) + d(fp, fy) \leq L + 8L + L = 10L.$$

Thus $d(gvx, fgvx) = d(fgvx, f^2gvx) \leq 10L$ hence

$$f \in \text{Stab}_{10L}(gx, gvx)$$

so by Lemma 5.2 there are only $N = N_W(10L)$ possible choices of f , as claimed. \square

Proof of Proposition 6.1. Let γ be a $(1, K_1)$ -quasigeodesic in X which joins α and β . By definition, if g belongs to $\mathcal{O}_{R,v}(\alpha, \beta)$, then gx lies within distance $\leq L$ of γ . Then one can pick points $(z_n)_{n \in \mathbb{Z}}$ along γ such that any point of γ is within distance $\leq L$ of some z_n . Then, any ball of radius r contains at most cr of such z_n , where c depends only on L and the quasigeodesic constant of γ . The claim then follows from Lemma 6.2. \square

We now turn to the proof of Theorem 1.8. By Theorem 2.5, we know that since both μ and its reflected measure $\check{\mu}$ are non-elementary, both the forward random walk and the backward random walk converge almost surely to points on the boundary of X . Thus, one defines the two boundary maps $\partial_{\pm} : (G^{\mathbb{Z}}, \mu^{\mathbb{Z}}) \rightarrow \partial X$ as follows. Let $\omega = (g_n)_{n \in \mathbb{Z}}$ be a bi-infinite sequence of increments, and define

$$\partial_+(\omega) := \lim_{n \rightarrow \infty} g_1 \dots g_n x, \quad \partial_-(\omega) := \lim_{n \rightarrow \infty} g_0^{-1} g_{-1}^{-1} \dots g_{-n}^{-1} x$$

the two endpoints of, respectively, the forward random walk and the backward random walk. Then choose $R \geq R_0$ as in Proposition 2.7 and $v = h^M$ as in Proposition 6.1. Define

$$\mathcal{O}(\omega) := \mathcal{O}_{R,v}(\partial_+(\omega), \partial_-(\omega))$$

the set of bounded geometry elements along the $(1, K_1)$ -quasigeodesic which joins $\partial_+(\omega)$ and $\partial_-(\omega)$. Note that, if $T : G^{\mathbb{Z}} \rightarrow G^{\mathbb{Z}}$ is the shift in the space of increments, we have

$$\mathcal{O}(T^n \omega) = \mathcal{O}(w_n^{-1} \partial_+(\omega), w_n^{-1} \partial_-(\omega)) = w_n^{-1} \mathcal{O}(\omega).$$

Now we will show that for almost every bi-infinite sample path ω the set $\mathcal{O}(\omega)$ is non-empty and has at most linear growth. In fact, by definition of bounded geometry,

$$p := \mathbb{P}(1 \in \mathcal{O}(\omega)) = \nu(\overline{S})\check{\nu}(\overline{S'}) > 0,$$

where $S = S_{v_x}(x, R)$ and $S' = S_x(vx, R)$, and their measures are positive by Proposition 2.7. Moreover, since the shift map T preserves the measure in the space of increments, we also have for any n

$$\mathbb{P}(w_n \in \mathcal{O}(\omega)) = \mathbb{P}(1 \in \mathcal{O}(T^n \omega)) = p > 0.$$

Thus, by the ergodic theorem, the number of times w_n belongs to $\mathcal{O}(\omega)$ grows almost surely linearly with n : namely, for a.e. ω

$$\lim_{n \rightarrow \infty} \frac{\#\{1 \leq i \leq n : w_i \in \mathcal{O}(\omega)\}}{n} = p > 0.$$

Hence the set $\mathcal{O}(\omega)$ is almost surely non-empty (in fact, it contains infinitely many elements). On the other hand, by Proposition 6.1 the set $\mathcal{O}(\omega)$ has at most linear growth, i.e. there exists $C > 0$ such that for any $z \in X$ we have

$$\#\mathcal{O}(\omega) \cap B_G(z, r) \leq Cr \quad \forall r > 0. \quad (10)$$

The Poisson boundary result now follows from the strip criterion (Theorem 2.12). Let $P(G)$ denote the set of subsets of G . Then, we define the strip map $S : \partial X \times \partial X \rightarrow P(G)$ as $S(\alpha, \beta) := \mathcal{O}_{R,v}(\alpha, \beta)$; hence, applying equation (10) with $z = x$, $r = d(w_n x, x)$ we obtain

$$\#|S(\alpha, \beta)g \cap B_G(w_n)| \leq Cd(w_n x, x).$$

Then, since μ has finite logarithmic moment, one has almost surely

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log d(w_n x, x) \rightarrow 0,$$

which verifies the criterion of Theorem 2.12, establishing that the Gromov boundary of X is a model for the Poisson boundary of the random walk.

Remark 6.3. We would like to thank the referee for pointing out an alternative approach if the action of G on X is cobounded. In this case, Osin's [Osi16] construction of the projection complex Y may be realized as a quotient of X , and the action of G on Y is acylindrical. Hence, one can use [MT18] to identify the Poisson boundary of (G, μ) with the hitting measure on ∂Y . One may verify that the Lipschitz map from X to Y is alignment preserving as defined by Dowdall and Taylor [DT17], whose work then shows that the subset of ∂X consisting of quasigeodesic rays with infinite diameter image in Y maps injectively into ∂Y . A quasigeodesic ray in X has infinite image in Y if it fellow travels with infinitely many distinct translates of a quasixis for the chosen WPD element, and this happens for a full measure subset of ∂X with respect to the hitting measure. Therefore ∂X with the hitting measure is a model for the Poisson boundary. It is not clear to the authors how to extend this argument to the non-cobounded case.

7. GENERICITY OF WPD ELEMENTS

Let G be a group acting by isometries on a hyperbolic space X , and let μ be a countable, non-elementary, bounded, WPD probability distribution on G . Let h be a WPD element in Γ_μ . We start by showing that the probability that a random walk gives a WPD element tends to one exponentially quickly, and furthermore that the probability that a quasixis of the WPD element fellow travels with a translate of a quasixis of h also tends to one exponentially quickly.

Before proceeding, we give a brief overview of the argument. We wish to show that a random walk on G gives rise to a WPD element with probability tending to one exponentially quickly. Given a WPD element h with quasixis α_h , one may construct a *projection complex* P on which G also acts. The projection complex is a quasi-tree, hence hyperbolic, and has the property that any element g of G which acts loxodromically on P acts as a WPD element on X , and furthermore, any quasixis α_g of g has a large subsegment which fellow travels with a translate of a quasixis α_h of h . Furthermore, this gives control over the size of the joint stabilizer $\text{Stab}_K(x, gx)$, a property we call *asymptotic acylindricity* (see Section 8). If G acts non-elementarily on X , then it also acts non-elementarily on P , so the fact that random walks on groups acting on hyperbolic spaces give loxodromic elements with probability tending to one, applied to the action on P , gives the required result.

The property that two axes have subsets that fellow travel each other will be useful, and so we will use the following definition from [CM15], see also [MS19].

Definition 7.1. Let G be a group acting on a Gromov hyperbolic space X . Given constants K and L we say that two geodesics γ and γ' in X have an (L, K) -match if there exist geodesic subsegments $\alpha \subseteq \gamma$ and $\alpha' \subseteq \gamma'$ of length $\geq L$ and some $g \in G$ such that $g\alpha$ and α' have Hausdorff distance $\leq K$.

The main result of this section, from which we will derive Theorem 1.11, is the following.

Theorem 7.2. *Let G be a group with a non-elementary action by isometries on a hyperbolic space X , and let h be a WPD element for this action. Then there is a constant K with the following properties. For any L there exists a non-elementary acylindrical action of G on a quasi-tree Y such that if $g \in G$ acts loxodromically on Y , then the action of g on X is loxodromic and WPD. Furthermore, a quasiaxis for g in X has an (L, K) -match with a quasiaxis for h in X .*

This result is implicit in the constructions of projection complexes in [BBF15, Osi16, DGO17, Bal17, BBFS20] and thus is likely well-known. As we are unable to find a reference in the literature, in the next two subsections we provide a proof using published results of [BBFS20].

7.1. Projection complexes. We now review the projection complex construction from [BBF15], [BBFS20]. We do not give complete details, but we state precisely the properties we use.

Let h be a WPD element. In general, h may only have an invariant quasiaxis in X , which is coarsely preserved by $E_G(h)$. However, X embeds quasi-isometrically inside a hyperbolic space X' such that h has an invariant *geodesic* axis A_h which is preserved by $E_G(h)$, see for example [BBF13, Lemma 4.9]¹. In the rest of this section, we will assume that the action of G on X has the property that $E_G(h)$ preserves a geodesic axis. At the end, we will remark that we can obtain the same results for general actions X with different constants by using the quasi-isometry between X and X' .

Note that, as in this section we need to consider distances in several metric spaces, we will write d_X instead of d for the distance in X . We say that a collection of geodesics in X has *D -bounded projections* if for any two distinct geodesics A and B in the collection, the nearest point projection $\pi_A(B)$ has diameter at most D . Let \mathcal{A} be the set of distinct translates of A_h under G . As h is a WPD element, there is a constant D such that \mathcal{A} has D -bounded projections, see for example [BBF15, Theorem H]. Given three distinct elements A, B and C of \mathcal{A} , define

$$d_C(A, B) := \text{diam}\{\pi_C(A) \cup \pi_C(B)\}.$$

Bestvina, Bromberg and Fujiwara [BBF15] define a *projection complex* $P_L(\mathcal{A})$, which is a graph whose vertices are elements of \mathcal{A} , and in which two distinct vertices A and B are connected by an edge if $d_C(A, B) \leq L$ for all $C \in \mathcal{A} \setminus \{A, B\}$. In fact, their construction is more general, but we shall restrict attention to a version that applies in the context of WPD actions. We shall give $P_L(\mathcal{A})$ the natural path metric in which every edge has length one, and we shall denote this metric by d_{P_L} .

Bestvina, Bromberg and Fujiwara [BBF15] showed that $P_L(\mathcal{A})$ is a quasi-tree for all L sufficiently large. Osin [Osi16] defined a slightly different space on which the action of G is acylindrically hyperbolic, and Balasubramanya [Bal17] showed

¹This result appears in the initial arXiv version but was omitted from the published version [BBF16].

that this construction could be modified to guarantee that the space is a quasi-tree. In fact, we shall use the version from Bestvina, Bromberg, Fujiwara and Sisto [BBFS20], which also construct a projection complex which is a quasi-tree and on which G acts acylindrically.

The result below summarizes the properties of the construction that we use.

Theorem 7.3. [BBFS20, Theorem 4.1, Theorem 3.10] *Let G be a group acting by isometries on a δ -hyperbolic space X . Let h be a WPD element such that $E_G(h)$ preserves a geodesic axis A_h . Let \mathcal{A} be the set of distinct translates of A_h under G , and suppose that \mathcal{A} has D -bounded projections. The nearest point projection maps π_A may be replaced with maps π'_A such that for all A and B in \mathcal{A} , $\pi'_A(B) \subseteq N_D(\pi_A(B))$, and for all L sufficiently large, the projection complex $P_L(\mathcal{A})$, constructed using the modified projection maps π'_A , is a quasi-tree on which G acts acylindrically. Furthermore, if G acts non-elementarily on X , then it acts non-elementarily on $P_L(\mathcal{A})$.*

We will also use the following result from [BBFS20], which shows that the distance in $P_L(\mathcal{A})$ between two axes A and B is coarsely equivalent to the number of other axes to which A and B have large diameter projections.

Theorem 7.4. [BBFS20, Corollary 3.7] *Let G be a group acting by isometries on a δ -hyperbolic space X . Let h be a WPD element such that $E_G(h)$ preserves a geodesic axis A_h . Let \mathcal{A} be the set of distinct translates of A_h under G . For $A, B \in \mathcal{A}$, denote*

$$Y_L(A, B) := \{C \in \mathcal{A} \setminus \{A, B\} \mid \pi'_C(A, B) \geq L\}.$$

Then there is a constant L_0 such that for all $L \geq L_0$, the metric on $P_L(\mathcal{A})$ is coarsely equivalent to the number of elements in $Y_L(A, B)$. In fact, for $A \neq B$,

$$\lfloor \frac{1}{2}(\#|Y_L(A, B)| + 1) \rfloor + 1 \leq d_{P_L}(A, B) \leq \#|Y_L(A, B)| + 1.$$

We next show that loxodromic isometries of the projection complex act as WPD elements on X .

7.2. Loxodromic isometries of the projection complex. We will make use of the following elementary result.

Lemma 7.5. *Let α and β be two geodesic segments in a δ -hyperbolic space X , of length at least L , contained in K -neighbourhoods of each other. Then*

$$\text{diam } \pi_\alpha(\beta) \geq L - 4K.$$

Proof. Let a be an endpoint of α . As $\alpha \subset N_K(\beta)$, there is a point $b \in \beta$ such that $d_X(a, b) \leq K$. Let a' be a nearest point on α to b , so $d_X(a', b) \leq K$. By the triangle inequality, $d_X(a, a') \leq 2K$. Applying the same argument to the other endpoint of α implies that the diameter of $\pi_\alpha(\beta)$ is at least $L - 4K$. \square

We now show that distance in $P_L(\mathcal{A})$ is a coarse lower bound for the distance between elements of \mathcal{A} in X .

Proposition 7.6. *Let G be a group acting on a δ -hyperbolic space X , and let h be a WPD isometry such that $E_G(h)$ preserves a geodesic axis A_h . Let \mathcal{A} be the collection of distinct translates of A_h under G , with D -bounded projections. Then there are constants K and $Q > 0$ with the following properties.*

There exists L_0 such that for all $L \geq L_0$, and for any A and B in \mathcal{A} , the distance d_{P_L} in the projection complex $P_L(\mathcal{A})$ is a coarse lower bound for distance in X , i.e.

$$d_X(A, B) \geq Qd_{P_L}(A, B) - Q. \quad (11)$$

Furthermore, any shortest geodesic $[a, b]$ from A to B in X has an (L, K) -match with the axis of h .

Proof. We give a brief outline of the argument. Let A and B be two elements of \mathcal{A} , and let γ_1 be a shortest path from A to B in X . Their distance $d_{P_L}(A, B)$ in the projection complex is coarsely equal to the number of elements in $Y_L(A, B)$, the collection of $C \in \mathcal{A}$ to which the projections of A and B are distance at least L apart. This means that the nearest point projection path from A to B via C in X is a quasigeodesic, and so the shortest path γ_1 from A to B fellow travels with C distance roughly L . However, as the collection of geodesics in \mathcal{A} has D -bounded projections, the fellow travelling segments of translates of A_h can't overlap too much along γ_1 , so this gives a lower bound on the length of γ_1 , which is linear in the number of elements in $Y_L(A, B)$, and hence linear in $d_{P_L}(A, B)$.

We now give the details of this argument. Recall that Proposition 2.3 says that if two points have nearest point projections to a geodesic that are distance at least L_1 apart, then the nearest point projection path is a $(1, K_1)$ -quasigeodesic, where K_1 and L_1 depend only on δ . Furthermore, by Proposition 2.2, there is a constant K_2 such that if two $(1, K_1)$ -quasigeodesics have common endpoints, then their Hausdorff distance is at most K_2 . Here K_2 depends on δ and K_1 , but as K_1 only depends on δ , K_2 only depends on δ .

Choose $L_0 = 9D + 8K_2 + L_1$, and let $L \geq L_0$. Let $\gamma_1 = [a, b]$ be a shortest path from A to B in X . We may assume that $A \neq B$ and so $d_{P_L}(A, B) \geq 1$, thus by the definition of $P_L(\mathcal{A})$ there is at least one $C \in \mathcal{A}$ such that $d_C(A, B) \geq L$. This implies that $d_X(\pi_C(a), \pi_C(b)) \geq L - 2D \geq L_1$, so by Proposition 2.3, the nearest point projection path $\gamma_2 = [a, \pi_C(a)] \cup [\pi_C(a), \pi_C(b)] \cup [\pi_C(b), b]$ is a $(1, K_1)$ -quasigeodesic.

By our choice of K_2 , γ_1 and γ_2 are contained in K_2 -neighbourhoods of each other. The segment $[\pi'_C(a), \pi'_C(b)]$ has length at least L , and so $[\pi_C(a), \pi_C(b)]$ has length at least $L - 2D > L_1$. As the nearest point projection path is a $(1, K_1)$ -quasigeodesic it is contained in a K_2 -neighbourhood of γ_1 , and so $[\pi'_C(a), \pi'_C(b)]$ is contained in a $(K_2 + D)$ -neighbourhood of γ_1 . As C is a translate of the axis A_h , this implies that the geodesic $\gamma_1 = [a, b]$ has an (L, K) -match with A_h , giving the final statement of the result with $K = K_2 + D$.

The choice of C in $Y_L(A, B)$ was arbitrary, so for every C in $Y_L(A, B)$, the geodesic $\gamma_1 = [a, b]$ K -fellow travels with C distance at least L . If the number of elements of $Y_L(A, B)$ is at least $2Dd_X(A, B)/L + 1$, then there are at least two distinct translates C and C' of A_h which have subsegments of length at least $L/2$ which K -fellow travel. By Lemma 7.5, the nearest point projection of C to C' has diameter at least $L/2 - 4K$. Our choice of L_0 ensures that $L/2 - 4K > D$, which contradicts the fact that elements of \mathcal{A} have D -bounded projections. Therefore

$$d_X(A, B) \geq \frac{L}{2D}(\#|Y_L(A, B)| - 1),$$

and so the result follows by choosing Q equal to $L/2D$. \square

We now show that if an isometry acts loxodromically on the projection complex $P_L(\mathcal{A})$, then it acts loxodromically on X .

Corollary 7.7. *Let G be a group acting by isometries on a hyperbolic space X , with a WPD element h such that $E_G(h)$ preserves a geodesic axis. Let $P_L(\mathcal{A})$ be the corresponding projection complex determined by h . Then for all L sufficiently large, if g acts loxodromically on $P_L(\mathcal{A})$, then g acts loxodromically on X .*

Proof. Recall that if g is a loxodromic isometry of $P_L(\mathcal{A})$, then the translation length of g is positive, i.e. $\tau_{P_L}(g) > 0$. Let $A \in \mathcal{A}$ and a be a point on the axis A . We observe that $d_X(a, g^n a) \geq d_X(A, g^n A)$, as a lies in A . Choosing $L \geq L_0$, where L_0 is the constant from Proposition 7.6, we may apply (11) to the pair A and $g^n A$ and obtain $d_X(A, g^n A) \geq Qd_{P_L}(A, g^n A) - Q$. Moreover, by definition of translation length (3), $d_{P_L}(A, g^n A) \geq n\tau_{P_L}(g)$ for any $A \in \mathcal{A}$ and any $n \geq 0$. Hence

$$d_X(a, g^n a) \geq Qn\tau_{P_L}(g) - Q.$$

Dividing by n and taking the limit as $n \rightarrow \infty$ shows that $\tau_X(g) \geq Q\tau_{P_L}(g) > 0$, and so the action of g on X is loxodromic, as required. \square

Corollary 7.8. *Let G be a group acting by isometries on a hyperbolic space X , with a WPD element h such that $E_G(h)$ preserves a geodesic axis, and let $P_L(\mathcal{A})$ be the corresponding projection complex determined by h . Then there are constants K and L_0 , such that for all $L \geq L_0$, if g acts loxodromically on $P_L(\mathcal{A})$, then g acts loxodromically on X . Furthermore, a quasixis of g has an (L, K) -match with the axis of h .*

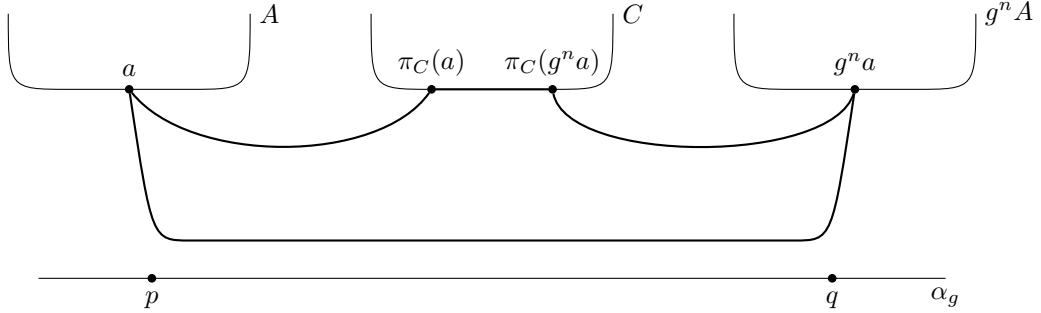


FIGURE 3. The geodesic from a to $g^n a$, and the nearest point projection path via C .

Proof. Let L_1 be the maximum of the fellow travelling constant L from Proposition 2.2 for $(1, K_1)$ -quasigeodesics, and the constant L from Proposition 2.3, such that if two points have nearest point projections to a geodesic distance at least L apart, then the nearest point projection path is a $(1, K_1)$ -quasigeodesic. Let D be a constant such that the geodesics in \mathcal{A} have D -bounded projections. Finally, choose L_0 sufficiently large such that Corollary 7.7 holds, and furthermore, choose $L_0 \geq 2D + L_1$.

Let A be an axis for h , and let a be a point on A . As g acts loxodromically on $P_L(\mathcal{A})$, the distance $d_{P_L}(A, g^n A)$ tends to infinity as n tends to infinity. By (11), there is an n sufficiently large such that $Y_L(A, g^n A)$ is non-empty. Let C be an element of $Y_L(A, g^n A)$. Recall that by the definition of the projection complex, the diameter of $\pi'_C(A) \cup \pi'_C(g^n C)$ is at least L . The image of the

modified projection maps π'_C is contained within a D -neighbourhood of the nearest point projection maps π_C , so $d_X(\pi_C(a), \pi_C(g^n a)) \geq L - 2D$. By our choice of L_0 , $L - 2D \geq L_1$, so by Proposition 2.3, the nearest point projection path $\eta = [a, \pi_C(a)] \cup [\pi_C(a), \pi_C(g^n a)] \cup [\pi_C(g^n a), g^n a]$ is a $(1, K_1)$ -quasigeodesic. In particular, there is a segment $[\pi_C(a), \pi_C(g^n a)]$ of length at least $L - 2D$ contained in an L_1 -neighbourhood of any geodesic $[a, g^n a]$.

By Proposition 2.1, there is a $(1, K_1)$ -quasigeodesic α_g in X , which is a quasixis for g acting on X . Let p be a nearest point on α_g to a , and let q be a nearest point on α_g to $g^n a$. As g is an isometry, $d_X(a, p) = d_X(g^n a, g^n p)$. The point $g^n p$ lies on $g^n \alpha_g$, which by Proposition 2.2, is contained in an L_1 -neighbourhood of α_g , and so $d_X(g^n a, q) \leq d_X(a, p) + L_1$, which in particular is independent of n .

Therefore, by Proposition 2.2, outside an $(d_X(a, p) + L_1)$ -neighbourhood of its endpoints, the geodesic $[a, g^n a]$ is contained in an L_1 -neighbourhood of α_g . By (11) the number of geodesics in \mathcal{A} which may have segments of length L which K -fellow travel a geodesic of length $(d_X(a, p) + L_1)$ is at most $(d_X(a, p) + L_1)/Q$. In particular, for n sufficiently large, there is an element C in $Y_L(A, g^n A)$ which has a subsegment of length at least $L - 2D$ contained in an L_1 -neighbourhood of $[a, g^n a]$, distance at least $d_X(a, p) + L_1$ from its endpoints, and hence contained in an $2L_1$ -neighbourhood of α_g . The translate C of A_h then has an (L, K) -match with α_g for $K = 2L_1 + 2D$. \square

Recall that the following (a priori weaker) definition, which we shall refer to as *axial WPD*, is equivalent to WPD.

Definition 7.9. Let G be a group acting on a δ -hyperbolic space X , and let h be a loxodromic isometry with a quasixis α_h . Then h is an *axial WPD* if there exists $p \in \alpha_h$ such that for any constant $K \geq 0$, there is an $M > 0$, such that

$$\#|\text{Stab}_K(p) \cap \text{Stab}_K(h^M p)| < \infty.$$

Lemma 7.10. Let G be a group acting on a δ -hyperbolic space X , and let h be a loxodromic isometry. Then h is an axial WPD if and only if h is WPD.

Proof. If h is WPD, then it is an axial WPD. We now show the other direction. By the triangle inequality, for any $x, y \in X$, $g \in G$, and $K \geq 0$

$$\text{Stab}_K(y) \cap \text{Stab}_K(h^M y) \subseteq \text{Stab}_{K'}(x) \cap \text{Stab}_{K'}(h^M x)$$

where $K' = K + 2d(x, y)$. \square

We now show that for L sufficiently large, loxodromics on $P_L(\mathcal{A})$ act as WPD elements on X .

Proposition 7.11. Let G be a group acting by isometries on a hyperbolic space X , with a WPD element h so that $E_G(h)$ preserves a geodesic axis, and let $P_L(\mathcal{A})$ be the corresponding projection complex determined by h . Then there is a constant L_0 such that for all $L \geq L_0$, if g acts loxodromically on $P_L(\mathcal{A})$, then g acts as a WPD element on X .

Proof. Let A_h be the geodesic axis of h in X , and let α_g be a $(1, K_1)$ -quasixis for g in X . Let p be a nearest point on α_g to A_h , and let K be a constant.

The group G acts on both X and $P_L(\mathcal{A})$. We will write $\text{Stab}_K^X(x)$ for the coarse stabilizer of a point $x \in X$ and $\text{Stab}_K^P(A)$ for the coarse stabilizer of a point $A \in P_L(\mathcal{A})$.

Let f be an isometry such that $f \in \text{Stab}_K^X(p) \cap \text{Stab}_K^X(g^m p)$. In particular, by the triangle inequality and the fact that p is a nearest point projection, $d_X(A_h, fA_h) \leq 2d_X(A_h, \alpha_g) + K$, and similarly, $d_X(g^m A_h, fg^m A_h) \leq 2d_X(A_h, \alpha_g) + K$. Using (11) implies that for $K' = (2d_X(A_h, \alpha_g) + K)/Q$,

$$f \in \text{Stab}_{K'}^P(A_h) \cap \text{Stab}_{K'}^P(g^m A_h).$$

The isometry g acts as a WPD element on $P_L(\mathcal{A})$, and let M_W and N_W be the corresponding functions from Lemma 5.2. For all $m \geq M_W(K')$ there are at most $N_W(K')$ elements f . Therefore g acts as an axial WPD element on X , hence by Lemma 7.10 as a WPD element, as required. \square

Theorem 7.2 now follows immediately from Corollary 7.8 and Proposition 7.11 in the case that $E_G(h)$ preserves a geodesic axis; the general case follows as discussed by replacing X with a quasi-isometric space X' on which $E_G(h)$ preserves a geodesic axis.

7.3. WPD isometries are generic. We may now prove the following slightly stronger form of Theorem 1.11.

Theorem 7.12. *Let G be a group acting on a Gromov hyperbolic space X , and let μ be a countable, non-elementary, bounded, WPD probability measure on G . Then there exist constants $B > 0$, $c < 1$ such that the probability that w_n is WPD satisfies*

$$\mathbb{P}(w_n \text{ is WPD}) \geq 1 - Bc^n$$

for any n .

Furthermore, for any WPD element $h \in \Gamma_\mu$, there is a constant K , such that for all $L \geq 0$, the probability that a quasixis for w_n has an (L, K) -match with a quasixis for h tends to one as $n \rightarrow \infty$, with exponential decay.

Proof. Let $h \in \Gamma_\mu$ be a WPD element and let K be given by Theorem 7.2. For any $L \geq 0$, let Y be the quasi-tree given by Theorem 7.2. As μ is bounded in X , it is also bounded in Y . As Γ_μ contains h and acts non-elementarily on X , it also acts non-elementarily on Y . A bounded non-elementary random walk on a group acting on a Gromov hyperbolic space gives rise to a loxodromic element with probability tending to one with exponential decay, by [MT18]. If w_n is loxodromic on Y , then it is WPD on X , as required. The final statement follows immediately from the final statement in Theorem 7.2. \square

8. ASYMPTOTIC ACYLINDRICALITY

We say that a group G acting by isometries on a Gromov hyperbolic space X is *acylindrical* if for all $K \geq 0$, there are constants $R \geq 0$ and $N \geq 0$, such that for all points x and y in X , with $d(x, y) \geq R$, one has the bound

$$\#|\text{Stab}_K(x) \cap \text{Stab}_K(y)| \leq N.$$

Definition 8.1. Let μ be a probability measure on a group G acting by isometries on a metric space X , and let $x \in X$. We say that the random walk generated by μ is *asymptotically acylindrical* if there is a function $N_{ac}: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that for all $K \geq 0$, the probability that

$$\#|\text{Stab}_K(x) \cap \text{Stab}_K(w_n x)| \leq N_{ac}(K)$$

tends to one as n tends to infinity.

8.1. More matching estimates. We now show that for any WPD element h in Γ_μ , the probability that $[x, w_n x]$ has an (L, K) -match with a translate of a quasiaxis α_h of h tends to one as n tends to infinity.

The following results are analogous to [MS19, Propositions 3.2], where the action is assumed to be acylindrical.

Proposition 8.2. *Let G be a group acting by isometries on a Gromov hyperbolic space X with a WPD element h , with quasiaxis α_h . Let x be a basepoint in X . Then there is a constant K_0 such that for any countable, non-elementary, WPD probability distribution μ on G , which is bounded in X , the following properties hold.*

- (1) *If w_n is loxodromic, then let α_{w_n} be a quasiaxis for w_n , and let p be a nearest point on α_{w_n} to the basepoint x . Then for any $K \geq K_0$ and any $L \geq 0$, there are constants $B_1 > 0$ and $c_1 < 1$ such that the probability that w_n is loxodromic and that $[p, w_n p]$ has an (L, K) -match with α_h is at least $1 - B_1 c_1^n$.*
- (2) *There is a constant K such that for any $L \geq 0$, there are constants $B_2 > 0$ and $c_2 < 1$ such that the probability that $\gamma_n = [x, w_n x]$ has an (L, K) -match with α_h is at least $1 - B_2 c_2^n$.*

Proof. Let K be the constant from Theorem 7.12. Then, for any $L \geq 0$, Theorem 7.12 implies that the probability that α_{w_n} and α_h have a (L, K) -match tends to one with exponential decay. This is illustrated in Figure 4 below.

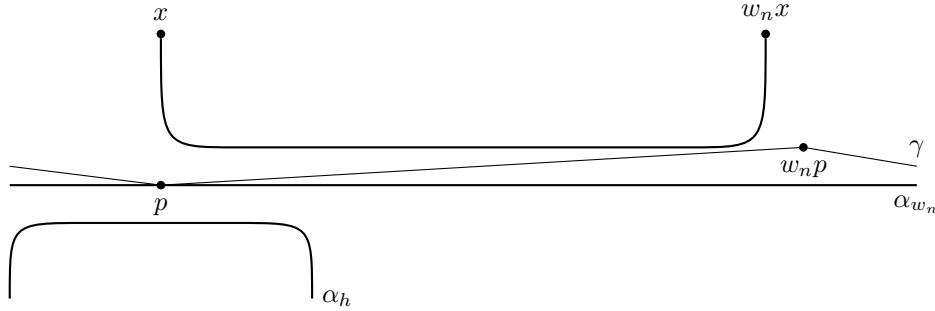


FIGURE 4. The quiaxis α_{w_n} has an (L, K) -match with the quiaxis α_h .

By Proposition 2.2, there exists L_1 , which only depends on δ , such that any two $(1, K_1)$ -quasigeodesics with common endpoints are contained in L_1 -neighbourhoods of each other. The point $w_n p$ lies on the quiaxis $w_n \alpha_{w_n}$, which is contained in an L_1 -neighbourhood of α_{w_n} . In particular, the distance from $w_n p$ to α_{w_n} is at most L_1 , and so again, by Proposition 2.2, the geodesic $[p, w_n p]$ is contained in a $2L_1$ -neighbourhood of α_{w_n} . Let γ be the orbit of $[p, w_n p]$ under powers of w_n . Then γ is a connected bi-infinite quiaxis for w_n , contained in an $2L_1$ -neighbourhood of α_{w_n} . Let q be a nearest point projection of x to γ , and let q' be a nearest point projection of $w_n x$ to γ . As $\alpha_{w_n}, w_n \alpha_{w_n}$ and γ are all contained in $2L_1$ -neighbourhoods of each other, $d_X(p, q) \leq 2L_1$ and $d_X(w_n p, q') \leq 2L_1$.

By Proposition 2.3, there are L_2 and K_2 , which only depend on δ , such that if $d_X(q, q') \geq L_2$, then the nearest point projection path $[x, q] \cup [q, q'] \cup [q', w_n x]$ is a $(1, K_2)$ -quasigeodesic. As $[q, q']$ and $[p, w_n p]$ are Hausdorff distance $2L_1$ apart, there are constants K_3 and L_3 , which only depend on δ , such that if $d_X(p, w_n p) \geq L_3$, then the path $[x, p] \cup [p, w_n p] \cup [w_n p, w_n x]$ is a $(1, K_3)$ -quasigeodesic.

The distance $d_X(p, w_n p)$ is at least the translation length $\tau(w_n)$. By Theorem 2.5, the translation length grows linearly with exponential decay, so the probability that $d_X(p, w_n p) \geq L_3$ tends to one with exponential decay. Therefore the probability that the path $[x, p] \cup [p, w_n p] \cup [w_n p, w_n x]$ is a $(1, K_3)$ -quasigeodesic tends to one with exponential decay.

If α_h has an (L, K) -match with α_{w_n} , then it has an $(L, K + 2L_1)$ -match with γ . If this match is disjoint from the orbit of p under powers of w_n , then we are done. If the orbit of p is contained in the match, then, at worst, p divides the subsegment of γ realizing the match in two equal parts, so the probability that $[p, w_n p]$ has an $(L/2, K + 2L_1)$ -match with α_h tends to one exponentially quickly. This gives the first statement of the result, for appropriate choices of constants.

For the second statement, the path $[p, w_n p]$ is a subsegment of the $(1, K_3)$ -quasigeodesic $[x, p] \cup [p, w_n p] \cup [w_n p, w_n x]$. By Proposition 2.2, there is a constant L_4 , which only depends on δ , such that $[p, w_n p]$ is contained in an L_4 -neighbourhood of $[x, w_n x]$. Therefore, the $(L/2, K + 2L_1)$ -match with $[p, w_n p]$ gives an $(L/2, K + 2L_1 + L_4)$ -match with $[x, w_n x]$, as required. \square

Finally, we show:

Lemma 8.3. *Let G be a group acting on a Gromov hyperbolic space X . Let μ be a countable, non-elementary, bounded, WPD probability distribution on G , and let h be a WPD element in G which lies in Γ_μ . Then there is a constant K_0 such that for any $\epsilon > 0$, any $K \geq K_0$, and any $L > 0$ there are constants $B > 0$ and $c < 1$ such that the probability that every segment $[w_i x, w_{i+\epsilon n} x]$ for $0 \leq i \leq n(1 - \epsilon)$ has a (L, K) -match with a translate of a quasiaxis of h is at least $1 - Bc^n$.*

Proof. By Proposition 8.2, for each i the probability that $[w_i x, w_{i+\epsilon n} x]$ does not have a (L, K) -match with a translate of a quasiaxis of h is at most $B_1 c_1^{\epsilon n}$ for some $c_1 < 1$, and there are at most $n(1 - \epsilon)$ possible values of i , hence the total probability is at most $B_1(1 - \epsilon)n c_1^{\epsilon n}$. The result then follows for suitable choices of B and c . \square

8.2. Proof of asymptotic acylindricity. We now show that if Γ_μ contains a WPD element, then the random walk determined by μ is asymptotically acylindrical with exponential decay, which is Theorem 1.12 in the Introduction.

Theorem 8.4. *Let G be a group acting by isometries on a Gromov hyperbolic space X , let $x \in X$, and let μ be countable, non-elementary, bounded, WPD probability distribution on G . Then for any $K \geq 0$, there are constants $N > 0$, $B > 0$ and $c < 1$ such that*

$$\mathbb{P}(\#\text{Stab}_K(x, w_n x) \leq N) \geq 1 - Bc^n.$$

Proof. Without loss of generality we assume that $E_G(h)$ preserves a geodesic axis; the general case follows as before by replacing the space X by a quasi-isometric space X' and changing constants. Recall that distance between elements of A in X is a coarse upper bound for the distance in $P_L(A)$. So if an isometry coarsely stabilizes x in X , then it coarsely stabilizes A_h in $P_L(A)$. By linear progress with exponential decay, the distance $d_{P_L}(A_h, w_n A_h)$ grows linearly with exponential decay. As the

action of G on $P_L(\mathcal{A})$ is acylindrical, the probability that the coarse stabilizer of A_h and $w_n A_h$ is bounded tends to one exponentially quickly, so this also holds for the coarse stabilizer of x and $w_n x$.

We now make this precise. As the action of G on the projection complex $P_L(\mathcal{A})$ is acylindrical, there are functions R_{ac} and N_{ac} such that for all $K \geq 0$, and all A and B in $P_L(\mathcal{A})$ with $d_{P_L}(A, B) \geq R_{ac}(K)$, we have

$$\#|\text{Stab}_K^P(A, B)| \leq N_{ac}(K).$$

Let A_h be the geodesic axis of h . Then if $d_X(x, fx) \leq K$, then by the triangle inequality $d_X(A_h, fA_h) \leq K + 2d_X(x, A_h)$. Recall that by Proposition 7.6, distance between elements of A in X is a coarse upper bound for the distance in $P_L(\mathcal{A})$. In particular, if we set $K' = (K + 2d_X(x, A_h))/Q$, where Q is from Proposition 7.6, then $d_{P_L}(A_h, fA_h) \leq K'$.

This implies that if $f \in \text{Stab}_K^X(x, w_n x)$, then $f \in \text{Stab}_{K'}^P(A_h, w_n A_h)$. By linear progress with exponential decay (Proposition 2.10), the probability that $d_{P_L}(A_h, w_n A_h) \geq R_{ac}(K')$ tends to one exponentially quickly. Therefore the probability that $\#|\text{Stab}_{K'}^P(A_h, w_n A_h)| \leq N_{ac}(K')$ tends to one with exponential decay, and so the probability that $\#|\text{Stab}_K^X(x, w_n x)| \leq N_{ac}(K')$ also tends to one exponentially quickly, as required. \square

9. NON-MATCHING ESTIMATES

So far, we have established generic properties of our random walks by proving *matching estimates*, i.e. by showing that with high probability there is a subsegment of the sample path that follow travels some given element. However, in order to establish our results on the normal closure, we need to prove that the probability of such a matching to occur too often is not so high: we call this a *non-matching estimate*. Note that, while matching happens for random walks on any group of isometries of a hyperbolic space, to prove non-matching one uses crucially the WPD property (and in fact, non-matching may not hold in the non-WPD case, for example, for a dense subgroup of $SL(2, \mathbb{R})$ acting on \mathbb{H}^2).

We now define notation for the nearest point projection of a location $w_m x$ of the random walk to a geodesic γ_n from x to $w_n x$.

Definition 9.1. Given integers $0 \leq m \leq n$, let γ_n be a geodesic from x to $w_n x$, and let $\gamma_n(t_m)$ be a nearest point on γ_n to $w_m x$.

The main non-matching estimate is the following proposition, which says that the probability that γ_n contains in its neighbourhood a translate of a given geodesic segment η starting at $\gamma_n(t_m)$ is bounded above by an exponential function of $|\eta|$. We will prove it by using the asymptotic acylindricity property established in the previous section.

Proposition 9.2. *Given a constant $\delta \geq 0$ there is a constant $K_0 \geq 0$ with the following properties. Let G be a group which acts by isometries on the δ -hyperbolic space X , and let μ be a countable, bounded probability distribution on G , such that the random walk generated by μ is asymptotically acylindrical with exponential decay.*

Then for any constant $K \geq K_0$ there are constants $B > 0$ and $c < 1$, such that for any geodesic segment η and any integers $m \geq 0$, $n \geq 0$, the probability that a

G -translate of η is contained in a K -neighbourhood of $[\gamma_n(t_m), \gamma_n(t_m + |\eta|)]$ is at most $Bc^{|\eta|}$.

Before embarking on the details, we give a brief overview of the contents of this section. Fix a geodesic segment η of length $2s$. We wish to estimate the probability that some translate of η is contained in a neighbourhood of $[\gamma_n(t_m), \gamma_n(t_m + 2s)]$. Let $U \subset (G, \mu)^{\mathbb{Z}}$ be the event that some translate of η is contained in a neighbourhood of $[\gamma_n(t_m), \gamma_n(t_m + 2s)]$, and let V be the event that some translate of the first half of η is contained in a neighbourhood of $[\gamma_n(t_m), \gamma_n(t_m + s)]$. Since $U \subseteq V$, the conditional probability of U given V satisfies $\mathbb{P}(U) = \mathbb{P}(U \cap V) \leq \mathbb{P}(U \mid V)$. Let U_g be the event that a specific translate $g\eta$ is contained in a neighbourhood of $[\gamma_n(t_m), \gamma_n(t_m + 2s)]$, and let V_g be the event that the first half of $g\eta$ is contained in a neighbourhood of $[\gamma_n(t_m), \gamma_n(t_m + s)]$. The event U is the union of the events U_g , and the event V is the union of the events V_g . It follows from exponential decay of shadows that $\mathbb{P}(U_g \mid V_g)$ decays exponentially in s . In order to use this fact to estimate $\mathbb{P}(U \mid V)$ we need the following extra information: it follows from asymptotic acylindricity that with high probability any point of V is contained in a bounded number of sets V_g , and this is enough for the exponential decay in s of $\mathbb{P}(U_g \mid V_g)$ to imply exponential decay in s of $\mathbb{P}(U \mid V)$.

We now give the details of the results discussed above. We will need information about the distribution of the nearest point projections of the locations $w_m x_0$ of the random walk to the geodesic γ_n , and we start with the following estimate on Gromov products, which follows directly from exponential decay of shadows.

Proposition 9.3. *Let G be a group acting by isometries on a Gromov hyperbolic space X , and let μ be a countable, non-elementary, bounded probability distribution on G . Then there are constants B and $c < 1$ such that for all $0 \leq i \leq n$ and for any $r \geq 0$,*

$$\mathbb{P}((x \cdot w_n x)_{w_i x} \geq r) \leq Bc^r.$$

Proof. If $(x \cdot w_n x)_{w_i x} \geq r$, then x lies in a shadow $S_{w_i}(w_n x, R)$, with $d(w_i x, w_n x) - R \geq r + O(\delta)$. The random variables w_i and $w_i^{-1} w_n$ are independent, so by exponential decay of shadows [MT18, eq. (16)], this occurs with probability at most $Bc^{r+O(\delta)}$. \square

Linear progress for the locations of the sample path $w_m x_0$ in X , and exponential decay for the distribution of the Gromov products $(x_0 \cdot w_n x_0)_{w_m x_0}$ imply that the points $\gamma_n(t_m)$ are reasonably evenly distributed along $\gamma_n = [x_0, w_n x_0]$. We now make this precise. As μ has bounded support in X , there is a constant D such that any point in γ_n lies within distance at most D from a nearest point projection $\gamma_n(t_i)$ of one of the locations of the random walk $w_i x$, for $0 \leq i \leq n$, and furthermore, we may choose D to be an upper bound for the diameter of the support of μ in X . For any constant $s \geq 0$, let P_s be the collection of indices $0 \leq i \leq n$ such that $t_i \in [s, s + D]$. This collection is non-empty if $s \leq |\gamma_n|$. We emphasize that P_s only contains indices between 0 and n , there may be other locations of the bi-infinite random walk which have nearest point projections to γ_n contained in $[\gamma(s), \gamma(s + D)]$, and we consider this separately in Proposition 9.5 below.

Proposition 9.4. *Let G be a group which acts by isometries on the hyperbolic space X , and let μ be a countable, non-elementary, bounded probability distribution*

on G . Then there are constants $0 < L_1 \leq L_2$, $B \geq 0$ and $c < 1$ such that for any $s > 0$ and any $n \geq 0$,

$$\mathbb{P}(P_s \subseteq [L_1s, L_2s]) \geq 1 - Bc^s.$$

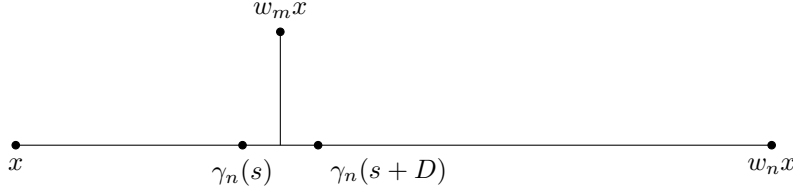


FIGURE 5. The set P_s defined right before Proposition 9.4. The index m belongs to P_s as its projection to $[x, w_n x]$ lies within distance s and $s + D$ from the basepoint.

Proof. If $s > d(x, w_n x)$, then $P_s = \emptyset$, and the statement follows immediately, so we may assume that $\gamma_n(s)$ determines a point in γ_n .

By linear progress with exponential decay (Proposition 2.10), there are constants $L > 0, B_1 \geq 0$ and $c_1 < 1$ such that for any $m \geq 0$

$$\mathbb{P}(d(x, w_m x) \leq Lm) \leq B_1 c_1^m.$$

Therefore, by summing the geometric series we get

$$\mathbb{P}(d(x, w_m x) \leq Lm \text{ for any } m \geq N) \leq \frac{B_1}{1 - c_1} c_1^N.$$

In particular, there are constants B_2 and $c_2 < 1$ such that

$$\mathbb{P}(d(x, w_m x) \geq Lm \text{ for all } m \geq 2s/L) \geq 1 - B_2 c_2^s. \quad (12)$$

If (12) holds, and if $m \geq 2s/L$, then $d(x, w_m x) \geq Lm \geq 2s$, so by thin triangles and the definition of the Gromov product, if the nearest point projection $\gamma(t_m)$ of $w_m x$ lies in $[\gamma_n(s), \gamma_n(s + D)]$, then

$$(x \cdot w_n x)_{w_m x} \geq d(x, w_m x) - s - D - O(\delta). \quad (13)$$

By exponential decay for Gromov products (Proposition 9.3), there are constants B_3 and c_3 such that $\mathbb{P}((x \cdot w_n x)_{w_m x} \geq r) \leq B_3 c_3^r$. In particular,

$$\mathbb{P}((x \cdot w_n x)_{w_m x} \geq Lm - s - D - O(\delta)) \leq B_3 c_3^{Lm - s - D - O(\delta)}.$$

This implies that there are constants B_4 and $c_4 < 1$ such that for any n

$$\mathbb{P}((x \cdot w_n x)_{w_m x} \geq Lm - s - D - O(\delta) \text{ for any } m \geq 2s/L) \leq B_4 c_4^s. \quad (14)$$

Except for a set of probability at most $B_2 c_2^s + B_4 c_4^s$, we may assume that (12) holds, and (14) does not hold. Equation (13) then implies that $\gamma(t_m)$ does not lie in $[\gamma_n(s), \gamma_n(s + D)]$ for all $m \geq 2s/L$. This gives the required upper bound, with $L_2 = 2/L$, and suitable choices of B and c . As μ has bounded support in X , the lower bound may be chosen to be $L_1 = 1/D$. \square

We now obtain estimates for the nearest point projections of the remaining locations of the random walk $w_m x$ to a geodesic $\gamma_n = [x, w_n x]$, i.e. for those indices $m \leq 0$ and $m \geq n$.

Proposition 9.5. *Let G be a group which acts by isometries on the hyperbolic space X , and let μ be a countable, non-elementary, bounded probability distribution on G . Then there are constants B and c such that for all $s \geq 0$ the probability that all of the nearest point projections of $\{w_m x : m \leq 0\}$ to $\gamma_n = [x, w_n x]$ are contained within distance s of the initial point x , and all of the nearest point projections of $\{w_m x : m \geq n\}$ to γ_n are contained within distance s of the terminal point $w_n x$, is at least $1 - Bc^s$.*

Proof. By the Markov property, the backward random walk $(w_{-n}x)_{n \in \mathbb{N}}$ is independent of γ_n . Similarly, the forward random walk starting at $w_n x$ is also independent of γ_n . More precisely, applying the isometry w_n^{-1} , the random walk $w_n^{-1}(w_m x)_{m \geq n}$ starting at x , is independent of $w_n^{-1}\gamma_n$. Therefore, it suffices to show that for any geodesic ray γ starting at x , a random walk has nearest point projection to an initial segment of γ with high probability.

Let γ be a geodesic ray starting at x , with unit speed parameterization, and consider the forward locations of the random walk $(w_n x)_{n \in \mathbb{N}}$. Let $\gamma(t_n)$ be the nearest point projection of a location $w_n x$ to γ . If $t_n \geq s$, then $w_n x$ lies in the shadow $S_x(\gamma(s), R)$, for some R which only depends on δ . By (5) the probability that $(w_n)_{n \in \mathbb{Z}}$ ever hits $S_x(\gamma(s), R)$ is at most Bc^s . Therefore the probability that this does not occur for any index n is at least $1 - Bc^s$. \square

We now consider the following situation: we have chosen an index $0 \leq m \leq n$, and a constant $s \geq 0$. We wish to estimate the probability that there is a translate of a geodesic η of length $2s$ close to γ_n starting at $\gamma_n(t_m)$. In order to do this, it will be convenient to have information about the distribution of the nearest point projections of $w_k x_0$ to γ_n , and in particular, the sets P_{t_m+s} and P_{t_m+2s} . Proposition 9.6 below assembles the geometric information we need from all of the results above, and in particular shows that with high probability, there are linear bounds on the sizes of P_{t_m+s} and P_{t_m+2s} , and that these sets are disjoint.

Proposition 9.6. *Let G be a group which acts by isometries on the hyperbolic space X , and let μ be a countable, non-elementary, bounded probability distribution on G . Then there are constants $0 < L_1 \leq L_2$, such that for any $0 < \epsilon < 1$, there are constants $B \geq 0$ and $c < 1$ such that for any $0 \leq m \leq n$ and $s > 0$, the probability that all of the following events occur is at least $1 - Bc^s$:*

$$(x \cdot w_n x)_{w_m x} \leq \epsilon s \tag{9.6.1}$$

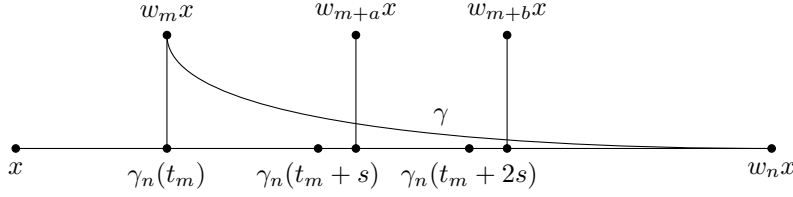
$$L_1 s \leq \min P_{t_m+s} \leq \max P_{t_m+s} \leq L_2 s \tag{9.6.2}$$

$$2L_1 s \leq \min P_{t_m+2s} \leq \max P_{t_m+2s} \leq 2L_2 s \tag{9.6.3}$$

$$(x \cdot w_n x)_{w_i x} \leq \epsilon s \text{ for all } i \in P_{t_m+s} \cup P_{t_m+2s} \tag{9.6.4}$$

$$\max P_{t_m+s} \leq \min P_{t_m+2s} \tag{9.6.5}$$

The proposition is illustrated in Figure 6 below, where the index $m+a$ belongs to P_{t_m+s} , and $m+b$ belongs to P_{t_m+2s} .

FIGURE 6. Nearest point projections relative to $\gamma_n(t_m)$.

Proof. We say that a function $\mathcal{E}(s): \mathbb{R} \rightarrow \mathbb{R}$ is *exponential in s* if there are constants $B \geq 0$ and $c < 1$ such that $\mathcal{E}(s) \leq Bc^s$ for all $s \geq 0$. We observe that the sum of any two functions which are exponential in s is exponential in s , and if $p(s)$ is a polynomial in s , and $\mathcal{E}(s)$ is exponential in s , then $p(s)\mathcal{E}(s)$ is also exponential in s .

By exponential decay for Gromov products (Proposition 9.3), eq. (9.6.1) holds with probability at least $1 - \mathcal{E}_1(s)$, where $\mathcal{E}_1(s) = Bc^s$.

Let γ be a geodesic from $w_m x$ to $w_n x$, with unit speed parameterization, and write $\gamma(t_k)$ for a nearest point projection of $w_k x$ to γ . By the Markov property, we may apply Proposition 9.5 to γ , and so there are constants $B \geq 0$ and $c < 1$ such that the probability that

$$\{\gamma(t_k) : k \in \mathbb{Z}, k \leq m\} \subset [w_m x, \gamma(s/2)] \quad (15)$$

holds with probability at least $1 - \mathcal{E}_2(s)$, where $\mathcal{E}_2(s) = Bc^s$.

By thin triangles and assuming that $(x \cdot w_n)_{w_m x} \leq \epsilon s$, if the nearest point projection to γ_n of a location $w_{m+a} x$ lies in $[\gamma_n(t_m + s), \gamma_n(t_m + s) + D]$, then the nearest point projection of $w_{m+a} x$ to γ lies in $[\gamma(s), \gamma(s + \epsilon s + D + \delta)]$. Proposition 9.4 applied to each of the $(\epsilon s + \delta)/D$ subsegments of $[s, s + \epsilon s + D + \delta]$ of length D implies that $L_1 s \leq a \leq L_2(s + \epsilon s + D + \delta)$ with probability at least $1 - \mathcal{E}_3(s)$, where $\mathcal{E}_3(s) = ((\epsilon s + \delta)/D)Bc^s$. Therefore (9.6.2) holds (with a slightly larger value of L_2). Furthermore, by (15) there are no locations $w_k x$ with $k \leq m$ or $k \geq n$ which have nearest point projections in $[\gamma(s), \gamma(s + \epsilon s + D + \delta)]$.

The exact same argument works for (9.6.3), as long as $t_m + 5s/2 \leq |\gamma|$.

Exponential decay for Gromov products then implies (9.6.4) with probability at least $1 - \mathcal{E}_4(s)$, where $\mathcal{E}_4(s) = 3(L_2 - L_1)sBc^{\epsilon s}$. The constant $3(L_2 - L_1)s$ here derives from the cardinality of $P_{t_m+s} \cup P_{t_m+2s}$ when (9.6.2) and (9.6.3) hold.

Finally, if there is some $b < a$, then $(x \cdot w_{m+a} x)_{w_{m+b} x} \geq s - D + O(\delta)$, and so the probability that this does not occur for any a and b (i.e. (9.6.5) holds) is at least $1 - \mathcal{E}_5(s)$, where $\mathcal{E}_5(s) = 3(L_2 - L_1)sBc^{s-D+O(\delta)}$.

Therefore all equations (9.6.1)–(9.6.5) hold with probability at least $1 - \mathcal{E}(s)$, where $\mathcal{E}(s)$ is the sum of the functions $\mathcal{E}_1(s)$ – $\mathcal{E}_5(s)$ above. All of these functions are exponential in s , so $\mathcal{E}(s)$ is also exponential in s , as required. \square

We now show that for any fixed translate $g\eta$ of a geodesic η of length $2s$, if the first half of η is contained in a neighbourhood of $[\gamma_n(t_m), \gamma_n(t_m + s)]$, then the probability that η is contained in a neighbourhood of $[\gamma_n(t_m), \gamma_n(t_m + 2s)]$ decays exponentially in s .

Proposition 9.7. *Let G be a group which acts by isometries on the hyperbolic space X , and let μ be a countable, non-elementary, bounded probability distribution*

on G . Then there are constants $B \geq 0$ and $c < 1$ such that for any geodesic segment η of length $2s$ with initial half-segment η_1 of length s , if there is an isometry $g \in G$ such that $g\eta_1$ is contained in a K -neighbourhood of $[\gamma_n(t_m), \gamma_n(t_m + s)]$, then the probability that $g\eta$ is contained in a K -neighbourhood of $[\gamma_n(t_m), \gamma_n(t_m + 2s)]$ is at most Bc^s .

Proof. By Proposition 9.6, there are constants B_1 and $c_1 < 1$ such that (9.6.1)–(9.6.5) hold, with probability at least $1 - B_1c_1^s$.

If $g\eta_1$ is contained in a K -neighbourhood of $[\gamma_n(t_m), \gamma_n(t_m + s)]$, then in order for η to be contained in a K -neighbourhood of $[\gamma_n(t_m), \gamma_n(t_m + 2s)]$, for any index $m + b \in P_{t_m+2s}$ the point $w_{m+b}x$ must lie in a shadow $S_{w_{m+a}x}(g\eta(2s), R)$, where R depends only on K and δ . As w_{m+a} and $w_{m+a}^{-1}w_{m+b}$ are independent, and there are at most $2(L_2 - L_1)s$ elements of P_{t_m+2s} , this happens with probability at most $2(L_2 - L_1)sB_2c_2^s$, by exponential decay for shadows. The result then follows for suitable choices of B and c . \square

Proposition 9.7 above only holds for a *fixed* translate $g\eta$. We will use asymptotic acylindricity to extend this result to hold for *some* translate $g\eta$, where g runs over all elements of G . We start with a result from Calegari and Maher [CM15], which says that every point in γ_n is close to some location w_kx_0 . We say that a point $\gamma(t) \in \gamma_n$ is K -close if $d(\gamma(t), w_ix) \leq K$ for some $0 \leq i \leq n$. We shall denote the set of K -close points by $\gamma_{n,K}$.

Lemma 9.8. [CM15, Lemma 5.13] *Given $\delta \geq 0$ and positive constants D, L and ϵ , there is a constant $K \geq 0$ such that for any sequence of points x_0, x_1, \dots, x_n in a δ -hyperbolic space X , with $d(x_i, x_{i+1}) \leq D$, and $d(x_0, x_n) \geq Ln$, and for any geodesic γ_n from x_0 to x_n , the total length of $\gamma_{n,K}$ is at least*

$$|\gamma_{n,K}| \geq (1 - \epsilon)|\gamma_n|.$$

Let U be the event that some translate of η is contained in a neighbourhood of $[\gamma_n(t_m), \gamma_n(t_m + 2s)]$, and let V be the event that the first half of some translate of η is contained in a neighbourhood of $[\gamma_n(t_m), \gamma_n(t_m + s)]$. We wish to estimate $\mathbb{P}(U)$. However, as $U \subset V$, the formula for conditional probability implies that $\mathbb{P}(U) \leq \mathbb{P}(U | V)$, so it suffices to estimate $\mathbb{P}(U | V)$.

Let U_g be the event that the translate $g\eta$ is contained in a neighbourhood of $[\gamma_n(t_m), \gamma_n(t_m + 2s)]$, and let V_g be the event that the first half of the translate $g\eta$ is contained in a neighbourhood of $[\gamma_n(t_m), \gamma_n(t_m + s)]$. The set U is equal to the union of the U_g , and similarly V is equal to the union of the V_g . For each g , we have $\mathbb{P}(U_g | V_g) \leq Bc^s$, by Proposition 9.7. We wish to use this information to estimate $\mathbb{P}(U | V)$. The key property is that asymptotic acylindricity implies that with high probability each point of V is contained in a bounded number of sets V_g , and so exponential decay for the individual conditional probabilities $\mathbb{P}(U_g | V_g)$ gives exponential decay for $\mathbb{P}(U | V)$. We now give the details of this argument.

Let V and $\{V_i\}_{i \in I}$ be a collection of subsets of a probability space. We say that the collection of sets $\{V_i\}_{i \in I}$ covers the set V if $V \subset \bigcup_{i \in I} V_i$. We say that the covering depth of the $\{V_i\}_{i \in I}$ is $\sup_{v \in V} \#\{i \in I : v \in V_i\}$. If the covering depth of $\{V_i\}_{i \in I}$ is N , and all sets are measurable, then $\mathbb{P}(V) \leq \sum_{i \in I} \mathbb{P}(V_i) \leq N\mathbb{P}(V)$.

We will also make use of the following definition:

Definition 9.9. We say that a pair of points x and y are (K, N) -stable if

$$\#\text{Stab}_K(x) \cap \text{Stab}_K(y) \leq N.$$

We say that a geodesic segment η is (K, N) -stable if its endpoints are (K, N) -stable.

Proof (of Proposition 9.2). Let $s := |\eta|/2$. We wish to estimate the probability that a translate of η is contained in a K -neighbourhood of $[\gamma(t_m), \gamma(t_m + 2s)]$. Let η_1 be the initial subsegment of η with length $|\eta_1| = |\eta|/2 = s$. By Proposition 9.6, we may assume that (9.6.1)–(9.6.5) hold, with probability at least $1 - Bc^s$.

Let us suppose now that a translate $g\eta$ is contained in a K -neighbourhood of $[\gamma(t_m), \gamma(t_m + 2s)]$. By thin triangles, the geodesic $g\eta_1$ is contained in a $(K + 2\delta)$ -neighbourhood of the geodesic $[w_mx, w_{m+a}x]$. By Lemma 9.8, choosing $\epsilon = 1/8$, there is a constant K_1 such that there are indices i and j , with w_ix within distance $K_2 = K_1 + K + 2\delta$ of $[g\eta_1(0), g\eta_1(s/4)]$ and w_jx within distance K_2 of $[g\eta_1(3s/4), g\eta_1(s)]$. In particular $d(w_ix, w_jx) \geq s/2 - 2K_2$, and so

$$|i - j| \geq (s/2 - 2K_2)/D. \quad (16)$$

Set $K_3 = \max\{K_2, 5K\}$ and $K_4 = K_3 + 2K + 2\delta$.

Let $U \subseteq (G, \mu)^{\mathbb{N}}$ be the set of sample paths for which a translate of η is contained in a K -neighbourhood of $[\gamma(t_m), \gamma(t_m + 2s)]$, and let U_g be the set of sample paths for which $g\eta$ is contained in a K -neighbourhood of $[\gamma(t_m), \gamma(t_m + 2s)]$. Let $V \subseteq (G, \mu)^{\mathbb{N}}$ be the set of sample paths for which a translate of η_1 is contained in a K -neighbourhood of $[\gamma(t_m), \gamma(t_m + s)]$, and let V_g be the set of sample paths for which $g\eta_1$ is contained in a K -neighbourhood of $[\gamma(t_m), \gamma(t_m + s)]$. As $U \subseteq V$, the conditional probability $\mathbb{P}(U|V)$ satisfies $\mathbb{P}(U) \leq \mathbb{P}(U|V)$.

Proposition 9.7 shows that for any g the conditional probability $\mathbb{P}(U_g|V_g)$ decays exponentially in n . The sets $\{U_g\}_{g \in G}$ cover U , in fact $U = \bigcup_{g \in G} U_g$, and similarly $V = \bigcup_{g \in G} V_g$. The covering depth of $\{V_g\}$ is an upper bound on the covering depth of $\{U_g\}$. We now show that with high probability the covering depth of $\{V_g\}$ is bounded, i.e. there exists a set S of large measure such that the covering depth of $\{V_g \cap S\}$ is bounded.

We now have two cases. If η_1 is not $(K_3, N_{ac}(K_4))$ -stable, then w_ix and w_jx are not $(K_4, N_{ac}(K_4))$ -stable, where $N_{ac}(K)$ is the function from asymptotic acylindricity. Then by Theorem 8.4 the probability that, given i and j , the points w_ix and w_jx are not $(K_4, N_{ac}(K_4))$ -stable is at most $Bc^{|j-i|} \leq B_3c_3^{s/2D}$ for some constants B_3 and $c_3 < 1$, where we used eq. (16). Recall that by construction $m \leq i \leq j \leq m + a$, and by (9.6.2) we have $a \leq L_2s$, hence there are at most $(L_2s)^2$ such choices of i, j . Hence, the probability that there are such indices i and j is at most $2(L_2s)^2 B_3c_3^{s/2D}$.

If η_1 is $(K_4, N_{ac}(K_4))$ -stable, then by definition the covering depth of V_g is at most $N_{ac}(K_4)$. By Proposition 9.7, there are constants B_4 and $c_4 < 1$ such that $\mathbb{P}(U_g|V_g) \leq B_4c_4^s$. As $U_g \subseteq V_g$, this implies $\mathbb{P}(U_g) \leq B_4c_4^s \mathbb{P}(V_g)$. Therefore

$$\mathbb{P}(U) \leq \sum_{g \in G} \mathbb{P}(U_g) \leq B_4c_4^s \sum_{g \in G} \mathbb{P}(V_g) \leq N_{ac}(K_4) B_4c_4^s \mathbb{P}(V) \leq N_{ac}(K_4) B_4c_4^s.$$

Therefore, the probability that a translate of η is contained in a K -neighbourhood of $[\gamma_n(t_m), \gamma_n(t_m + s)]$ is at most $Bc^s + 2(L_2s)^2 B_3c_3^{s/2D} + N_{ac}(K_4) B_4c_4^s$, which has exponential decay in s , as required. \square

We are now interested in the particular case of matching between two subsegments of a given geodesic segment. We call this phenomenon a *self-match*. Here is the precise definition.

Definition 9.10. We say that a geodesic segment γ has an (L, K) -self match if there exist two disjoint subsegment $\eta, \eta' \subseteq \gamma$ of length L and an element $g \in G \setminus \{1\}$ such that the Hausdorff distance between $g\eta$ and η' is at most K .

Proposition 9.11. *Let G be a group acting by isometries on a Gromov hyperbolic space X , and let μ be a countable, non-elementary, bounded probability distribution on G , such that the random walk generated by μ is asymptotically acylindrical with exponential decay. Then there is a constant K_0 , depending only on δ , such that for any $K \geq K_0$, there exists $B > 0$ such that for any $L \geq 0$ and any $n \geq 0$ the probability that γ_n has an (L, K) -self match is at most $n^3 B c^L$.*

Proof. Suppose that γ_n has an (L, K) -self-match. Then there is a subgeodesic $\eta = [\gamma_n(t), \gamma_n(t+L)]$ such that a translate $g\eta$ is contained in a K -neighbourhood of γ_n , and the nearest point projection of $g\eta$ to γ_n is disjoint from η . Without loss of generality, we may assume that the translate of η is contained in a K -neighbourhood of $[\gamma_n(t+L), \gamma_n(|\gamma_n|)]$.

There is a constant D such that the nearest point projection of the sample path $\{w_m x : 0 \leq m \leq n\}$ to γ_n is D -coarsely onto, and the diameter of the support of μ in X is at most D . Let $w_m x$ be a location of the random walk such that the nearest point projection $\gamma_n(t_m)$ lies within distance D of the interval of γ_n between η and the nearest point projection of $g\eta$.

Then η is contained in a $(K + D + \delta)$ -neighbourhood of $[x, w_m x]$, and $g\eta$ is contained in a $(K + D + \delta)$ -neighbourhood of $[w_m x, w_n x]$. We do not need to consider all possible subsegments of $[x, w_m x]$, as it suffices to consider those whose endpoints are integer distances from x . More precisely, there is a subsegment $\eta_- = [\gamma_n(a), \gamma_n(b)]$ of η , for integers $a \leq b$, with $|\eta_-| \geq |\eta| - 2$. If we set $K_1 := K + D + \delta + 1$, then the geodesic η_- K_1 -matches $\gamma' = [w_m x, w_n x]$ at distance $\gamma'(c)$ from $w_m x$, where c is also an integer.

There are at most n choices for m , at most $d(x, w_m x) \leq Dm \leq Dn$ choices for a , and at most $d(w_m x, w_n x) \leq D(n - m) \leq Dn$ choices for c , so in total at most $D^2 n^3$ choices for the triple (m, a, c) . Given a triple of choices m, a and c , and the constant K_1 , Proposition 9.2 implies that there are constants B_1 and c_1 such that the probability that a translate of η_- is contained in a K_1 -neighbourhood of $[w_m x, w_n x]$ is at most $B_1 c_1^{L-2D}$. Therefore the probability that γ_n has an (L, K) -self-match is at most $D^2 n^3 B_1 c_1^{L-2D}$, and the result follows by suitable choices of B and c (since D is a constant). \square

We will use the following result due to Dahmani and Horbez [DH18, Proposition 2.5]: they do not explicitly state the rate, but it follows immediately from the proof.

Proposition 9.12. *Given δ and K_1 there is a constant K with the following properties. Let G be a group acting on a δ -hyperbolic space X , and let μ be a countable, non-elementary, bounded probability distribution on G . Let $\ell > 0$ be the drift of the random walk generated by μ . If w_n is loxodromic, let p denote a nearest point projection of x to a quasi-axis for w_n . Then there exist constants $B > 0$, $c < 1$ such that for any $\epsilon > 0$ we have*

$$\mathbb{P}(\gamma_n \text{ has a } ((\ell - \epsilon)n, K)\text{-match with } [p, w_n p]) \geq 1 - Bc^{\epsilon n}.$$

Finally, we record the following result, which is an immediate consequence of Propositions 9.11 and Proposition 9.12 above.

Corollary 9.13. *For any $\delta \geq 0$, there is a constant K_0 with the following properties. Let G be a group acting by isometries on a δ -hyperbolic space X , and let μ be a countable, non-elementary, bounded probability distribution on G , such that the random walk generated by μ is asymptotically acylindrical with exponential decay. Let $\ell > 0$ be the drift for μ , and let p be a point on a quasixis for w_n .*

Then for any $K \geq K_0$ and $\epsilon > 0$, there are constants B and $c < 1$ such that for any $n \geq 0$ the probability that either $\gamma_n = [x, w_n x]$ or $[p, w_n p]$ has an $(\epsilon n, K)$ -self match is at most Bc^n .

10. ASYMMETRIC ELEMENTS

We now use the non-matching results to show that a generic element is asymmetric in the following sense. This definition is a variation of the one used in [MS19], where similar results are obtained in the case that the action is acylindrical.

Definition 10.1. We say that a loxodromic isometry $g \in G$ is (ϵ, L, K) -asymmetric if for any subsegment $[p, q] \subset \alpha_g$ of length at least $\epsilon d(p, gp)$, and any group element h , if $h[p, q]$ is contained in an L -neighbourhood of α_g , then there is an $i \in \mathbb{Z}$ such that $d(hp, g^i p) \leq K$ and $d(hq, g^i q) \leq K$.

Proposition 10.2. *Given a constant $\delta \geq 0$, for any constants $\epsilon > 0$ and $L \geq 0$, there is a constant K such that if G is a group acting on a δ -hyperbolic space X , and μ is a countable, non-elementary, bounded probability distribution on G , such that the random walk generated by μ is asymptotically acylindrical with exponential decay, then there are constants B and $c < 1$ such that the probability that w_n is (ϵ, L, K) -asymmetric is at least $1 - Bc^n$.*

We first recall the following useful fact about isometries of Gromov hyperbolic spaces.

Proposition 10.3. *Given $\delta \geq 0$ there is a constant K_0 such that for any $K \geq K_0$, if X is a δ -hyperbolic space, and g is an isometry for which there is a point $x \in X$ such that $d(x, gx) \geq 3K$ and $(x \cdot g^2 x)_{gx} \leq K$, then g is loxodromic, and any quasixis α_g of g passes within distance $2K$ of gx .*

Proof. This follows from the following estimate for the translation length of an isometry:

$$\tau(g) \geq d(x, gx) - 2(x \cdot g^2 x)_{gx} - O(\delta),$$

see for example [MT18, Proposition 5.8]. As long as $\tau(g) \geq O(\delta)$, then any path $[x, gx]$ has a subsegment which is contained in an L_1 -neighbourhood of α_g , and so by thin triangles, the distance from gx to α_g is at most $(x \cdot g^2 x)_{gx} + L_1 + O(\delta)$. \square

Let γ_1 and $\gamma_2 = [x, y]$ be two $(1, K_1)$ -quasigeodesics. Parameterizations $\gamma_1: I_1 \rightarrow X$ and $\gamma_2: I_2 \rightarrow X$ determine orientations of γ_1 and γ_2 . Let $x' = \gamma_1(s)$ be a nearest point on γ_1 to x , and let $y' = \gamma_1(t)$ be a nearest point on γ_1 to y . We say these orientations agree if $s < t$ for any choice of nearest points $x' = \gamma_1(s)$ and $y' = \gamma_1(t)$, and we say they disagree if $s > t$ for any choice of nearest points $x' = \gamma_1(s)$ and $y' = \gamma_1(t)$. In any other case we say that the orientation of γ_2 is not well-defined with respect to γ_1 . We omit the proof of the following basic fact.

Proposition 10.4. *Given constants δ, K_1 and L , there is a constant L' with the following properties. Let X be a δ -hyperbolic space, and let γ_1 and γ_2 be $(1, K_1)$ -quasigeodesics in X such that γ_2 is contained in an L -neighbourhood of γ_1 . If the*

length of γ_2 is at least L' , then the orientation of γ_2 either agrees or disagrees with that of γ_1 .

Recall that we say a function $\mathcal{E}(n): \mathbb{N} \rightarrow \mathbb{N}$ is exponential in n if there are constants B and $c < 1$ such that $\mathcal{E}(n) \leq Bc^n$ for all $n \geq 0$. Clearly, if $\mathcal{E}_1(n)$ is exponential in n , and $\mathcal{E}_2(n)$ is exponential in n , then the sum of these two functions is exponential in n .

We may now complete the proof of Proposition 10.2.

Proof of Proposition 10.2. If $L' \geq L$, then $N_L(\alpha_g) \subseteq N_{L'}(\alpha_g)$, so if the result holds for some K' and L' , it also holds for K' and L . Therefore, without loss of generality we may assume that $L \geq 1 + \delta$.

Let α_{w_n} be a quasiaxis for w_n , and let x' be the nearest point projection of the basepoint x to α_{w_n} . If the result holds for some $\epsilon > 0$, it holds for any larger value of ϵ , so we may assume that $\epsilon \leq 1$. Furthermore, as α_{w_n} is w_n -invariant, after translating by a power of w_n , and possibly replacing ϵ by $\epsilon/2$, we may assume that $w_n^i[p, q]$ is contained in $[x', w_n x']$. By abuse of notation, we will relabel $w_n^i[p, q]$ as $[p, q]$.

If $h[p, q]$ is contained in a L -neighbourhood of α_{w_n} , then as α_{w_n} is w_n -invariant, then after replacing h by $w_n^k h$, we may assume that the nearest point projection of $h[p, q]$ to α_{w_n} is contained in $[x', w_n^2 x']$. By abuse of notation, we will relabel $w_n^k h$ as h .

Given L , let L' be the constant from Proposition 10.4. As $d(x', w_n x')$ tends to infinity almost surely as n tends to infinity, we may assume that $d(x', w_n x') \geq L'/\epsilon$, and so $d(p, q) \geq L'$. In particular, the orientation of $h[p, q]$ is well defined with respect to α_{w_n} , and either agrees, or disagrees with the orientation of α_{w_n} .

First consider the case in which h reverses the orientation of $[p, q]$ with respect to α_{w_n} , as illustrated below in Figure 7. We will show that if this occurs, it gives a self-match for γ_n which occurs with probability which is at most exponential in n .

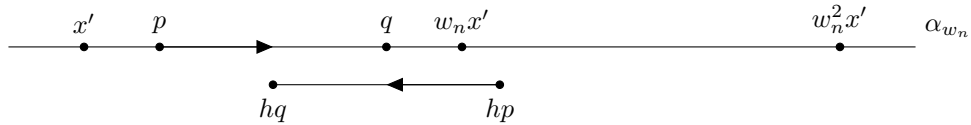


FIGURE 7. An orientation reversing translate of $[p, q]$ close to α_{w_n} .

By replacing $[p, q]$ by either its initial half, or terminal half, we may assume that either $[p, q]$ or $w_n^{-1}[p, q]$ has nearest point projection to α_{w_n} contained in $[x', w_n x']$. Again replacing $[p, q]$ by either its initial half, or terminal half, we may assume that $h[p, q]$ lies within distance K of a disjoint subsegment of $[x', w_n x']$ of length at least $\epsilon d(x', w_n x')/4$. This gives rise to an $(\epsilon d(x', w_n x')/4, K)$ -self match for $[x', w_n x']$.

Let $\ell > 0$ be the linear progress constant for μ , and fix some $0 < \epsilon' < \min\{\ell, 1\}/2$.

The subsegment $[x', w_n x']$ of α_{w_n} is contained in an L_1 -neighbourhood of $[x, w_n x]$, and by Proposition 9.12, given $\epsilon' > 0$, there are constants B_1 and $c_1 < 1$ such that the probability that the length of $[x', w_n x']$ is at least $(\ell - \epsilon')n$ is at least $1 - \mathcal{E}_1(n)$, where $\mathcal{E}_1(n) = B_1 c_1^n$, where ℓ is the linear progress constant for μ .

This gives an $(\epsilon(\ell - \epsilon')n/4, K)$ -self match for $[x, w_n x]$, and by Proposition 9.11, there are constants B_2 and $c_2 < 1$ such that the probability that this occurs is at most $\mathcal{E}_2(n) = B_2 c_2^n$.

Therefore, the existence of an orientation reversing translate of $[p, q]$ occurs with probability at most $\mathcal{E}_1(n) + \mathcal{E}_2(n)$, which is exponential in n , as required.

We now consider the case in which the orientation of $h[p, q]$ agrees with that of α_{w_n} . We may replace $[p, q]$ by either its initial half or terminal half subinterval (in which case replace ϵ by $\epsilon/2$), and possibly replace h by $w_n^{-1}h$, to ensure that the nearest point projection of $h[p, q]$ to α_{w_n} is contained in $[x', w_n x']$. This is illustrated below in Figure 8.

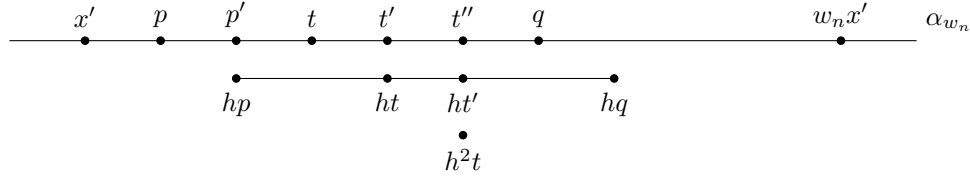


FIGURE 8. An orientation preserving translate of $[p, w_n p]$ close to α_{w_n} .

Let p' be a nearest point on α_{w_n} to hp . If $d(p, p') \geq \epsilon \ell n/10$, then this gives a linear size self-match of $[x, w_n x]$, and again by Proposition 9.11 there are constants B_3 and $c_3 < 1$ such that the probability that this occurs is at most $\mathcal{E}_3(n) = B_3 c_3^n$.

We shall choose a constant $K = 4L + O(\delta)$, but in order to guarantee that there is no circularity in our choice of constants, we now recall some basic facts about Gromov hyperbolic spaces and give an explicit choice of the $O(\delta)$ term in terms of geometric constants which only depend on δ .

Recall that every quasixis is a $(1, K_1)$ -quasigeodesic, where K_1 only depends on δ . Let L_1 be a Morse constant for $(1, K_1)$ -quasigeodesics, i.e. any geodesic $[x, y]$ with endpoints in a $(1, K_1)$ -quasigeodesic α is contained in an L_1 -neighbourhood of α . As K_1 only depends on δ , the Morse constant L_1 also only depends on δ .

Given constants $\delta \geq 0$ and $K_1 \geq 0$ there are constants K_2 and K_3 , such that for any $(1, K_1)$ -quasigeodesic α , and any two points x and y in X , if x' is the nearest point projection of x to α and y' is the nearest point projection of y to α , then if x' and y' are distance at least K_2 apart, then the geodesic from x to y is Hausdorff distance at most K_3 from the piecewise geodesic path $[x, x'] \cup [x', y'] \cup [y', y]$. Furthermore

$$d(x', y') \geq d(x, y) - d(x, x') - d(y, y') - K_3. \quad (17)$$

As K_1 only depends on δ , the constants K_2 and K_3 also only depend on δ . We may now set $K = 4L + 2K_1 + 3K_2 + 3K_3 + 6\delta$.

Now suppose that p' is close to p and the length of $[p, p']$ is greater than K but less than $\epsilon \ell n/10$. Let t be any point in $[p', q]$. Let t' be a nearest point on $[p, q]$ to ht , and let t'' be a nearest point on $[p, q]$ to ht' .

Claim 10.5. We have chosen K sufficiently large such that $d(t, t') \geq K_2$.

Proof. By (17),

$$d(p', t') \geq d(hp, ht) - d(hp, p') - d(ht, t') - K_3.$$

As h is an isometry, and $d(hp, p')$ and $d(ht, t')$ are at most L , this gives

$$d(p', t') \geq d(p, t) - 2L - K_3.$$

The points p, p', t and t' all lie on the $(1, K_1)$ -quasigeodesic α_{w_n} , which implies $d(p', t) + d(t, t') \geq d(p', t') - K_1$, and $d(p, t) \geq d(p, p') + d(p', t) - K_1$. This yields

$$d(t, t') \geq d(p, p') - 2L - 2K_1 - K_3.$$

Our choice of K therefore guarantees that $d(t, t') \geq K_2$, as required. In fact $d(t, t') \geq 2L + K_2 + K_3 \geq K_2$, and we will now use this stronger bound to obtain a bound on $d(t', t'')$. \square

Claim 10.6. We have chosen K sufficiently large such that $d(t', t'') \geq K_2$.

Proof. By (17),

$$d(t', t'') \geq d(ht, ht') - d(ht, t') - d(ht', t'') - K_3.$$

as h is an isometry, and $d(ht, t')$ and $d(ht', t'')$ are at most L , this gives

$$d(t', t'') \geq d(t, t') - 2L - K_3.$$

Our choice of K then implies that $d(t', t'') \geq K_2$, as required. \square

As $d(t', t'') \geq K_2 + L$, the geodesic from ht to h^2t passes within distance K_3 of $[t', t'']$, the Gromov product $(t \cdot h^2t)_{ht}$ is at most $K_4 := L + K_2 + K_3 + 2\delta$. We have chosen K sufficiently large such that $d(t, ht) \geq 3K_4$, and so Proposition 10.3 implies that h is loxodromic, and any quasixis of h passes within distance $2K_4$ of α_{w_n} .

As we have assumed that $\tau(h) \leq \epsilon n/10$, this gives a $(\epsilon n/10, 2K_4)$ -self match of $[x', w_n x']$, and hence of $\gamma_n = [x, w_n x]$, and so again by Proposition 9.11 there are constants B_4 and $c_4 < 1$ such that the probability that this occurs is at most $\mathcal{E}_4(n) = B_4 c_4^n$.

Therefore, we have shown that the case of an orientation preserving translate of $[p, q]$ occurs with probability at most $\mathcal{E}_3(n) + \mathcal{E}_4(n)$, which is exponential in n , as required. \square

11. SMALL CANCELLATION AND NORMAL CLOSURE

We will now prove results on the normal closure (Theorems 1.4 and 1.5 in the Introduction). In order to do so, we will use the following notions of *small cancellation* from [DGO17]. We remark that the small cancellation results in this section were previously obtained in the case of acylindrical actions by Maher and Sisto [MS19], using work of Hull [Hul16], and we further extend their methods to the case of WPD actions. If $H \subseteq G$ is a subgroup, we define its *injectivity radius* as

$$\text{inj}(H) := \inf\{d(gx, x) : g \in H \setminus \{1\}, x \in X\}.$$

Let \mathcal{R} be a family of loxodromic elements which is closed under conjugation. We define its *injectivity radius* as

$$\text{inj}(\mathcal{R}) := \inf_{g \in \mathcal{R}} \inf\{d(g^k x, x), k \in \mathbb{Z} \setminus \{0\}, x \in X\}.$$

In particular, if g is loxodromic and $\mathcal{R} := \{hgh^{-1}, h \in G\}$ is the set of conjugates of g , then

$$\text{inj}(\mathcal{R}) \geq \tau(g).$$

Following [DGO17], for a loxodromic element g , let $\text{Ax}(g)$ be the 20δ -neighbourhood of set of points x for which $d(x, gx) \leq \inf_{y \in X} d(y, gy) + \delta$. If $\tau(g)$ is sufficiently large, then this set is contained in a bounded neighbourhood of a quasixis α_g for g .

Proposition 11.1. *Given $\delta \geq 0$, there are constants A and K , such that if g is a loxodromic isometry of δ -hyperbolic space X with quasixis α_g and $\tau(g) \geq A$, then $\text{Ax}(g) \subset N_K(\alpha_g)$. Furthermore, $\text{Ax}(g)$ is 10δ -quasiconvex.*

Proof. Let x be a point in X , and let p be a nearest point on α_g to x . As we may assume that α_g is g -invariant, gp is a nearest point on α_g to gx , and $d(p, gp) \geq \tau(g)$. Given δ , there are constants A_1 and K_1 such that if $d(p, gp) \geq A_1$, then the union of the three geodesic segments $[x, p]$, $[p, gp]$ and $[gp, gx]$ is contained in a bounded neighbourhood of a geodesic $[x, gx]$, and in particular,

$$d(x, gx) \geq d(x, p) + d(p, gp) + d(gp, gx) - K_1.$$

This is an elementary application of thin triangles, see for example [MT18, Proposition 2.3] for the geodesic case. As the quasigeodesics constants for the quasixis α_g only depend on δ , A_1 and K_1 may also be chosen to only depend on δ . Therefore, if $d(x, p) \geq B_1 + \delta$ then x does not lie in $\text{Ax}(x)$, so we may choose $A = A_1$ and $K = K_1 + \delta$.

For the final statement, see for example Coulon [Cou16, Proposition 3.10]. \square

We also define, for g and h loxodromic,

$$\Delta(g, h) := \text{diam} (N_{20\delta}(\text{Ax}(g)) \cap N_{20\delta}(\text{Ax}(h)))$$

where $N_R(Y)$ denotes the R neighbourhood of the set Y in X .

Recall that $E_G(h)$ is the maximal virtually cyclic subgroup containing h , which is equal to the stabilizer of the endpoints $\{\lambda_h^-, \lambda_h^+\}$ of h in ∂X . We now record the following elementary property of $E_G(h)$, that the image of this group in X under the orbit map intersects any bounded set in only finitely many points.

Lemma 11.2. *Let G be a group acting on a Gromov hyperbolic space X which contains a loxodromic isometry h , and let H be a subgroup of G which contains $\langle h \rangle$ as a finite index subgroup. Then for any $x \in X$ and $K \geq 0$, there is an N such that $\#|Hx \cap B_K(x)| \leq N$.*

Proof. As $\langle h \rangle$ is a finite index subgroup of H , there is a finite set of group elements F such that H is a finite union of right cosets $\langle h \rangle f$, for $f \in F$. In particular, any element $g \in H$ may be written as $g = h^k f$, for some $k \in \mathbb{N}$ and $f \in F$. By the triangle inequality, $d(x, gx) \geq d(x, h^k x) - d(x, fx)$. The distances $d(x, fx)$ have an upper bound depending on F and x , and $d(x, h^k x) \geq k\tau(h)$, so there are only finitely many group elements $g \in H$ with $d(x, gx) \leq K$. \square

Let g be a loxodromic element in G . We shall write $E_G^\pm(g)$ for the orientation preserving subgroup of $E_G(g)$, i.e. the subgroup which stabilizes λ_g^+ and λ_g^- pointwise. This group is either equal to $E_G(g)$ or has index two in $E_G(g)$. There are elements g with $E_G(g) = E_G^\pm(g)$, and in fact they are generic.

Corollary 11.3. *Let G be a group acting by isometries on a δ -hyperbolic space X , and let μ be a countable, non-elementary, bounded probability distribution on G . Then there are constants B and $c < 1$ such that the probability that w_n is loxodromic with $E_G(w_n) = E_G^\pm(w_n)$ is at least $1 - Bc^n$.*

Proof. If $E_G^+(w_n)$ is index two in $E_G(w_n)$, then there is an element f which reverses the orientation of α_{w_n} . This gives an $(\ell n/4, K)$ -self match of $[p, w_n p]$, where $\ell > 0$ is the positive drift constant for μ , and K is the fellow travelling constant from Proposition 2.2. However by Corollary 9.13, there are constants B and $c < 1$ such that the probability that this occurs is at most Bc^n . \square

An essential feature of asymmetric elements is the following.

Proposition 11.4. *Given $\delta \geq 0$, there are constants K and L such that if g is a WPD element of G which is $(1, L, K)$ -asymmetric, with translation length $\tau(g) > 3L + 2K$, then there is a surjective homomorphism $\phi: E_G^+(g) \rightarrow \mathbb{Z}$ with $\phi(g) = 1$. In particular,*

$$E_G^+(g) = \langle g \rangle \rtimes \ker \phi,$$

where $\ker \phi$ is finite and consists precisely of the elliptic elements of $E_G^+(g)$.

Note that the proposition is not true if one replaces $E_G^+(g)$ by $E_G(g)$, as the latter may contain infinitely many elliptic elements (think of the action of the infinite dihedral group on \mathbb{Z}).

Proof. Let p be a point on a quasiaxis α_g . Let L be the fellow travelling constant from Proposition 2.2. The quasiaxis α_g is L -coarsely preserved by $E_G^+(g)$. As g is $(1, L, K)$ -asymmetric, the set $\{g^i p: i \in \mathbb{Z}\}$ is K -coarsely preserved by $E_G^+(g)$. As elements act by isometries, this gives an action of $E_G^+(g)$ on \mathbb{Z} , defined as follows. If $f \in E_G^+(g)$, $\phi(f)$ sends $g^i p$ to the nearest $g^j p$ to $f g^i p$. As g is WPD, the group $E_G^+(g)$ is virtually cyclic, so $\ker \phi$ is finite. The element $g \in E_G^+(g)$ maps to $1 \in \mathbb{Z}$ and gives a splitting, so $E_G^+(g) = \langle g \rangle \rtimes \ker \phi$.

As $\ker \phi$ is a finite subgroup of G , all elements of $\ker \phi$ are elliptic. If $\phi(f) \neq 0$, then as $\tau(g) \geq 3L + 2K$, the three points p, fp and $f^2 p$ satisfy $d(p, fp) \geq 3L$, $d(fp, f^2 p) \geq 3L$ and $(p \cdot f^2 p)_{fp} \leq L$, and so f is loxodromic by Proposition 10.3. \square

Let G_{WPD} denote the set of WPD elements of G , and let $H \leq G$ be a subgroup of G which contains an element of G_{WPD} . Define

$$E_G^+(H) := \bigcap_{g \in H \cap G_{\text{WPD}}} E_G^+(g).$$

and an equivalent definition holds for $E_G(H)$. We will also use the notation $E(G) := E_G(G)$ when G and H are equal.

Recall that two elements h_1, h_2 of G are *commensurable* if some power of h_1 is conjugate to some power of h_2 , and *non-commensurable* otherwise. The result below follows from the arguments in [DGO17, Lemma 6.17], but we give the details for the convenience of the reader.

Proposition 11.5. *Let G be a group acting by isometries on a Gromov hyperbolic space X , and let H be a non-elementary subgroup of G which contains an element of G_{WPD} . Then there exist two independent, WPD elements h_1, h_2 in H such that*

$$E_G^+(h_1) \cap E_G^+(h_2) = E_G^+(H).$$

Moreover, for any $K \geq 0$ there exists an element f in H such that for any $z \in \alpha_f$ one has

$$\text{Stab}_K(z, fz) \subseteq E_G^+(H).$$

Proof. By [DGO17, Corollary 6.12], there exist two non-commensurable, loxodromic, WPD elements h_1, h_2 in H (pick h_1 as one such element, then apply Corollary 6.12 with the subgroup called G in Corollary 6.12 chosen to be H , the subgroup called H in the Corollary 6.12 chosen to be $E_G(h_1)$ and $a \in H \setminus E_G(h_1)$). Let N be the normalizer of H in G , i.e.

$$N := \{x \in G : xHx^{-1} = H\}$$

which contains the group H . Denote as $T(h_i)$ the set of finite order elements in $E_G^+(h_i)$. In $E_G^+(h_i)$ every conjugacy class is finite (since all conjugate elements have equal translation length), so a result of Neumann [Neu51] then implies that the set $T(h_i)$ of finite order elements is a finite group. Let us suppose that for any $x \in N$ we have

$$E_G^+(xh_1x^{-1}) \cap E_G^+(h_2) \neq E_G^+(H).$$

Note moreover that

$$E_G^+(xh_1x^{-1}) \cap E_G^+(h_2) = xT(h_1)x^{-1} \cap T(h_2).$$

Given $(s, t) \in P := T(h_1) \times (T(h_2) \setminus E^+(H))$, we pick $y \in N$ such $ysy^{-1} = t$, if it exists, and $y(s, t) = 1$ otherwise. Let $C_N(t)$ be the centralizer of t in N . Now, we claim that

$$N = \bigcup_{(s,t) \in P} y(s, t)C_N(t).$$

Indeed, let $x \in N$. Then since $xT(h_1)x^{-1} \cap T(h_2) \neq E_G^+(H)$, then there exists $s \in T(h_1)$ and $t \in T(h_2) \setminus E^+(H)$ such that $s = x^{-1}tx \in T(h_1)$. Thus if $y = y(s, t)$ then $s = x^{-1}tx = y^{-1}ty$, so $xy^{-1} \in C_N(t)$. This means that there is a finite collection of cosets of the subgroups $C_N(t)$, with $t \in T(h_2) \setminus E^+(H)$, which covers N , and a theorem of Neumann [Neu54] then implies that at least one of these subgroups has finite index in N . Therefore, there is a $t \in T(h_2) \setminus E_G^+(H)$ such that $C_N(t)$ has finite index in N . Hence, if $h \in N$ is a WPD element, then there exists $k > 0$ such that $h^k t = t h^k$, hence $t \in E_G^+(h)$. Thus, $t \in E_G^+(N) \subseteq E_G^+(H)$, which is a contradiction. Finally, let us note that the claim implies that h_1 and h_2 are independent. In fact, as both h_1 and h_2 are WPD, the fixed point sets of h_1 and h_2 cannot have a common point. This is because in this case both h_1 and h_2 would coarsely stabilize a large segment of the quasixis of h_1 , which by Theorem 5.3, would imply that $E_G^+(h_1) = E_G^+(h_2)$, contradicting the non-commensurability of h_1 and h_2 .

We now prove the second claim. As h_1 and h_2 are independent loxodromic isometries, the ping-pong lemma implies that for any $n > 0$ sufficiently large, the orbit map gives a quasi-isometric embedding of the free group $\langle h_1^n, h_2^n \rangle$ in X . In particular, for all $m > 0$, the element $f := h_1^{nm} h_2^{nm}$ is loxodromic.

Fix some $K \geq 0$, and let L_1 be the fellow travelling constant for $(1, K_1)$ -quasigeodesics from Proposition 2.4. Let L_2 be the constant given by Theorem 5.3 using the constant $K + 2\delta + L_1$. We may choose m sufficiently large so that there are two segments $\eta_1 \subseteq \alpha_{h_1}$ and $\eta_2 \subseteq \alpha_{h_2}$ of length $\geq L_2$, and a segment $\eta \subseteq \alpha_f$ such that

$$\eta_1 \cup \eta_2 \subseteq N_{L_1}(\eta).$$

Thus, if h belongs to $\text{Stab}_K(z, fz)$, then for some $k \in \mathbb{Z}$ the isometry $f^k h f^{-k}$ $(K + 2\delta)$ -coarsely stabilizes the segment η , hence it also $(K + 2\delta + L_1)$ -coarsely

stabilizes both η_1 and η_2 , and preserves the orientation of the axes. Then by Theorem 5.3 it is contained in

$$E_G^+(h_1) \cap E_G^+(h_2) = E_G^+(H).$$

Thus, h belongs to $f^{-k}E_G^+(H)f^k = E_G^+(H)$, as required. \square

From now on we shall assume that the probability distribution μ is reversible, so Γ_μ is a group. We will use the notation $E_\mu := E_G^+(\Gamma_\mu)$.

Corollary 11.6. *Given $\delta \geq 0$ there are constants K and L with the following properties. Let G be a group acting by isometries on a δ -hyperbolic space X , and let μ be a countable, non-elementary, reversible, bounded, WPD probability distribution on G . Then there are constants B and $c < 1$ such that the probability that w_n is loxodromic, $(1, L, K)$ -asymmetric, WPD with*

$$E_G(w_n) = E_G^+(w_n) = \langle w_n \rangle \rtimes E_\mu$$

is at least $1 - Bc^n$. In particular, if E_μ is trivial, then $E_G(w_n)$ is cyclic with probability at least $1 - Bc^n$.

Proof. We are left with proving the last claim. By Proposition 10.2, we know that there are constants B_1 and $c_1 < 1$ such that the probability that w_n is $(1, L, K)$ -asymmetric is at least $1 - B_1c_1^n$, hence

$$E_G^+(w_n) = \langle w_n \rangle \rtimes \ker \phi$$

where $\phi : E_G^+ \rightarrow \mathbb{Z}$ is the homomorphism given in Proposition 11.4. Now, since w_n is asymmetric, we have that $\ker \phi$ is the (finite) set of elliptic elements in $E_G^+(w_n)$, hence it is contained in $\text{Stab}_K(p, w_np)$ where p is some point on the quasixis of w_n . Let $f \in \Gamma_\mu$ be given by Proposition 11.5. By Proposition 8.2, there are constants B_2 and $c_2 < 1$ such that the probability the quasixis of w_n has a (L, K) -match with a translate of the quasixis of f is at least $1 - B_2c_2^n$. Therefore, for $K' = 2K + 2\delta$ we get for some $z \in \alpha_f$

$$\ker \phi \subseteq \text{Stab}_K(p, w_np) \subseteq g\text{Stab}_{K'}(z, fz)g^{-1} \subseteq E_G^+(\Gamma_\mu) = E_\mu.$$

The result then holds for suitable choices of B and $c < 1$. \square

Given $g \in G$ a loxodromic element, let us define the *fellow travelling constant* for g as

$$\Delta(g) := \sup_{h \in G \setminus E(g)} \Delta(g, hgh^{-1})$$

where $E(g)$ is the maximal elementary subgroup which contains g .

Definition 11.7. ([DGO17, Definition 6.25]) Let X be a δ -hyperbolic space with $\delta > 0$, and let \mathcal{R} be a family of loxodromic isometries of X which is closed under conjugation. Then we say that \mathcal{R} satisfies the (A, ϵ) -small cancellation condition if the following holds:

- (1) $\text{inj}(\mathcal{R}) \geq A\delta$
- (2) $\Delta(g, h) \leq \epsilon \cdot \text{inj}(\mathcal{R})$ for all $g \neq h^{\pm 1} \in \mathcal{R}$.

We will now prove that the cyclic subgroup generated by a power of w_n satisfies the small cancellation condition. First of all, we show that the fellow travelling constant between translates of the quasixis is sublinear in n .

Proposition 11.8. *Let G be a group of isometries of a δ -hyperbolic metric space X , and μ a countable, non-elementary, reversible, bounded, WPD probability measure on G . Let $\ell > 0$ be the drift of the random walk. Then for any $0 < \epsilon < 1$, there are constants B and $c < 1$ such that for all n the fellow travelling constant of w_n satisfies*

$$\mathbb{P}(\Delta(w_n) \geq \ell n) \leq Bc^n.$$

Proof. By Proposition 11.1, there is an L such that $N_{20\delta}(\text{Ax}(w_n)) \subset N_{L/2}(\alpha_{w_n})$. Therefore, if $\Delta(w_n) \geq \ell n$, there is a translate $h\alpha_{w_n}$, with $h \notin E(w_n)$, such that α_{w_n} and $h\alpha_{w_n}$ have a $(\ell n, L)$ -match. This by definition means that there is a segment $\eta = [p, q] \subseteq \alpha_{w_n}$ with $|\eta|$ equal to ℓn , such that $h\eta$ is contained in an L -neighbourhood of α_{w_n} . By replacing η with $w_n^i \eta$ for some $i \in \mathbb{Z}$ and replacing ϵ by $\epsilon/2$, we can assume that $\eta \subseteq [x', w_n x']$ where x' is a nearest point projection of the basepoint x to α_{w_n} .

By Proposition 10.2, there are constants B_1 and $c_1 < 1$ such that the element w_n is (ϵ, L, K) -asymmetric with probability at least $1 - B_1 c_1^n$. Thus there is a K , depending on ϵ and L , such that up to replacing h by $w_n^j h$ for some $j \in \mathbb{Z}$, we may assume that $d(p, hp) \leq K$ and $d(q, hq) \leq K$.

Let f be given as in the second part of Proposition 11.5. As $[p, q]$ has length ℓn and is contained in $[x', w_n x']$, by Lemma 8.3 there are constants B_2 and $c_2 < 1$ such that the probability that it contains a match with a large subsegment of a translate $g\alpha_f$ of a quasiaxis α_f (where $g \in \Gamma_\mu$) is at least $1 - B_2 c_2^n$.

As h K -coarsely stabilizes this subsegment, this implies that there exists $z \in \alpha_f$ such that by Proposition 11.5,

$$h \in \text{Stab}_K(gz, gfgz) = g\text{Stab}_K(z, fz)g^{-1} \subseteq gE_G^+(\Gamma_\mu)g^{-1} = E_G^+(\Gamma_\mu),$$

hence, since by construction $E_G^+(\Gamma_\mu) \subseteq E_G^+(w_n)$ and, by Corollary 11.3, there are constants B_3 and $c_3 < 1$ such that the probability that $E_G^+(w_n) = E_G(w_n)$ is at least $1 - B_3 c_3^n$. Therefore, by suitable choices of B and $c < 1$, any such h must lie in $E_G(w_n)$ with probability at least $1 - Bc^n$. However, this contradicts our initial choice of h , and implies that $\Delta(w_n) \geq \ell n$ with probability at most Bc^n , as required. \square

11.1. The structure of the normal closure. The last step we need to understand the structure of the normal closure $\langle\langle w_n \rangle\rangle$ of w_n in G is to take care of the fact that the elementary subgroup $E_G^+(w_n)$ need not be cyclic, so we may have to pass to a power of w_n . However, the power may be chosen to be a constant which only depends on G and μ , as we now explain.

Let Γ_μ be the group generated by the support of μ , and let $E_\mu := E_G^+(\Gamma_\mu)$. By definition, E_μ is a normal subgroup of Γ_μ , hence one has the homomorphism

$$\varphi : \Gamma_\mu \rightarrow \text{Aut } E_\mu \tag{18}$$

given by conjugation: $g \mapsto (k \mapsto gkg^{-1})$. We will denote as $H_\mu := \varphi(\Gamma_\mu)$ the image of φ .

Lemma 11.9. *The image of φ in $\text{Aut } E_\mu$ is trivial if and only if $E_\mu = Z(\Gamma_\mu)$.*

Proof. First note that $Z(\Gamma_\mu) \subseteq E_\mu$. In fact, let $g \in Z(\Gamma_\mu)$ and let $h \in \Gamma_\mu$ be a loxodromic, WPD element. Then $ghg^{-1} = h$, hence $\text{Fix}(ghg^{-1}) = g\text{Fix}(h) = \text{Fix}(h)$, hence $g \in E_G(h)$. Since this is true for any h WPD, then $g \in E_\mu$.

Moreover, the kernel of φ is the set of g which commute with every element of E_μ , hence the image is trivial if and only if every element of E_μ commutes with every element of Γ_μ , which means that $E_\mu \subseteq Z(\Gamma_\mu)$. \square

Now, by Corollary 11.6, with probability which tends to 1, $E_G(w_n)$ is the semidirect product

$$E_G(w_n) = \langle w_n \rangle \rtimes E_\mu$$

and the group structure of $E_G(w_n)$ is determined by the map $\langle w_n \rangle \rightarrow \text{Aut } E_\mu$, hence by the image $\varphi(w_n)$ in $\text{Aut } E_\mu$.

Lemma 11.10. *Let K be a finite group, let $\psi \in \text{Aut } K$, and consider the semidirect product*

$$H = \mathbb{Z} \rtimes_\psi K$$

where we denote as t a generator for \mathbb{Z} , so that $tkt^{-1} = \psi(k)$ for any $k \in K$. Then:

- (1) for any $a \in \mathbb{Z} \setminus \{0\}$, if $\psi(t^a) = 1$, then the normal closure of t^a in H is cyclic and equal to $\langle t^a \rangle$;
- (2) if $\psi(t) \neq 1$, then the normal closure of t in H is not cyclic and not free;

Proof. Let $u = t^a$, and suppose that $\psi(u) = 1$. Then for any $k \in K$ we have $kuk^{-1} = u$ and since by construction u commutes with t , then u commutes with H , hence the normal closure $\langle\langle u \rangle\rangle = \langle u \rangle$ is infinite cyclic.

Now, since H is virtually cyclic and the subgroup of a free group is free, then the normal closure $N := \langle\langle t \rangle\rangle$ is free if and only if it is infinite cyclic. Moreover, since t generates \mathbb{Z} , the only cyclic group which contains $\langle t \rangle$ is $\langle t \rangle$ itself. Hence $\langle\langle t \rangle\rangle$ is free if and only if it coincides with $\langle t \rangle$. If the image $\phi(t)$ is not trivial, then there exists $k \in K$ such that $ktk^{-1} \neq t$, hence the normal closure is larger than $\langle t \rangle$, hence not free. \square

Lemma 11.11. *Let $h \in G$ be a loxodromic, WPD element, and let $g \in G$. Then if $ghg^{-1} \in E_G(h)$, then $g \in E_G(h)$.*

Proof. Suppose that $ghg^{-1} \in E_G(h)$, and let $\Lambda := \{\lambda^+, \lambda^-\}$ be the set of fixed points of h on ∂X . Then by the assumption ghg^{-1} also fixes Λ , hence by conjugating h fixes $g^{-1}\Lambda$. Since h fixes exactly two points on the boundary, then $\Lambda = g^{-1}\Lambda$, which implies that $g \in E_G(h)$. \square

We are now ready to present the main Theorem (Theorems 1.5 and 1.4) and its proof.

Theorem 11.12. *Let G be a group acting on a Gromov hyperbolic space X , and let μ be a countable, non-elementary, reversible, bounded, WPD probability measure on G . Let $k = k(\mu)$ be the characteristic index of μ . Then:*

- (1) the probability that the normal closure $\langle\langle w_n \rangle\rangle$ of w_n in G is free satisfies

$$\mathbb{P}(\langle\langle w_n \rangle\rangle \text{ is free}) \rightarrow \frac{1}{k}$$

as $n \rightarrow \infty$. As a corollary, this probability tends to 1 if and only if $E_\mu = Z(\Gamma_\mu)$.

- (2) Moreover,

$$\mathbb{P}(\langle\langle w_n^k \rangle\rangle \text{ is free}) \rightarrow 1$$

as $n \rightarrow \infty$, and indeed there exist constant $B > 0, c < 1$ such that

$$\mathbb{P}(\langle\langle w_n^k \rangle\rangle \text{ is free}) \geq 1 - Bc^n$$

for any n .

- (3) Finally, if $N_n := \langle\langle w_n^k \rangle\rangle$, then for any $R > 0$ the injectivity radius of N_n satisfies for any n

$$\mathbb{P}(\text{inj}(N_n) \geq R) \geq 1 - Bc^n.$$

Proof. Let us choose $\alpha > 0$. Then by [DGO17, Proposition 6.23] there exist constants (A, ϵ) such that if a family $\{N_\lambda\}_{\lambda \in \Lambda}$ of subgroups, closed under conjugation, satisfies the small cancellation condition, then $\{N_\lambda\}$ is α -rotating on a hyperbolic graph X' . Note that X' is obtained from X in the following way. First, one chooses a hyperbolic graph X'' which is equivariantly quasi-isometric to X . This is chosen once and for all; let K be the Lipschitz constant of the map $X \rightarrow X''$. Now, the coned off space X' is obtained by coning off certain quasiconvex subsets of a rescaled copy $\lambda X''$. However, by looking at the proof one realizes that one can make sure that $\lambda \leq 1$ in all cases (indeed, in the language of [DGO17, Proposition 6.23], the correct choice is $\lambda = \min\left(\frac{\delta_c}{\delta}, \frac{\Delta_c}{\Delta}, 1\right)$, with $A = \max\left(\frac{\text{inj}_c(r_0)}{\delta_c}, \frac{\text{inj}_c(r_0)}{\delta}\right)$ and $\epsilon = \frac{\Delta_c}{\text{inj}_c(r_0)}$.) Thus, the map $X \rightarrow X'$ is K -Lipschitz, where K only depends on X and not on the constant α .

Let us fix $\alpha \geq 200$, and let (A, ϵ) chosen as above. Let $\ell > 0$ be the drift of the random walk. Then by Theorem 2.5 (3), there are constants B_1 and $c_1 < 1$ such that

$$\mathbb{P}\left(\tau(w_n) \geq \frac{\ell n}{2}\right) \geq 1 - B_1 c_1^n.$$

Moreover, by Proposition 11.8, there are constants B_2 and $c_2 < 1$ such that

$$\mathbb{P}\left(\Delta(w_n) \leq \frac{\epsilon \ell n}{2}\right) \geq 1 - B_2 c_2^n.$$

Now by Corollary 11.6, there are constants B_3 and $c_3 < 1$ such that

$$\mathbb{P}(E_G^+(w_n) = \langle w_n \rangle \times E_\mu) \geq 1 - B_3 c_3^n.$$

Thus, for suitable choices of B_4 and $c_4 < 1$,

$$\mathbb{P}\left(\tau(w_n) \geq A\delta, \Delta(w_n) \leq \epsilon\tau(w_n) \text{ and } E_G^+(w_n) = \langle w_n \rangle \times E_\mu\right) \geq 1 - B_4 c_4^n. \quad (19)$$

In particular, with probability which tends to 1 we have

$$E_G(w_n) = \langle w_n \rangle \times_{\varphi_n} E_\mu$$

where $\varphi_n = \varphi(w_n)$ is the image of w_n under the homomorphism

$$\varphi : \Gamma_\mu \rightarrow \text{Aut } E_\mu.$$

Now, we have two cases.

- (1) if $\varphi(w_n) = 1$, then all conjugates of w_n in G belong to different elementary subgroups.

In fact, suppose that there exists $g \in G$ such that $gw_n g^{-1} \in E_G(g)$. Then, by Lemma 11.11 one has $g \in E_G(w_n)$, and by Lemma 11.10 one has $gw_n g^{-1} = w_n$.

Now, consider the family of subgroups $\mathcal{R}_n := \{gw_n g^{-1}\}_{g \in G}$. Finally, let $N_n = \langle\langle H_n \rangle\rangle$ be the normal closure of H_n . By equation (19) above, with probability at least $1 - B_4 c_4^n$, the family \mathcal{R}_n satisfies the (A, ϵ) -small cancellation condition, hence it is an α -rotating family. Then by [DGO17,

Corollary 5.4], the normal closure of w_n is the free product of conjugates of $\langle w_n \rangle$, hence it is free.

- (2) if $\varphi(w_n) \neq 1$, then there exists $g \in \Gamma_\mu$ such that $gw_n g^{-1} \neq w_n$. This implies that the intersection

$$\langle\langle w_n \rangle\rangle \cap E_G(w_n)$$

is larger than $\langle w_n \rangle$, hence the normal closure $\langle\langle w_n \rangle\rangle$ cannot be a free group.

By the above discussion, the probability that the normal closure of w_n in G is free converges to the probability that w_n maps to the identity in E_μ . In order to compute such probability, note that under the map

$$\varphi : \Gamma_\mu \rightarrow \text{Aut } E_\mu$$

the random walk on Γ_μ pushes forward to a random walk on $\text{Aut } E_\mu$, which is a finite group. Hence, the random walk equidistributes on the elements of the image of φ into $\text{Aut } E_\mu$, hence the probability that $\varphi(w_n) = 1$ converges to $\frac{1}{\#H_\mu}$, where $\#H_\mu$ is the cardinality of the image of φ . That is, the normal closure of w_n is free if and only if the image $\varphi(w_n) = 1$, and the probability of this happening tends to $\frac{1}{\#H_\mu}$, so

$$\mathbb{P}(\langle\langle w_n \rangle\rangle \text{ is free}) \rightarrow \frac{1}{\#H_\mu}.$$

Hence, this probability tends to 1 if and only if the image group $H_\mu = \varphi(\Gamma_\mu)$ is the trivial group, hence by Lemma 11.9 if and only if $E_\mu = Z(\Gamma_\mu)$.

To prove (ii), if $k = \#H_\mu$, then every element in the image of φ has order which divides k , hence $\varphi(w_n^k) = \varphi(w_n)^k = 1$. Thus, as in the previous argument, if one defines $H_n := \langle w_n^k \rangle$, the probability that the family $\mathcal{R}_n := \{gw_n^k g^{-1}\}_{g \in G}$ satisfies the small cancellation condition tends to 1, hence the probability that the normal closure $N_n := \langle\langle w_n^k \rangle\rangle$ is free satisfies

$$\mathbb{P}(\langle\langle w_n^k \rangle\rangle \text{ is free}) \geq 1 - Bc^n$$

for suitable choices of $B > 0, c < 1$.

Now, to prove (iii), given $R > 0$ let α be such that $\frac{\delta\alpha}{K} = R$. Then one can choose (A, ϵ) as before for such α . Then with probability at least $1 - B_4 c_4^n$, the family \mathcal{R}_n is α -rotating. Hence, by [DGO17, Theorem 5.3], for each $g \in N_n$, either g belongs to some conjugate of H_n or is loxodromic on X' with translation length at least $\alpha\delta$. Then since the map $X \rightarrow X'$ is K -Lipschitz, such elements have translation length on X at least $\frac{\alpha\delta}{K}$. On the other hand, by Theorem 2.5 (3) we know that with probability at least $1 - B_1 c_1^n$, the isometry w_n^k is loxodromic on X with translation length $\geq R$. Therefore for suitable choices of B_5 and $c_5 < 1$, the probability that the injectivity radius of N_n is at least R is at least $1 - B_5 c_5^n$. The stated result then follows for suitable choices of B and $c < 1$. \square

Corollary 11.13. *Let G be a group acting on a Gromov hyperbolic space, and let μ be a countable, non-elementary, reversible, bounded, WPD probability measure on G . Let $k = k(\mu)$ be the characteristic index of μ , and let $N_n(\omega) := \langle\langle w_n^k \rangle\rangle$ be the normal closure of w_n^k in G . Then for almost every sample path ω , the sequence*

$$(N_1(\omega), N_2(\omega), \dots, N_n(\omega), \dots)$$

contains infinitely many different normal subgroups of G .

Proof. Fix $M > 0$, and consider the set

$$A_M := \{\omega : \sup_n \text{inj}(N_n(\omega)) \leq M\}.$$

We claim that $\mathbb{P}(A_M) = 0$. Indeed, suppose $\mathbb{P}(A_M) = \epsilon > 0$. Then by Theorem 11.12, there exists n_0 such that for $n \geq n_0$

$$\mathbb{P}(\text{inj}(N_n) \geq M + 1) > 1 - \epsilon$$

which is a contradiction because such a set must be disjoint from A_M . Then for almost every ω we have

$$\limsup_{n \rightarrow \infty} \text{inj}(N_n(\omega)) = +\infty,$$

which implies the claim. \square

This completes the proof of Theorem 1.4 in the Introduction.

11.2. Application to the mapping class group. In the case of the mapping class group, we may answer [Mar19, Problem 10.11] and establish Theorem 1.9, as we now explain.

Corollary 11.14. *Let S be a surface of finite type whose mapping class group $\text{Mod}(S)$ is infinite. Let μ be a probability distribution on $\text{Mod}(S)$ such that the support of μ has bounded image in the curve complex under the orbit map, and for which $\Gamma_\mu = \text{Mod}(S)$. Then there are constants $B > 0$ and $c < 1$ such that the probability that the normal closure $\langle\langle w_n \rangle\rangle$ is a free subgroup of $\text{Mod}(S)$ is at least $1 - Bc^n$.*

This follows immediately from Theorem 11.12, and the fact that if $G = \text{Mod}(S_{g,n})$ is the mapping class group of a surface of finite type, the group $E_G^+(G)$ is equal to the center of G , as we now explain.

We shall write $S_{g,n}$ for the surface of genus g with n punctures. The mapping class groups $S_{0,n}$ with $n \leq 3$ are finite, so the results of this paper do not apply to them, and we shall ignore them for the purposes of this section.

Proposition 11.15. *Let $S_{g,n}$ be a surface of genus g with n punctures, and suppose that its mapping class group $G = \text{Mod}(S_{g,n})$ is infinite. Then $E_G^+(G)$ is equal to the center of G .*

Proof. If the mapping class group $G = \text{Mod}(S_{g,n})$ is infinite, then its center is trivial, unless $S_{g,n}$ is one of the following four surfaces: $S_{1,0}, S_{1,1}, S_{1,2}$ or $S_{2,0}$, in which case the center $Z(G)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, generated by the hyperelliptic involution, see for example [Iva92, Remark 8.15] or [FM12, Section 3.4].

Recall that $E_G^+(G)$ is a subgroup of $E_G(G)$, which is equal to the maximal finite normal subgroup of G . By Ivanov [Iva92, Section 11, Exercise 5.b] any finite normal subgroup of the mapping class group is contained in the center $Z(G)$. In the cases in which the center is non-trivial, it is generated by the hyperelliptic involution, which acts trivially on the boundary, and so fixes pointwise the endpoints of all pseudo-Anosov elements. In particular, the groups $Z(G)$, $E_G(G)$ and $E_G^+(G)$ are all equal. \square

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