Smoothable zero dimensional schemes and special projections of algebraic varieties

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Abstract

We study the degrees of generators of the ideal of a projected Veronese variety $v_2(\mathbb{P}^3) \subset \mathbb{P}^9$ to $\mathbb{P}^6$ depending on the center of projection. This is related to the geometry of zero dimensional schemes of length 8 in $\mathbb{A}^4$, Cremona transforms of $\mathbb{P}^6$, and the geometry of Tonoli Calabi-Yau threefolds of degree 17 in $\mathbb{P}^6$.

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1 Introduction

The aim of the paper is to find and investigate a relation between the following three a priori distinct subjects:

- analysing smoothability of finite degree 8 subschemes in $\mathbb{A}^4$.
- describing special projections of the double Veronese embedding of $\mathbb{P}^3$ to $\mathbb{P}^6$,
- studying the action of special $(2, 4)$ Cremona transformations $\mathbb{P}^6 \dashrightarrow \mathbb{P}^6$,

For a linear subspace $L \subset \operatorname{Sym}^2 V$ let us denote by $\pi_L \colon \mathbb{P}(\operatorname{Sym}^2 V) \dashrightarrow \mathbb{P}(\operatorname{Sym}^2 V/L)$ the projection. Let $v_2 \colon \mathbb{P}(V) \to \mathbb{P}(\operatorname{Sym}^2 V)$ be the second Veronese embedding. We prove the following theorem.

Theorem 1. *Let $V = \mathbb{A}^4$ and $L \subset \operatorname{Sym}^2 V$ be a linear subspace of dimension 3. Assume that the composition $\pi_L \circ v_2 \colon \mathbb{P}(V) \dashrightarrow \mathbb{P}(\operatorname{Sym}^2 V/L)$ is regular and an embedding. Let X_L be its image. Then the following are equivalent*

(a) $\pi_L \circ v_2 : \mathbb{P}^3 \to \mathbb{P}^6$ *is a restriction of a Cremona transformation* $\mathbb{P}^6 \dashrightarrow \mathbb{P}^6$ *of type* $(2, 4)$ *based in a rational octic surface or a limit of such restrictions,*

(b) X_L *is contained in a cubic hypersurface,*

(c) X_L *is contained in a three-dimensional space of cubic hypersurfaces,*

(d) the scheme $R = \operatorname{Spec} \operatorname{Apolar}(L)$ *is smoothable,*

(e) L *is spanned by partial derivatives of a cubic form* $F \in \operatorname{Sym}^3 V$.

The above equivalent conditions describe a closed, irreducible subset of $\operatorname{Gr}(3, \operatorname{Sym}^2 V)$. *For a general* L *in this subset the space of cubics containing* X_L *is exactly three-dimensional and it is spanned by Segre cubics.*

Our original motivation for studying these projections is related to understanding the geometry of constructions of Calabi–Yau threefolds in $\mathbb{P}^6$, some of which are related to special projections of the Veronese embedding of $\mathbb{P}^3$. More precisely, in [KK16, KK18], the second and third author study the projections

$$\pi_L(v_2(\mathbb{P}^3)) = X_L^8 \subset \mathbb{P}^6$$

from $\mathbb{P}(L)$ for L of dimension 3. For a generic L one has $H^0(\mathcal{I}_{X_L^8}(3)) = 0$. However, using *Macaulay2* with a lot of random choices it was proven that one can find L such that X_L^8 is smooth and $H^0(\mathcal{I}_{X_L^8}(3)) = 3$. For such special projections, by the bilinkage construction one can construct degenerated Tonoli Calabi-Yau threefolds of degree 17 in $\mathbb{P}^6$ (cf. [KK16]). In the present paper we explain the geometric meaning of these special exceptional centers. Our problem is related to the following more general subject. Let $X \subset \mathbb{P}(W) \simeq \mathbb{P}^r$ be an algebraic variety, consider the projection $\pi_L \colon X \to X_L \subset \mathbb{P}^{r-t}$ from the center

$$\mathbb{P}(L) = \mathbb{P}^{t-1} \subset \mathbb{P}(W)$$

such that π_L is an isomorphism.

Problem 2. *What are the possible Betti numbers of the ideal of $X_L \subset \mathbb{P}^{r-t}$ when we move the center of projection $L \in \mathrm{Gr}(t, W)$?*

The study of the geometry of central projections is a classical topic that was widely studied for generic projections [Rob71], [GP13], [BE12]. The study of Betti numbers of projected varieties was discussed in [AK11], [HK12], [AR02]. Using the mapping cone construction the authors were able under some conditions to relate the Betti numbers of the variety before and after the projection. In particular in [AK11, Prop. 4.11] it is described how the number of quadrics in the ideal of a centrally projected variety changes when we move the center of projection.

The case of projections with center being a point is also considered in [AR02]. The authors describe sub-schemes $Z_k(X) \subset \mathbb{P}^r$ being the loci of points such that the ideal of the projected variety admits more generators of a given degree. The aim of this paper is to study the case when the center of projection has dimension ≥ 1. The first case to consider are the projections of the second Veronese embeddings $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ and its projections to smooth surfaces in $\mathbb{P}^4$. However, it is easy to see that all such projections give projectively isomorphic images. The cases of projection of $v_2(\mathbb{P}^3)$ to $\mathbb{P}^8$ and $\mathbb{P}^7$ are treated in [AR02]. Our situation is hence the next case to check.

The problem is unexpectedly related to the theory of zero-dimensional schemes, which seems a purely algebraic one. For such a scheme $R \subset \mathbb{A}^n$ we say that R is *smoothable* if it is a limit of smooth subschemes (tuples of points). More precisely, let Hilb_d be the Hilbert scheme of d points on $\mathbb{A}^n$, so that (closed) points of Hilb_d are finite degree d subschemes of $\mathbb{A}^n$. Let $\mathrm{Hilb}_d^\circ \subset \mathrm{Hilb}_d$ be an open subset consisting of tuples of d distinct points on $\mathbb{A}^n$. Let Hilb_d^{sm} be the closure of Hilb_d°. Then $R \subset \mathbb{A}^n$ is smoothable if and only if the corresponding point lies in Hilb_d^{sm}. Whether a given R is smoothable is a difficult question, see [CEVV09, EV10, CN09, CN11, CJN15, DJNT17, Šiv12, BCR17a, BCR17b, BdSHJ13]. It is connected with the search for equations of secant varieties [BB14, BGL13]. An important aspect of Theorem 1 is that it gives a geometrical interpretation of smoothability, in a special case.

Our special projections are also related to the theory of Cremona transformations, i.e., birational self maps of the projective space. Such self maps are induced by systems of homogeneous polynomials of equal degree called the *degree* of the Cremona transformation. These polynomials define the indeterminacy of the map called the *base* of the Cremona transformation. One also defines the *type* of a Cremona transformation as a pair consisting of its degree and the degree of its inverse. The problem of classification of Cremona transformations with smooth base loci was considered in [CK89, PR14, HKS92, Sta18]. In particular, Cremona transformations with base loci being smooth surfaces was classified in [CK89]. In this work we investigate and exploit the geometry of one of the five types of transformations with base loci being smooth surfaces: Cremona transformations of type $(2, 4)$ in $\mathbb{P}^6$ based in a rational surface embedded as a surface of degree 8 and sectional genus 3 in $\mathbb{P}^6$.

The paper is centered around the proof of Theorem 1. In Section 2 we discuss general results on base loci and zero-dimensional schemes. Section 3 applies them to the special case of quadric embeddings $\mathbb{P}^3 \to \mathbb{P}^6$. In Subsection 3.1 we discuss deformations of zero-dimensional, degree 8 subschemes of $\mathbb{A}^4$ and give a geometric proof of the equivalence of Condition (e) and Condition (d) from Theorem 1 (the equivalence was first proven in [CEVV09]). In Subsection 3.2 we prove equivalence of Conditions (d), (b) and (c). Finally, in Subsection 3.3 we prove equivalence of Conditions (a) and (d) and provide a formal proof of Theorem 1.

2 Preliminaries

We work over an algebraically closed base field $\mathbf{k}$ of characteristic $\neq 2, 3$. For a vector space V, by $\mathbb{P}(V)$ we mean the scheme $\operatorname{Proj} \operatorname{Sym}(V^\vee)$. Then the cone over $\mathbb{P}(V)$ is identified with V. When speaking about a rational map (or a morphism) $\varphi : \mathbb{P}(V) \dashrightarrow \mathbb{P}(W)$ we always implicitly fix a morphism $\hat{\varphi} : V \to W$ inducing φ.

Definition 3. *We define the* jump locus *inside the Grassmannian* $J^l_k \subset \operatorname{Gr}(t, r+1)$ *by the following* jump condition*: a point* $L \in \operatorname{Gr}(t, r+1)$ *is in the set* J^l_k *if* $H^0(\mathcal{I}_{X_L}(k))$ *has dimension higher then the generic value and the difference is* l*.*

By the semicontinuity theorem we deduce that $l \geq 0$.

It follows from [AK11, Prop. 4.1] that the isomorphic projection of a m-normal variety from a center of dimension $t-1$ is still m-normal for $m \geq t+1$. In the case where $X \subset \mathbb{P}^r$ is projectively normal it follows from [AR02, Prop. 2.1] that the number of hypersurfaces of degree $k \geq t+1$ in the ideal of the projected variety is uniquely determined. So we are interested in the sets J^l_k for $k \leq t$.

On the other hand from [AK11, Prop. 4.11], the number of generators of degree 2 of the centrally projected variety (from a point) is uniquely determined by the dimension of the secant locus of the projection in the case $X \subset \mathbb{P}^r$.

Natural problems occur:

- Are the loci J^l_k related to the secant loci of $X \subset \mathbb{P}^r$,
- For a given $X \subset \mathbb{P}^r$ and $2 < k \leq t$ what are the possible values of l such that J^l_k is non-empty.
- Are there jumps i.e $0 < l < p < m$ such that $J^l_k \neq \emptyset$ and $J^m_k \neq \emptyset$ but $J^p_k = \emptyset$ for some k.

In this paper we address all those questions in our example.

Note that we cannot describe J^l_k directly by induction using projections from points studied in [AR02] since the schemes $Z_k(X_p)$ vary when we move the center $p \in \mathbb{P}^r - \operatorname{Sec}(X)$.

2.1 Affine base loci

One of the main objects in our study of rational maps are the affine base loci, that we define below. Recall that in our convention a rational map $\varphi \colon \mathbb{P}(V) \dashrightarrow \mathbb{P}(W)$ comes with a fixed map $\hat{\varphi} \colon V \to W$ on the level of cones.

Definition 4. *Let $\varphi : \mathbb{P}(V) \dashrightarrow \mathbb{P}(W)$ be a rational map. The* affine base locus *is the affine scheme $\hat{\varphi}^{-1}(0) \subset V$. We denote it by $\mathrm{Bs}_{\mathrm{aff}}(\varphi)$.*

For every φ, the affine base locus is invariant under homothety and its image in $\mathbb{P}(V)$ is the base locus for φ. If φ is regular, then $\mathrm{Bs}_{\mathrm{aff}}(\varphi)$ is supported at $0 \in V$, hence it is zero-dimensional. Note that $\mathrm{Bs}_{\mathrm{aff}}(\varphi)$ may be non-empty even for a regular φ.

Recall that each rational map between projective spaces is given by a d-th Veronese embedding composed with a linear projection and that such map is regular (respectively, isomorphism onto the image) if and only if the center of the linear projection does not intersect the image of $\mathbb{P}(V)$ in $\mathbb{P}(\mathrm{Sym}^d V)$ (respectively, the secant variety of the image).

Example 5. *Let $\mathbb{P}^3 \to \mathbb{P}^6$ be a morphism given by seven quadrics. The algebra $A = H^0(\mathcal{O}_{\mathrm{Bs}_{\mathrm{aff}}(\varphi)}) = \mathbf{k}[x_0, x_1, x_2, x_3]/(q_1, \ldots, q_7)$ is zero-dimensional and graded. For a general enough choice of quadrics we have $(x_0, x_1, x_2, x_3)(q_1, \ldots, q_7) = (x_0, x_1, x_2, x_3)^3$, hence A is spanned by unity, linear forms and three complementary quadrics, so it has degree $1 + 4 + 3$.*

We now aim at describing the geometry behind the affine base locus of a *morphism* of projective spaces. This sends us to the world of finite schemes and Hilbert schemes of points.

2.2 Apolarity

Recall that we have assumed that $\mathbf{k}$ has characteristic not equal to two or three. For a characteristic-free description of apolarity and further information see e.g. [IK99, Jel17].

We recall a very useful parameterization tool for finite schemes, called apolarity or Macaulay inverse systems. Namely, $\mathrm{Sym}\, V^\vee$ acts on $\mathrm{Sym}\, V$, where elements of $V^\vee$ act as partial derivatives. For a finite dimensional subspace $L \subset \mathrm{Sym}\, V$ we may consider the ideal $\mathrm{Ann}\,(L)$ of all operators from $\mathrm{Sym}\, V^\vee$ annihilating L and the quotient

$$\mathrm{Apolar}(L) = \mathrm{Sym}\, V^\vee / \mathrm{Ann}\,(L)$$

which is a local zero-dimensional $\mathbf{k}$-algebra with residue field $\mathbf{k}$ and of rank equal to $\dim_{\mathbf{k}}(\mathrm{Sym}\, V^\vee \circ L)$. We will be mostly interested in the case when $\dim V = 4$ and L is a three-dimensional space of quadrics.

Example 6. *Let $V = \mathrm{span}\,(x, y, z, t)$ and $L = \mathrm{span}\left(x^2, y^2, z^2 - t^2\right)$. Let $V^\vee = \mathrm{span}\,(\partial_x, \partial_y, \partial_z, \partial_t)$ be the dual basis. Then*

$$\mathrm{Ann}\,(L) = \mathrm{span}\left(\partial_x\partial_y,\ \partial_x\partial_z,\ \partial_x\partial_t,\ \partial_y\partial_z,\ \partial_y\partial_t,\ \partial_z\partial_t,\ \partial_z^2 + \partial_t^2\right) + \mathrm{Sym}^{\geq 3} V^\vee.$$

Consequently, $\mathrm{Apolar}(L) \simeq \mathrm{span}\left(\partial_x^2,\ \partial_y^2,\ \partial_z^2 - \partial_t^2,\ \partial_x,\ \partial_y,\ \partial_z,\ \partial_t, 1\right)$ *as linear spaces.*

A theorem of Macaulay asserts that $L \mapsto \mathrm{Apolar}(L)$ induces a bijection between zero-dimensional subschemes of V supported at the origin and finite dimensional subspaces $L' \subset \mathrm{Sym}\, V$ which are closed under the action of $\mathrm{Sym}\, V^\vee$, i.e., which are $(\mathrm{Sym}\, V^\vee)$-submodules. Clearly subspaces spanned by homogeneous elements give $\mathbf{k}^*$-invariant schemes and conversely. Moreover, principal $(\mathrm{Sym}\, V^\vee)$-submodules correspond precisely to Gorenstein schemes. For example, a general cubic gives a graded Gorenstein subscheme with Hilbert function $(1, n, n, 1)$, where $n = \dim V$. Note that $\mathrm{Apolar}(-)$ is order preserving: a subscheme corresponds to a smaller linear space.

2.3 Geometry of zero-dimensional schemes

For a zero-dimensional scheme, its *length* is the linear dimension of its algebra of global sections. If the scheme is embedded into a projective space, then it is the same as its degree. In this subsection we fix the length d of considered schemes. If the scheme is irreducible, then it corresponds to a *local* algebra $(A, \mathfrak{m}, \mathbf{k})$. In this case by the *Hilbert function* we denote $H(i) = \dim_{\mathbf{k}} \mathfrak{m}^i/\mathfrak{m}^{i+1}$. This function is usually written as a vector of its non-zero values, e.g. for a first infinitesimal neighbourhood of a point in $\mathbb{A}^3$ one gets $(1, 3)$. For a general reference on finite schemes, see [BJ17].

For a general reference on the Hilbert schemes of points, see [FGI+05, HS04, Har10].

The functor of embedded flat families of length d zero-dimensional subschemes of $\mathbb{A}^n$ is represented by the *Hilbert scheme of points* of $\mathbb{A}^n$, which we denote $\mathrm{Hilb}_d \mathbb{A}^n$ or shortly Hilb_d. It is a connected, quasi-projective scheme; in fact it is an open subset of the Hilbert scheme of points on n-dimensional projective space (working with an affine instead of a projective space is natural, since we look towards analysing affine base loci). Closed points of Hilb_d correspond to zero-dimensional schemes; we will denote by $[R]$ the point corresponding to a subscheme $R \subset \mathbb{A}^n$.

The most natural zero-dimensional subscheme of $\mathbb{A}^n$ of length d is just a d-tuple of points. Denote by $\mathrm{Hilb}_d^\circ \subset \mathrm{Hilb}_d$ the subset corresponding to all tuples of points. A scheme is a tuple of points precisely when it is smooth, hence Hilb_d° is open in Hilb_d. Its closure is then an irreducible component of Hilb_d, denoted by Hilb_d^{sm}. The points of Hilb_d^{sm} are limits of smooth schemes, hence are called *smoothable* schemes and Hilb_d^{sm} is called the *smoothable component*. This component has dimension nd.

Smoothability has a down-to-earth characterisation, at least in the graded case and generically. For a scheme $R \subset \mathbb{A}^n$ we say that it is a $\mathbf{k}^*$*-limit* if $R = \lim_{t\to 0} t\Gamma$ for a tuple Γ of d points of $\mathbb{A}^n$. Then R is $\mathbf{k}^*$-invariant and smoothable. The notion of $\mathbf{k}^*$-limits may be formulated differently as follows. Fix a $\mathbf{k}^*$-limit $R = \lim_{t\to 0} t\Gamma$. Compactify $\mathbb{A}^n$ to a $\mathbb{P}^n$ by adding a "time coordinate". Then Γ becomes as a set of d points of $\mathbb{P}^n$ and R is the hyperplane section of the cone over Γ by the hyperplane corresponding to $t = 0$. We say that an irreducible scheme $R \subset \mathbb{A}^n$ is *compressed* if there is an s such that the Hilbert function of R satisfies $H(i) = \dim \mathrm{Sym}^i \mathbf{k}^n$ for all $i < s$ and $H(i) = 0$ for all $i > s$. For example if $R \subset \mathbb{A}^3$ is an irreducible scheme of length 12, then it is compressed if and only if its Hilbert function is $(1, 3, 6, 2)$.

Corollary 7. *The set of smoothable compressed subschemes of $\mathbb{A}^n$ is irreducible and its general member is a $\mathbf{k}^*$-limit.*

Proof. This follows from [CEVV09, Lemma 5.4]. □

Corollary 8. *Let $\varphi : \mathbb{P}(V) \to \mathbb{P}(W)$ be a morphism and $R \subset V$ be its affine base locus. If R is a $\mathbf{k}^*$-limit then there exists an inclusion $V \subset V'$ with one dimensional cokernel and an extension $\psi : \mathbb{P}(V') \dashrightarrow \mathbb{P}(W)$. Conversely, if ψ exists and its base locus Γ is non-empty and smooth of degree $\deg R$, then R is $\mathbf{k}^*$-limit.*

Proof. If $R \subset V$ is a $\mathbf{k}^*$-limit of Γ then one may compactify V to $\mathbb{P}(V')$ by adding a "time coordinate" as above. Then R is a hyperplane section of the cone over Γ and the equations of φ defining Γ lift to equations of Γ, which induce a rational map $\psi : \mathbb{P}(V') \dashrightarrow \mathbb{P}(W)$ with base locus Γ. Conversely, if ψ exists and its base locus Γ is smooth non empty, then it is also zero-dimensional. Let $\hat{\Gamma}$ be the affine base locus of ψ. Then $\hat{\Gamma}$ is cone over Γ, perhaps with some embedded component at the origin (due to lack of saturation). Let $\hat{\Gamma}' \subset \hat{\Gamma}$ be the cone given by saturation $I(\Gamma)$. The scheme $R' = \hat{\Gamma}' \cap V$ is a hyperplane section of $\hat{\Gamma}'$, hence has degree $\deg \Gamma$, which is equal to $\deg R$ by assumption. The scheme R' is a $\mathbf{k}^*$-limit of the affine scheme Γ in $\mathbb{P}(V') \setminus \mathbb{P}(V)$. By definition, $R = V \cap \hat{\Gamma}$, thus $R' \subset R$ are two zero-dimensional schemes of the same degree, so $R' = R$. □

For $n \leq 2$ the smoothable component is the unique component and in fact Hilb_d is smooth [Fog68]. This is no longer the case for $n \geq 3$. If the dimension is at least three and d is large enough, then the Hilbert scheme is reducible and singular ([CEVV09, Erm12, Iar84] and also [IE78][1]). Not much is known about the additional components of the Hilbert scheme, in fact their mere presence seems to discourage investigators.

It is known that for $d \leq 7$, any n or for $d = 8$, $n \leq 3$ the Hilbert scheme is irreducible [CEVV09]. This scheme is reducible for $d = 8$ and $n \geq 4$. In this paper we are interested in the "smallest" reducible example: the Hilbert scheme of $d = 8$ points on $\mathbb{A}^4$.

3 Projections of $v_2(\mathbb{P}^3)$

In this section we study the geometry of projections of $v_2(\mathbb{P}^3) \subset \mathbb{P}^9$ to $\mathbb{P}^6$. Let us introduce some useful tools.

3.1 Hilbert scheme of eight points on affine four-space

The paper by Cartwright, Erman, Velasco and Viray [CEVV09] is devoted to the analysis of the Hilbert scheme of eight points on affine 4-space. Most of the facts below are found there; our contributions are Proposition 9 and Remark 10.

The Hilbert scheme of 8 points on $V := \mathbb{A}^4$ has two irreducible components. The smoothable component has dimension $8 \cdot 4 = 32$. The other component, Hilb_{143}, has dimension 25 and is isomorphic to $V \times \mathrm{Gr}(7, \mathrm{Sym}^2 V^\vee)$. The isomorphism is given by sending (p, N) to the irreducible scheme $\mathrm{Sym}\, V^\vee/(N) + (V^\vee)^3$ translated so that its support is p. Hence the schemes corresponding to points of Hilb_{143} are irreducible and have Hilbert function $(1, 4, 3)$. By apolarity we may equivalently parameterise this component as $V \times \mathrm{Gr}(3, \mathrm{Sym}^2 V)$, by sending (p, L) to $\mathrm{Apolar}(L + V)$ supported at p. Note that if $\mathrm{Apolar}(L+V) \neq \mathrm{Apolar}(L)$, then the partials of forms of L span a proper subspace W of V, hence $L \subset \mathrm{Sym}^2 W$. In this case $\mathbb{P}(L)$ intersects the secant variety of $v_2(\mathbb{P}(W))$. Thus, for our purposes, the difference between $\mathrm{Apolar}(L)$ and $\mathrm{Apolar}(L + V)$ is negligible.

The two components intersect along an irreducible 24-dimensional set, which has the form $V \times D$, where $D \subset \mathrm{Gr}(3, \mathrm{Sym}^2 V)$ is a divisor of degree two on the Grassmannian, which we call the *smoothable divisor*. Hence smoothability is independent of the embedding (this is true for all finite schemes [BJ17, Theorem 1.1]). The equation of D is obtained as follows: let $W \subset \mathrm{Sym}^2 V$ be a 3-dimensional space spanned by q_1, q_2, q_3 that correspond to 4×4 matrices A_1, A_2, A_3. Then the equation is the Pfaffian of the 12×12 matrix

$$\begin{bmatrix} 0 & A_1 & -A_2 \\ -A_1 & 0 & A_3 \\ A_2 & -A_3 & 0 \end{bmatrix}. \tag{1}$$

Salmon gave a geometric description of the smoothable divisor, by showing that its equation vanishes on q_1, q_2, q_3 if and only if there exists a cubic F and linear differential operators d_1, d_2, d_3 such that $d_i F = q_i$. Below we give another proof of this fact.

Proposition 9. *Let $L \subset \mathrm{Sym}^2 V$ be 3-dimensional and $R = \mathrm{Spec}\,\mathrm{Apolar}(L + V)$ be the corresponding zero-dimensional scheme of degree eight. The following conditions are equivalent:*

1. *there is a cubic F such that L is spanned by three partial derivatives of F,*
2. *the scheme R is smoothable.*

[1] Note: there is a known numerical mistake in the computation on page 169, compare [CN09].

Proof. The implication 1 $\implies$ 2 is noted in [CEVV09, Rmk 5.9] and we refer to this work for details (see also Remark 10).

We will prove the implication 2 $\implies$ 1. By the obvious parameterisation, the set of L satisfying Condition 1 is closed. Thus it is enough to show that each point $[R]$ is a limit of points in the smoothable divisor satisfying 1. By Corollary 7 the set of smoothable, irreducible schemes with Hilbert function $(1, 4, 3)$ is irreducible and the set of $\mathbf{k}^*$-limits is dense inside it. Hence we may consider only $\mathbf{k}^*$-limits.

Let $R = \lim_{t \to 0} t\Gamma$ be a $\mathbf{k}^*$-limit in $\mathbb{A}^4$ with Hilbert function $(1, 4, 3)$. Compactify $\mathbb{A}^4$ to $\mathbb{P}^4$ and consider $\Gamma \subset \mathbb{P}^4$. By [EP00, Theorem 8.6] this set can be enlarged to a tuple Γ' of ten arithmetically Gorenstein points of $\mathbb{P}^4$. Since $R = \hat{\Gamma} \cap H$ is a hyperplane section of the cone over Γ, it lies in a hyperplane section $S = \hat{\Gamma}' \cap H$ of the cone over Γ'. This section is zero-dimensional Gorenstein with Hilbert series $(1, 4, 4, 1)$, provided that H intersects Γ' properly. This condition can be always achieved by perturbing H, which perturbs R in the smoothable divisor. Hence, at least in any neighbourhood of R, we get a point $R_\varepsilon \subset S_\varepsilon$, where S_ε is Gorenstein with Hilbert function $(1, 4, 4, 1)$. We conclude that $S_\varepsilon = \operatorname{Apolar}(F_\varepsilon)$ for some cubic and that L_ε is contained in the partials of this cubic. □

Remark 10. *In the proof of Proposition 9 we got an inclusion of R into S whose Hilbert function is $(1, 4, 4, 1)$ and even an inclusion of curves whose hyperplane sections are R and S. Moreover, Condition 1 asserts that for a given S every $R \subset S$ is smoothable. But for a given S and a curve C_S smoothing it as above R is not necessarily smoothed by a sub-curve C_R. Indeed, for a fixed C_S the existence of C_R is equivalent to the existence of $C_{R'}$, where $R' \subset S$ is the residuum of R in S. In our case R' is a tangent vector in S. One sees immediately that the tangent vectors which do lift lie in the planes spanned by the two of the 10 points of C_S. In particular not all tangent vectors lift.*

3.2 Smoothable schemes of degree 8 and special quadric morphisms $\mathbb{P}^3 \to \mathbb{P}^6$

Let V have dimension four. In this section we consider morphisms $\mathbb{P}(V) \to \mathbb{P}^6$ given by quadrics. They factor as second Veronese v_2 composed with a projection

$$\pi_L \colon \mathbb{P}(\operatorname{Sym}^2 V) \dashrightarrow \mathbb{P}^6$$

from an $L \subset \operatorname{Sym}^2 V$. Let $X_L = \pi_L(v_2(\mathbb{P}^3))$. A general $\mathbb{P}(L)$ does not intersect the secant variety of $v_2(\mathbb{P}^3)$, hence for such L the variety X_L is isomorphic to $\mathbb{P}(V)$ via $\pi_L \circ v_2$.

Denote by $L^\perp \subset \operatorname{Sym}^2 V^\vee$ the space perpendicular to L, then naturally $\mathbb{P}^6 = \mathbb{P}((L^\perp)^\vee)$. The pullback of sections from $\mathbb{P}^6$ to X_L and then to $\mathbb{P}(V)$ gives a restriction map $H^0(\mathcal{O}_{\mathbb{P}^6}(3)) \to H^0(\mathcal{O}_{\mathbb{P}(V)}(3))$, which algebraically reads

$$\operatorname{Sym}^3 L^\perp \to \operatorname{Sym}^6 V^\vee. \tag{2}$$

The spaces on both sides of this map have dimension 84 and in fact for a general choice of L this morphism is an isomorphism. Let $\mathrm{Gr} := \mathrm{Gr}(3, \operatorname{Sym}^2 V)$ be the parameter space for such L. The above discussion globalises as follows: the tautological sequence

$$0 \to \mathcal{T} \to \mathcal{O}_{\mathrm{Gr}} \otimes \operatorname{Sym}^2 V \to \mathcal{Q} \to 0$$

dualises to $0 \to \mathcal{Q}^\vee \to \mathcal{O}_{\mathrm{Gr}} \otimes \operatorname{Sym}^2 V^\vee \to \mathcal{T}^\vee \to 0$. The multiplication $\operatorname{Sym}^3 \operatorname{Sym}^2 V^\vee \to \operatorname{Sym}^6 V^\vee$ induces a map

$$\mu \colon \operatorname{Sym}^3 \mathcal{Q}^\vee \to \mathcal{O}_{\mathrm{Gr}} \otimes \operatorname{Sym}^6 V^\vee, \tag{3}$$

whose restriction to a point $[\mathbb{P}(L)] \in \mathrm{Gr}$ is exactly (2). Thus the jump locus has a natural scheme structure at the degeneracy locus of μ. A Schubert calculation shows that the degree of the jump locus is 36.

As described in Section 2.2, a point $[\mathbb{P}(L)]$ of $\mathrm{Gr}(3, \mathrm{Sym}^2 V)$ has an associated zero-dimensional scheme $R = \mathrm{Spec}\,\mathrm{Apolar}(L+V)$, which is of length 8 and for a general L it is equal to $\mathrm{Spec}\,\mathrm{Apolar}(L)$ (so that adding V may be thought of as stabilisation of length).

Now comes the Leitmotiv of this work: we show that smoothability implies the jump and even more. The authors discussed the problem just before the conference dinner at Schreyerfest and the first computational evidence for Theorem 11 was obtained just before the dessert. Hence we call the following *Schreyerfest-dinner theorem*.

For a three-dimensional $L \subset \mathrm{Sym}^2 V$ we say that L *satisfies the jump condition* if the image X_L of $v_2\mathbb{P}(V)$ under the projection from $\mathbb{P}(L)$ lies on a cubic. We say that the jump is at least by l if X_L lies on an (at least) l-dimensional space of cubics.

Theorem 11 (Schreyerfest-dinner)**.** *Let $L \subset \mathrm{Sym}^2 V$ be a linear subspace of dimension three. Suppose that $R = \mathrm{Spec}\,\mathrm{Apolar}(L+V)$ is smoothable. The $\mathbb{P}(L)$ satisfies the jump condition and the jump is at least by three.*

Proof. The jump condition is closed, so it is enough to prove it for a general L. Hence, by Corollary 7 we may assume that R is a $\mathbf{k}^*$-limit and that its ideal is generated by quadrics. By Corollary 8 the map $\pi_L \circ v_2 : \mathbb{P}(V) = \mathbb{P}^3 \to \mathbb{P}^6$ extends to a rational map $\mathbb{P}^4 \dashrightarrow \mathbb{P}^6$ whose base locus Γ_8 is a tuple of 8 points. Since a general tuple of such 8 points gives R as above, we may assume that Γ_8 is general. Also the ideal of Γ_8 is generated by seven quadrics.

Now we apply Gale duality. We refer the reader to the beautiful paper [EP00]. Since $\Gamma_8 \subset \mathbb{P}^4$ is general, its Gale dual exists by [EP00, Corollary 2.4] and is a 8-tuple $\Gamma_8' \subset \mathbb{P}^2$, which we may also assume to be general. Hence, there exists $q \in \mathbb{P}^2$ such that $\Gamma_8' \cup \{q\}$ is a complete intersection of two elliptic curves. Also, in this case the Gale duality can be made explicit: starting from $\Gamma_8' \subset \mathbb{P}^2$ one takes the second Veronese reembedding and projects from q, obtaining $\Gamma_8 \subset \mathbb{P}^4$, see [EP00, Corollary 2.6 and p. 138-9]. In any case, the pencil of elliptic curves passing through Γ_8' gives a pencil $\mathcal{E}$ of projectively normal elliptic curves through Γ_8, [EP00, Corollary 3.2].

Each curve E in this pencil is arithmetically Gorenstein and cut out by 5 quadrics. These quadrics define a rational map $c_E : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ which is *birational* with inverse c_E' given by cubics [CK89]. Now the image $c_E'(c_E(\mathbb{P}^3))$ spans at most a $\mathbb{P}^3$, hence the coordinates of c_E' give a cubic equation between five of the quadrics defining $\mathbb{P}(V) = \mathbb{P}^3 \to \mathbb{P}^6$. Such relation gives a cubic equation $C_E \in \mathbb{P}\left(\mathrm{Sym}^3 L^\perp\right)$ of the image. The situation is summarised in the following diagram

$$\begin{array}{ccc} \mathbb{P}^3 & \xrightarrow{\pi_L \circ v_2} & \mathbb{P}^6 \\ \big\downarrow {\scriptstyle lin} & & \big\downarrow {\scriptstyle lin} \\ \mathbb{P}^4 & \overset{c_E}{\dashrightarrow} & \mathbb{P}^4 \end{array} \tag{4}$$

The argument can be made global to obtain a map $syz : \mathbb{P}^1 \to \mathbb{P}\left(\mathrm{Sym}^3 L^\perp\right)$.

It remains to show that there are at least *three* linearly independent cubic equations. We will do this by showing that syz is not constant or a line. This requires working with all curves from the pencil $\mathcal{E}$. These curves sweep out a rational cubic scroll which is the image of $\mathbb{P}^2$. The ideal of this scroll is generated by three quadrics — maximal minors of a 2×3 matrix. Hence the containments between the quadrics are as follows:

$$\{3 \text{ in the ideal of scroll}\} \subset \{5 \text{ cutting out an elliptic curve}\} \subset \{7 \text{ defining the 8-points}\}.$$

Then any elliptic curve defines a point in $\mathrm{Gr}(2,4)$ corresponding to the space of its quadric equations modulo the quadrics vanishing on the scroll. We obtain a map $eq : \mathbb{P}^1 \to \mathrm{Gr}(2,4)$.

If syz was constant, there would be a cubic relation between the three quadrics defining the scroll. But no such relation exists. Clearly syz factors through eq. If the image of syz was a line, also the image of eq would be a line in the Plücker embedding. But such a line corresponds to a family of projective lines on a plane passing through a point. Such a point would give a quadric not in the ideal of the scroll but vanishing on all elliptic curves. This contradicts the fact that the elliptic curves fill the scroll. □

Remark 12. *The cubics C_E appearing in the proof of Theorem 11 are called Segre cubics, see [Dol16, p. 4].*

One wonders to what extent the statement of Theorem 11 can be reversed. For this consider first the $\mathbb{P}(L) \subset \mathbb{P}(\mathrm{Sym}^2 V)$ which intersect the secant (then $\mathrm{Sym}^3 L^\perp \to \mathrm{Sym}^6 V^\vee$ is not an isomorphism).

Lemma 13. *Let $\mathbb{P}(L)$ be a projective plane intersecting the second secant of $v_2(\mathbb{P}(V))$. Then X_L lies on an at least three dimensional space of cubics.*

Proof. The secant is the locus of quadrics of rank at most two. The jump locus is closed, so we may assume that $\mathbb{P}(L)$ intersects the secant in a rank two quadric xy. Then no form of $L^\perp$ contains the monomial $\partial_x\partial_y$. Enlarge x, y to a basis x, y, z, t of V. Then $L^\perp \subset \mathrm{span}\left(\partial_x^2, \partial_y^2\right) + \mathrm{span}\left(\partial_z, \partial_t\right) \cdot V^\vee$. Observe that $(L^\perp)^3 \subset \mathrm{Sym}^6 V^\perp$ annihilates the forms x^5y, x^3y^3 and xy^5. Hence,

$$\dim((L^\perp)^3) \leq \dim(\mathrm{Sym}^6 V^\perp) - 3 = \dim(\mathrm{Sym}^3 L^\perp) - 3$$

and we obtain an (at least) three dimensional space of cubics containing X_L. □

Example 14. *Note that when $L \subset \mathrm{Sym}^2 W$ for some three-dimensional subspace $W \subset V$, then $\mathbb{P}(L)$ insersects the secant variety of $v_2(\mathbb{P}(W))$, because the latter is a divisor in $\mathbb{P}(\mathrm{Sym}^2 W)$. Thus, $\mathbb{P}(L)$ intersects the secant variety of $v_2(\mathbb{P}(V))$.*

Now we prove a stronger form of Theorem 11. For the definition of jump condition, see the paragraph above Theorem 11.

Theorem 15. *Let $[\mathbb{P}(L)] \in \mathrm{Gr}(3, \mathrm{Sym}^2 V)$ be a projective plane. The following conditions are equivalent:*

1. *$\mathbb{P}(L)$ satisfies the jump condition,*
2. *$\mathbb{P}(L)$ satisfies the jump condition and the jump is at least by three,*
3. *either $\mathbb{P}(L)$ intersects the secant variety of $v_2(\mathbb{P}(V))$ or the zero-dimensional scheme associated to L is smoothable.*

Proof. Clearly, (2) implies (1). By Theorem 11 and Lemma 13 we have (3) implies (2). It remains to prove that (1) implies (3). These conditions are both divisorial (for smoothability, see Section 3.1), so it remains to check their degrees.

The (Plücker) degree of the locus of spaces intersecting the secant variety is 10 and the degree of the smoothable divisor is 2, see (1). By the discussion below Map (3) (before Theorem 11) , the jump locus has degree 36. The jump on the secant and smoothable parts is both at least by three, so these divisors contribute to the jump locus with multiplicity at least three, hence in total they contribute by $3 \cdot (10 + 2) = 36$ to the degree. Thus the jump locus is equal to the union of those divisors. □

3.3 Cremona transformations associated to octic surfaces

Let us describe further the geometry of the Cremona transformations described in the proof of Theorem 11. We claim that these transformations can be lifted to the Cremona transformations defined by octic surfaces, which we now recall. Consider a set $Z \subset \mathbb{P}^2$ of eight points. The quartics through Z define an embedding $\mathrm{Bl}_Z \mathbb{P}^2 \to \mathbb{P}^6$. The embedded surface $S_8 = S_8(Z) \subset \mathbb{P}^6$ is a rational *octic surface*. One checks that S_8 is contained in a singular scroll X and in fact $S_8 = D_1 \cap D_2$ is a complete intersection of two linearly equivalent divisors inside the scroll [HKS92, Section 3].

Moreover [HKS92, Thm 3.2], the quadrics through S_8 define a Cremona transformation $c_{S_8} \colon \mathbb{P}^6 \dashrightarrow \mathbb{P}^6$, whose inverse is given by quartics through a model of $\mathbb{P}^4$ blown in 8 points [ST70]. If one takes a general $\mathbb{P}^4$-section in $\mathbb{P}^6$ then $S_8 \cap \mathbb{P}^4$ is a tuple of eight points lying on a pencil of elliptic normal curves generated by $D_1 \cap \mathbb{P}^4$ and $D_2 \cap \mathbb{P}^4$ and this pencil fills a cubic scroll $X \cap \mathbb{P}^4$. Each Cremona transformation related to an elliptic curve in this pencil is obtained from $\mathbb{P}^6 \dashrightarrow \mathbb{P}^6$ by composing with a linear embedding and a linear projection. In consequence we obtain a further geometric description of our projection from L.

Proposition 16. *Let $\mathbb{P}(L) \subset \mathbb{P}(\mathrm{Sym}^2 V)$ be a $\mathbb{P}^2$ corresponding to a general element of the smoothable divisor. Then there exists an octic surface S_8 and a linear embedding $\ell \colon \mathbb{P}^3 \to \mathbb{P}^6$ such that $\pi_L = \ell \circ c_{S_8}$:*

$$\begin{array}{ccc} \mathbb{P}^3 & \xrightarrow{\pi_L \circ v_2} & \mathbb{P}^6 \\ \downarrow{\scriptstyle lin} & & \| \\ \mathbb{P}^6 & \overset{c_{S_8}}{\dashrightarrow} & \mathbb{P}^6 \end{array}$$

Proof. Since $\mathbb{P}(L)$ corresponds to a general element of the smoothable divisor, there is a (suitably general) set of eight points $\Gamma \subset \mathbb{P}^4$ such that $\pi_L \circ v_2$ lifts to a map $\mathbb{P}^4 \dashrightarrow \mathbb{P}^6$ given by quartics through Γ, see Corollaries 7-8. The claim follows if we prove that Γ is a linear section of an octic surface S_8, which we do below. Let $\Gamma' \subset \mathbb{P}^2$ be the Gale transform of $\Gamma \subset \mathbb{P}^4$. By Gale duality [EP00, Corollary 3.2], the cubics containing $\Gamma' \subset \mathbb{P}^2$ give a pencil of elliptic normal curves containing Γ, which fill a smooth cubic scroll $X \subset \mathbb{P}^4$ containing Γ. The Picard group of X is generated by the hyperplane section of the scroll H and the fiber R of the scroll. From the adjunction formula, the pencil of elliptic curves is a subsystem the system $|2H - R|$. Let D_1, D_2 be two elements from this subsystem. Since $D_1.D_2 = 8$ we infer that $D_1 \cap D_2 = \Gamma$.

Let us consider cubic scrolls $Y \subset \mathbb{P}^5$ and $Z \subset \mathbb{P}^6$ such that Z restricts to Y and Y to X. We can assume that Y is smooth and Z is a cone over Y. From the restriction exact sequence

$$0 \to \mathcal{O}_Y(H-R) \to \mathcal{O}_Y(2H-R) \to \mathcal{O}_X(2H-R) \to 0$$

and the fact that $h^1(\mathcal{O}_Y(H-R)) = h^1(\mathcal{O}_Z(H-R)) = 0$ we find divisors B_1 and B_2 on Z in the linear system $|2H - R|$ such that $B_1|_X = D_1$ and $B_2|_X = D_2$ (in fact we have a freedom of choice of B_1 and B_2). Then $B_1 \cap B_2 \cap \mathbb{P}^4 = \Gamma$, so $B_1 \cap B_2$ is a complete intersection. By [HKS92, Section 3], the intersection of two divisors from $|2H - R|$ is a rational octic surface $S_8 \subset \mathbb{P}^6$. Now, Γ is the intersection of this S_8 with the linear subspace spanned by X. □

Remark 17. *By Proposition 16 the elliptic curves E constructed in the proof of Theorem 11 appear as linear sections of S_8 and the corresponding Cremona transformations c_E come from c_{S_8} composed with a linear embedding and projection. Note that c_{S_8} has type $(2,4)$ while all c_E have type $(2,3)$.*

Now we formally summarize how do the previous results add up to give our main result.

Proof of Theorem 1. First, since L does not intersect the secant of $v_2(\mathbb{P}(V))$, by Subsection 3.1 we have $\mathrm{Apolar}(L) = \mathrm{Apolar}(L+V)$. Let us recapitulate the proof of equivalence of Conditions (a)-(e). The equivalence of (d) and (e) is proven in 9. The equivalence of Conditions (b), (c) and (d) is given in Theorem 15. By Proposition 16, Condition (d) implies Condition (a). Finally, if $\pi_L \circ v_2$ is a general restriction of a Cremona based in a rational octic surface, then $\mathrm{Spec}(\mathrm{Apolar}(L))$ is a section of a cone over 8 points of that surface, hence is smoothable. Thus, Condition (a) implies (d) and the proof of equivalences is concluded. The irreducibility of the smoothable divisor follows from the parametrization from (e) and was proven in [CEVV09]. For a general L in this divisor, the existence of a three-dimensional space of Segre cubics follows from Remark 12. If $I(X_L)_3$ was at least four-dimensional for all L in the smoothable divisor, then the contribution of the smoothable divisor to the jump locus would be higher than allowed, see the proof of Theorem 15. □

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