

TOWARDS TRANSVERSE TORIC GEOMETRY

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ABSTRACT. We study certain foliated complex manifolds that behave similarly to complete nonsingular toric varieties. We classify them by combinatorial objects that we call marked fans. We describe the basic cohomology algebras of them in terms of corresponding marked fans. We also study the basic Dolbeault cohomology algebras of them.

1. INTRODUCTION

A toric variety is a normal algebraic variety over the complex numbers \mathbb{C} with an effective action of an algebraic torus having an open dense orbit. On the other hand, a fan is a collection of cones in a real vector space with the origin as vertex satisfying certain conditions. A fundamental theorem in toric geometry states that the category of toric varieties is equivalent to the category of rational fans.

In this paper, we study certain foliated manifolds that behave similarly to complete nonsingular toric varieties. As a combinatorial counterpart to such a foliated manifold, we introduce the notion of *marked fan*. A marked fan is a fan in a finite dimensional vector space equipped with information of generators of 1-cones.

The main concerns in this paper are the followings.

- (1) A compact connected manifold M equipped with a maximal action of a compact torus G .
- (2) The canonical foliation F on a compact connected complex manifold M .

We say that an effective action of a compact torus G on a smooth connected manifold M is *maximal* if there exists a point $x \in M$ such that $\dim G + \dim G_x = \dim M$, where G_x denotes the isotropy subgroup at $x \in M$ of G . All compact connected complex manifolds equipped with maximal actions of compact tori are completely classified in [12]. There are rich examples of such manifolds. One can see that compact complex tori, complete nonsingular toric varieties, Hopf manifolds and Calabi-Eckmann manifolds admit maximal actions of compact tori. More interesting examples are

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LVM manifolds, LVMB manifolds and moment-angle manifolds with complex structures. LVM manifolds are non-Kähler compact complex manifolds, constructed in [16] and the acronyms stand for López de Medrano-Verjovsky [15] and Meersseman. LVMB manifolds are generalizations of LVM manifolds, constructed by Bosio in [4]. LVMB manifolds are obtained as the quotients of complements of coordinate subspace arrangements in projective spaces $\mathbb{C}P^n$ by free, proper, and holomorphic \mathbb{C}^m -actions. They are naturally equipped with actions of compact tori which are maximal. Moment-angle manifolds are topological manifolds equipped with actions of compact tori, and constructed from simplicial complexes. In [18] and [19], it is shown that even dimensional moment-angle manifolds coming from star-shaped spheres carry invariant complex structures.

The canonical foliation F on a complex manifold M is a holomorphic foliation generated by a local free action of a certain subgroup H' of the group $\text{Aut}(M)$ of all biholomorphisms on M . For more precise, let \mathfrak{g} be the Lie algebra of a maximal compact torus G of $\text{Aut}(M)$ and J the complex structure of $\text{Aut}(M)$. The Lie subalgebra $\mathfrak{h}' := \mathfrak{g} \cap J\mathfrak{g}$ is a complex subalgebra. One can see that the corresponding Lie group $H' := \exp(\mathfrak{h}') \subset \text{Aut}(M)$ acts on M local freely and does not depend on the choice of the maximal compact torus G . Each leaf of the canonical foliation F is an H' -orbit and vice versa. The canonical foliations on LVMB manifolds have been studied in [2] from a different viewpoint. In [2], under some assumptions, it has been shown that the basic betti numbers of an LVMB manifold with respect to the canonical foliation coincides with the h -vector of a certain simplicial sphere. It also has been shown that the basic cohomology algebras of certain LVMB manifolds are generated by degree 2 elements, as well as the cohomology algebras of complete nonsingular toric varieties.

If the action of a compact torus G on a compact connected complex manifold M is maximal and preserves the complex structure on M , then the torus G can be regarded as a maximal torus of the group $\text{Aut}(M)$ of all biholomorphisms of M .

Example 1.1 (Complete nonsingular toric varieties). Let M be a complete nonsingular toric variety of complex dimension n . Let $G^{\mathbb{C}}$ be the algebraic torus of complex dimension n acting on M . Let G be the maximal compact torus of $G^{\mathbb{C}}$. G is a maximal compact torus of $\text{Aut}(M)$. Since the action of $G^{\mathbb{C}}$ is effective, we have that $\mathfrak{h}' = \mathfrak{g} \cap J\mathfrak{g} = \{0\}$. Therefore, every leaf of the canonical foliation F on M is a point. The leaf space M/F is a complete nonsingular toric variety M itself.

Example 1.2 (Compact complex tori). Let Γ be a lattice of \mathbb{C}^n . Let G be the compact complex torus \mathbb{C}^n/Γ of real dimension $2n$. Let $M = G$. We define the action of G on M is given as the group operation. Then the action of G on M is maximal and preserves the complex structure. For a fundamental vector field X_v on M generated by an element $v \in \mathbb{C}^n$, we have that JX_v is the fundamental vector field $X_{\sqrt{-1}v}$. Therefore we have $\mathfrak{h}' = \mathfrak{g} \cap J\mathfrak{g} = \mathfrak{g}$. Thus the canonical foliation F on

M has exactly one leaf M . The leaf space M/F is a point and it can be regarded as the toric variety of dimension 0.

Example 1.3 (Hopf surfaces). Let $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $1 < |\alpha_1| \leq |\alpha_2|$. Define an action of \mathbb{Z} on $\mathbb{C}^2 \setminus \{0\}$ to be $k \cdot (z_1, z_2) := (\alpha_1^k z_1, \alpha_2^k z_2)$ for $k \in \mathbb{Z}$, $(z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}$. The quotient manifold $M := \mathbb{C}^2 \setminus \{0\} / \mathbb{Z}$ is a complex manifold that is diffeomorphic to $S^3 \times S^1$. We equip the action of $(\mathbb{R}/\mathbb{Z})^3 =: G$ on M as follows. Let $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathbb{C}$ such that $e^{2\pi\sqrt{-1}\tilde{\alpha}_i} = \alpha_i$ for $i = 1, 2$. We define an action of \mathbb{R}^3 on M by

$$(t_1, t_2, s) \cdot [z_1, z_2] := [e^{2\pi\sqrt{-1}(t_1 + \tilde{\alpha}_1 s)} z_1, e^{2\pi\sqrt{-1}(t_2 + \tilde{\alpha}_2 s)} z_2]$$

for $(t_1, t_2, s) \in \mathbb{R}^3$ and $[z_1, z_2] \in M$, where $[z_1, z_2]$ denotes the equivalence class of $(z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}$. The action of \mathbb{R}^3 above on M preserves the complex structure on M and it has the global stabilizers \mathbb{Z}^3 . Thus the action of \mathbb{R}^3 descends to an action of the compact torus G that preserves the complex structure on M . One can see that the orbit through $[1, 0]$ has dimension 2. Thus, the action of G on M is maximal. By definition of \mathfrak{h}' , we have that $\mathfrak{h}' \subset \mathfrak{g} = \mathbb{R}^3$ has the basis vectors $v_1 = (\operatorname{Re}(\tilde{\alpha}_1), \operatorname{Re}(\tilde{\alpha}_2), -1)$ and $v_2 = (\operatorname{Im}(\tilde{\alpha}_1), \operatorname{Im}(\tilde{\alpha}_2), 0)$. On the other hand, $\mathfrak{h}' = \{v \in \mathbb{R}^3 \mid v \cdot (v_1 \times v_2) = 0\}$. Thus \mathfrak{h}' is a Lie algebra of a subtorus of G if and only if the fractions of entries of $v_1 \times v_2$ are rational. One can see that the fractions of entries of $v_1 \times v_2$ are rational if and only if there exist positive integers n_1, n_2 such that $\alpha_1^{n_1} = \alpha_2^{n_2}$. In other words, if there are no positive integers n_1, n_2 such that $\alpha_1^{n_1} = \alpha_2^{n_2}$, then the canonical foliation F on M has a leaf that is not closed. The leaf space M/F is not Hausdorff unless $\alpha_1^{n_1} = \alpha_2^{n_2}$ for some positive integers n_1 and n_2 .

There are 3 main results in this paper. The first main result states that complex manifolds with maximal torus actions (up to an equivalence relation which is slightly stronger than the notion of transversally equivalence on foliated manifolds) can be described by the corresponding marked fans. The second main result states that the basic cohomology with respect to the canonical foliation of a complex manifold with a maximal torus action admits the same formula as complete nonsingular toric varieties. To show this, we introduce the *basic forgetful map* $\operatorname{for}_B: H_G^*(M) \rightarrow H_B^*(M)$, where $H_G^*(M)$ denotes the equivariant cohomology of the G -manifold M . We will see that for_B is surjective and describe the kernel of for_B . The third main result states that the basic Dolbeault cohomology group $H_B^{p,q}(M)$ of our space concentrates to the center. Namely, we will see that $H_B^{p,q}(M) = 0$ if $p \neq q$. To see this complex geometric result, we use Minkowsky sum of polytopes and localization of equivariant cohomologies.

This paper is organized as follows. Section 2 is preliminaries. In Subsection 2.1, we review the canonical foliations. In Subsection 2.2, we briefly review the equivalence of categories between \mathcal{C}_1 and \mathcal{C}_2 , where \mathcal{C}_1 is the category of complex manifolds with maximal torus actions, and \mathcal{C}_2 is the combinatorial counter part of \mathcal{C}_1 . In Section 3, we study the notion of transverse equivalence in \mathcal{C}_1 . In Section 4, we introduce marked fans and show the first main result. Section 5 is preliminaries for the second and third main results. In Subsection 5.1, we review the complex structure of a tubular

neighborhood of a minimal orbit. In Subsection 5.2, we briefly review transverse Kähler forms and moment maps. In Section 6, we study the additive structure of the basic cohomology and the relation between the basic cohomology and the equivariant cohomology through the basic forgetful map. In Section 7, we show the second main result. We describe the basic cohomology algebra in terms of the corresponding marked fan. In Section 8, we show the third main result.

2. PRELIMINARIES 1

2.1. Canonical foliations. Let M be a compact connected complex manifold and G a maximal compact torus of the group $\text{Aut}(M)$ of all automorphisms on M . $\text{Aut}(M)$ is a complex Lie group and its Lie algebra can be identified with the vector space $\mathfrak{X}(M)$ of all holomorphic vector fields on M (see [3] for detail). Denote by \mathfrak{g} the Lie algebra of G and by J the complex structure on M . Put $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ and denote by $v + \sqrt{-1}v'$ instead of $v \otimes 1 + v' \otimes \sqrt{-1}$ for $v, v' \in \mathfrak{g}$ for short. We define the \mathbb{C} -subspace \mathfrak{h} of $\mathfrak{g}^{\mathbb{C}}$ by

$$(2.1) \quad \mathfrak{h} := \{v + \sqrt{-1}v' \in \mathfrak{g}^{\mathbb{C}} \mid X_v + JX_{v'} = 0\},$$

where X_v denotes the fundamental vector field generated by $v \in \mathfrak{g}$. Let $p: \mathfrak{g}^{\mathbb{C}} \rightarrow \mathfrak{g}$ be the first projection $(\mathfrak{g} \otimes 1) \oplus (\mathfrak{g} \otimes \sqrt{-1}) \rightarrow \mathfrak{g}$. We think of \mathfrak{g} as a real Lie subalgebra of $\mathfrak{X}(M)$ via $v \mapsto X_v$. Then, $p(\mathfrak{h})$ is a commutative subalgebra of $\mathfrak{X}(M)$ invariant under J because $p(\mathfrak{h}) = \mathfrak{g} \cap J\mathfrak{g}$. For $v \in p(\mathfrak{h})$ and $x \in M$, we denote by $(X_v)_x$ the value of the fundamental vector field X_v at $x \in M$.

Proposition 2.1. *For $v \in p(\mathfrak{h})$ and $x \in M$, $(X_v)_x = 0$ if and only if $X_v = 0$.*

Proof. Suppose that $(X_v)_x = 0$. Let φ_s be the partial flow of X_v at time s and ψ_t the partial flow of JX_v at time t . Denote by T the closure of the complex Lie group $H' := \{\varphi_s\psi_t \mid s, t \in \mathbb{R}\}$. Since $p(\mathfrak{h}) \subseteq \mathfrak{g}$, we have that T is a subtorus of G . Since the action of H' fixes $x \in M$, the action of T also fixes $x \in M$. We consider the isotropy representation $T_x M$ of T . Since T is compact, we have that $T_x M$ is a unitary representation of T . In particular, $T_x M$ is a unitary representation of H' . Since H' is a complex Lie group and the action of H' on M is holomorphic, we have that $T_x M$ is a holomorphic representation of H' . For a connected complex Lie group, any holomorphic unitary representation is trivial. Therefore $T_x M$ is a trivial representation of H' . Since T is a closure of H' , we have that $T_x M$ is a trivial representation of T . This together with the connectedness of M yields that T -fixes all $y \in M$ by slice theorem. Since T is a subtorus of $\text{Aut}(M)$, the action of T on M is effective. Therefore T is the trivial subgroup of $\text{Aut}(M)$. Hence H' is trivial. Thus we have that if $(X_v)_x = 0$ for $v \in p(\mathfrak{h})$ then $X_v = 0$. This shows the “if” part. The “only if” part is obvious. The proposition is proved. \square

Proposition 2.2. *The followings hold.*

- (1) *Elements in $p(\mathfrak{h})$ centralize $\mathfrak{X}(M)$.*

(2) $p(\mathfrak{h})$ does not depend on the choice of a maximal compact torus G of $\text{Aut}(M)$.

Proof. Let $\exp_G: \mathfrak{g} \rightarrow G$ denote the exponential map. We define the subgroup H' of G by $H' := \exp_G(p(\mathfrak{h}))$. Since G is compact, $\mathfrak{X}(M)$ is a unitary representation of G . In particular, $\mathfrak{X}(M)$ is a unitary representation of H' . Since H' is a holomorphic subgroup of $\text{Aut}(M)$, we have that $\mathfrak{X}(M)$ is a holomorphic representation of H' . Therefore $\mathfrak{X}(M)$ is a trivial representation of H' , showing Part (1).

Let G' be another maximal compact torus of $\text{Aut}(M)$ and \mathfrak{g}' the Lie algebra of G' . As before, we think of \mathfrak{g} and \mathfrak{g}' as real Lie subalgebras of $\mathfrak{X}(M)$. Since $\text{Aut}(M)$ is a Lie group, there exists $g \in \text{Aut}(M)$ such that $gGg^{-1} = G'$ (see [11, Chapter XV, Section 3]). In particular we have $\text{Ad}_g(\mathfrak{g}) = \mathfrak{g}'$. Since Ad_g is \mathbb{C} -linear, we have that $J(\text{Ad}_g(\mathfrak{g})) = \text{Ad}_g(J\mathfrak{g})$. Since $p(\mathfrak{h}) = \mathfrak{g} \cap J\mathfrak{g}$, we have that $\text{Ad}_g(p(\mathfrak{h})) = p'(\mathfrak{h}')$, where $p': \mathfrak{g}'^{\mathbb{C}} \rightarrow \mathfrak{g}'$ denotes the projection. On the other hand, by (1), we have that Ad_g is the identity on $p(\mathfrak{h})$. Therefore $p(\mathfrak{h})$ does not depend on the choice of G , proving Part (2). \square

By Proposition 2.1, we have a holomorphic foliation F on M whose leaves are generated by $p(\mathfrak{h})$. The definition of F is intrinsic by Proposition 2.2. We call F *the canonical foliation* on M .

2.2. Complex manifolds with maximal torus actions. In this subsection, we briefly recall the classification of complex manifolds with maximal torus actions given in [12] for reader's convenience. Let M be a connected smooth manifold equipped with an effective action of a compact torus G . We say that the G -action on M is *maximal* if there exists a point $x \in M$ such that $\dim G + \dim G_x = \dim M$. If the action of G on M is maximal, then we can think of G as a maximal compact torus of the group of diffeomorphisms on M (see [12, Lemma 2.2]).

Let \mathcal{C}_1 denote the class that consists of all triples (M, G, y) satisfying the followings.

- (1) M is a compact connected complex manifold.
- (2) G is a compact torus acting on M . The G -action on M is maximal and preserves the complex structure on M . In particular, G is a maximal compact torus of $\text{Aut}(M)$.
- (3) $y \in M$ satisfies that $G_y = \{1\}$.

For $(M_1, G_1, y_1), (M_2, G_2, y_2) \in \mathcal{C}_1$, the hom-set $\text{Hom}_{\mathcal{C}_1}((M_1, G_1, y_1), (M_2, G_2, y_2))$ is defined to be the set of pairs (f, α) satisfying the followings.

- (1) $\alpha: G_1 \rightarrow G_2$ is a smooth homomorphism.
- (2) f is an α -equivariant holomorphic map. Namely, for all $x \in M_1$ and all $g \in G_1$, we have $f(g \cdot x) = \alpha(g) \cdot f(x)$.
- (3) $f(y_1) = y_2$.

Then the class \mathcal{C}_1 and the hom-sets $\text{Hom}_{\mathcal{C}_1}(-, -)$ form a category.

As a combinatorial counterpart of \mathcal{C}_1 , we consider the followings. Let \mathcal{C}_2 denote the class that consists of triples $(\Delta, \mathfrak{h}, G)$ satisfying the followings.

- (1) G is a compact torus.
- (2) Let \mathfrak{g} be the Lie algebra of G and $\exp_G: \mathfrak{g} \rightarrow G$ the exponential map. We think of $\ker \exp_G \subset \mathfrak{g}$ a lattice of \mathfrak{g} . Then, Δ is a nonsingular fan in \mathfrak{g} with respect to the lattice $\ker \exp_G$.
- (3) As before, we denote by $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ and decompose $\mathfrak{g}^{\mathbb{C}}$ as $\mathfrak{g}^{\mathbb{C}} = (\mathfrak{g} \otimes 1) \oplus (\mathfrak{g} \otimes \sqrt{-1})$. Let $p: (\mathfrak{g} \otimes 1) \oplus (\mathfrak{g} \otimes \sqrt{-1}) \rightarrow \mathfrak{g}$ be the first projection. Then the restriction $p|_{\mathfrak{h}}: \mathfrak{h} \rightarrow \mathfrak{g}$ of p to \mathfrak{h} is injective.
- (4) Let $q: \mathfrak{g} \rightarrow \mathfrak{g}/p(\mathfrak{h})$ be the quotient map. Then

$$q(\Delta) := \{q(\sigma) \subset \mathfrak{g}/p(\mathfrak{h}) \mid \sigma \in \Delta\}$$

is a complete fan and the map $\Delta \rightarrow q(\Delta)$ given by $\sigma \mapsto q(\sigma)$ is bijective.

For $(\Delta_1, \mathfrak{h}_1, G_1), (\Delta_2, \mathfrak{h}_2, G_2) \in \mathcal{C}_2$, the hom-set $\text{Hom}_{\mathcal{C}_2}((\Delta_1, \mathfrak{h}_1, G_1), (\Delta_2, \mathfrak{h}_2, G_2))$ is defined to be the set of all smooth homomorphism $\alpha: G_1 \rightarrow G_2$ satisfying the followings.

- (1) Let \mathfrak{g}_1 and \mathfrak{g}_2 denote the Lie algebras of G_1 and G_2 , respectively. The differential $d\alpha: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ induces a morphism of fans Δ_1 and Δ_2 . Namely, for any $\sigma_1 \in \Delta_1$, there exists $\sigma_2 \in \Delta_2$ such that $d\alpha(\sigma_1) \subset \sigma_2$.
- (2) The linear map $d\alpha^{\mathbb{C}} := d\alpha \otimes \text{id}_{\mathbb{C}}: \mathfrak{g}_1^{\mathbb{C}} \rightarrow \mathfrak{g}_2^{\mathbb{C}}$ satisfies that $d\alpha^{\mathbb{C}}(\mathfrak{h}_1) \subset \mathfrak{h}_2$.

Then the class \mathcal{C}_2 and the hom-sets $\text{Hom}_{\mathcal{C}_2}(-, -)$ form a category.

To $(M, G, y) \in \mathcal{C}_1$, we can assign $(\Delta, \mathfrak{h}, G) =: \mathcal{F}_1(M, G, y) \in \mathcal{C}_2$ as follows. A *characteristic submanifold* of M is a complex codimension 1 connected component of the fixed point set by the action restricted to a 1-dimensional subtorus of G . If there is no characteristic submanifold (in this case, the action of G on M is free and simply transitive), then Δ is the fan that consists of the 0-dimensional cone $\{0\}$ in \mathfrak{g} only. Let M_1, \dots, M_k be characteristic submanifolds of M . Let G_i denote the 1-dimensional subtorus of G that fixes M_i pointwise. Since the codimension of M_i is 1, the normal bundle $TM|_{M_i}/TM_i$ is a complex line bundle on M_i . For any $x \in M_i$, the normal space $(TM|_{M_i}/TM_i)_x$ is a representation of G_i . Since G_i fixes M_i pointwise and the action is effective, we have that $(TM|_{M_i}/TM_i)_x$ is a faithful G_i -representation. We denote by $\mu_i: G_i \rightarrow S^1$ the character of the representation $(TM|_{M_i}/TM_i)_x$ of G_i . Since μ_i is a faithful representation, there exists the inverse $\lambda_i: S^1 \rightarrow G_i$ of μ_i . Since each G_i is a subtorus of G , we obtain an element $\lambda_i \in \text{Hom}(S^1, G)$ for each i . We identify $\text{Hom}(S^1, G)$ with $\ker \exp_G$ as follows. First, we have a smooth isomorphism $\psi: \mathbb{R}/\mathbb{Z} \rightarrow S^1$ given by $[t] \mapsto e^{2\pi\sqrt{-1}t}$ for $[t] \in \mathbb{R}/\mathbb{Z}$. For $\gamma \in \ker \exp_G$, we have a smooth homomorphism $\lambda_\gamma: \mathbb{R}/\mathbb{Z} \rightarrow G$ given by $[t] \mapsto \exp_G(t\gamma)$. The composition $\lambda_\gamma \circ \psi^{-1}$ is an element in $\text{Hom}(S^1, G)$ and we have an isomorphism $\ker \exp_G \rightarrow \text{Hom}(S^1, G)$ given by $\gamma \mapsto \lambda_\gamma \circ \psi^{-1}$. Through this isomorphism, we identify $\text{Hom}(S^1, G)$ with $\ker \exp_G$. As a result, we have elements $\lambda_1, \dots, \lambda_k \in \mathfrak{g}$.

The fan Δ is given by

$$\Delta := \left\{ \text{pos}(\lambda_i \mid i \in I) \mid \bigcap_{i \in I} M_i \neq \emptyset, I \subset \{1, \dots, k\} \right\},$$

where $\text{pos}(\lambda_i \mid i \in I)$ denotes the cone spanned by $\lambda_i, i \in I$. \mathfrak{h} is given as (2.1). Then one can see that $(\Delta, \mathfrak{h}, G) = \mathcal{F}_1(M, G, y) \in \mathcal{C}_2$. For $(M_1, G_1, y_1), (M_2, G_2, y_2) \in \mathcal{C}_1$ and a morphism $(f, \alpha) \in \text{Hom}_{\mathcal{C}_1}((M_1, G_1, y_1), (M_2, G_2, y_2))$, one can see that $\alpha =: \mathcal{F}_1(f, \alpha) \in \text{Hom}_{\mathcal{C}_2}(\mathcal{F}_1(M_1, G_1, y_1), \mathcal{F}_2(M_2, G_2, y_2))$. As a result, we have a covariant functor $\mathcal{F}_1: \mathcal{C}_1 \rightarrow \mathcal{C}_2$.

Conversely, we construct an object $(M, G, y) =: \mathcal{F}_2(\Delta, \mathfrak{h}, G)$ in \mathcal{C}_1 from $(\Delta, \mathfrak{h}, G) \in \mathcal{C}_2$ as follows. Let $X(\Delta)$ denote the toric variety associated with Δ and $G^{\mathbb{C}}$ the complexification of the compact torus G . Let $\exp_{G^{\mathbb{C}}}: \mathfrak{g}^{\mathbb{C}} \rightarrow G^{\mathbb{C}}$ denote the exponential map. The quotient M of $X(\Delta)$ by $H := \exp_{G^{\mathbb{C}}}(\mathfrak{h}) \subset G^{\mathbb{C}}$ is a compact complex manifold and the action of G on $X(\Delta)$ descends to the maximal action on $X(\Delta)/H$. Let $\pi: X(\Delta) \rightarrow M$ be the quotient map. Then we put $y := \pi(1_{G^{\mathbb{C}}})$, where $1_{G^{\mathbb{C}}}$ is the unit of $G^{\mathbb{C}} \subset X(\Delta)$. Let $(\Delta_1, \mathfrak{h}_1, G_1), (\Delta_2, \mathfrak{h}_2, G_2) \in \mathcal{C}_2$. For a morphism $\alpha \in \text{Hom}((\Delta_1, \mathfrak{h}_1, G_1), (\Delta_2, \mathfrak{h}_2, G_2))$, since $d\alpha$ is a morphism of fans Δ_1 and Δ_2 , we have an α -equivariant holomorphic map (toric morphism) $\varphi_\alpha: X(\Delta_1) \rightarrow X(\Delta_2)$ such that $\varphi_\alpha(1_{G_1^{\mathbb{C}}}) = 1_{G_2^{\mathbb{C}}}$. φ_α descends to an α -equivariant holomorphic map $f_\alpha: X(\Delta_1)/H_1 \rightarrow X(\Delta_2)/H_2$. Namely, $(f_\alpha, \alpha) =: \mathcal{F}_2(\alpha) \in \text{Hom}(\mathcal{F}_2(\Delta_1, \mathfrak{h}_1, G_1), \mathcal{F}_2(\Delta_2, \mathfrak{h}_2, G_2))$. As a result, we have a covariant functor $\mathcal{F}_2: \mathcal{C}_2 \rightarrow \mathcal{C}_1$.

It has been shown that the functors \mathcal{F}_1 and \mathcal{F}_2 are (weak) inverses to each other. In particular, \mathcal{C}_1 and \mathcal{C}_2 are equivalent as categories.

3. TRANSVERSALLY EQUIVALENCE

Let (M_1, F_1) and (M_2, F_2) be smooth manifolds with foliations F_1 on M_1 and F_2 on M_2 . We say that (M_1, F_1) and (M_2, F_2) are *transversally equivalent* if there exist a foliated manifold (M_0, F_0) and a surjective submersion $f_i: M_0 \rightarrow M_i$ for $i = 1, 2$ such that

- (1) $f_i^{-1}(x_i)$ is connected for all $x_i \in M_i$ and
- (2) the preimage of every leaf of F_i by f_i is a leaf of F_0

(see [17, Definition 2.1] for detail). The notion of transversally equivalence is an equivalence relation on foliated manifolds. We restrict our attention to the category \mathcal{C}_1 and canonical foliations.

Definition 3.1. Let $(M_1, G_1, y_1), (M_2, G_2, y_2) \in \mathcal{C}_1$. We say that (M_1, G_1, y_1) and (M_2, G_2, y_2) are *equivalent* if there exist $(M_0, G_0, y_0) \in \mathcal{C}_1$ and morphisms $(f_i, \alpha_i) \in \text{Hom}_{\mathcal{C}_1}((M_0, G_0, y_0), (M_i, G_i, y_i))$ for $i = 1, 2$ satisfying the followings.

- (1) $\ker \alpha_i$ is connected.
- (2) $f_i: M_0 \rightarrow M_i$ is a principal $\ker \alpha_i$ -bundle.

We remark that (M_i, G_i, y_i) and (M_0, G_0, y_0) are also equivalent in Definition 3.1. Definition 3.1 determines an equivalence relation on \mathcal{C}_1 .

Proposition 3.2. *Let $(M, G, y), (M_0, G_0, y_0) \in \mathcal{C}_1$. Let F and F_0 be the canonical foliation on M and M_0 , respectively. Let a morphism*

$$(f, \alpha) \in \text{Hom}_{\mathcal{C}_1}((M_0, G_0, y_0), (M, G, y))$$

satisfy the followings.

- (1) $\ker \alpha$ is connected.
- (2) $f: M_0 \rightarrow M$ is a principal $\ker \alpha$ -bundle.

Then,

- (1) $f^{-1}(x)$ is connected for all $x \in M$ and
- (2) the preimage of every leaf of F by f is a leaf of F_0 .

Before the proof of Proposition 3.2, we recall the following:

Theorem 3.3 ([12, Theorem 11.1]). *Let $(M, G, y), (M_0, G_0, y_0) \in \mathcal{C}_1$. Let $(\Delta, \mathfrak{h}, G) := \mathcal{F}_1(M, G, y)$ and $(\Delta_0, \mathfrak{h}_0, G_0) := \mathcal{F}_1(M_0, G_0, y_0)$. Let*

$$(f, \alpha) \in \text{Hom}_{\mathcal{C}_1}((M, G, y), (M_0, G_0, y_0)).$$

Let $\Delta^{(1)}$ and $\Delta_0^{(1)}$ denote the set of 1-cones in Δ and Δ_0 , respectively. Then, $f: M \rightarrow M_0$ is a principal $\ker \alpha$ -bundle if and only if α is surjective and $d\alpha: \mathfrak{g} \rightarrow \mathfrak{g}_0$ induces a one-to-one correspondence from the primitive generators of 1-cones in Δ to the primitive generators of 1-cones in Δ' .

proof of Proposition 3.2. Since $\ker \alpha$ is connected and $f: M_0 \rightarrow M$ is a principal $\ker \alpha$ -bundle, we have that $f^{-1}(x)$ is connected for all $x \in M$.

Let $p_0: \mathfrak{g}_0^{\mathbb{C}} \rightarrow \mathfrak{g}_0$ and $p: \mathfrak{g}^{\mathbb{C}} \rightarrow \mathfrak{g}$ be the projections. Let $q_0: \mathfrak{g}_0 \rightarrow \mathfrak{g}_0/p_0(\mathfrak{h}_0)$ and $q: \mathfrak{g} \rightarrow \mathfrak{g}/p(\mathfrak{h})$ be the quotient maps. Since $d\alpha^{\mathbb{C}}(\mathfrak{h}_0) \subset \mathfrak{h}$ and $\overline{p \circ d\alpha^{\mathbb{C}}} = d\alpha \circ p_0$, we have $d\alpha(p_0(\mathfrak{h}_0)) \subset p(\mathfrak{h})$. Thus $d\alpha: \mathfrak{g}_0 \rightarrow \mathfrak{g}$ induces a linear map $\overline{d\alpha}: \mathfrak{g}_0/p_0(\mathfrak{h}_0) \rightarrow \mathfrak{g}/p(\mathfrak{h})$. We show that $\overline{d\alpha}$ is an isomorphism. Since α is surjective, so is $d\alpha$. Since $d\alpha$ is surjective, so is $\overline{d\alpha}$. Let $\bar{v} \in \mathfrak{g}_0/p_0(\mathfrak{h}_0)$ be a nonzero element. Since $q_0(\Delta_0)$ is complete, we have that there exists $\sigma \in \Delta_0$ such that $\bar{v} \in q_0(\sigma)$. Let $v \in \sigma$ such that $q_0(v) = \bar{v}$. Let $\lambda_1, \dots, \lambda_k$ be the primitive generators of σ . Since σ is nonsingular, $\lambda_1, \dots, \lambda_k$ are linearly independent. Therefore there uniquely exist $c_1, \dots, c_k \in \mathbb{R}$ such that $v = \sum_j c_j \lambda_j$ and one of c_1, \dots, c_k is nonzero. By Theorem 3.3, each $d\alpha(\lambda_j)$ is a primitive generator of 1-cone in Δ . Since $d\alpha$ induces a morphism of fans Δ_0 and Δ , $d\alpha(\sigma)$ should be a cone in Δ and $d\alpha(\lambda_1), \dots, d\alpha(\lambda_k)$ are primitive generators of $d\alpha(\sigma)$. Therefore $d\alpha(v) = \sum_j c_j d\alpha(\lambda_j)$ is nonzero, because $d\alpha(\lambda_1), \dots, d\alpha(\lambda_k)$ are linearly independent and one of c_1, \dots, c_k is nonzero. Thus $d\alpha(v) \in d\alpha(\sigma) \setminus \{0\}$. Since q gives a one-to-one correspondence between Δ and $q(\Delta)$, it follows from $d\alpha(v) \in d\alpha(\sigma) \setminus \{0\}$ that $q \circ d\alpha(v) \neq 0$. It turns out that $\overline{d\alpha}(\bar{v}) \neq 0$. Therefore $\overline{d\alpha}$ is injective.

We show that $d\alpha^{-1}(p(\mathfrak{h})) = p_0(\mathfrak{h}_0)$. Since $d\alpha(p_0(\mathfrak{h}_0)) \subset p(\mathfrak{h})$, we have $d\alpha^{-1}(p(\mathfrak{h})) \supset p_0(\mathfrak{h}_0)$. To show the opposite inclusion, let $v \in d\alpha^{-1}(p(\mathfrak{h}))$. Then $d\alpha(v) \in p(\mathfrak{h})$. Thus

$q \circ d\alpha(v) = 0$. On the other hand, we have $q \circ d\alpha(v) = \overline{d\alpha} \circ q_0(v)$. Therefore $\overline{d\alpha} \circ q_0(v) = 0$. Since $\overline{d\alpha}$ is injective, we have $q_0(v) = 0$. This yields that $v \in p_0(\mathfrak{h}_0)$. Therefore $d\alpha^{-1}(p(\mathfrak{h})) = p_0(\mathfrak{h}_0)$.

Let L be a leaf of F . Since each leaf of F is an orbit with respect to the action of G restricted to $\exp_G(p(\mathfrak{h})) =: H'$, we have $L = H' \cdot x$ for some $x \in L$. Since $f: M_0 \rightarrow M$ is a principal $\ker \alpha$ -bundle, there exists $x_0 \in M_0$ such that $f(x_0) = x$. The leaf through x_0 is the orbit of the action of G_0 restricted to $\exp_{G_0}(p_0(\mathfrak{h}_0)) =: H'_0$. We show that $f^{-1}(L) = H'_0 \cdot x_0$. Let $x'_0 \in H'_0 \cdot x_0$. Then there exists $h'_0 \in H'_0$ such that $x'_0 = h'_0 \cdot x_0$. Since f is α -equivariant, we have $f(x'_0) = \alpha(h'_0) \cdot f(x_0) = \alpha(h'_0) \cdot x$. Since $d\alpha^{-1}(p(\mathfrak{h})) = p_0(\mathfrak{h}_0)$, we have $\alpha(h'_0) \in H'$ and hence $f(x'_0) \in L$. Thus we have the inclusion $f^{-1}(L) \supset H'_0 \cdot x_0$. To show the opposite inclusion, let $x''_0 \in f^{-1}(L)$. Then $f(x''_0) \in L$. Thus there exists $h' \in H'$ such that $x = h' \cdot f(x''_0)$. Since $\ker \alpha$ is connected and $d\alpha^{-1}(p(\mathfrak{h})) = p_0(\mathfrak{h}_0)$, there exists $\tilde{h}'_0 \in H'_0$ such that $\alpha(\tilde{h}'_0) = h'$. Then we have $f(\tilde{h}'_0 \cdot x''_0) = h' \cdot f(x''_0) = x = f(x_0)$. Since f is a principal $\ker \alpha$ -bundle, there exists $k \in \ker \alpha$ such that $x_0 = k \cdot (\tilde{h}'_0 \cdot x''_0)$. Since $\ker \alpha$ is connected and $d\alpha^{-1}(p(\mathfrak{h})) = p_0(\mathfrak{h}_0)$, we have $\ker \alpha \subset H'_0$. Therefore $k \cdot \tilde{h}'_0 \in H'_0$. Since $x_0 = k \cdot (\tilde{h}'_0 \cdot x''_0) = (k \cdot \tilde{h}'_0) \cdot x''_0$, we have $x''_0 = (k \cdot \tilde{h}'_0)^{-1} \cdot x_0 \in H'_0 \cdot x_0$. Therefore $f^{-1}(L) = H'_0 \cdot x_0$. It turns out that the preimage of every leaf of F by f is a leaf of F_0 . The theorem is proved. \square

Theorem 3.4. *Let $(M_1, G_1, y_1), (M_2, G_2, y_2) \in \mathcal{C}_1$. Let F_1 and F_2 be the canonical foliation on M_1 and M_2 , respectively. If (M_1, G_1, y_1) and (M_2, G_2, y_2) are equivalent, then (M_1, F_1) and (M_2, F_2) are transversally equivalent.*

Proof. It follows from Proposition 3.2 immediately. \square

4. THE FUNDAMENTAL THEOREMS

Definition 4.1. A *marked fan* is a quadruple $(\tilde{V}, \tilde{\Gamma}, \tilde{\Delta}, \tilde{\lambda})$ satisfying the followings.

- (1) \tilde{V} is a finite dimensional \mathbb{R} -vector space.
- (2) $\tilde{\Gamma}$ is a finitely generated subgroup of \tilde{V} that spans \tilde{V} linearly.
- (3) $\tilde{\Delta}$ is a fan and each cone in $\tilde{\Delta}$ is generated by finite elements in $\tilde{\Gamma}$.
- (4) Let $\tilde{\Delta}^{(1)}$ denote the set of all 1-cones in $\tilde{\Delta}$. $\tilde{\lambda}$ is a function $\tilde{\lambda}: \tilde{\Delta}^{(1)} \rightarrow \tilde{\Gamma}$ such that $\tilde{\lambda}(\rho)$ is a generator of ρ for 1-cone $\rho \in \tilde{\Delta}$.

Moreover if the fan $\tilde{\Delta}$ is simplicial, we say that the marked fan $(\tilde{V}, \tilde{\Gamma}, \tilde{\Delta}, \tilde{\lambda})$ is simplicial. If $\tilde{\Delta}$ is complete, we say that the marked fan $(\tilde{V}, \tilde{\Gamma}, \tilde{\Delta}, \tilde{\lambda})$ is complete. We denote by $\tilde{\mathcal{C}}_2$ the class that consists of complete simplicial marked fans.

Definition 4.2. We say that marked fans $(\tilde{V}_1, \tilde{\Gamma}_1, \tilde{\Delta}_1, \tilde{\lambda}_1)$ and $(\tilde{V}_2, \tilde{\Gamma}_2, \tilde{\Delta}_2, \tilde{\lambda}_2)$ are *isomorphic* if there exists a linear isomorphism $\varphi: \tilde{V}_1 \rightarrow \tilde{V}_2$ that satisfies the followings.

- (1) $\varphi(\tilde{\Gamma}_1) = \tilde{\Gamma}_2$.

- (2) φ induces an isomorphism of fans $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$. Namely, $\varphi(\tilde{\sigma}_1) \in \tilde{\Delta}_2$ for all $\tilde{\sigma}_1 \in \tilde{\Delta}_1$ and the map $\tilde{\Delta}_1 \rightarrow \tilde{\Delta}_2$ given by $\tilde{\sigma}_1 \mapsto \varphi(\tilde{\sigma}_1)$ is bijective. We denote the bijection $\tilde{\Delta}_1 \rightarrow \tilde{\Delta}_2$ by the same symbol φ .
- (3) $\tilde{\lambda}_2 \circ \varphi|_{\tilde{\Delta}_1^{(1)}} = \varphi \circ \tilde{\lambda}_1$.

To each $(M, G, y) \in \mathcal{C}_1$, we can assign a complete simplicial marked fan $(\tilde{V}, \tilde{\Gamma}, \tilde{\Delta}, \tilde{\lambda}) \in \tilde{\mathcal{C}}_2$ as follows: Let $(\Delta, \mathfrak{h}, G) := \mathcal{F}_1(M, G, y)$. As before, we denote by $p: \mathfrak{g}^{\mathbb{C}} \rightarrow \mathfrak{g}$ the projection and $q: \mathfrak{g} \rightarrow \mathfrak{g}/p(\mathfrak{h})$ the quotient map. For each 1-cone $\rho \in \Delta^{(1)}$, we denote by $\lambda(\rho) \in \ker \exp_G$ the primitive generator of ρ . Under this notation,

- (1) $\tilde{V} := \mathfrak{g}/p(\mathfrak{h})$.
- (2) $\tilde{\Gamma} := q(\ker \exp_G)$.
- (3) $\tilde{\Delta} := q(\Delta)$.
- (4) $\tilde{\lambda}(q(\rho)) := q(\lambda(\rho))$ for $\rho \in \Delta^{(1)}$.

We denote by $\tilde{\mathcal{F}}_1: \mathcal{C}_1 \rightarrow \tilde{\mathcal{C}}_2$ the assignment above. Theorems 4.3 and 4.4 below tell us that $\tilde{\mathcal{F}}_1$ gives an interpretation between \mathcal{C}_1 and $\tilde{\mathcal{C}}_2$.

Theorem 4.3. *Let $(M_1, G_1, y_1), (M_2, G_2, y_2) \in \mathcal{C}_1$. Then, (M_1, G_1, y_1) and (M_2, G_2, y_2) are equivalent if and only if $\tilde{\mathcal{F}}_1(M_1, G_1, y_1)$ and $\tilde{\mathcal{F}}_1(M_2, G_2, y_2)$ are isomorphic.*

Theorem 4.4. *$\tilde{\mathcal{F}}_1$ is essentially surjective. Namely, for any $(\tilde{V}, \tilde{\Gamma}, \tilde{\Delta}, \tilde{\lambda}) \in \tilde{\mathcal{C}}_2$, there exists $(M, G, y) \in \mathcal{C}_1$ such that $(\tilde{V}, \tilde{\Gamma}, \tilde{\Delta}, \tilde{\lambda})$ and $\tilde{\mathcal{F}}_1(M, G, y)$ are isomorphic.*

Proof of Theorem 4.3. Suppose that $(M_1, G_1, y_1), (M_2, G_2, y_2) \in \mathcal{C}_1$ are equivalent. Then there exist $(M_0, G_0, y_0) \in \mathcal{C}_1$ and $(f_i, \alpha_i) \in \text{Hom}_{\mathcal{C}_1}((M_0, G_0, y_0), (M_i, G_i, y_i))$ for $i = 1, 2$ such that

- (1) $\ker \alpha_i$ is connected and
- (2) $f_i: M_0 \rightarrow M_i$ is a principal $\ker \alpha_i$ -bundle.

We show that $\tilde{\mathcal{F}}_1(M_0, G_0, y_0)$ and $\tilde{\mathcal{F}}_1(M_i, G_i, y_i)$ are isomorphic.

For $j = 0, 1, 2$, let $(\Delta_j, \mathfrak{h}_j, G_j) := \mathcal{F}_1(M_j, G_j, y_j)$. Let $p_j: \mathfrak{g}_j^{\mathbb{C}} \rightarrow \mathfrak{g}_j$ be the projection, $q_j: \mathfrak{g}_j \rightarrow \mathfrak{g}_j/p_j(\mathfrak{h}_j)$ the quotient map and $\exp_{G_j}: \mathfrak{g}_j \rightarrow G_j$ the exponential map. We have that $\overline{d\alpha_i}: \mathfrak{g}_0/p_0(\mathfrak{h}_0) \rightarrow \mathfrak{g}_i/p_i(\mathfrak{h}_i)$ is an isomorphism (see the proof of Proposition 3.2). By Theorem 3.3, we have that α_i is surjective. Since $\ker \alpha_i$ is connected and α_i is surjective, we have $d\alpha_i(\ker \exp_{G_0}) = \ker \exp_{G_i}$. Since $q_i \circ d\alpha_i = \overline{d\alpha_i} \circ q_0$, we have $\overline{d\alpha_i}(q_0(\ker \exp_{G_0})) = q_i(\ker \exp_{G_i})$. Let $\sigma_0 \in \Delta_0$. Since $d\alpha_i$ induces a morphism of fans Δ_0 and Δ_i and one-to-one correspondence between primitive generators by Theorem 3.3, we have $d\alpha_i(\sigma_0) \in \Delta_i$ and $d\alpha_i$ induces a bijection $\Delta_0 \rightarrow \Delta_i$ via $\sigma_0 \mapsto d\alpha_i(\sigma_0)$. Since $q_j: \mathfrak{g}_j \rightarrow \mathfrak{g}_j/p_j(\mathfrak{h}_j)$ induces a bijection $\Delta_j \rightarrow q_j(\Delta_j)$ via $\sigma_j \mapsto q_j(\sigma_j)$, we have $\overline{d\alpha_i}: \mathfrak{g}_0/p_0(\mathfrak{h}_0) \rightarrow \mathfrak{g}_i/p_i(\mathfrak{h}_i)$ induces a bijection $q_0(\Delta_0) \rightarrow q_i(\Delta_i)$. For $\rho_j \in \Delta_j^{(1)}$, we denote by $\lambda_j(\rho_j)$ the primitive generator of ρ_j . Since $d\alpha_i$ induces a one-to-one

correspondence between primitive generators, we have

$$(4.1) \quad d\alpha_i(\lambda_0(\rho_0)) = \lambda_i(d\alpha_i(\rho_0))$$

for $\rho_0 \in \Delta_0^{(1)}$. Applying q_i to the left hand side of (4.1), we have $q_i(d\alpha_i(\lambda_0(\rho_0))) = \overline{d\alpha_i}(q_0(\lambda_0(\rho_0)))$. Thus we have

$$(4.2) \quad \overline{d\alpha_i}(q_0(\lambda_0(\rho_0))) = q_i(\lambda_i(d\alpha_i(\rho_0))).$$

Let $\tilde{\lambda}_j: q_j(\Delta_j)^{(1)} \rightarrow q_j(\ker \exp_{G_j})$ be the map given by $\tilde{\lambda}_j(q_j(\rho_j)) = q_j(\lambda_j(\rho_j))$ for $\rho_j \in \Delta_j^{(1)}$. By (4.2), we have

$$\overline{d\alpha_i} \circ \tilde{\lambda}_0(q_0(\rho_0)) = \tilde{\lambda}_i(q_i(d\alpha_i(\rho_0))) = \tilde{\lambda}_i(\overline{d\alpha_i}(q_0(\rho_0))),$$

that is, $\overline{d\alpha_i} \circ \tilde{\lambda}_0 = \tilde{\lambda}_i \circ \overline{d\alpha_i}$. Therefore $\tilde{\mathcal{F}}_1(M_0, G_0, y_0)$ and $\tilde{\mathcal{F}}_1(M_i, G_i, y_i)$ are isomorphic.

Conversely, suppose that $\tilde{\mathcal{F}}_1(M_1, G_1, y_1)$ and $\tilde{\mathcal{F}}_1(M_2, G_2, y_2)$ are isomorphic. Then there exists a linear isomorphism $\varphi: \mathfrak{g}_1/p_1(\mathfrak{h}_1) \rightarrow \mathfrak{g}_2/p_2(\mathfrak{h}_2)$ satisfying the followings.

- (1) $\varphi(q_1(\ker \exp_{G_1})) = q_2(\ker \exp_{G_2})$.
- (2) φ induces an isomorphism of fans $q_1(\Delta_1) \rightarrow q_2(\Delta_2)$.
- (3) $\tilde{\lambda}_2 \circ \varphi|_{q_1(\Delta_1)^{(1)}} = \varphi \circ \tilde{\lambda}_1$.

We construct $(\Delta_0, \mathfrak{h}_0, G_0) \in \mathcal{C}_2$ and apply Theorem 3.3 in order to show that (M_1, G_1, y_1) and (M_2, G_2, y_2) are equivalent. Define

$$\Gamma_0 := \{(\gamma_1, \gamma_2) \in \ker \exp_{G_1} \times \ker \exp_{G_2} \mid \varphi(q_1(\gamma_1)) = q_2(\gamma_2)\}$$

and denote by \mathfrak{g}_0 the linear hull of Γ_0 in $\mathfrak{g}_1 \times \mathfrak{g}_2$. Let $\exp_{G_1 \times G_2}: \mathfrak{g}_1 \times \mathfrak{g}_2 \rightarrow G_1 \times G_2$ be the exponential map. Then $G_0 := \exp_{G_1 \times G_2}(\mathfrak{g}_0)$ is a subtorus of $G_1 \times G_2$ and Γ_0 coincides with the kernel of the exponential map $\exp_{G_0}: \mathfrak{g}_0 \rightarrow G_0$. For each $\sigma_1 \in \Delta_1$, there uniquely exists $\sigma_2 \in \Delta_2$ such that $\varphi(q_1(\sigma_1)) = q_2(\sigma_2)$. We denote such σ_2 by $\Phi(\sigma_1)$ for $\sigma_1 \in \Delta_1$. Then, for each $\rho_1 \in \Delta_1^{(1)}$, the element $\lambda_0(\rho_1) := (\lambda_1(\rho_1), \lambda_2(\Phi(\rho_1)))$ is a primitive element in Γ_0 . Suppose that a cone $\sigma_1 \in \Delta_1$ is the Minkowsky sum $\sigma_1 = \rho_{1,1} + \cdots + \rho_{1,k}$ of 1-cones $\rho_{1,1}, \dots, \rho_{1,k} \in \Delta_1^{(1)}$. Then we denote by $\Psi(\sigma_1)$ the cone in \mathfrak{g}_0 spanned by $\lambda_0(\rho_{1,1}), \dots, \lambda_0(\rho_{1,k})$. Under this notation, we have a nonsingular fan $\Delta_0 := \{\Psi(\sigma_1) \subset \mathfrak{g}_0 \mid \sigma_1 \in \Delta_1\}$. Let $\alpha_i: G_0 \rightarrow G_i$ be the projection $G_1 \times G_2 \rightarrow G_i$ restricted to $G_0 \subset G_1 \times G_2$ for $i = 1, 2$. Then $\alpha_1: G_0 \rightarrow G_1$ and $\alpha_2: G_0 \rightarrow G_2$ induce morphisms $d\alpha_1: \Delta_0 \rightarrow \Delta_1$ and $d\alpha_2: \Delta_0 \rightarrow \Delta_2$ of fan. In addition, these morphisms of fan induce bijections between cones in fans. Moreover, $d\alpha_i: \mathfrak{g}_0 \rightarrow \mathfrak{g}_i$ induces a one to one correspondence from the primitive generators of 1-cones in Δ_0 and the primitive generators of 1-cones in Δ_i .

Define

$$\mathfrak{h}_0 := \{(v_1, v_2) \in \mathfrak{h}_1 \times \mathfrak{h}_2 \mid (p_1(v_1), p_2(v_2)) \in \mathfrak{g}_0\}.$$

Then \mathfrak{h}_0 is a \mathbb{C} -subspace of $\mathfrak{g}_0^{\mathbb{C}} \subset \mathfrak{g}_1^{\mathbb{C}} \times \mathfrak{g}_2^{\mathbb{C}}$. Moreover, the restriction $p_0|_{\mathfrak{h}_0}$ of the projection $p_0: \mathfrak{g}_0^{\mathbb{C}} \rightarrow \mathfrak{g}_0$ is injective because $p_1|_{\mathfrak{h}_1}$ and $p_2|_{\mathfrak{h}_2}$ both are injective. Since

q_i and $d\alpha_i$ both are surjective, we have that $q_i \circ d\alpha_i: \mathfrak{g}_0 \rightarrow \mathfrak{g}_i/p_i(\mathfrak{h}_i)$ is surjective for $i = 1, 2$. We claim that

$$(4.3) \quad \ker q_1 \circ d\alpha_1 = \ker q_2 \circ d\alpha_2 = p_0(\mathfrak{h}_0).$$

Since $q_2 \circ d\alpha_2 = \varphi \circ q_1 \circ d\alpha_1$ and φ is an isomorphism, we have that the first equality of (4.3) holds. For the second equality of (4.3), let $(\gamma_1, \gamma_2) \in \ker q_2 \circ d\alpha_2$. Then we have $q_2(\gamma_2) = 0$ and $\gamma_2 \in p_2(\mathfrak{h}_2)$. Since $q_2 \circ d\alpha_2 = \varphi \circ q_1 \circ d\alpha_1$, we have $f \circ q_1(\gamma_1) = 0$. Since φ is an isomorphism, we have $q_1(\gamma_1) = 0$. Thus we have $\gamma_1 \in p_1(\mathfrak{h}_1)$. Therefore $(\gamma_1, \gamma_2) \in p_0(\mathfrak{h}_0)$. Therefore $\ker q_2 \circ d\alpha_2 \subset p_0(\mathfrak{h}_0)$. Conversely, let $(\gamma_1, \gamma_2) \in p_0(\mathfrak{h}_0)$. Then $q_2 \circ d\alpha_2(\gamma_1, \gamma_2) = q_2(\gamma_2)$. Since $\gamma_2 \in p_2(\mathfrak{h}_2)$, we have $q_2(\gamma_2) = 0$. Therefore $(\gamma_1, \gamma_2) \in \ker q_2 \circ d\alpha_2$, showing the opposite inclusion $\ker q_2 \circ d\alpha_2 \supset p_0(\mathfrak{h}_0)$.

Let $q_0: \mathfrak{g}_0 \rightarrow \mathfrak{g}_0/p_0(\mathfrak{h}_0)$ be the quotient map. Since $d\alpha_1: \mathfrak{g}_0 \rightarrow \mathfrak{g}_1$ satisfies that $\ker q_1 \circ d\alpha_1 = p_0(\mathfrak{h}_0)$, $d\alpha_1$ induces an isomorphism $\overline{d\alpha_1}: \mathfrak{g}_0/p_0(\mathfrak{h}_0) \rightarrow \mathfrak{g}_1/p_1(\mathfrak{h}_1)$ such that $\overline{d\alpha_1} \circ q_0 = q_1 \circ d\alpha_1$. Since the maps $\Delta_0 \rightarrow \Delta_1$ given by $\sigma_0 \mapsto d\alpha_1(\sigma_0)$ and $\Delta_1 \rightarrow q_1(\Delta_1)$ given by $\sigma_1 \mapsto q_1(\sigma_1)$ are bijective, we have that the composition $\Delta_0 \rightarrow q_1(\Delta_1)$ given by $\sigma_0 \mapsto q_1 \circ d\alpha_1(\sigma_0)$ is bijective. Since $q_0 = \overline{d\alpha_1}^{-1} \circ q_1 \circ d\alpha_1$ and $\overline{d\alpha_1}^{-1}$ is an isomorphism, we have that

$$q_0(\Delta_0) = \{q_0(\sigma_0) \mid \sigma_0 \in \Delta_0\}$$

is a complete fan in $\mathfrak{g}_0/p_0(\mathfrak{h}_0)$ and the map $\Delta_0 \rightarrow q_0(\Delta_0)$ given by $\sigma_0 \mapsto q_0(\sigma_0)$ is bijective. Therefore we have $(\Delta_0, \mathfrak{h}_0, G_0) \in \mathcal{C}_2$.

Since $\mathfrak{h}_0 \subset \mathfrak{h}_1 \times \mathfrak{h}_2 \subset \mathfrak{g}_1^{\mathbb{C}} \times \mathfrak{g}_2^{\mathbb{C}}$ and $d\alpha_i^{\mathbb{C}}$ is nothing but the restriction of the projection $\mathfrak{g}_1^{\mathbb{C}} \times \mathfrak{g}_2^{\mathbb{C}} \rightarrow \mathfrak{g}_i^{\mathbb{C}}$ to $\mathfrak{g}_0^{\mathbb{C}}$, we have $d\alpha_i^{\mathbb{C}}(\mathfrak{h}_0) \subset \mathfrak{h}_i$. Thus we have $\alpha_i \in \text{Hom}_{\mathcal{C}_2}((\Delta_0, \mathfrak{h}_0, G_0), (\Delta_i, \mathfrak{h}_i, G_i))$. Since α_i is surjective and $d\alpha_i: \mathfrak{g}_0 \rightarrow \mathfrak{g}_i$ induces a one-to-one correspondence from the primitive generator of 1-cones in Δ_0 to the primitive generators of 1-cones in Δ_i , there exist $(M_0, G_0, y_0) \in \mathcal{C}_1$ and α_i -equivariant holomorphic map $f_i: M_0 \rightarrow M_1$ such that $f_i(y_0) = y_i$ and f_i is a $\ker \alpha_i$ -principal bundle by Theorem 3.3 and by the equivalence of categories \mathcal{C}_1 and \mathcal{C}_2 .

It remains to show that $\ker \alpha_i$ is connected for $i = 1, 2$. Since $\alpha_i: G_0 \rightarrow G_i$ is surjective, we have that $\ker \alpha_i$ is connected if and only if $d\alpha_i(\ker \exp_{G_0}) = \ker \exp_{G_i}$. Remark that $\ker \exp_{G_0} = \Gamma_0$. Let $\gamma_1 \in \ker \exp_{G_1}$. Since $\varphi(q_1(\ker \exp_{G_1})) = q_2(\ker \exp_{G_2})$, there exists $\gamma_2 \in \ker \exp_{G_2}$ such that $q_2(\gamma_2) = \varphi \circ q_1(\gamma_1)$. Then we have $(\gamma_1, \gamma_2) \in \Gamma_0$, showing that $d\alpha_i(\ker \exp_{G_0}) = \ker \exp_{G_i}$. The same argument works for that $d\alpha_i(\ker \exp_{G_0}) = \ker \exp_{G_2}$, and we have that $\ker \alpha_i$ is connected. The theorem is proved. \square

Proof of Theorem 4.4. Let $(\tilde{V}, \tilde{\Gamma}, \tilde{\Delta}, \tilde{\lambda}) \in \tilde{\mathcal{C}}_2$. Let $\tilde{\rho}_1, \dots, \tilde{\rho}_m$ be all 1-dimensional cones in $\tilde{\Delta}$. Let K be the abstract simplicial complex given by

$$K := \{\{i_1, \dots, i_k\} \subset \{1, \dots, m\} \mid \rho_{i_1} + \dots + \rho_{i_k} \in \tilde{\Delta}\}.$$

For $j = 1, \dots, m$, put $\tilde{\gamma}_j := \tilde{\lambda}(\rho_j)$. Suppose that $\dim \tilde{V} = N_1$. Since $\tilde{\Gamma}$ is finitely generated, there exist a positive integer N_2 and $\tilde{\gamma}_{m+1}, \dots, \tilde{\gamma}_{N_2} \in \tilde{\Gamma}$ such that $\tilde{\gamma}_1, \dots, \tilde{\gamma}_m, \tilde{\gamma}_{m+1}, \dots, \tilde{\gamma}_{N_2}$

generates $\tilde{\Gamma}$, $m \leq N_2$ and $N_2 - N_1$ is nonnegative even integer. For $j = 1, \dots, N_2$, let e_j denote the j -th standard basis vector of \mathbb{R}^{N_2} . The collection of cones

$$\Delta := \{\mathbb{R}_{\geq 0}e_{i_1} + \dots + \mathbb{R}_{\geq 0}e_{i_k} \mid \{i_1, \dots, i_k\} \in K\}$$

is a nonsingular fan with respect to the lattice \mathbb{Z}^{N_2} .

Let $\psi: \mathbb{R}^{N_2} \rightarrow \tilde{V}$ be the linear map given by $\psi(e_j) = \tilde{\gamma}_j$ for $j = 1, \dots, N_2$. Then the map $\Delta \rightarrow \tilde{\Delta}$ given by $\sigma \mapsto \psi(\sigma)$ is bijective. Since $\tilde{\gamma}_1, \dots, \tilde{\gamma}_m$ generates $\tilde{\Gamma}$, we have that ψ is surjective. Put $N_3 := (N_2 - N_1)/2 = \dim \ker \psi/2$. Let $b_1, \dots, b_{N_3}, b'_1, \dots, b'_{N_3}$ be basis vectors of $\ker \psi$. Let \mathfrak{h} be the \mathbb{C} -subspace of $(\mathbb{R}^{N_2})^{\mathbb{C}} = \mathbb{C}^{N_2}$ spanned by $b_1 + \sqrt{-1}b'_1, \dots, b_{N_3} + \sqrt{-1}b'_{N_3}$. Let $p: (\mathbb{R}^{N_2})^{\mathbb{C}} = \mathbb{C}^{N_2} \rightarrow \mathbb{R}^{N_2}$ be the projection. We claim that the restriction $p|_{\mathfrak{h}}$ is injective. Suppose that

$$(4.4) \quad p \left(\sum_{j=1}^{N_3} \alpha_j (b_j + \sqrt{-1}b'_j) \right) = 0, \quad \alpha_j \in \mathbb{C}.$$

Let $\alpha_j = a_j + \sqrt{-1}a'_j$, $a_j, a'_j \in \mathbb{R}$. Then (4.4) yields that $\sum_{j=1}^{N_3} a_j b_j - a'_j b'_j = 0$. Since $b_1, \dots, b_{N_3}, b'_1, \dots, b'_{N_3}$ are basis vectors of $\ker \psi$, we have $\alpha_j = 0$ for all j . Therefore $p|_{\mathfrak{h}}$ is injective.

Since $p(b_j + \sqrt{-1}b'_j) = b_j$ and $p(-\sqrt{-1}(b_j + \sqrt{-1}b'_j)) = b'_j$, we have $p(\mathfrak{h}) \subset \ker \psi$. On the other hand, $p(v + \sqrt{-1}v')$ is a linear combination of $b_1, \dots, b_{N_3}, b'_1, \dots, b'_{N_3}$ for $v + \sqrt{-1}v' \in \mathfrak{h}$. Therefore we have $p(\mathfrak{h}) = \ker \psi$. Let $q: \mathbb{R}^{N_2} \rightarrow \mathbb{R}^{N_2}/p(\mathfrak{h})$. Then ψ induces the linear isomorphism $\bar{\psi}: \mathbb{R}^{N_2}/p(\mathfrak{h}) \rightarrow \tilde{V}$ such that $\bar{\psi} \circ q = \psi$. Therefore $q(\Delta) = \bar{\psi}^{-1}(\tilde{\Delta})$ is a complete fan and the map $\Delta \rightarrow q(\Delta)$ given by $\sigma \mapsto q(\sigma)$ is bijective. Thus we have $(\Delta, \mathfrak{h}, \mathbb{R}^{N_2}/\mathbb{Z}^{N_2}) \in \mathcal{C}_2$. $\mathcal{F}_2(\Delta, \mathfrak{h}, \mathbb{R}^{N_2}/\mathbb{Z}^{N_2}) \in \mathcal{C}_1$ is what we wanted, proving the theorem. \square

5. PRELIMINARIES 2

5.1. Structures around minimal orbits. Let M be a connected manifold equipped with an effective action of G . Since the action of G on M is effective, we have $\dim G_x + \dim G \leq \dim M$ for any x . This inequality is equivalent to the inequality

$$(5.1) \quad \dim G \cdot x \geq 2 \dim G - \dim M.$$

We say that a G -orbit $G \cdot x$ is *minimal* if the equality of (5.1) holds. Then, the effective action of G is maximal if and only if there exists a minimal orbit $G \cdot x$.

Lemma 5.1 ([12, Lemma 2.3]). *Let M be a connected manifold equipped with a maximal action of a compact torus G . Let $G \cdot x$ be a minimal orbit. Then the followings hold.*

- (1) *The isotropy subgroup G_x of G at x is connected.*
- (2) *$G \cdot x$ is a connected component of the fixed point set of the action of G restricted to G_x on M .*

- (3) *Each minimal orbit is isolated. In particular, there are finitely many minimal orbits if M is compact.*

Until the end of this subsection, let $(M, G, y) \in \mathcal{C}_1$ and let $G \cdot x$ be a minimal orbit. By Lemma 5.1 (2), we have that $G \cdot x$ is a connected component of the fixed point set of the action of G restricted to G_x on M . This together with the assumption that the G -action on M preserves the complex structure implies that $G \cdot x$ is a holomorphic submanifold of M . Let $G^{\mathbb{C}}$ be the complexification of G . Since M is compact and the action of G preserves the complex structure J on M , we have the holomorphic $G^{\mathbb{C}}$ -action on M . The global stabilizers form a holomorphic closed subgroup

$$H := \{g \in G^{\mathbb{C}} \mid g \cdot x' = x' \text{ for all } x' \in M\}$$

of $G^{\mathbb{C}}$. The Lie algebra of H is nothing but \mathfrak{h} given by (2.1). By definition, we have an effective holomorphic action of the quotient group $G^{\mathbb{C}}/H$ on M . We denote by G^M the quotient group $G^{\mathbb{C}}/H$.

Lemma 5.2 ([12, Lemma 4.6 and Equation (4.4)]). *The followings hold.*

- (1) *There exists an open dense $G^{\mathbb{C}}$ -orbit in M . In particular, we have $\dim H = 2 \dim G - \dim M = \dim G \cdot x$.*
- (2) *We have the decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus \sqrt{-1}\mathfrak{g}_x \oplus \mathfrak{h}$.*

Since $G \cdot x$ is a holomorphic submanifold of M , we have that the vector field JX_v tangents to $G \cdot x$ for all $v \in \mathfrak{g}$, where X_v denotes the fundamental vector field generated by $v \in \mathfrak{g}$. Thus $G \cdot x$ is invariant under the action of G^M . Since G acts on $G \cdot x$ transitively, G^M also acts on $G \cdot x$ transitively. Therefore we have that $G \cdot x$ is biholomorphic to the homogeneous space $G^M/(G^M)_x$.

Lemma 5.3 ([12, Lemma 4.8]). *Let \mathfrak{g}_x be the Lie algebra of G_x and $(\mathfrak{g}^M)_x$ the Lie algebra of the isotropy subgroup $(G^M)_x$ of G^M . The natural map $(\mathfrak{g}_x)^{\mathbb{C}} \hookrightarrow \mathfrak{g}^{\mathbb{C}}/\mathfrak{h}$ induces an isomorphism $(G_x)^{\mathbb{C}} \rightarrow (G^M)_x$.*

The normal space $T_x M/T_x(G \cdot x)$ is a faithful representation space of $(G_x)^{\mathbb{C}}$. Thus there exist $\alpha_1, \dots, \alpha_k \in \text{Hom}((G_x)^{\mathbb{C}}, \mathbb{C}^*)$ such that

$$T_x M/T_x(G \cdot x) \cong \mathbb{C}_{\alpha_1} \oplus \dots \oplus \mathbb{C}_{\alpha_k}$$

as $(G_x)^{\mathbb{C}}$ -representations, where \mathbb{C}_{α} denotes the complex 1-dimensional representation of the weight $\alpha \in \text{Hom}((G_x)^{\mathbb{C}}, \mathbb{C}^*)$. Since $\dim T_x M/T_x(G \cdot x) = \dim (G_x)^{\mathbb{C}}$ and $T_x M/T_x(G \cdot x)$ is faithful, we have that $\alpha_1, \dots, \alpha_k$ form a basis of $\text{Hom}((G_x)^{\mathbb{C}}, \mathbb{C}^*)$.

Proposition 5.4 ([12, Proposition 4.9]). *There uniquely exists a minimal G^M -invariant open neighborhood $N(G \cdot x) \subset M$ of $G \cdot x$ that is G^M -equivariantly biholomorphic to*

$$G^M \times_{(G_x)^{\mathbb{C}}} \bigoplus_{i=1}^k \mathbb{C}_{\alpha_i}.$$

5.2. Transverse Kähler structures. Let M be a smooth manifold. A *presymplectic form* is a closed 2-form on M . Let F be a smooth foliation on M . A presymplectic form ω is said to be *transverse symplectic with respect to F* if the kernel of ω coincides with the subbundle TF of TM that consists of vectors tangent to leaves of F . Suppose that a torus G acts on M smoothly and a transverse symplectic form ω with respect to F is G -invariant. Then, the Cartan formula tells us that the 1-form $i_{X_v}\omega$ is a closed 1-form, where X_v denotes the fundamental vector field generated by an element v of the Lie algebra \mathfrak{g} of G and $i_{X_v}\omega$ denotes the interior product. A *moment map* $\Phi: M \rightarrow \mathfrak{g}^*$ is a map satisfying that $dh_v = -i_{X_v}\omega$, where $h_v: M \rightarrow \mathbb{R}$ is a function given by $h_v(x) = \langle \Phi(x), v \rangle$ for all $x \in M$ and $v \in \mathfrak{g}$.

Suppose that M is a complex manifold and F is a holomorphic foliation on M . Let J be the complex structure on M . A *transverse Kähler form* ω on M with respect to F is a closed 2-form that satisfies the followings.

- (1) $\omega(JX, JY) = \omega(X, Y)$ for all $X, Y \in T_xM$.
- (2) $\omega(X, JX) \geq 0$ for all $X \in T_xM$. The equality holds if and only if $X \in T_xF$.

By definition, a transverse Kähler form ω with respect to F is a transverse symplectic form with respect to F . The canonical foliation relates the existence of a transverse Kähler form that has a moment map, see [13, Proposition 4.2].

Until the end of this subsection, let $(M, G, y) \in \mathcal{C}_1$ and let F be the canonical foliation on M . Let $(\Delta, \mathfrak{h}, G) := \mathcal{F}_1(M, G, y) \in \mathcal{C}_2$. As before, let $p: \mathfrak{g}^{\mathbb{C}} \rightarrow \mathfrak{g}$ be the projection and $q: \mathfrak{g} \rightarrow \mathfrak{g}/p(\mathfrak{h})$ the quotient map.

Theorem 5.5 ([13, Theorems 2.7, 4.3 and 5.8 and Proposition 4.1]). *The followings hold.*

- (1) *Let ω be a G -invariant transverse Kähler form with respect to F . Then there exists a moment map $\Phi: M \rightarrow \mathfrak{g}^*$. Moreover, there exist $\alpha \in \mathfrak{g}^*$ and a smooth map $\tilde{\Phi}: M \rightarrow (\mathfrak{g}/p(\mathfrak{h}))^*$ such that $\Phi = q^* \circ \tilde{\Phi} + \alpha$.*
- (2) *Let $\tilde{\Phi}$ be as (1). Let Z_1, \dots, Z_l be the connected components of the set of common critical points of h_v for all $v \in \mathfrak{g}$. Then $\tilde{\Phi}(Z_j)$ is a point c_j for $j = 1, \dots, l$ and $\tilde{\Phi}(M)$ is a convex hull of c_1, \dots, c_N . The fan $q(\Delta)$ is an inner normal fan of the polytope $\tilde{\Phi}(M)$.*
- (3) *If $q(\Delta)$ is polytopal, then there exists a G -invariant transverse Kähler form with respect to F .*
- (4) *For a transverse Kähler form ω with respect to F , the average $\int_{g \in G} g^*\omega dg$ is a G -invariant transverse Kähler form with respect to F .*

In particular, there exists a transverse Kähler form with respect to F if and only if $q(\Delta)$ is polytopal.

In Section 8, we will see how we construct a transverse Kähler form with respect to F from a polytopal fan $q(\Delta)$.

Let ω be a G -invariant transverse Kähler form with respect to F and $\tilde{\Phi}$ as above. We shall describe the set $Z := Z_1 \sqcup \cdots \sqcup Z_N$ of common critical points of h_v for all $v \in \mathfrak{g}$. Since $dh_v = -i_{X_v}\omega$ and the kernel of ω coincides with TF , we have that $x \in Z$ if and only if $(X_v)_x \in T_x F$ for all $v \in \mathfrak{g}$. Namely, we have

$$Z = \{x \in M \mid (X_v)_x \in T_x F \text{ for all } v \in \mathfrak{g}\}.$$

Therefore Z does not depend on ω and $\tilde{\Phi}$.

Proposition 5.6. *Z is the union of all minimal orbits. Each connected component of Z is a minimal orbit. Conversely, each minimal orbit is a connected component of Z .*

Proof. Let $x \in Z$ and $v \in \mathfrak{g}$. Since $(X_v)_x \in T_x F$, there uniquely exists $v' \in \mathfrak{p}(\mathfrak{h})$ such that $(X_v)_x = (X_{v'})_x$. Then we have $(X_{v-v'})_x = (X_v)_x - (X_{v'})_x = 0$. Therefore $v - v' \in \mathfrak{g}_x$. Thus we have the decomposition $\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{p}(\mathfrak{h})$. Therefore we have $\dim G \cdot x = \dim \mathfrak{g} - \dim \mathfrak{g}_x = \dim \mathfrak{p}(\mathfrak{h})$. Since $p|_{\mathfrak{h}}$ is injective, we have $\dim G \cdot x = \dim H = 2 \dim G - \dim M$ by Lemma 5.2 (1). Namely, $G \cdot x$ is a minimal orbit. Therefore Z is contained in the union of minimal orbits. Conversely, if $G \cdot x$ is a minimal orbit, then $\dim G - \dim G_x = \dim G \cdot x = 2 \dim G - \dim M$. By Lemma 5.2 (1), we have $2 \dim G - \dim M = \dim H$. Thus we have $\dim G - \dim G_x = \dim H$. Since $p|_{\mathfrak{h}}$ is injective and $\dim G - \dim G_x = \dim H$, we have $\dim \mathfrak{p}(\mathfrak{h}) + \dim \mathfrak{g}_x = \dim \mathfrak{g}$. This together with $\mathfrak{g}_x \cap \mathfrak{p}(\mathfrak{h}) = \{0\}$ yields that $\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{p}(\mathfrak{h})$. Therefore we have $(X_v)_x \in T_x F$ for any v . This shows that $x \in Z$. Therefore Z is the union of all minimal orbits.

By Lemma 5.1 (3), each minimal orbit is isolated. Therefore each connected component of Z is a minimal orbit and vice versa. The proposition is proved. \square

Lemma 5.7 ([13, Lemma 2.2]). *Let $v \in \mathfrak{g}$. Let $x \in M$ be a critical point of h_v . Let $\alpha_1, \dots, \alpha_k \in \text{Hom}(G_x, S^1)$ be the nonzero weights of the G_x -representation $T_x M$. The followings hold.*

- (1) h_v is nondegenerate. Namely, each component of the critical set is a submanifold and the second derivative is a nondegenerate quadric form in the transverse direction.
- (2) There exist $v_x \in \mathfrak{g}_x$ and $v' \in \mathfrak{p}(\mathfrak{h})$ such that $v = v_x + v'$. The index of h_v at x is twice as many as the number of α_i such that $\langle d\alpha_i, v_x \rangle < 0$.

Lemma 5.8. *Let $G \cdot x$ be a minimal orbit. Then there exists $v \in \mathfrak{g}$ such that $G \cdot x$ is a critical submanifold and h_v attains the minimum on $G \cdot x$.*

Proof. Let $G \cdot x$ be a minimal orbit. Let $\alpha_1, \dots, \alpha_k \in \text{Hom}(G_x, S^1)$ be the nonzero weights of $T_x M$. It follows from Lemma 5.1 (2) that the codimension of $G \cdot x$ is equal to $2k$. Thus we have that $\alpha_1, \dots, \alpha_k$ form a \mathbb{Z} -basis of $\text{Hom}(G_x, S^1)$; otherwise, the action of G is not effective. We identify $\text{Hom}(S^1, G_x)$ with $\ker \exp_{G_x} \subset \mathfrak{g}_x$ as before. Let $v_1, \dots, v_k \in \mathfrak{g}_x$ be the dual basis of $\alpha_1, \dots, \alpha_k$. Put $v := v_1 + \cdots + v_k$. Since

$\langle d\alpha_i, v \rangle = 1 > 0$ for all i , we have that $G \cdot x$ is a critical submanifold and the index of h_v on $G \cdot x$ is 0. Therefore h_v attains a local minimum on $G \cdot x$. Put $c := h_v(x)$. By Lemma 5.7, all indices of h_v are even. Thus we have that $h_v^{-1}(c)$ is connected. Therefore h_v attains the global minimum on $G \cdot x$, as required. \square

Proposition 5.9. *Let Z_1, \dots, Z_N be minimal orbits. Each image $\tilde{\Phi}(Z_j)$ of Z_j by $\tilde{\Phi}: M \rightarrow (\mathfrak{g}/p(\mathfrak{h}))^*$ is a vertex of $\tilde{\Phi}(M)$ and $\tilde{\Phi}(Z_i) \neq \tilde{\Phi}(Z_j)$ if $i \neq j$.*

Proof. Let Z_j be a minimal orbit. By Proposition 5.6, we have that Z_j is a connected component of the set of common critical points of h_v . By Theorem 5.5 (2), we have that $\tilde{\Phi}(Z_j)$ is a point c_j . By Lemma 5.8, there exists $v \in \mathfrak{g}$ such that Z_j is a critical submanifold and h_v attains the minimum on Z_j . Let $c \in \mathbb{R}$ be the minimum value of h_v . Since

$$(5.2) \quad h_v(x) = \langle \Phi(x), v \rangle = \langle q^* \circ \tilde{\Phi}(x) + \alpha, v \rangle = \langle \tilde{\Phi}(x), q(v) \rangle + \langle \alpha, v \rangle$$

and h_v attains the minimum c on Z_j , we have that the half space

$$\{y \in (\mathfrak{g}/p(\mathfrak{h}))^* \mid \langle y, q(v) \rangle + \langle \alpha, v \rangle \geq c\} \supset \tilde{\Phi}(M)$$

and

$$\{y \in (\mathfrak{g}/p(\mathfrak{h}))^* \mid \langle y, q(v) \rangle + \langle \alpha, v \rangle = c\} \cap \tilde{\Phi}(M) = \tilde{\Phi}(Z_j).$$

Therefore $\tilde{\Phi}(Z_j)$ is a vertex of $\tilde{\Phi}(M)$.

Let Z_i be another minimal orbit. Suppose that $\tilde{\Phi}(Z_i) = \tilde{\Phi}(Z_j)$. Then, by (5.2), we have $h_v(Z_i) = h_v(Z_j)$. Since $h_v^{-1}(c) = Z_j$, we have $Z_i = Z_j$. It turns out that $\tilde{\Phi}(Z_i) \neq \tilde{\Phi}(Z_j)$ if $i \neq j$. The proposition is proved. \square

Lemma 5.10. *There exists $v \in \mathfrak{g}$ that satisfies the followings.*

- (1) *The set of critical points of h_v coincides with $Z = Z_1 \sqcup \dots \sqcup Z_l$.*
- (2) *$h_v(Z_i) \neq h_v(Z_j)$ if $i \neq j$.*

Proof. By Proposition 5.9, the subset

$$H_{i,j} := \{v \in \mathfrak{g} \mid \langle \tilde{\Phi}(Z_i) - \tilde{\Phi}(Z_j), v \rangle = 0\}$$

is a hyperplane for $1 \leq i < j \leq l$. By Proposition 5.6, we have that $\mathfrak{g}_x \oplus p(\mathfrak{h}) = \mathfrak{g}$ if and only if $x \in Z$. Since M is compact, the set $\{\mathfrak{g}_x \mid x \in M\}$ is finite. Thus the set

$$A := \mathfrak{g} \setminus \left(\bigcup_{1 \leq i < j \leq N} H_{i,j} \cup \bigcup_{x \notin Z} (\mathfrak{g}_x \oplus p(\mathfrak{h})) \right)$$

is not empty. Any element $v \in A$ satisfies the conditions (1) and (2), proving the lemma. \square

6. BASIC COHOMOLOGY AND EQUIVARIANT COHOMOLOGY

Let M be a smooth manifold equipped with an effective action of a compact torus G . Let a subspace $\mathfrak{h}' \subset \mathfrak{g}$ generate a smooth foliation F on M . Namely, $H' := \exp_G(\mathfrak{h}')$ acts on M local freely and each leaf of F is an H' -orbit. Since G is commutative, TF is a G -invariant subbundle of TM .

We denote by $\Omega^*(M)$ the DGA (differential graded algebra) that consists of all differential forms on M . Since G acts on M smoothly, we have that $\Omega^*(M)$ is a G -representation space. We denote by $\Omega^*(M)^G$ the sub-DGA of all G -invariant differential forms. We denote by $H^*(M)$ the cohomology $H^*(\Omega^*(M))$ of $\Omega^*(M)$.

Let $\alpha \in \Omega^*(M)$. We say that α is *basic* if α satisfies that $i_{X_v}\alpha = 0$ and $L_{X_v}\alpha = 0$ for all $v \in \mathfrak{h}'$, where $L_{X_v}\alpha$ denotes the Lie derivative of α by X_v . We denote by $\Omega_B^*(M)$ the set of all basic forms. By definition, $\Omega_B^*(M)$ becomes a sub-DGA of $\Omega^*(M)$. Namely, the differential of a basic form is basic. Since the G -action commutes with the H' -action, $\Omega_B^*(M)$ is a sub-representation of $\Omega^*(M)$. We denote by $\Omega_B^*(M)^G$ the sub-DGA of all G -invariant basic forms on M . We denote by $H_B^*(M)$ the cohomology $H^*(\Omega_B^*(M))$ of $\Omega_B^*(M)$ and call it *the basic cohomology of M for F* .

Lemma 6.1. *Let M, G, \mathfrak{h}', F be as above. Then the followings hold.*

- (1) $H^*(M)$ is isomorphic to $H^*(\Omega^*(M)^G)$.
- (2) $H_B^*(M)$ is isomorphic to $H^*(\Omega_B^*(M)^G)$.

Proof. In case when $\mathfrak{h}' = \{0\}$, any differential form is basic. Therefore it suffices to show Part (2).

First, we show that the inclusion $\Omega_B^*(M)^G \hookrightarrow \Omega^*(M)^G$ induces an injective homomorphism $H^*(\Omega_B^*(M)^G) \rightarrow H^*(\Omega^*(M)^G)$. Let $I: \Omega_B^*(M)^G \rightarrow \Omega^*(M)^G$ be the linear map given by

$$I(\alpha) := \int_{g \in G} g^* \alpha dg, \quad \alpha \in \Omega_B^*(M)^G,$$

where dg denotes the normalized Haar measure on G . Then the composition of the inclusion $\Omega_B^*(M)^G \hookrightarrow \Omega^*(M)^G$ with I is the identity. Thus, the composition induces an cohomology isomorphism $H^*(\Omega_B^*(M)^G) \rightarrow H^*(\Omega^*(M)^G)$. Therefore the inclusion $\Omega_B^*(M)^G \hookrightarrow \Omega^*(M)^G$ induces an injective homomorphism $H^*(\Omega_B^*(M)^G) \rightarrow H^*(M)$.

We show that the induced homomorphism $H^*(\Omega_B^*(M)^G) \rightarrow H_B^*(M)$ is surjective. Let $[\alpha] \in H_B^*(M)$ be a cohomology class represented by a closed basic form $\alpha \in \Omega_B^*(M)$. Let $\gamma_1, \dots, \gamma_n$ be basis vectors of $\ker \exp_G$. Define

$$D := \left\{ v = \sum_{i=1}^n a_i \gamma_i \mid 0 \leq a_i < 1 \right\}.$$

Then the exponential map restricted to D gives a bijection $\exp_G|_D: D \rightarrow G$. For $v \in D$ and $t \in \mathbb{R}$, we define $g_t := \exp_G(tv) \in G$ and

$$\theta_{g_1} := \int_0^1 g_t^*(i_{X_v}\alpha) dt \in \Omega_B^*(M).$$

We claim that $d\theta_{g_1} = g_1^*\alpha - \alpha$. Since

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{g_{t+h}^*\alpha - g_t^*\alpha}{h} &= L_{X_v}g_t^*\alpha \\ &= i_{X_v}dg_t^*\alpha + di_{X_v}g_t^*\alpha \\ &= di_{X_v}g_t^*\alpha \\ &= dg_t^*i_{X_v}\alpha, \end{aligned}$$

we have

$$\begin{aligned} d\theta_{g_1} &= d \int_0^1 g_t^*i_{X_v}\alpha dt \\ &= \int_0^1 dg_t^*i_{X_v}\alpha dt \\ &= g_1^*\alpha - g_0^*\alpha \\ &= g_1^*\alpha - \alpha. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{g_1 \in G} g_1^*\alpha - \alpha dg &= \int_{g_1 \in G} d\theta_{g_1} dg \\ &= d \int_{g_1 \in G} \theta_{g_1} dg. \end{aligned}$$

On the other hand,

$$\int_{g_1 \in G} g_1^*\alpha - \alpha dg = I(\alpha) - \alpha.$$

Therefore $I(\alpha)$ and α determine the same cohomology class in $H_B^*(M)$. This together with $I(\alpha) \in \Omega_B^*(M)^G$ yields that the induced homomorphism $H^*(\Omega_B^*(M)^G) \rightarrow H_B^*(M)$ is surjective, proving the lemma. \square

The *Cartan model* of the equivariant de Rham complex is given by

$$\Omega_G^*(M) := (S^*(\mathfrak{g}^*) \otimes \Omega^*(M))^G$$

where $S^*(\mathfrak{g}^*)$ denotes the symmetric tensor algebra of \mathfrak{g}^* . The degree of elements in \mathfrak{g}^* is 2 and the degree of an element in $\Omega^*(M)$ is as usual. An element in $\Omega_G^*(M)$

can be think of a G -equivariant $\Omega^*(M)$ -valued polynomial function on \mathfrak{g} . Since G is abelian, we have that $S^*(\mathfrak{g}^*)$ is a trivial G -representation. Thus we have

$$\begin{aligned}\Omega_G^*(M) &:= (S^*(\mathfrak{g}^*) \otimes \Omega^*(M))^G \\ &= S^*(\mathfrak{g}^*) \otimes \Omega^*(M)^G.\end{aligned}$$

Namely, an element in $\Omega_G^*(M)$ is a $\Omega^*(M)^G$ -valued polynomial function on \mathfrak{g} . The differential $d_G: \Omega_G^*(M) \rightarrow \Omega_G^*(M)$ is given by

$$(d_G \alpha)(v) := d(\alpha(v)) - i_{X_v} \alpha(v)$$

for $\alpha \in \Omega_G^*(M)$ and $v \in \mathfrak{g}$. The cohomology of the DGA $(\Omega_G^*(M), d_G)$ is called *the equivariant cohomology of M* and denoted by $H_G^*(M)$. $H_G^*(M)$ is not only an \mathbb{R} -algebra but also an $S^*(\mathfrak{g}^*)$ -algebra.

Lemma 6.2 ([14, Lemma 4.4 and Proposition 4.6]). *Suppose that M is paracompact. The followings hold.*

- (1) *There exists a \mathfrak{h}' -valued G -invariant 1-form θ on M such that $i_{X_v} \theta = v$ for all $v \in \mathfrak{h}'$.*
- (2) *Let v_1, \dots, v_k be basis vectors of \mathfrak{h}' . Write*

$$\theta = \sum_{i=1}^k \theta_i \otimes v_i, \quad \theta_i \in \Omega^1(M)^G.$$

Let W be the subspace of $\Omega^1(M)^G$ spanned by $\theta_1, \dots, \theta_k$. Then we have the decomposition

$$\Omega^*(M)^G = \Omega_B^*(M)^G \otimes \bigwedge W.$$

Suppose that M is paracompact and let W be as in Lemma 6.2. Then we have the decomposition

$$(6.1) \quad \Omega_G^*(M) = S^*(\mathfrak{g}^*) \otimes \Omega_B^*(M)^G \otimes \bigwedge W.$$

Since $\mathfrak{h}' \subset \mathfrak{g}'$, we have that the inclusion induces the surjective homomorphism $S^*(\mathfrak{g}^*) \rightarrow S^*(\mathfrak{h}'^*)$. We think of elements in $S^*(\mathfrak{h}'^*) \otimes \Omega_B^*(M)^G \otimes \bigwedge W$ as $\Omega_B^*(M)^G \otimes \bigwedge W$ -valued polynomial functions on \mathfrak{h}' . For short, we denote $S^*(\mathfrak{h}'^*) \otimes \Omega_B^*(M)^G \otimes \bigwedge W$ by $\Omega_{\mathfrak{h}'}^*(M)$. Let

$$d_{\mathfrak{h}'}: \Omega_{\mathfrak{h}'}^*(M) \rightarrow \Omega_{\mathfrak{h}'}^*(M)$$

be the linear map given by

$$d_{\mathfrak{h}'} \alpha(u') = d(\alpha(u')) - i_{X_{u'}} \alpha(u'), \quad \alpha \in \Omega_{\mathfrak{h}'}^*(M), u' \in \mathfrak{h}'.$$

Then $d_{\mathfrak{h}'}^2 = 0$. Let x_1, \dots, x_k denote the dual basis vectors of the basis vectors v_1, \dots, v_k of \mathfrak{h}' . Then $d_{\mathfrak{h}'}$ is represented as

$$d_{\mathfrak{h}'} \beta = d\beta - \sum_{i=1}^k x_i \otimes i_{X_{v_i}} \beta$$

for $\beta \in \Omega_B^*(M)^G$ and $d_{\mathfrak{h}'}$ is a homomorphism of $S^*(\mathfrak{h}')$ -algebra. We define an \mathbb{R} -algebra homomorphism

$$f: \Omega_{\mathfrak{h}'}^*(M) \rightarrow \Omega_B^*(M)^G$$

given by

$$f(x_i \otimes 1 \otimes 1) = d\theta_i, \quad f(1 \otimes \beta \otimes 1) = \beta, \quad f(1 \otimes 1 \otimes w_i) = 0$$

for $i = 1, \dots, k$ and $\beta \in \Omega_B^*(M)^G$.

Lemma 6.3. *The map $f: \Omega_{\mathfrak{h}'}^*(M) \rightarrow \Omega_B^*(M)^G$ above is a DGA homomorphism.*

Proof. We need to show that $d \circ f = f \circ d_{\mathfrak{h}'}$. Let n_1, \dots, n_k be nonnegative integers and $\beta \in \Omega_B^*(M)^G$. Then

$$x_1^{n_1} \dots x_k^{n_k} \otimes \beta \otimes \theta_{j_1} \wedge \dots \wedge \theta_{j_m} \in S^*(\mathfrak{h}') \otimes \Omega_B^*(M)^G \otimes \bigwedge W.$$

By definition of f , we have

$$\begin{aligned} d \circ f(x_1^{n_1} \dots x_k^{n_k} \otimes \beta \otimes \theta_{j_1} \wedge \dots \wedge \theta_{j_m}) &= d(\beta \wedge d\theta_1^{n_1} \wedge \dots \wedge d\theta_k^{n_k}) \\ &= d\beta \wedge d\theta_1^{n_1} \wedge \dots \wedge d\theta_k^{n_k}. \end{aligned}$$

On the other hand,

$$\begin{aligned} & d_{\mathfrak{h}'}(x_1^{n_1} \dots x_k^{n_k} \otimes \beta \otimes \theta_{j_1} \wedge \dots \wedge \theta_{j_m}) \\ &= x_1^{n_1} \dots x_k^{n_k} \otimes d\beta \otimes \theta_{j_1} \wedge \dots \wedge \theta_{j_m} \\ (6.2) \quad & + \sum_{i=1}^m x_1^{n_1} \dots x_k^{n_k} \otimes (-1)^{\deg \beta + i - 1} \beta \wedge d\theta_{j_i} \otimes \theta_{j_1} \wedge \dots \wedge \widehat{\theta}_{j_i} \wedge \dots \wedge \theta_{j_m} \\ & - \sum_{i=1}^m x_{j_i} \cdot x_1^{n_1} \dots x_k^{n_k} \otimes (-1)^{\deg \beta + i - 1} \beta \otimes \theta_{j_1} \wedge \dots \wedge \widehat{\theta}_{j_i} \wedge \dots \wedge \theta_{j_m}. \end{aligned}$$

Therefore we have

$$f \circ d_{\mathfrak{h}'}(x_1^{n_1} \dots x_k^{n_k} \otimes \beta \otimes \theta_{j_1} \wedge \dots \wedge \theta_{j_m}) = d\beta \wedge d\theta_1^{n_1} \wedge \dots \wedge d\theta_k^{n_k}.$$

These computations show $d \circ f = f \circ d_{\mathfrak{h}'}$, as required. \square

f is the *Cartan operator*. Namely, the following holds:

Theorem 6.4 ([10, Chapter 5]). *f induces an isomorphism*

$$H^*(\Omega_{\mathfrak{h}'}^*(M), d_{\mathfrak{h}'}) \rightarrow H^*(\Omega_B^*(M)^G, d_B) \cong H_B^*(M).$$

As a conclusion, we have a DGA homomorphism $\Omega_G^*(M) \rightarrow \Omega_B^*(M)^G$ given by $\alpha \mapsto f(\alpha|_{\mathfrak{h}'})$. We denote by $\text{for}_B: H_G^*(M) \rightarrow H_B^*(M)$ the induced homomorphism and call it *the basic forgetful map*.

Remark 6.5. One can construct the basic forgetful map $\text{for}_B: H_G^*(M) \rightarrow H_B^*(M)$ via the *equivariant basic cohomology* introduced in [9]. To prevent confusing, by $\Omega^*(M, F)$ we mean the basic complex $\Omega_B^*(M)$ and by $H^*(M, F)$ we mean the basic cohomology $H_B^*(M)$ for a moment. Let \mathfrak{g}' be a complement of \mathfrak{h}' in \mathfrak{g} . Then we have a transverse action of \mathfrak{g}' on the foliated manifold (M, F) . The cohomology $H_{\mathfrak{g}'}^*(M, F)$ of the DGA $\Omega_{\mathfrak{g}'}^*(M, F) = (S^*(\mathfrak{g}'^*) \otimes \Omega^*(M, F))^{\mathfrak{g}'}$ is the \mathfrak{g}' -equivariant F -basic cohomology. One can see that $H_G^*(M)$ is isomorphic to $H_{\mathfrak{g}'}^*(M, F)$ (see [9, Example 4.3]). The basic forgetful map is the composition of the isomorphism $H_G^*(M) \cong H_{\mathfrak{g}'}^*(M, F)$ with the natural map $H_{\mathfrak{g}'}^*(M, F) \rightarrow H^*(M, F)$. We note that equivariant basic cohomology has been constructed for not only abelian but also arbitrary transverse actions. See [9] for details.

6.1. Local computations. Let G be a compact torus and G' a subtorus of G . Let \mathfrak{h}' be a complement of \mathfrak{g}' in \mathfrak{g} , that is, \mathfrak{h}' is a linear subspace of \mathfrak{g} such that $\mathfrak{g}' \oplus \mathfrak{h}' = \mathfrak{g}$. Let Y be a smooth manifold equipped with an action of G' . We define the right G' -action on $G \times Y$ by

$$(g, y) \cdot g' := (gg', g'^{-1} \cdot y)$$

for $(g, y) \in G \times Y$ and $g' \in G'$. We define the left G -action on $G \times Y$ by

$$\tilde{g} \cdot (g, y) := (\tilde{g}g, y)$$

for $(g, y) \in G \times Y$ and $\tilde{g} \in G$. Then the left G -action on $G \times Y$ descends to the left G -action on the quotient manifold $G \times_{G'} Y$. Since $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{h}'$, we have a foliation F on $G \times_{G'} Y$ whose leaves are generated by \mathfrak{h}' . We denote by $\Omega_B^*(G \times_{G'} Y)$ the basic complex with respect to F .

We denote

- by X_v^L the fundamental vector field on $G \times Y$ generated by $v \in \mathfrak{g}$ with respect to the left G -action,
- by $X_{v'}^R$ the fundamental vector field on $G \times Y$ generated by $v' \in \mathfrak{g}'$ with respect to the right G' -action,
- by X_v^G the fundamental vector field on G generated by $v \in \mathfrak{g}$ and
- by $X_{v'}^Y$ the fundamental vector field on Y generated by $v' \in \mathfrak{g}'$.

Let $\omega \in \Omega^*(G \times Y)$. We say that ω is *right \mathfrak{g}' -basic* if $i_{X_{v'}^R} \omega = 0$ and $L_{X_{v'}^R} \omega = 0$ for all $v' \in \mathfrak{g}'$. We say that ω is *left \mathfrak{h}' -basic* if $i_{X_{u'}^L} \omega = 0$ and $L_{X_{u'}^L} \omega = 0$ for all $u' \in \mathfrak{h}'$. Let $p_G: G \times Y \rightarrow G$ and $p_Y: G \times Y \rightarrow Y$ be the projections. Since p_G and p_Y both have global sections, the pull-back maps p_G^* and p_Y^* are injective. For a differential form $\alpha \in \Omega^*(G)$ and $\beta \in \Omega^*(Y)$, we have $p_G^*(\alpha) \wedge p_Y^*(\beta) \in \Omega^*(G \times Y)$ and such elements generates $\Omega^*(G \times Y)$. Let $\alpha \in \Omega^*(G)$ and $\beta \in \Omega^*(Y)$. Then we have

$$i_{X_v^L}(p_G^*(\alpha) \wedge p_Y^*(\beta)) = p_G^*(i_{X_v^G} \alpha) \wedge p_Y^*(\beta)$$

for $v \in \mathfrak{g}$ and

$$i_{X_{v'}^R}(p_G^*(\alpha) \wedge p_Y^*(\beta)) = p_G^*(i_{X_{v'}^G} \alpha) \wedge p_Y^*(\beta) + (-1)^{\deg \alpha} p_G^*(\alpha) \wedge p_Y^*(i_{-X_{v'}^Y} \beta)$$

for $v' \in \mathfrak{g}'$.

Lemma 6.6. *Let G, G', \mathfrak{h}', Y be as above. Then there exists an isomorphism $\eta: \Omega_B^*(G \times_{G'} Y)^G \rightarrow \Omega^*(Y)^{G'}$ such that $\eta \circ i_{X_v} = i_{X_{v'}} \circ \eta$ for $v = v' + u'$, $v' \in \mathfrak{g}'$, $u' \in \mathfrak{h}'$.*

Proof. Let $\omega' \in \Omega_B^*(G \times_{G'} Y)^G$. Let $\pi: G \times Y \rightarrow G \times_{G'} Y$ be the quotient map. Then the pull-back $\pi^*(\omega') \in \Omega^*(G \times Y)$ is G -invariant, right \mathfrak{g}' -basic and left \mathfrak{h}' -basic form. Conversely, such a differential form on $G \times Y$ descends to a differential form in $\Omega_B^*(G \times_{G'} Y)^G$. These correspondence gives an isomorphism between $\Omega_B^*(G \times_{G'} Y)^G$ and

$$A^* := \{\omega \in \Omega^*(G \times Y)^G \mid \omega \text{ is left } \mathfrak{h}'\text{-basic and right } \mathfrak{g}'\text{-basic}\}.$$

Let $\omega \in A^*$. Suppose that

$$\omega = \sum_{i=1}^k p_G^*(\alpha_i) \wedge p_Y^*(\beta_i), \quad \alpha_i \in \Omega^*(G), \beta_i \in \Omega^*(Y).$$

Since ω is G -invariant, by averaging with the G -action, we may assume that all $\alpha_i \in \Omega^*(G)$ is G -invariant. Since $\Omega^*(G)^G \cong \bigwedge \mathfrak{g}^*$ and the pull-back maps are injective, we have that A^* is isomorphic to

$$\{\omega \in \bigwedge \mathfrak{g}^* \otimes \Omega^*(Y) \mid \omega \text{ is left } \mathfrak{h}'\text{-basic and right } \mathfrak{g}'\text{-basic}\}.$$

But being left \mathfrak{h}' -basic implies that this DGA is isomorphic to

$$\{\omega \in \bigwedge (\mathfrak{g}/\mathfrak{h}')^* \otimes \Omega^*(Y) \mid \omega \text{ is right } \mathfrak{g}'\text{-basic}\}.$$

Since $\mathfrak{g}' \oplus \mathfrak{h}' = \mathfrak{g}$, we have

$$\{\omega \in \bigwedge (\mathfrak{g}/\mathfrak{h}')^* \otimes \Omega^*(Y) \mid \omega \text{ is right } \mathfrak{g}'\text{-basic}\} \cong \Omega^*(Y)^{G'}.$$

Thus we obtain the isomorphism $\eta: \Omega_B^*(G \times_{G'} Y)^G \rightarrow \Omega^*(Y)^{G'}$. It follows from the construction of η that $\eta \circ i_{X_v} = i_{X_{v'}} \circ \eta$ for $v = v' + u'$, $v' \in \mathfrak{g}'$, $u' \in \mathfrak{h}'$. The lemma is proved. \square

Lemma 6.7. *Let G, G', \mathfrak{h}', Y be as above. We think of $S^*((\mathfrak{g}/\mathfrak{h}')^*)$ as a subalgebra of $S^*(\mathfrak{g}^*)$ via the dual of the quotient map $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}'$. We identify \mathfrak{g}'^* with $(\mathfrak{g}/\mathfrak{h}')^*$ via the decomposition $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{h}'$. Then the followings hold.*

- (1) $H_B^*(G \times_{G'} Y)$ is isomorphic to $H^*(Y)$ as \mathbb{R} -algebras.
- (2) $H_G^*(G \times_{G'} Y)$ is isomorphic to $H_{G'}^*(Y)$ as $S^*((\mathfrak{g}/\mathfrak{h}')^*)$ -algebras.

Proof. Part (1) follows from Lemmas 6.1 and 6.6 immediately.

We show Part (2). Let $\pi: G \times_{G'} Y \rightarrow G/G'$ be the projection induced by the first projection $G \times Y \rightarrow G$. π is a G -equivariant map. Since H' acts on G/G' transitively and local freely, there exists a \mathfrak{h}' -valued G -invariant 1-form θ on G/G' such that $i_{X_v'}\theta = v$ for all $v \in \mathfrak{h}'$, where X_v' denotes the fundamental vector field on G/G' generated by v . Since π is G -equivariant, we have that the pull-back $\pi^*\theta$ is a \mathfrak{h}' -valued G -invariant 1-form on $G \times_{G'} Y$ such that $i_{X_v}\pi^*\theta = v$. Thus we have a Cartan

operator $f: S^*(\mathfrak{g}') \otimes \Omega_{\mathfrak{h}'}^*(G \times_{G'} Y) \rightarrow S^*(\mathfrak{g}') \otimes \Omega_B^*(G \times_{G'} Y)$ that induces a cohomology isomorphism. By Lemma 6.6 and facts $\Omega_G^*(G \times_{G'} Y) = S^*(\mathfrak{g}'^*) \otimes \Omega_{\mathfrak{h}'}^*(G \times_{G'} Y)$ and $\Omega_{G'}^*(Y) = S(\mathfrak{g}'^*) \otimes \Omega^*(Y)^{G'}$, we have that $H_G^*(G \times_{G'} Y)$ is isomorphic to $H_{G'}^*(Y)$ as $S^*((\mathfrak{g}/\mathfrak{h}')^*)$ -algebras, as required. \square

6.2. Global computations. Until the end of this subsection, let $(M, G, y) \in \mathcal{C}_1$. Let F be the canonical foliation on M . Let $(\Delta, \mathfrak{h}, G) = \mathcal{F}_1(M, G, y) \in \mathcal{C}_2$. We denote by $p: \mathfrak{g}^{\mathbb{C}} \rightarrow \mathfrak{g}$ the projection, $\mathfrak{h}' := p(\mathfrak{h})$ and $q: \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}'$ the quotient map. We assume that $q(\Delta)$ is polytopal. In particular, M admits a transverse Kähler form with respect to F .

By Theorem 5.5, there exist a G -invariant transverse Kähler form ω and a smooth function $h_v: M \rightarrow \mathbb{R}$ such that $dh_v = -i_{X_v}\omega$ for any $v \in \mathfrak{g}$. By Proposition 5.6 and Lemma 5.10, there exists $v \in \mathfrak{g}$ such that the set of critical points of h_v coincides with the union Z of minimal orbits of M . Let Z_1, \dots, Z_l be the minimal orbits of M . By renumbering them if necessary, we may assume that

$$h_v(Z_1) < h_v(Z_2) < \dots < h_v(Z_l).$$

Throughout this subsection, we fix such $v \in \mathfrak{g}$.

By Proposition 5.4, there uniquely exists a G^M -invariant open neighborhood U_j of Z_j that is G^M -equivariantly biholomorphic to

$$G^M \times_{G_j^{\mathbb{C}}} \bigoplus_{i=1}^k \mathbb{C}_{\alpha_{j,i}}$$

where G_j denotes the isotropy subgroup of G at a point x in Z_j and $\alpha_{j,1}, \dots, \alpha_{j,k} \in \text{Hom}(G_j^{\mathbb{C}}, \mathbb{C}^*)$ are the weights of the $G_j^{\mathbb{C}}$ -representation $T_x M / T_x Z_j$. We define

$$U_j^* := \begin{cases} \emptyset & \text{for } j = 1, \\ \bigcup_{i=1}^{j-1} U_i \cap U_j & \text{for } j = 2, \dots, l. \end{cases}$$

Let J be the complex structure on M . We denote by γ_t the partial flow of JX_v . For $x \in M$, consider the set

$$W_x := \bigcap_{t_0 \in \mathbb{R}} \overline{\{\gamma_t(x) \mid t \geq t_0\}}.$$

Lemma 6.8. *For any $x \in M$, the set W_x is a nonempty subset of a minimal orbit in M .*

Proof. Let $(t_i)_{i=1, \dots}$ be a monotonic increasing sequence of real numbers diverges to ∞ . Since M is compact, the sequence $(\gamma_{t_i}(x))_{i=1, \dots}$ has a subsequence that converges to a point in M . This shows that W_x is not empty.

Let $f: M \rightarrow \mathbb{R}$ be a function defined by $f = \omega(X_v, JX_v)$. Since

$$L_{JX_v} h_v = i_{JX_v} dh_v = i_{JX_v}(-i_{X_v}\omega) = \omega(-X_v, JX_v) \leq 0,$$

the function $\mathbb{R} \rightarrow \mathbb{R}$ given by $t \mapsto h_v(\gamma_t(x))$ is monotone decreasing. Since M is compact, we have that h_v is bounded. Therefore we have

$$\lim_{t \rightarrow \infty} \frac{d}{dt} h_v(\gamma_t(x)) = 0.$$

Suppose that the sequence $(\gamma_{t_i}(x))_{i=1, \dots}$ converges to a point $x_0 \in M$. Since f is continuous, we have $\lim_{i \rightarrow \infty} f(\gamma_{t_i}(x)) = f(x_0)$. On the other hand, we have

$$\lim_{i \rightarrow \infty} f(\gamma_{t_i}(x)) = \lim_{t \rightarrow \infty} f(\gamma_t(x)) = \lim_{t \rightarrow \infty} \frac{d}{dt} h_v(\gamma_t(x)) = 0.$$

Therefore we have $f(x_0) = 0$. Since $f(x') = 0$ if and only if $(X_v)_{x'} \in T_{x'}F$ for $x' \in M$, we have $x_0 \in Z_j$ for some j . On the other hand, $\lim_{t \rightarrow \infty} h_v(\gamma_t(x)) = h_v(x_0) = h_v(Z_j)$. Since $\lim_{t \rightarrow \infty} h_v(\gamma_t(x))$ does not depend on the sequence $(\gamma_{t_i}(x))_{i=1, \dots}$ and $h_v(Z_i) \neq h_v(Z_j)$ if $i \neq j$, we have $W_x \subset Z_j$. The lemma is proved. \square

Lemma 6.9. *Let $x \in M$. For $j = 1, \dots, l$, the followings hold.*

- (1) *If $W_x \subset Z_j$, then $x \in U_j$.*
- (2) *If $x \in U_j$, then $W_x \subset \bigcup_{i \leq j} Z_i$.*

Proof. Suppose that $W_x \subset Z_j$. Since U_j is a neighborhood of Z_j , we have that there exists $t \in \mathbb{R}$ such that $\gamma_t(x) \in U_j$. Since U_j is G^M -invariant, we have $x \in U_j$, proving (1).

Suppose that $j = l$. Since $Z = Z_1 \sqcup \dots \sqcup Z_l$, we have $W_x \subset \bigcup_{i \leq j} Z_i$ by Lemma 6.8. For $j = 1, \dots, l-1$, we define

$$V_j := \{x' \in U_j \mid |h_v(x') - h_v(Z_j)| < h_v(Z_{j+1}) - h_v(Z_j)\}.$$

V_j is an open neighborhood of Z_j . Since $V_j \subset U_j$ and U_j is the minimal G^M -invariant neighborhood of Z_j , we have

$$U_j = \bigcup_{g \in G^M} g \cdot V_j.$$

Suppose that $x \in U_j$. Then we have that there exists $g \in G^M$ such that $g \cdot x \in V_j$. Thus we have $|h_v(g \cdot x) - h_v(Z_j)| < h_v(Z_{j+1}) - h_v(Z_j)$. In particular, we have $h_v(g \cdot x) < h_v(Z_{j+1})$. Since γ_t is G^M -equivariant, we have $g \cdot W_x = W_{g \cdot x}$. Since each minimal orbit is G^M -invariant, we have that W_x and $g \cdot W_x$ is contained in the same minimal orbit. Therefore we have

$$h_v(W_x) = \lim_{t \rightarrow \infty} h_v(\gamma_t(x)) = \lim_{t \rightarrow \infty} h_v(\gamma_t(g \cdot x)) \leq h_v(g \cdot x) < h_v(Z_{j+1})$$

because $t \mapsto h_v(\gamma_t(x))$ is monotone decreasing. It turns out that $W_x \subset \bigcup_{i \leq j} Z_i$, proving (2). \square

Lemma 6.10. *Let $v_j \in \mathfrak{g}_j$ and $v' \in \mathfrak{h}'$ such that $v = v_j + v'$. Then, U_j^* is G^M -equivariantly biholomorphic to $G_M \times_{G_j^c} Y_j$, where*

$$Y_j = \bigoplus_{\langle d\alpha_{j,i}, v_j \rangle \leq 0} \mathbb{C}_{\alpha_{j,i}} \times \left(\left(\bigoplus_{\langle d\alpha_{j,i}, v_j \rangle > 0} \mathbb{C}_{\alpha_{j,i}} \right) \setminus \{0\} \right) \subset \bigoplus_{i=1}^k \mathbb{C}_{\alpha_{j,i}}.$$

Proof. Let $x \in U_j$. By Lemma 6.9 (2), we have $W_x \subset \bigcup_{i \leq j} Z_i$. If $W_x \subset \bigcup_{i < j} Z_i$, then $x \in \bigcup_{i < j} U_i$ by Lemma 6.9 (1). Therefore we have

$$U_j^* = U_j \setminus \{x \in U_j \mid W_x \subset Z_j\}.$$

Let $\phi: U_j \rightarrow G^M \times_{G_j^c} \bigoplus_{i=1}^k \mathbb{C}_{\alpha_{j,i}}$ be a G^M -equivariant biholomorphism. Suppose that $[g, w_1, \dots, w_k] = \phi(x)$. Then we have

$$\begin{aligned} \phi(\gamma_t(x)) &= [\exp_{G^M}(tJv)g, w_1, \dots, w_k] \\ &= [\exp_{G^M}(tJv')g, e^{t\langle d\alpha_{j,1}, v_j \rangle} w_1, \dots, e^{t\langle d\alpha_{j,k}, v_j \rangle} w_k]. \end{aligned}$$

Since $\phi(Z_j)$ is represented as $\{[g, w_1, \dots, w_k] \mid w_1 = \dots = w_k = 0\}$, we have that $W_x \subset Z_j$ if and only if $e^{t\langle d\alpha_{j,1}, v_j \rangle} w_1, \dots, e^{t\langle d\alpha_{j,k}, v_j \rangle} w_k$ converge to 0 as t reaches ∞ . It turns out that $W_x \subset Z_j$ if and only if $w_i = 0$ for $\langle d\alpha_{j,i}, v_j \rangle > 0$, proving the lemma. \square

Now we are in a position to compute the basic betti numbers. A similar result can be found in [2, Theorem 3.1] for certain LVMB manifolds.

Proposition 6.11. *The followings hold.*

- (1) $H_B^{\text{odd}}\left(\bigcup_{i \leq j} U_i\right) = 0$ for all $j = 1, \dots, l$.
- (2) $\dim H_B^{2d}(M)$ coincides with the number of critical submanifolds whose co-index is $2d$, where co-index is defined to be the index of $-h_v$.

Remark 6.12.

- (1) By the Poincaré duality of the basic cohomology and Proposition 6.11 (2), we have that the number of critical submanifolds whose co-index is $2d$ coincides with the number of critical submanifolds whose index is $2d$.
- (2) The basic cohomologies of critical submanifolds Z_1, \dots, Z_l of h_v are trivial. Proposition 6.11 means that the basic Morse-Bott function h_v is perfect.
- (3) In case when closed leaves of F are Z_1, \dots, Z_l only, we can apply [9, Theorem 6.4] to (M, F) and we obtain Proposition 6.11 (2) immediately. However, not every M satisfies this condition.
- (4) Proposition 6.11 follows from the Morse inequality for the basic cohomology shown in [1] immediately. We will give an elementary proof of Proposition 6.11 below by using Mayer-Vietoris.

Proof of Proposition 6.11. By Mayer-Vietoris, we have a long exact sequence

$$\cdots \rightarrow H_B^q \left(\bigcup_{i \leq j} U_i \right) \rightarrow H_B^q \left(\bigcup_{i \leq j-1} U_i \right) \oplus H_B^q(U_j) \rightarrow H_B^q(U_j^*) \rightarrow \cdots$$

By Lemma 6.7, we have $H_B^q(U_j) = 0$ and

$$H_B^{q-1}(U_j^*) \rightarrow H_B^q \left(\bigcup_{i \leq j} U_i \right) \rightarrow H_B^q \left(\bigcup_{i < j} U_i \right) \rightarrow H_B^q(U_j^*)$$

is exact for $q \geq 1$. By Lemmas 6.7 and 6.10, we have

$$H_B^q(U_j^*) \cong \begin{cases} \mathbb{R} & \text{if } q = 0 \text{ or the co-index of } Z_j \text{ is } q + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have $H_B^q \left(\bigcup_{i \leq j} U_i \right) \cong H_B^q \left(\bigcup_{i \leq j-1} U_i \right)$ unless q and $q+1$ are not the co-index on Z_j . If q is the co-index of Z_j , then we have an exact sequence

$$(6.3) \quad H^{q-1} \left(\bigcup_{i \leq j-1} U_i \right) \rightarrow \mathbb{R} \rightarrow H^q \left(\bigcup_{i \leq j} U_i \right) \rightarrow H^q \left(\bigcup_{i \leq j-1} U_i \right) \rightarrow 0.$$

We show that $H^{\text{odd}} \left(\bigcup_{i \leq j} U_i \right) = 0$ by induction on j . If $j = 1$, then $H^{2d-1}(U_1) = 0$ for any $d \in \mathbb{N}$. Suppose that $H^{\text{odd}} \left(\bigcup_{i \leq j-1} U_i \right) = 0$. Since $H_B^{2d-1} \left(\bigcup_{i \leq j} U_i \right) \cong H_B^{2d-1} \left(\bigcup_{i \leq j-1} U_i \right)$ unless $2d-1$ and $2d$ are not the co-index on Z_j and all co-indices are even, we have $H^{2d-1} \left(\bigcup_{i \leq j} U_i \right) = 0$ for $d \in \mathbb{N}$ except the case when $2d$ is the co-index on Z_j . If $2d$ is the co-index on Z_j , by (6.3) and induction hypothesis we have $H^{2d-1} \left(\bigcup_{i \leq j} U_i \right) = 0$, showing (1).

By (6.3), we have a short exact sequence

$$0 \rightarrow \mathbb{R} \rightarrow H^{2d} \left(\bigcup_{i \leq j} U_i \right) \rightarrow H^{2d} \left(\bigcup_{i \leq j-1} U_i \right) \rightarrow 0.$$

if $2d$ is the co-index of Z_j . Therefore we have that $\dim H_B^{2d}(M)$ coincides with the number of critical submanifolds whose co-index is $2d$, showing (2). \square

Let $\mathfrak{g}' \subset \mathfrak{g}$ be a complement of \mathfrak{h}' in \mathfrak{g} . Then the decomposition $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{h}'$ induces a decomposition $S^*(\mathfrak{g}^*) = S^*(\mathfrak{g}'^*) \otimes S^*(\mathfrak{h}'^*)$. This allows us to think of elements in $\Omega_G^*(M)$ as $\Omega_{\mathfrak{h}'}^*(M)$ -valued polynomial functions on \mathfrak{g}' . Namely, we have the decomposition $\Omega_G^*(M) = S^*(\mathfrak{g}'^*) \otimes \Omega_{\mathfrak{h}'}^*(M)$. This gives us to define the following double complex structure. For $p, q \in \mathbb{Z}$, we set

$$C^{p,q} := S^p(\mathfrak{g}'^*) \otimes \Omega_{\mathfrak{h}'}^{q-p}(M)$$

so that $\Omega_G^*(M) = \bigoplus_{p,q \in \mathbb{Z}} C^{p,q}$. Define $d: C^{p,q} \rightarrow C^{p,q+1}$ by $d := 1 \otimes d_{\mathfrak{h}'}$ and $\delta: C^{p,q} \rightarrow C^{p+1,q}$ by

$$((\delta\alpha)(v'))(u') = -i_{X_{v'+u'}}((\alpha(v'))(u'))$$

for $\alpha \in \Omega_G^*(M)$ and $v' \in \mathfrak{g}'$ and $u' \in \mathfrak{h}'$. Here, α is regarded as an $\Omega_{\mathfrak{h}'}^*(M)$ -valued polynomial functions on \mathfrak{g}' . Then we have $d_G = d + \delta$ and $d^2 = \delta^2 = d\delta + \delta d = 0$.

Proposition 6.13. *The E_1 term of the spectral sequence of the double complex $(\bigoplus C^{p,q}, d, \delta)$ is*

$$E_1 = S^*(\mathfrak{g}'^*) \otimes H^*(\Omega_{\mathfrak{h}'}^*(M), d_{\mathfrak{h}'}).$$

More explicitly,

$$E_1^{p,q} = S^p(\mathfrak{g}'^*) \otimes H^{q-p}(\Omega_{\mathfrak{h}'}^*(M), d_{\mathfrak{h}'}).$$

Proof. This follows from the definition of d immediately. \square

Proposition 6.14. *The spectral sequence of the double complex $(\bigoplus C^{p,q}, d, \delta)$ collapses at the E_1 term.*

Proof. By Theorem 6.4, we have that $H^*(\Omega_{\mathfrak{h}'}^*(M), d_{\mathfrak{h}'})$ is isomorphic to $H_B^*(M)$. By Proposition 6.11, we have $H_B^{\text{odd}}(M) = 0$. It turns out that $E_1^{p,q} = 0$ when $p + q$ is odd. Therefore the spectral sequence of $(\bigoplus C^{p,q}, d, \delta)$ collapses at the E_1 term. \square

As a conclusion, we have the following:

Theorem 6.15. *$H_G^*(M)$ is isomorphic to $S^*(\mathfrak{g}'^*) \otimes H^*(\Omega_{\mathfrak{h}'}^*(M), d_{\mathfrak{h}'})$ as $S^*(\mathfrak{g}'^*)$ -modules.*

Remark 6.16. The assertion of Theorem 6.15 is equivalent to that the transverse action of \mathfrak{g}' on (M, F) is *equivariantly formal* in the sense of [9].

Theorem 6.15 yields that the DGA homomorphism $\Omega_G^*(M) \rightarrow \Omega_{\mathfrak{h}'}^*(M)$ given by $\alpha \mapsto \alpha(0)$ induces a surjective homomorphism $H_G^*(M) \rightarrow H^*(\Omega_{\mathfrak{h}'}^*(M), d_{\mathfrak{h}'})$ and its kernel is the ideal generated by elements in $S^2(\mathfrak{g}'^*) = \mathfrak{g}'^*$. Since the Cartan operator $f: \Omega_{\mathfrak{h}'}^*(M) \rightarrow \Omega_B^*(M)$ given as Lemma 6.3 above induces an isomorphism $H^*(\Omega_{\mathfrak{h}'}^*(M), d_{\mathfrak{h}'}) \rightarrow H_B^*(M)$, we have the following.

Theorem 6.17. *We think of elements in $\Omega_G^*(M)$ as $\Omega^*(M)^G$ -valued polynomial functions on \mathfrak{g} . Then, the map $\Omega_G^*(M) \rightarrow \Omega_B^*(M)$ given by*

$$\alpha \mapsto f(\alpha|_{\mathfrak{h}'}) \quad \text{for } \alpha \in \Omega_G^*(M)$$

induces a surjective homomorphism $\text{for}_B: H_G^(M) \rightarrow H_B^*(M)$. The kernel of for_B is generated by elements in $q^*((\mathfrak{g}/\mathfrak{h}')^*) \subset \mathfrak{g}^* = S^2(\mathfrak{g}^*)$.*

6.3. Localization to minimal orbits. We use the same notations as the last subsection.

Proposition 6.18. *Let $\iota_j: \bigcup_{i \leq j} Z_i \rightarrow \bigcup_{i \leq j} U_i$ be the inclusion. Then the induced homomorphism*

$$\iota_j^*: H_G^* \left(\bigcup_{i \leq j} U_i \right) \rightarrow H_G^* \left(\bigcup_{i \leq j} Z_i \right) = \bigoplus_{i \leq j} H_G^*(Z_i)$$

is injective.

Proof. It is obvious that $H_G^0 \left(\bigcup_{i \leq j} U_i \right) \rightarrow H_G^0 \left(\bigcup_{i \leq j} Z_i \right)$ is injective. Let $q > 0$. Induction on j . By Proposition 5.4 and Lemma 6.7, we have $H_G^q(U_1) \cong H_G^q(Z_1)$. Suppose that the proposition holds for $j-1$. By Mayer-Vietoris sequences of equivariant cohomologies, we have a commutative diagram

$$\begin{array}{ccccccc} H_G^{q-1}(U_j^*) & \longrightarrow & H_G^q(\bigcup_{i \leq j} U_i) & \longrightarrow & H_G^q(\bigcup_{i \leq j-1} U_i) \oplus H_G^q(U_j) & \longrightarrow & H_G^q(U_j^*) \\ \downarrow & & \downarrow \iota_j^* & & \downarrow & & \downarrow \\ H_G^{q-1}(\emptyset) & \longrightarrow & H_G^q(\bigcup_{i \leq j} Z_i) & \longrightarrow & H_G^q(\bigcup_{i \leq j-1} Z_i) \oplus H_G^q(Z_j) & \longrightarrow & H_G^q(\emptyset) \end{array}$$

whose horizontal lines are exact. Since $q > 0$, we have that $H_G^{q-1}(\emptyset)$ and $H_G^q(\emptyset)$ vanish. Thus it is enough to show that $H_G^{\text{odd}}(\bigcup_{i \leq j} U_i) = 0$ and $H_G^{\text{odd}}(U_j^*) = 0$. By Lemmas 6.7, 6.10 and [5, Corollary 2.18], we have $H_G^{\text{odd}}(U_j^*) = 0$. We show that $H_G^{\text{odd}}(\bigcup_{i \leq j} U_i) = 0$ by induction on j . Since $H_G^*(U_1) \cong H_G^*(Z_1) \cong S^*(\mathfrak{g}/\mathfrak{h}^*)^*$, we have $H_G^{\text{odd}}(U_1) = 0$. Suppose that $H_G^{2d+1}(\bigcup_{i \leq j-1} U_i) = 0$ for all nonnegative integer d . By Mayer-Vietoris, we have an exact sequence

$$(6.4) \quad H_G^{2d} \left(\bigcup_{i \leq j-1} U_i \right) \oplus H_G^{2d}(U_j) \rightarrow H_G^{2d}(U_j^*) \rightarrow H_G^{2d+1} \left(\bigcup_{i \leq j} U_i \right) \rightarrow 0.$$

Since the inclusion $U_j^* \rightarrow U_j$ induces a surjective map $H_G^*(U_j) \rightarrow H_G^*(U_j^*)$, we have that the first arrow $H_G^{2d} \left(\bigcup_{i \leq j-1} U_i \right) \oplus H_G^{2d}(U_j) \rightarrow H_G^{2d}(U_j^*)$ in (6.4) is surjective. Therefore we have that the second arrow $H_G^{2d}(U_j^*) \rightarrow H_G^{2d+1} \left(\bigcup_{i \leq j} U_i \right)$ in (6.4) is 0. Therefore $H_G^{2d+1} \left(\bigcup_{i \leq j} U_i \right) = 0$, as required. \square

Corollary 6.19. *Let $\iota: Z \rightarrow M$ be the inclusion. Then the induced homomorphism $\iota^*: H_G^*(M) \rightarrow H_G^*(Z)$ is injective.*

Proof. It follows from Proposition 6.18 immediately. \square

We shall see the degree 2 part of ι^* for later use. Since

$$\Omega_G^2(M) = S^0(\mathfrak{g}^*) \otimes \Omega^2(M)^G \oplus S^2(\mathfrak{g}^*) \otimes \Omega^0(M)^G$$

and $S^2(\mathfrak{g}^*) = \mathfrak{g}^*$, we have that every element in $\Omega_G^2(M)$ can be written as $1 \otimes \beta + \Psi$ for some $\beta \in \Omega^2(M)^G$ and G -invariant smooth map $\Psi: M \rightarrow \mathfrak{g}^*$. Let Z_j be a connected component of Z and $\kappa_j: Z_j \rightarrow M$ the inclusion.

Lemma 6.20. *Let $1 \otimes \beta + \Psi \in \Omega_G^2(M)$. If β is basic with respect to F , then $\Psi|_{Z_j}$ is constant and*

$$\kappa_j^*(1 \otimes \beta + \Psi) = \Psi(Z_j) \otimes 1.$$

Proof. Since Z_j is a connected component of Z , we have that Z_j is a minimal orbit by Proposition 5.6. Thus $\beta|_{Z_j} = 0$ and $\Psi|_{Z_j}$ is constant. \square

7. THE DANILOV-JURKIEWICZ TYPE FORMULA

Let $(M, G, y) \in \mathcal{C}_1$. Let F be the canonical foliation of M . Let $(\Delta, \mathfrak{h}, G) = \mathcal{F}_1(M, G, y) \in \mathcal{C}_2$. As before, we denote by $p: \mathfrak{g}^{\mathbb{C}} \rightarrow \mathfrak{g}$ the projection and by $q: \mathfrak{g} \rightarrow \mathfrak{g}/p(\mathfrak{h})$ the quotient map. Let $(\tilde{V}, \tilde{\Gamma}, \tilde{\Delta}, \tilde{\lambda}) = \tilde{\mathcal{F}}_1(M, G, y) \in \tilde{\mathcal{C}}_2$. The purpose of this subsection is to describe the basic cohomology $H_B^*(M)$ explicitly in terms of the corresponding marked fan $(\tilde{V}, \tilde{\Gamma}, \tilde{\Delta}, \tilde{\lambda})$.

Suppose that Δ has m 1-cones ρ_1, \dots, ρ_m . Let K be the abstract simplicial complex given by

$$K := \{\{i_1, \dots, i_k\} \subset \{1, \dots, m\} \mid \rho_{i_1} + \dots + \rho_{i_k} \in \Delta\}.$$

K is the underlying simplicial complex of Δ . Let $\lambda(\rho_i)$ be the primitive generator of ρ_i for $i = 1, \dots, m$. We take a positive integer $N \in \mathbb{N}$ and a linear map $\varphi: \mathbb{R}^N \rightarrow \mathfrak{g}$ such that $\varphi(e_i) = \lambda(\rho_i)$ for $i = 1, \dots, m$, where e_i denotes the standard basis vector of \mathbb{R}^N , $\varphi(\mathbb{Z}^N) = \ker \exp_G$ and $\dim \ker \varphi$ is even. These assumptions imply that $\varphi: \mathbb{R}^N \rightarrow \mathfrak{g}$ induces a surjective homomorphism $\alpha: \mathbb{R}^N/\mathbb{Z}^N \rightarrow G$ whose kernel is an even dimensional subtorus. Put $\mathfrak{g}_0 = \mathbb{R}^N$ and $G_0 = \mathbb{R}^N/\mathbb{Z}^N$. The collection of cones

$$\Delta_0 := \{\mathbb{R}_{\geq 0}e_{i_1} + \dots + \mathbb{R}_{\geq 0}e_{i_k} \mid \{i_1, \dots, i_k\} \in K\}$$

is a nonsingular fan in \mathfrak{g}_0 with respect to the lattice $\ker \exp_{G_0} = \mathbb{Z}^N$. K is also the underlying simplicial complex of Δ_0 . Let v_1, \dots, v_l be \mathbb{C} -basis vectors of \mathfrak{h} . Then, $p(v_1), \dots, p(v_l), p(-\sqrt{-1}v_1), \dots, p(-\sqrt{-1}v_l)$ are \mathbb{R} -basis vectors of $p(\mathfrak{h})$. Since φ is surjective, there exist $b_j, b'_j \in \mathfrak{g}_0$ such that $\varphi(b_j) = p(v_j)$, $\varphi(b'_j) = p(-\sqrt{-1}v_j)$ for $j = 1, \dots, l$. Let $c_1, \dots, c_{\nu}, c'_1, \dots, c'_{\nu}$ be \mathbb{R} -basis vectors of $\ker \varphi$. Let \mathfrak{h}_0 be the \mathbb{C} -subspace of \mathbb{C}^N spanned by $b_1 + \sqrt{-1}b'_1, \dots, b_l + \sqrt{-1}b'_l, c_1 + \sqrt{-1}c'_1, \dots, c_{\nu} + \sqrt{-1}c'_{\nu}$. Then by definition, we have $(\Delta_0, \mathfrak{h}_0, G_0) \in \mathcal{C}_2$ and $\alpha \in \text{Hom}_{\mathcal{C}_2}((\Delta_0, \mathfrak{h}_0, G_0), (\Delta, \mathfrak{h}, G))$. Let $(M_0, G_0, y_0) = \mathcal{F}_2(\Delta_0, \mathfrak{h}_0, G_0)$.

Lemma 7.1. *There exists $f: M_0 \rightarrow M$ satisfying the followings.*

- (0) $(f, \alpha) \in \text{Hom}_{\mathcal{C}_1}((M_0, G_0, y_0), (M, G, y))$.
- (1) $\ker \alpha$ is connected.
- (2) $f: M_0 \rightarrow M$ is a principal $\ker \alpha$ -bundle.

In particular, (M, G, y) and (M_0, G_0, y_0) are equivalent, see Definition 3.1.

Proof. It follows from Theorem 3.3 and the definition of $(\Delta_0, \mathfrak{h}_0, G_0)$ immediately. \square

M_0 is the quotient space $X(\Delta_0)/H_0$ of the toric variety $X(\Delta_0)$ by the action of $H_0 = \exp_{G_0^{\mathbb{C}}}(\mathfrak{h}_0)$. By definition of Δ_0 , the toric variety $X(\Delta_0)$ is the complement of the coordinate subspace arrangement

$$X(\Delta_0) = \mathbb{C}^N \setminus \bigcup_{I \notin K} L_I,$$

where

$$L_I = \{(z_1, \dots, z_N) \in \mathbb{C}^N \mid z_i = 0 \text{ for all } i \in I\}.$$

The action of $G_0 = (S^1)^N$ on \mathbb{C}^N is given by coordinatewise multiplication. Let x_1, \dots, x_N be the dual basis vectors of $e_1, \dots, e_N \in \mathfrak{g}_0$. Then

$$H_{G_0}^*(\mathbb{C}^N) \cong S^*(\mathfrak{g}_0^*) = \mathbb{R}[x_1, \dots, x_N].$$

By [5, Corollary 2.18] and [6, Theorem 4.8 and Remark 4.10], we have that the inclusion $X(\Delta_0) \rightarrow \mathbb{C}^N$ induces a surjective homomorphism $\mathbb{R}[x_1, \dots, x_N] \rightarrow H_{G_0}^*(X(\Delta_0))$ and the kernel \mathcal{I} is the Stanley-Reisner ideal $\mathcal{I} = \langle x_{i_1} \dots x_{i_k} \mid \{i_1, \dots, i_k\} \notin K \rangle$. In particular, $H_{G_0}^*(X(\Delta_0)) \cong \mathbb{R}[x_1, \dots, x_N]/\mathcal{I}$.

Since H_0 is contractible and the action of H_0 on $X(\Delta_0)$ is free, we have $H_{G_0}^*(M_0) \cong H_{G_0}^*(X(\Delta_0))$. To describe $H_B^*(M_0)$, we shall see the image of $(\mathfrak{g}_0/p_0(\mathfrak{h}_0))^*$ by the dual of the quotient map $q_0: \mathfrak{g}_0 \rightarrow \mathfrak{g}_0/p_0(\mathfrak{h}_0)$, where $p_0: \mathfrak{g}_0^{\mathbb{C}} \rightarrow \mathfrak{g}_0$ denotes the projection. Let $\tilde{x} \in (\mathfrak{g}_0/p_0(\mathfrak{h}_0))^*$. Then

$$(7.1) \quad q_0^*(\tilde{x}) = \sum_{i=1}^N \langle q_0^*(\tilde{x}), e_i \rangle x_i = \sum_{i=1}^N \langle \tilde{x}, q_0(e_i) \rangle x_i.$$

Proposition 7.2. *Let $(M_0, G_0, y_0) \in \mathcal{C}_1$ be as above. Let $(\tilde{V}_0, \tilde{\Gamma}_0, \tilde{\Delta}_0, \tilde{\lambda}_0) = \tilde{\mathcal{F}}_1(M_0, G_0, y_0) \in \tilde{\mathcal{C}}_2$ be the corresponding marked fan. Let $\tilde{\rho}_1, \dots, \tilde{\rho}_m$ be 1-cones of $\tilde{\Delta}_0$. Suppose that M_0 is transverse Kähler with respect to the canonical foliation F_0 on M_0 . Then we have an isomorphism*

$$H_B^*(M_0) \cong \mathbb{R}[x_1, \dots, x_m]/\mathcal{I}' + \mathcal{J}',$$

where the degree of x_i is 2 for $i = 1, \dots, m$, \mathcal{I}' is the Stanley-Reisner ideal of the underlying simplicial complex of $\tilde{\Delta}_0$, that is,

$$\mathcal{I}' = \langle x_{i_1} \dots x_{i_k} \mid \tilde{\rho}_{i_1} + \dots + \tilde{\rho}_{i_k} \notin \tilde{\Delta}_0 \rangle$$

and

$$\mathcal{J}' = \left\langle \sum_{i=1}^m \langle \tilde{x}, \tilde{\lambda}_0(\tilde{\rho}_i) \rangle x_i \mid \tilde{x} \in \tilde{V}_0^* \right\rangle.$$

Proof. By Theorem 6.17, there exists a surjective homomorphism

$$\text{for}_B: H_G^*(M_0) \cong \mathbb{R}[x_1, \dots, x_N]/\mathcal{I} \rightarrow H_B^*(M_0)$$

whose kernel is generated by elements in the image of $(\mathfrak{g}_0/p_0(\mathfrak{h}_0))^*$ by the dual of the quotient map $q_0: \mathfrak{g}_0 \rightarrow \mathfrak{g}_0/p_0(\mathfrak{h}_0)$, where \mathcal{I} denotes the Stanley-Reisner ideal. By (7.1), we have that $\ker \text{for}_B$ is the image of the ideal

$$\mathcal{J} = \left\langle \sum_{i=1}^N \langle \tilde{x}, q_0(e_i) \rangle x_i \mid \tilde{x} \in \tilde{V}_0^* \right\rangle.$$

By renumbering, we may assume that $\tilde{\rho}_i = q_0(\mathbb{R}_{\geq 0}e_i)$ for $i = 1, \dots, m$ without loss of generality. Then, the underlying simplicial complex K of Δ is also the one of $\tilde{\Delta}_0$. More precisely, $\{i_1, \dots, i_k\} \in K$ if and only if $1 \leq i_j \leq m$ for all j and $\tilde{\rho}_{i_1} + \dots + \tilde{\rho}_{i_k} \in \tilde{\Delta}_0$. Since singletons $\{m+1\}, \dots, \{N\} \notin K$, we have $x_{m+1}, \dots, x_N \in \mathcal{I}$. The images of \mathcal{I} and \mathcal{J} by the quotient map

$$\mathbb{R}[x_1, \dots, x_N] \rightarrow \mathbb{R}[x_1, \dots, x_N]/\langle x_{m+1}, \dots, x_N \rangle = \mathbb{R}[x_1, \dots, x_m]$$

are nothing but \mathcal{I}' and \mathcal{J}' , respectively. The proposition is proved. \square

The following formula is well known as the theorem of Danilov and Jurkiewicz in case of complete nonsingular toric varieties.

Theorem 7.3. *Let $(M, G, y) \in \mathcal{C}_1$. Let $(\tilde{V}, \tilde{\Gamma}, \tilde{\Delta}, \tilde{\lambda}) = \tilde{F}_1(M, G, y) \in \tilde{\mathcal{C}}_2$. Let $\tilde{\rho}_1, \dots, \tilde{\rho}_m$ be 1-cones of $\tilde{\Delta}$. Suppose that M is transverse Kähler with respect to the canonical foliation F on M . Then we have an isomorphism*

$$H_B^*(M) \cong \mathbb{R}[x_1, \dots, x_m]/\mathcal{I}' + \mathcal{J}',$$

where \mathcal{I}' is the Stanley-Reisner ideal of the underlying simplicial complex of $\tilde{\Delta}$, that is,

$$\mathcal{I}' = \langle x_{i_1} \dots x_{i_k} \mid \tilde{\rho}_{i_1} + \dots + \tilde{\rho}_{i_k} \notin \tilde{\Delta} \rangle$$

and

$$\mathcal{J}' = \left\langle \sum_{i=1}^m \langle \tilde{x}, \tilde{\lambda}(\tilde{\rho}_i) \rangle x_i \mid \tilde{x} \in \tilde{V}^* \right\rangle.$$

Proof. Let $(M_0, G_0, y_0) \in \mathcal{C}_1$ be as above. By Lemma 7.1, we have that (M, G, y) and (M_0, G_0, y_0) are equivalent. Let F and F_0 be the canonical foliations on M and M_0 , respectively. By Theorem 3.4, we have that (M, F) and (M_0, F_0) are transversally equivalent. Thus we have an isomorphism $H_B^*(M_0) \rightarrow H_B^*(M)$.

It follows from Theorem 4.3 that $\tilde{F}_1(M, G, y)$ and $\tilde{F}_1(M_0, G_0, y_0)$ are isomorphic. This together with Proposition 7.2 yields that $H_B^*(M_0)$ and $H_B^*(M)$ have the same description. \square

Corollary 7.4. *Let $(M, G, y) \in \mathcal{C}_1$. Let F be the canonical foliation on M . Suppose that M is transverse Kähler with respect to F . Then, the basic cohomology algebra $H_B^*(M)$ is generated by degree 2 elements.*

8. BASIC DOLBEAULT COHOMOLOGY

Let $(M, G, y) \in \mathcal{C}_1$. Let F be the canonical foliation on M . The canonical foliation F on a compact connected complex manifold M is homologically orientable (see [14, Proposition 4.8]). Suppose that M is transverse Kähler with respect to F . Then we have the Hodge decomposition

$$H_B^r(M) \otimes \mathbb{C} = \bigoplus_{p+q=r} H_B^{p,q}(M)$$

(see [7] for detail). The purpose of this section is to show the following.

Theorem 8.1. *Let $(M, G, y) \in \mathcal{C}_1$. Let F be the canonical foliation on M . Suppose that M is transverse Kähler with respect to F . Then,*

$$H_B^{p,q}(M) = \begin{cases} 0 & \text{if } p \neq q, \\ H_B^{2p}(M) \otimes \mathbb{C} & \text{if } p = q. \end{cases}$$

Remark 8.2. (1) Theorem 8.1 can be proved by applying [8, Theorem 7.5] with a small modification. We will show Theorem 8.1 by a totally different argument. We will show that $H_B^*(M)$ is generated by transverse Kähler forms with respect to F . Remark that a transverse Kähler form is a closed positive $(1, 1)$ -form and basic with respect to F .

(2) For toric Sasaki manifolds, a result similar to Theorem 8.1 has been shown, see [8, Corollary 8.4].

We use the same notation as Section 7. Let $(M_0, G_0, y_0) \in \mathcal{C}_1$ and $(\Delta_0, \mathfrak{h}_0, G_0) \in \mathcal{C}_2$ be as in Section 7. We assume that M_0 admits a transverse Kähler form with respect to the canonical foliation F_0 of M_0 . Let $(\tilde{V}_0, \tilde{\Gamma}_0, \tilde{\Delta}_0, \tilde{\lambda}_0) = \tilde{F}_1(M_0, G_0, y_0) \in \tilde{\mathcal{C}}_2$. As before, we denote by $p_0: \mathfrak{g}_0^{\mathbb{C}} \rightarrow \mathfrak{g}_0$ the projection and by $q_0: \mathfrak{g}_0 \rightarrow \mathfrak{g}_0/p_0(\mathfrak{h}_0)$ the quotient map. Δ_0 has m 1-cones $\rho_1 = \mathbb{R}_{\geq 0}e_1, \dots, \rho_m = \mathbb{R}_{\geq 0}e_m$, where $e_1, \dots, e_m, e_{m+1}, \dots, e_N$ denote the standard basis vectors of $\mathbb{R}^N = \mathfrak{g}_0$. Thus $\tilde{\Delta}_0 = q_0(\Delta_0)$ has m 1-cones $\tilde{\rho}_1 = q_0(\rho_1), \dots, \tilde{\rho}_m = q_0(\rho_m)$. Then we have $\tilde{\lambda}_0(\tilde{\rho}_i) = q_0(e_i)$. For short, we put $\mu_i := q_0(e_i)$ for $i = 1, \dots, N$. For $b \in \mathbb{R}$, we define

$$H_{i,b} := \{\tilde{x} \in \tilde{V}_0^* \mid \langle \tilde{x}, \mu_i \rangle \geq b\}.$$

If $\mu_i \neq 0$, then $H_{i,b}$ is a half space whose inner normal vector is μ_i . If $\mu_i = 0$, then

$$H_{i,b} = \begin{cases} \tilde{V}_0^* & \text{if } b \leq 0, \\ \emptyset & \text{if } b > 0. \end{cases}$$

It follows from Theorem 5.5 that $\tilde{\Delta}_0$ is polytopal. Let $P \subset \tilde{V}_0 = \mathfrak{g}_0/p_0(\mathfrak{h}_0)$ be an inner normal polytope of Δ_0 . Then there uniquely exist $b_1, \dots, b_m \in \mathbb{R}$ such that $P = \bigcap_{i=1}^m H_{i,b_i}$.

Let $\epsilon > 0$. We take $b_{m+1}, \dots, b_N \in \mathbb{R}$ so that $P \subset H_{j, b_j + \epsilon} \subset H_{j, b_j}$ for $j = m+1, \dots, N$. Then we have an embedding $\Psi: P \rightarrow \mathfrak{g}_0^* = (\mathbb{R}^N)^*$ given by

$$(8.1) \quad \Psi(\tilde{x}) = \sum_{i=1}^N (\langle \tilde{x}, \mu_i \rangle - b_i) x_i \quad \text{for } \tilde{x} \in P,$$

where x_1, \dots, x_N denote the dual basis vectors of e_1, \dots, e_N . In other words, $\Psi = q_0^* - \sum_{i=1}^N b_i x_i$. By definition of b_1, \dots, b_N , the coefficients $\langle \tilde{x}, \mu_i \rangle - b_i$ are nonnegative. Let $\mathfrak{h}'_0 := p_0(\mathfrak{h}_0)$, $H'_0 := \exp_{G_0}(\mathfrak{h}'_0)$ and $r_0: \mathfrak{h}'_0 \rightarrow \mathfrak{g}_0$ the inclusion. Then we have that $r_0^* \circ \Psi: \tilde{V}^* \rightarrow \mathfrak{h}'_0^*$ is constant and $r_0^* \circ \Psi(\tilde{x}) = -\sum_{i=1}^N b_i r_0^*(x_i)$. Let ω_{st} be the Kähler form on \mathbb{C}^N given by

$$\omega_{\text{st}} = -\frac{\sqrt{-1}}{2\pi} \sum_{i=1}^N dz_i \wedge d\bar{z}_i.$$

Then, a moment map $\Phi_{\text{st}}: \mathbb{C}^N \rightarrow \mathfrak{g}_0^* = (\mathbb{R}^N)^*$ is given by

$$\Phi_{\text{st}}(z) = \sum_{i=1}^N |z_i|^2 x_i \quad \text{for } z = (z_1, \dots, z_N) \in \mathbb{C}^N.$$

A moment map $\Phi_{\text{st}}^{H'_0}: \mathbb{C}^N \rightarrow \mathfrak{h}'_0^*$ with respect to the action restricted to H'_0 is given by composing Φ_{st} with the surjective map $r_0^*: \mathfrak{g}_0^* \rightarrow \mathfrak{h}'_0^*$ induced by the inclusion $r_0: \mathfrak{h}'_0 \rightarrow \mathfrak{g}_0$.

For short, we denote

$$\mathcal{Z}_{b_1, \dots, b_N} := (\Phi_{\text{st}}^{H'_0})^{-1} \left(r_0^* \left(-\sum_{i=1}^N b_i x_i \right) \right).$$

Lemma 8.3. *The followings hold.*

- (1) $\Phi_{\text{st}}(\mathcal{Z}_{b_1, \dots, b_N}) = \Psi(P)$.
- (2) $\mathcal{Z}_{b_1, \dots, b_N}$ is compact.
- (3) The element $r_0^*(-\sum_{i=1}^N b_i x_i)$ is a regular value of $\Phi_{\text{st}}^{H'_0}$.

In particular, $\mathcal{Z}_{b_1, \dots, b_N}$ is a compact smooth manifold equipped with an action of G_0 .

Proof. Let $(z_1, \dots, z_N) \in \mathbb{C}^N$ be a point in $\mathcal{Z}_{b_1, \dots, b_N}$. Let e'_1, \dots, e'_k be basis vectors of \mathfrak{h}'_0 . Since

$$\langle \Phi_{\text{st}}^{H'_0}(z_1, \dots, z_N), e'_j \rangle = \langle \Phi_{\text{st}}(z_1, \dots, z_N), r_0(e'_j) \rangle,$$

we have

$$(8.2) \quad \sum_{i=1}^N (|z_i|^2 + b_i) \langle x_i, e'_j \rangle = 0$$

for all $j = 1, \dots, k$. It follows from (8.2) that the element $\sum_{i=1}^N (|z_i|^2 + b_i)x_i \in \ker r_0^*$. Thus there uniquely exists $\tilde{x} \in \tilde{V}_0^*$ such that

$$q_0^*(\tilde{x}) = \sum_{i=1}^N (|z_i|^2 + b_i)x_i.$$

By pairing with e_i , we have

$$(8.3) \quad \langle \tilde{x}, \mu_i \rangle = |z_i|^2 + b_i$$

for all $i = 1, \dots, N$. Thus we have $\langle \tilde{x}, \mu_i \rangle \geq b_i$ for all i . It turns out that $\tilde{x} \in P$. Conversely, if $\tilde{x} \in P$, then we have that

$$(\sqrt{\langle \tilde{x}, \mu_1 \rangle - b_1}, \dots, \sqrt{\langle \tilde{x}, \mu_N \rangle - b_N}) \in \mathcal{Z}_{b_1, \dots, b_N}.$$

Thus we have $\Phi_{\text{st}}(\mathcal{Z}_{b_1, \dots, b_N}) = \Psi(P)$, where $\Psi: P \rightarrow (\mathbb{R}^N)^*$ is the embedding given by (8.1). Thus $\mathcal{Z}_{b_1, \dots, b_N}$ is a bounded subset of \mathbb{C}^N . Since $\mathcal{Z}_{b_1, \dots, b_N}$ is closed, we have that $\mathcal{Z}_{b_1, \dots, b_N}$ is compact, proving Part (1) and Part (2).

Now we show Part (3). Let ξ_i and η_i be the real and imaginary part of z_i , respectively. By (8.2), it is enough to show that the matrix

$$(8.4) \quad \begin{pmatrix} 2\xi_1 \langle x_1, e'_1 \rangle & 2\eta_1 \langle x_1, e'_1 \rangle & \cdots & 2\xi_N \langle x_N, e'_1 \rangle & 2\eta_N \langle x_N, e'_1 \rangle \\ \vdots & \vdots & & \vdots & \vdots \\ 2\xi_1 \langle x_1, e'_k \rangle & 2\eta_1 \langle x_1, e'_k \rangle & \cdots & 2\xi_N \langle x_N, e'_k \rangle & 2\eta_N \langle x_N, e'_k \rangle \end{pmatrix}$$

has rank k for some basis vectors e'_1, \dots, e'_k . Since P is a normal polytope of the simplicial fan $\tilde{\Delta}_0$, we have that P is a simple polytope of dimension $N - k$. We denote by F_i the facet of P given by

$$F_i := P \cap \{\tilde{x} \in \tilde{V}_0^* \mid \langle \tilde{x}, \mu_i \rangle = b_i\}$$

for $i = 1, \dots, m$. Define

$$I_{\tilde{x}} := \{i \in \{1, \dots, m\} \mid \tilde{x} \in F_i\}.$$

Since P is simple, we have $|I_{\tilde{x}}| \leq N - k$. By renumbering, we may assume that $I_{\tilde{x}} = \{1, \dots, N - k'\}$ for some $k' \geq k$ and $F_1 \cap \cdots \cap F_{N-k'} \cap F_{N-k'+1} \cdots \cap F_{N-k} \neq \emptyset$ without loss of generality. Then we have $|z_j|^2 > 0$ for $j = N - k' + 1, \dots, N$ by (8.3). In particular, either ξ_j or η_j are nonzero for $j = N - k + 1, \dots, N$. Thus, if the matrix

$$(8.5) \quad \begin{pmatrix} \langle x_{N-k+1}, e'_1 \rangle & \cdots & \langle x_N, e'_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle x_{N-k+1}, e'_k \rangle & \cdots & \langle x_N, e'_k \rangle \end{pmatrix}$$

is full-rank for some basis vectors e'_1, \dots, e'_k , then the matrix (8.4) has rank k . Since $F_1 \cap \cdots \cap F_{N-k'} \cap F_{N-k'+1} \cdots \cap F_{N-k} \neq \emptyset$, we have that $\mu_1 = q_0(e_1), \dots, \mu_{N-k} =$

$q_0(e_{N-k})$ are basis vectors of \tilde{V}_0 . Let $\tilde{x}_1, \dots, \tilde{x}_{N-k} \in \tilde{V}_0^*$ be the dual basis vectors of μ_1, \dots, μ_{N-k} . Then

$$(8.6) \quad q_0^*(\tilde{x}_i) = \sum_{j=1}^N \langle q_0^*(\tilde{x}_i), e_j \rangle x_j = x_i + \sum_{j=N-k+1}^N \langle q_0^*(\tilde{x}_i), e_j \rangle x_j$$

for $i = 1, \dots, N-k$. We claim that $r_0^*(x_{N-k+1}), \dots, r_0^*(x_N)$ are linearly independent. Suppose that

$$\sum_{j=N-k+1}^N c_j r_0^*(x_j) = 0, \quad c_j \in \mathbb{R}.$$

Then $\sum_{j=N-k+1}^N c_j x_j \in \ker r_0^*$. Therefore $\sum_{j=N-k+1}^N c_j x_j$ is a linear combination of $q_0^*(\tilde{x}_1), \dots, q_0^*(\tilde{x}_{N-k})$. By (8.6), we have $\sum_{j=N-k+1}^N c_j x_j = 0$. Thus we have $c_j = 0$ for all $j = N-k+1, \dots, N$. Thus $r_0^*(x_{N-k+1}), \dots, r_0^*(x_N)$ are linearly independent. By dimensional reason, we have that $r_0^*(x_{N-k+1}), \dots, r_0^*(x_N)$ form a basis of \mathfrak{h}_0^* . If e'_1, \dots, e'_k are the dual basis vectors of $r_0^*(x_{N-k+1}), \dots, r_0^*(x_N)$, then the matrix (8.5) is the identity matrix, proving Part (3). The lemma is proved. \square

Lemma 8.4. *Let $z \in \mathcal{Z}_{b_1, \dots, b_N}$. The followings hold.*

- (1) *The H_0 -orbit $H_0 \cdot z \subset \mathbb{C}^N$ through z intersect with $\mathcal{Z}_{b_1, \dots, b_N}$ at z transversely.*
- (2) *$\mathcal{Z}_{b_1, \dots, b_N} \cap H_0 \cdot z = \{z\}$.*

Proof. For $v \in \mathfrak{g}_0$, we denote by $h_v: \mathbb{C}^N \rightarrow \mathbb{R}$ the function given by $h_v(z) = \langle \Phi_{\text{st}}(z), v \rangle$ for $z \in \mathbb{C}^N$. Let $z \in \mathcal{Z}_{b_1, \dots, b_N}$. By definition of $\mathcal{Z}_{b_1, \dots, b_N}$, we have

$$\begin{aligned} T_z \mathcal{Z}_{b_1, \dots, b_N} &= \{Y \in T_z \mathbb{C}^N \mid dh_v(Y) = 0 \text{ for all } v \in \mathfrak{h}'_0\} \\ &= \{Y \in T_z \mathbb{C}^N \mid \omega_{\text{st}}(X_v, Y) = 0 \text{ for all } v \in \mathfrak{h}'_0\}. \end{aligned}$$

Let $u + \sqrt{-1}v \in \mathfrak{h}_0$, where $u, v \in \mathfrak{h}'_0$. Then, we have

$$\omega_{\text{st}}(X_v, X_{u+\sqrt{-1}v}) = \omega_{\text{st}}(X_v, JX_v) \geq 0$$

and the equality holds if and only if $v = 0$. Since the restriction of $p_0: \mathfrak{g}_0^{\mathbb{C}} \rightarrow \mathfrak{g}_0$ to \mathfrak{h}_0 is injective, we have $\omega_{\text{st}}(X_v, X_{u+\sqrt{-1}v}) > 0$ unless $u + \sqrt{-1}v = 0$. Thus we have $T_z \mathcal{Z}_{b_1, \dots, b_N} \cap T_z(H_0 \cdot z) = \{0\}$, proving Part (1).

Suppose that $g \cdot z \in \mathcal{Z}_{b_1, \dots, b_N}$ for some $g \in H_0$. Then there exists $u + \sqrt{-1}v \in \mathfrak{h}_0$ such that $\exp_{G_0^{\mathbb{C}}}(u + \sqrt{-1}v) = g$. Let γ_t denote the partial flow of $X_{u+\sqrt{-1}v} = X_u + JX_v$. Since

$$\begin{aligned} L_{X_u + JX_v} h_v &= i_{X_u + JX_v} dh_v \\ &= i_{X_u + JX_v} (-i_{X_v} \omega_{\text{st}}) \\ &= \omega_{\text{st}}(-X_v, X_u + JX_v) \\ &= -\omega_{\text{st}}(X_v, JX_v) \leq 0, \end{aligned}$$

we have that the function $\mathbb{R} \rightarrow \mathbb{R}$ given by $t \mapsto h_v(\gamma_t(v))$ is strictly monotone decreasing unless $v = 0$. Thus we have

$$\langle \Phi_{\text{st}}^{H'_0}(z), v \rangle = h_v(z) \geq h_v(\gamma_1(z)) = h_v(g \cdot z) = \langle \Phi_{\text{st}}^{H'_0}(g \cdot z), v \rangle$$

and the equality holds if and only if $v = 0$. Since $g \cdot z \in \mathcal{Z}_{b_1, \dots, b_N}$, we have $\langle \Phi_{\text{st}}^{H'_0}(z), v \rangle = \langle \Phi_{\text{st}}^{H'_0}(g \cdot z), v \rangle$. Thus we have $v = 0$. This together with the injectivity of the restriction of $p_0: \mathfrak{g}_0^{\mathbb{C}} \rightarrow \mathfrak{g}_0$ to \mathfrak{h}_0 implies that $u + \sqrt{-1}v = 0$ and hence g is the unit of H_0 . This shows that $\mathcal{Z}_{b_1, \dots, b_N} \cap H_0 \cdot z = \{z\}$, proving Part (2). The lemma is proved. \square

Lemma 8.5. *The followings hold.*

- (1) $\mathcal{Z}_{b_1, \dots, b_N} \subset X(\Delta_0)$.
- (2) Let $\varphi: \mathcal{Z}_{b_1, \dots, b_N} \rightarrow M_0 = X(\Delta_0)/H_0$ be the smooth map induced by the inclusion $\mathcal{Z}_{b_1, \dots, b_N} \hookrightarrow X(\Delta_0)$. Then φ is injective and a local diffeomorphism.

Proof. Let $z = (z_1, \dots, z_N) \in \mathcal{Z}_{b_1, \dots, b_N}$. By Lemma 8.3 (1), we have that there exists $\tilde{x} \in P$ such that $\langle \tilde{x}, \mu_i \rangle = |z_i|^2 + b_i$ for $i = 1, \dots, N$. Put

$$I_z = \{i \in \{1, \dots, N\} \mid z_i = 0\}.$$

Then $i \in I_z$ if and only if $\langle \tilde{x}, \mu_i \rangle - b_i = 0$. It turns out that $\tilde{x} \in \bigcap_{i \in I_z} F_i$. Let K be the underlying simplicial complex of Δ_0 . Since P is a normal polytope of the simplicial fan $\tilde{\Delta}_0$, we have that $I \in K$ if and only if $\bigcap_{i \in I} F_i \neq \emptyset$. Thus we have $I_z \in K$. Since

$$X(\Delta_0) = \mathbb{C}^N \setminus \bigcup_{I \notin K} L_I,$$

where

$$L_I := \{(z_1, \dots, z_n) \in \mathbb{C}^N \mid z_i = 0 \text{ for all } i \in I\},$$

we have $z \notin \bigcup_{I \notin K} L_I$ because $I_z \in K$. This shows Part (1).

Let $z \in \mathcal{Z}_{b_1, \dots, b_N}$. We denote by $[z] \in M_0 = X(\Delta_0)/H_0$ the H_0 -orbit through z . The differential $d\varphi_z: T_z \mathcal{Z}_{b_1, \dots, b_N} \rightarrow T_{[z]} M_0$ is the composition of the inclusion $T_z \mathcal{Z}_{b_1, \dots, b_N} \rightarrow T_z X(\Delta_0)$ and the quotient map $T_z X(\Delta_0) \rightarrow T_z X(\Delta_0)/T_z(H_0 \cdot z) = T_{[z]} M_0$. This together with Lemma 8.4 (1) yields that $d\varphi_z$ is an isomorphism. By inverse function theorem, we have that φ is a local diffeomorphism. It follows from Lemma 8.4 (2) that φ is injective, proving Part (2). The lemma is proved. \square

Corollary 8.6. *Let $\varphi: \mathcal{Z}_{b_1, \dots, b_N} \rightarrow M_0 = X(\Delta_0)/H_0$ be the smooth map induced by the inclusion $\mathcal{Z}_{b_1, \dots, b_N} \hookrightarrow X(\Delta_0)$. φ is a G_0 -equivariant diffeomorphism.*

Proof. Since $\mathcal{Z}_{b_1, \dots, b_N}$ is a G_0 -invariant subset of $X(\Delta_0)$, we have that φ is G_0 -equivariant. By Lemma 8.5 (2), we have that φ is an injective local diffeomorphism. Therefore φ is a diffeomorphism onto the open subset $\varphi(\mathcal{Z}_{b_1, \dots, b_N}) \subset M_0$. By Lemma 8.3 (2), we have that $\varphi(\mathcal{Z}_{b_1, \dots, b_N})$ is closed. Since M_0 is connected, we have that an open and closed subset of M_0 is \emptyset or M_0 . Since $\varphi(\mathcal{Z}_{b_1, \dots, b_N})$ is open and closed in M_0 , we have $\varphi(\mathcal{Z}_{b_1, \dots, b_N}) = M_0$. This shows that φ is surjective and hence $\varphi: \mathcal{Z}_{b_1, \dots, b_N} \rightarrow M_0$ is a G_0 -equivariant diffeomorphism. The corollary is proved. \square

Now we are in a position to construct a transverse Kähler form on M_0 with respect to the canonical foliation F_0 via the diffeomorphism $\varphi: \mathcal{Z}_{b_1, \dots, b_N} \rightarrow M_0$. We denote by $\omega_{\text{st}}|_{\mathcal{Z}_{b_1, \dots, b_N}}$ the restriction of ω_{st} to the submanifold $\mathcal{Z}_{b_1, \dots, b_N} \subset X(\Delta_0) \subset \mathbb{C}^N$. We denote by ω_{b_1, \dots, b_N} the pull-back $(\varphi^{-1})^* \omega_{\text{st}}|_{\mathcal{Z}_{b_1, \dots, b_N}}$ of $\omega_{\text{st}}|_{\mathcal{Z}_{b_1, \dots, b_N}}$ by $\varphi^{-1}: M_0 \rightarrow \mathcal{Z}_{b_1, \dots, b_N}$. By definition, ω_{b_1, \dots, b_N} is a closed 2-form on M_0 .

Lemma 8.7. ω_{b_1, \dots, b_N} is a transverse Kähler form on M_0 with respect to F_0 .

Proof. Let $z \in \mathcal{Z}_{b_1, \dots, b_N}$. Since

$$T_z \mathcal{Z}_{b_1, \dots, b_N} = \{Y \in T_z X(\Delta_0) \mid \omega_{\text{st}}(X_v, Y) = 0 \text{ for all } v \in \mathfrak{h}'_0\},$$

the kernel of $\omega_{\text{st}}|_{\mathcal{Z}_{b_1, \dots, b_N}}$ at z coincides with $T_z(H'_0 \cdot z)$. Since φ is a G_0 -equivariant diffeomorphism, we have that ω_{b_1, \dots, b_N} is a transverse symplectic form on M_0 with respect to the canonical foliation F_0 .

Let $\pi: X(\Delta_0) \rightarrow M_0$ be the quotient map. Let J_{st} and J_0 be the complex structure on $X(\Delta_0)$ and M_0 , respectively. Since π is holomorphic, we have $d\pi_z \circ J_{\text{st}} = J_0 \circ d\pi_z$. Let $T_z(H'_0 \cdot z)^\perp$ be the orthogonal complement of $T_z(H'_0 \cdot z)$ in $T_z \mathcal{Z}_{b_1, \dots, b_N}$ with respect to the inner product $\omega_{\text{st}}(-, J_{\text{st}}-)$. Then we have that

$$T_z(H'_0 \cdot z)^\perp = \{Y \in T_z X(\Delta_0) \mid \omega_{\text{st}}(X_v, Y) = \omega_{\text{st}}(X_v, J_{\text{st}}Y) = 0 \text{ for all } v \in \mathfrak{h}'_0\}$$

and hence $T_z(H'_0 \cdot z)^\perp$ is closed under J_{st} . Let $X^\perp, Y^\perp \in T_z(H'_0 \cdot z)^\perp$ and $X, Y \in T_z(H'_0 \cdot z)$. Since $d\varphi_z(T_z(H'_0 \cdot z)) = T_{\varphi(z)}F_0$ and $T_{\varphi(z)}F_0$ is closed under J_0 , we have

$$(8.7) \quad \begin{aligned} & \omega_{b_1, \dots, b_N}(J_0(d\varphi_z(X^\perp + X)), J_0(d\varphi_z(Y^\perp + Y))) \\ &= \omega_{b_1, \dots, b_N}(J_0(d\varphi_z(X^\perp)), J_0(d\varphi_z(Y^\perp))). \end{aligned}$$

Since $d\varphi_z$ coincides with the restriction of $d\pi_z$ to $T_z \mathcal{Z}_{b_1, \dots, b_N}$, we have

$$\begin{aligned} & \omega_{b_1, \dots, b_N}(J_0 d\varphi_z(X^\perp), J_0 d\varphi_z(Y^\perp)) \\ &= \omega_{b_1, \dots, b_N}(J_0 d\pi_z(X^\perp), J_0 d\pi_z(Y^\perp)) \\ &= \omega_{b_1, \dots, b_N}(d\pi_z(J_{\text{st}}X^\perp), d\pi_z(J_{\text{st}}Y^\perp)) \\ &= \omega_{b_1, \dots, b_N}(d\varphi_z(J_{\text{st}}X^\perp), d\varphi_z(J_{\text{st}}Y^\perp)) \\ &= \omega_{\text{st}}(J_{\text{st}}X^\perp, J_{\text{st}}Y^\perp) \\ &= \omega_{\text{st}}(X^\perp, Y^\perp) \\ &= \omega_{b_1, \dots, b_N}(d\varphi_z(X^\perp), d\varphi_z(Y^\perp)). \end{aligned}$$

This together with (8.7) yields that $\omega_{b_1, \dots, b_N}(J_0 X_0, J_0 Y_0) = \omega_{b_1, \dots, b_N}(X_0, Y_0)$ for any $X_0, Y_0 \in T_{\varphi(z)}M_0$. Namely, ω_{b_1, \dots, b_N} is of type $(1, 1)$.

The positivity of ω_{b_1, \dots, b_N} can be shown by the same argument. More precisely, we have

$$\begin{aligned}
& \omega_{b_1, \dots, b_N}(d\varphi_z(X^\perp + X), J_0(d\varphi_z(X^\perp + X))) \\
&= \omega_{b_1, \dots, b_N}(d\varphi_z(X^\perp), J_0(d\varphi_z(X^\perp))) \\
&= \omega_{b_1, \dots, b_N}(d\varphi_z(X^\perp), J_0(d\pi_z(X^\perp))) \\
&= \omega_{b_1, \dots, b_N}(d\varphi_z(X^\perp), d\pi_z(J_{\text{st}}X^\perp)) \\
&= \omega_{b_1, \dots, b_N}(d\varphi_z(X^\perp), d\varphi_z(J_{\text{st}}X^\perp)) \\
&= \omega_{\text{st}}(X^\perp, J_{\text{st}}X^\perp) \geq 0.
\end{aligned}$$

Thus $\omega_{b_1, \dots, b_N}(X_0, J_0X_0) \geq 0$ for any $X_0 \in T_{\varphi(z)}M_0$. The lemma is proved. \square

Lemma 8.8. *The composition $\Phi_{b_1, \dots, b_N} := \Phi_{\text{st}} \circ \varphi^{-1}: M_0 \rightarrow \mathfrak{g}_0^*$ is a moment map of M_0 with respect to the transverse symplectic form ω_{b_1, \dots, b_N} . The image $\Phi_{b_1, \dots, b_N}(M_0)$ coincides with $\Psi(P)$.*

Proof. Let $v \in \mathfrak{g}_0$. We show that $dh_v = -i_{X_v}\omega_{b_1, \dots, b_N}$, where $h_v: M_0 \rightarrow \mathbb{R}$ is given by $h_v(x) = \langle \Phi_{b_1, \dots, b_N}(x), v \rangle$. Since $\varphi^*h_v = \langle \varphi^*\Phi_{\text{st}}, v \rangle$, by taking differential we have $d\varphi^*h_v = \varphi^*(-i_{X_v}\omega_{\text{st}}|_{\mathcal{Z}_{b_1, \dots, b_N}})$. Since $d\varphi^*h_v = \varphi^*(dh_v)$ and φ is a G_0 -equivariant diffeomorphism, we have $dh_v = -i_{X_v}\omega_{b_1, \dots, b_N}$. Therefore Φ_{b_1, \dots, b_N} is a moment map.

By definition, we have $\Phi_{b_1, \dots, b_N}(M_0) = \Phi_{\text{st}}(\mathcal{Z}_{b_1, \dots, b_N})$. This together with Lemma 8.3 (1) yields that $\Phi_{b_1, \dots, b_N}(M_0) = \Psi(P)$, proving the lemma. \square

By Lemma 8.8 and $\Psi(P) = q_0^*(P) - \sum_{i=1}^N b_i x_i$, we have an induced moment map $\tilde{\Phi}_{b_1, \dots, b_N}: M_0 \rightarrow \tilde{V}_0^*$ that satisfies

$$q_0^* \circ \tilde{\Phi}_{b_1, \dots, b_N} - \sum_{i=1}^N b_i x_i = \Phi_{b_1, \dots, b_N}.$$

The image $\tilde{\Phi}_{b_1, \dots, b_N}(M_0)$ coincides with P and $q_0^* \circ \tilde{\Phi}_{b_1, \dots, b_N}: M_0 \rightarrow \mathfrak{g}_0^*$ is also a moment map. We say that $(b_1, \dots, b_N) \in \mathbb{R}^N$ is *admissible* if (b_1, \dots, b_N) satisfies the followings.

(1) $P_{b_1, \dots, b_m} := \bigcap_{i=1}^m H_{i, b_i}$ is a normal polytope of $\tilde{\Delta}_0$. In particular,

$$P \cap \{\tilde{x} \in V_0^* \mid \langle \tilde{x}, \mu_i \rangle = b_i\}$$

is a facet of P for $i = 1, \dots, m$.

(2) There exists $\epsilon > 0$ such that $P_{b_1, \dots, b_N} \subset H_{i, b_i + \epsilon} \subset H_{i, b_i}$ for $i = m+1, \dots, N$.

The set of all admissible elements in \mathbb{R}^N is nonempty and open in \mathbb{R}^N . For each admissible element (b_1, \dots, b_N) , we have a d_{G_0} -closed class $\omega_{b_1, \dots, b_N}^{G_0} := 1 \otimes \omega_{b_1, \dots, b_N} - q_0^* \circ \Phi_{b_1, \dots, b_N} \in S^0(\mathfrak{g}_0^*) \otimes \Omega_B^2(M)^{G_0} \oplus S^2(\mathfrak{g}_0^*) \otimes \Omega_B^0(M)^{G_0}$.

We will see the relation between the basic cohomology class $[\omega_{b_1, \dots, b_N}] \in H_B^2(M_0)$ and the equivariant cohomology class $[\omega_{b_1, \dots, b_N}^{G_0}] \in H_{G_0}^2(M_0)$ via the localization map. We need the following lemma.

Lemma 8.9. *Let Z_1, \dots, Z_l be minimal orbits of M_0 . Let $(b_1, \dots, b_N), (b'_1, \dots, b'_N) \in \mathbb{R}^N$ be admissible. For $i = 1, \dots, m$, we define facets*

$$\begin{aligned} F_i &:= P_{b_1, \dots, b_m} \cap \{\tilde{x} \in \tilde{V}_0^* \mid \langle \tilde{x}, \mu_i \rangle = b_i\}, \\ F'_i &:= P_{b'_1, \dots, b'_m} \cap \{\tilde{x}' \in \tilde{V}_0^* \mid \langle \tilde{x}', \mu_i \rangle = b'_i\} \end{aligned}$$

of $P_{b_1, \dots, b_m}, P_{b'_1, \dots, b'_m}$, respectively. Then, $\tilde{\Phi}_{b_1, \dots, b_m}(Z_j) \in F_i$ if and only if $\tilde{\Phi}_{b'_1, \dots, b'_m}(Z_j) \in F'_i$.

Proof. Let $\varphi: \mathcal{Z}_{b_1, \dots, b_N} \rightarrow M_0$ and $\varphi': \mathcal{Z}_{b'_1, \dots, b'_N} \rightarrow M_0$ be the G_0 -equivariant diffeomorphisms induced by the inclusions. Let $z = (z_1, \dots, z_N) \in \varphi^{-1}(Z_j)$. Put

$$I_z := \{i \in \{1, \dots, m\} \mid z_i = 0\}.$$

Then the isotropy subgroup of G_0 at z is

$$\{(g_1, \dots, g_N) \in (S^1)^N = G_0 \mid g_i = 1 \text{ for } i \notin I_z\}.$$

Since $(\varphi')^{-1} \circ \varphi: \mathcal{Z}_{b_1, \dots, b_N} \rightarrow \mathcal{Z}_{b'_1, \dots, b'_N}$ is a G_0 -equivariant diffeomorphism, we have $I_z = I_{z'}$, where $z' = (\varphi')^{-1} \circ \varphi(z)$. Let $\Psi: P_{b_1, \dots, b_m} \rightarrow \mathfrak{g}_0^*$ be the embedding given by

$$\Psi(\tilde{x}) = \sum_{i=1}^N (\langle \tilde{x}, \mu_i \rangle - b_i) x_i \quad \text{for } \tilde{x} \in P_{b_1, \dots, b_N}.$$

By Lemma 8.8, there exists $\tilde{x} \in P_{b_1, \dots, b_N}$ such that

$$\Psi(\tilde{x}) = \Phi_{b_1, \dots, b_N}(Z_j) = \Phi_{\text{st}} \circ \varphi^{-1}(Z_j).$$

Thus we have that $\langle \tilde{x}, \mu_i \rangle = b_i$ if and only if $i \in I_z$. Since $q_0^* \circ \tilde{\Phi}_{b_1, \dots, b_N}(Z_j) - \sum_{i=1}^m b_i x_i = \Phi_{b_1, \dots, b_N}(Z_j) = \Psi(\tilde{x})$, we have $q_0^* \circ \tilde{\Phi}_{b_1, \dots, b_N}(Z_j) = \sum_{i=1}^N \langle \tilde{x}, \mu_i \rangle x_i$. It turns out that $\tilde{\Phi}_{b_1, \dots, b_N}(Z_j) = \tilde{x}$. Since $\langle \tilde{x}, \mu_i \rangle = b_i$ if and only if $i \in I_z$, we have that $\tilde{\Phi}_{b_1, \dots, b_N}(Z_j) \in F_i$ if and only if $i \in I_z$. By the same argument, we also have that $\tilde{\Phi}_{b'_1, \dots, b'_N}(Z_j) \in F'_i$ if and only if $i \in I_{z'}$. Since $I_z = I_{z'}$, we have that $\tilde{\Phi}_{b_1, \dots, b_m}(Z_j) \in F_i$ if and only if $\tilde{\Phi}_{b'_1, \dots, b'_m}(Z_j) \in F'_i$, as required. \square

Lemma 8.10. *Let $(b_1, \dots, b_N), (b'_1, \dots, b'_N) \in \mathbb{R}^N$ be admissible. Then, $[\omega_{b_1, \dots, b_N}] = [\omega_{b'_1, \dots, b'_N}]$ if and only if there exists $\tilde{y} \in \tilde{V}_0^*$ such that $P_{b_1, \dots, b_m} = P_{b'_1, \dots, b'_m} + \tilde{y}$.*

Proof. Suppose that $[\omega_{b_1, \dots, b_N}] = [\omega_{b'_1, \dots, b'_N}]$. Then we have $\text{for}_B([\omega_{b_1, \dots, b_N}^{G_0}]) = \text{for}_B([\omega_{b'_1, \dots, b'_N}^{G_0}])$. By Theorem 6.17, we have that there exists an element $\tilde{y} \in \tilde{V}_0^*$ such that $[\omega_{b_1, \dots, b_N}^{G_0}] = [\omega_{b'_1, \dots, b'_N}^{G_0}] + [q_0^*(\tilde{y})]$.

Let Z_1, \dots, Z_l be minimal orbits. By Lemma 6.20, we have

$$[q_0^* \circ \tilde{\Phi}_{b_1, \dots, b_N}(Z_j)] = [q_0^* \circ \tilde{\Phi}_{b'_1, \dots, b'_N}(Z_j)] + [q_0^*(\tilde{y})].$$

By Lemma 6.7 (2), we have that $H_{G_0}^2(Z_j)$ is isomorphic to $q_0^*(\tilde{V}_0^*)$. Thus we have

$$\tilde{\Phi}_{b_1, \dots, b_N}(Z_j) = \tilde{\Phi}_{b'_1, \dots, b'_N}(Z_j) + \tilde{y}$$

for all j . By Theorem 5.5 (2), we have $P_{b_1, \dots, b_m} = P_{b'_1, \dots, b'_m} + \tilde{y}$.

Now we show the converse. Suppose that there exists $\tilde{y} \in \tilde{V}_0^*$ such that $P_{b_1, \dots, b_m} = P_{b'_1, \dots, b'_m} + \tilde{y}$. By Proposition 5.9 and Lemma 8.9, we have $\tilde{\Phi}_{b_1, \dots, b_N}(Z_j) = \tilde{\Phi}_{b'_1, \dots, b'_N}(Z_j) + \tilde{y}$ for all j . By Corollary 6.19 and Lemma 6.20, we have $[\omega_{b_1, \dots, b_N}^{G_0}] = [\omega_{b'_1, \dots, b'_N}^{G_0}] + [q_0^*(\tilde{y})]$. Since $\text{for}_B([q_0^*(\tilde{y})]) = 0$, $\text{for}_B([\omega_{b_1, \dots, b_N}^{G_0}]) = [\omega_{b_1, \dots, b_N}]$ and $\text{for}_B([\omega_{b'_1, \dots, b'_N}^{G_0}]) = [\omega_{b'_1, \dots, b'_N}]$, we have $[\omega_{b_1, \dots, b_N}] = [\omega_{b'_1, \dots, b'_N}]$, proving the lemma. \square

Lemma 8.11. *Let $(b_1, \dots, b_m), (b'_1, \dots, b'_m) \in \mathbb{R}^m$. Assume that P_{b_1, \dots, b_m} and $P_{b'_1, \dots, b'_m}$ are inner normal polytopes of $\tilde{\Delta}_0$. Let $\tilde{y} \in \tilde{V}_0^*$. Then, $P_{b_1, \dots, b_m} = P_{b'_1, \dots, b'_m} + \tilde{y}$ if and only if $\langle \tilde{y}, \mu_i \rangle = b_i - b'_i$ for $i = 1, \dots, m$.*

Proof. Let $\tilde{x} \in V_0^*$. Put $\tilde{x}' := \tilde{x} - \tilde{y}$. Then we have

$$\begin{aligned} \langle \tilde{x}, \mu_i \rangle \geq b_i &\iff \langle \tilde{x}' + \tilde{y}, \mu_i \rangle \geq b_i \\ &\iff \langle \tilde{x}', \mu_i \rangle \geq b_i - \langle \tilde{y}, \mu_i \rangle \end{aligned}$$

for $i = 1, \dots, m$. Thus, if $\langle \tilde{y}, \mu_i \rangle = b_i - b'_i$ for all $i = 1, \dots, m$, then the condition that $\tilde{x} \in P_{b_1, \dots, b_m}$ is equivalent to the condition that $\tilde{x}' \in P_{b'_1, \dots, b'_m}$.

Suppose that $P_{b_1, \dots, b_m} = P_{b'_1, \dots, b'_m} + \tilde{y}$. As before, for $i = 1, \dots, m$, we define

$$\begin{aligned} F_i &:= P_{b_1, \dots, b_m} \cap \{\tilde{x} \in V_0^* \mid \langle \tilde{x}, \mu_i \rangle = b_i\}, \\ F'_i &:= P_{b'_1, \dots, b'_m} \cap \{\tilde{x}' \in V_0^* \mid \langle \tilde{x}', \mu_i \rangle = b'_i\}. \end{aligned}$$

Since P_{b_1, \dots, b_m} and $P_{b'_1, \dots, b'_m}$ are inner normal polytopes of $\tilde{\Delta}_0$, we have that F_i and F'_i both are nonempty facets. Since $P_{b_1, \dots, b_m} = P_{b'_1, \dots, b'_m} + \tilde{y}$, we have $F_i = F'_i + \tilde{y}$. Let $\tilde{x} \in F_i$. Then we have $\tilde{x}' := \tilde{x} - \tilde{y} \in F'_i$. On the other hand,

$$\begin{aligned} \langle \tilde{x}, \mu_i \rangle = b_i &\iff \langle \tilde{x}' + \tilde{y}, \mu_i \rangle = b_i \\ &\iff \langle \tilde{x}', \mu_i \rangle = b_i - \langle \tilde{y}, \mu_i \rangle. \end{aligned}$$

This together with the fact that $\tilde{x}' := \tilde{x} - \tilde{y} \in F'_i$ yields that $b_i - \langle \tilde{y}, \mu_i \rangle = b'_i$. Namely, we have that if $P_{b_1, \dots, b_m} = P_{b'_1, \dots, b'_m} + \tilde{y}$, then $\langle \tilde{y}, \mu_i \rangle = b_i - b'_i$ for $i = 1, \dots, m$. The lemma is proved. \square

Lemma 8.12. *The followings hold.*

- (1) *Let $(b_1, \dots, b_N), (b'_1, \dots, b'_N) \in \mathbb{R}^N$ be admissible. Then $(b_1 + b'_1, \dots, b_N + b'_N)$ is admissible, $P_{b_1, \dots, b_m} + P_{b'_1, \dots, b'_m} = P_{b_1 + b'_1, \dots, b_N + b'_N}$ and $[\omega_{b_1, \dots, b_N}] + [\omega_{b'_1, \dots, b'_N}] = [\omega_{b_1 + b'_1, \dots, b_N + b'_N}]$.*
- (2) *Let $(b_1, \dots, b_N) \in \mathbb{R}^N$ be admissible and r a positive real number. Then (rb_1, \dots, rb_N) is admissible, $rP_{b_1, \dots, b_N} = P_{rb_1, \dots, rb_N}$ and $r[\omega_{b_1, \dots, b_N}] = [\omega_{rb_1, \dots, rb_N}]$.*

Proof. For (1), let $(b_1, \dots, b_N), (b'_1, \dots, b'_N) \in \mathbb{R}^N$ be admissible. Let $\tilde{x} \in P_{b_1, \dots, b_m}$ and $\tilde{x}' \in P_{b'_1, \dots, b'_m}$. Since $P_{b_1, \dots, b_m} = \bigcap_{i=1}^m H_{i, b_i}$ and $P_{b'_1, \dots, b'_m} = \bigcap_{i=1}^m H_{i, b'_i}$, we have $\langle \tilde{x}, \mu_i \rangle \leq b_i$ and $\langle \tilde{x}', \mu_i \rangle \leq b'_i$ for $i = 1, \dots, m$. Therefore we have $\langle \tilde{x} + \tilde{x}', \mu_i \rangle \leq b_i + b'_i$. This shows that $P_{b_1, \dots, b_m} + P_{b'_1, \dots, b'_m} \subset P_{b_1+b'_1, \dots, b_m+b'_m}$. For $\mu \in \tilde{V}_0$, we denote by F_μ, F'_μ, F''_μ the maximal faces of $P_{b_1, \dots, b_m}, P_{b'_1, \dots, b'_m}, P_{b_1, \dots, b_m} + P_{b'_1, \dots, b'_m}$ whose inner normal vector is μ . Let $\tilde{x}'' \in F''_\mu$. Then there exist $\tilde{x} \in P_{b_1, \dots, b_m}$ and $\tilde{x}' \in P_{b'_1, \dots, b'_m}$ such that $\tilde{x}'' = \tilde{x} + \tilde{x}'$. By definition of F''_μ , we have $\langle \tilde{y}'', \mu \rangle \leq \langle \tilde{x}'', \mu \rangle$ for any $\tilde{y}'' \in P_{b_1, \dots, b_m} + P_{b'_1, \dots, b'_m}$. Suppose that $\tilde{x} \notin F_\mu$. Then for $\tilde{y} \in F_\mu$ we have $\langle \tilde{x}, \mu \rangle < \langle \tilde{y}, \mu \rangle$. Thus we have $\langle \tilde{x}'', \mu \rangle < \langle \tilde{y} + \tilde{x}', \mu \rangle$. This contradicts to that $\langle \tilde{y}'', \mu \rangle \leq \langle \tilde{x}'', \mu \rangle$ for any $\tilde{y}'' \in P_{b_1, \dots, b_m} + P_{b'_1, \dots, b'_m}$. Therefore we have $\tilde{x} \in F_\mu$. Using the same argument, we also have that $\tilde{x}' \in F'_\mu$. As a conclusion, we have $F''_\mu = F_\mu + F'_\mu$. Since P_{b_1, \dots, b_m} and $P_{b'_1, \dots, b'_m}$ are inner normal polytopes of $\tilde{\Delta}_0$, F_μ and F'_μ have the same inner normal cone. Thus we have that F_μ, F'_μ and F''_μ have the same inner normal cone. Suppose that F''_μ is a facet. Then F_μ is also a facet of P_{b_1, \dots, b_N} . Therefore μ coincides with one of μ_1, \dots, μ_m up to positive scalar multiplication. Put $\mu = \mu_i, i = 1, \dots, m$. Since $F''_{\mu_i} = F_{\mu_i} + F'_{\mu_i}$, we have

$$F''_{\mu_i} = (P_{b_1, \dots, b_m} + P_{b'_1, \dots, b'_m}) \cap \{\tilde{x}'' \mid \langle \tilde{x}'', \mu_i \rangle = b_i + b'_i\}.$$

Since $F''_{\mu_1}, \dots, F''_{\mu_m}$ are all facets of $P_{b_1, \dots, b_m} + P_{b'_1, \dots, b'_m}$, we have

$$P_{b_1, \dots, b_m} + P_{b'_1, \dots, b'_m} = \bigcap_{i=1}^m H_{i, b_i + b'_i} = P_{b_1+b'_1, \dots, b_m+b'_m}.$$

This together with the fact that F''_μ and F_μ have the same inner normal cone yields that $P_{b_1+b'_1, \dots, b_m+b'_m}$ is an inner normal polytope of $\tilde{\Delta}_0$. Since (b_1, \dots, b_N) is admissible, there exists $\epsilon > 0$ such that $\langle \tilde{x}, \mu_i \rangle \geq b_i + \epsilon$ for any $\tilde{x} \in P_{b_1, \dots, b_m}$ and $i = m+1, \dots, N$. Since (b'_1, \dots, b'_N) is admissible, there exists $\epsilon' > 0$ such that $\langle \tilde{x}', \mu_i \rangle \geq b'_i + \epsilon'$ for any $\tilde{x}' \in P_{b'_1, \dots, b'_m}$ and $i = m+1, \dots, N$. Thus we have $\langle \tilde{x} + \tilde{x}', \mu_i \rangle \geq b_i + b'_i + \epsilon + \epsilon'$ for any $\tilde{x} + \tilde{x}' \in P_{b_1, \dots, b_m} + P_{b'_1, \dots, b'_m}$ and $i = m+1, \dots, N$. This together with the fact that $P_{b_1, \dots, b_m} + P_{b'_1, \dots, b'_m} = P_{b_1+b'_1, \dots, b_m+b'_m}$ yields that $(b_1 + b'_1, \dots, b_N + b'_N)$ is admissible. Let Z_1, \dots, Z_l be minimal orbits of M_0 . By Lemma 6.20, we have

$$\begin{aligned} \kappa_j^*([\omega_{b_1, \dots, b_N}^{G_0}]) &= [q_0^* \circ \tilde{\Phi}_{b_1, \dots, b_N}(Z_j)], \\ \kappa_j^*([\omega_{b'_1, \dots, b'_N}^{G_0}]) &= [q_0^* \circ \tilde{\Phi}_{b'_1, \dots, b'_N}(Z_j)], \\ \kappa_j^*([\omega_{b_1+b'_1, \dots, b_N+b'_N}^{G_0}]) &= [q_0^* \circ \tilde{\Phi}_{b_1+b'_1, \dots, b_N+b'_N}(Z_j)]. \end{aligned}$$

On the other hand, since $P_{b_1, \dots, b_m} + P_{b'_1, \dots, b'_m} = P_{b_1+b'_1, \dots, b_m+b'_m}$ and

$$\begin{aligned} \tilde{\Phi}_{b_1, \dots, b_N}(M_0) &= P_{b_1, \dots, b_m}, \\ \tilde{\Phi}_{b'_1, \dots, b'_N}(M_0) &= P_{b'_1, \dots, b'_m}, \\ \tilde{\Phi}_{b_1+b'_1, \dots, b_N+b'_N}(M_0) &= P_{b_1+b'_1, \dots, b_m+b'_m}, \end{aligned}$$

we have $\kappa_j^*([\omega_{b_1, \dots, b_N}^{G_0}]) + \kappa_j^*([\omega_{b'_1, \dots, b'_N}^{G_0}]) = \kappa_j^*([\omega_{b_1+b'_1, \dots, b_N+b'_N}^{G_0}])$ by Lemma 8.9. By Corollary 6.19 we have $[\omega_{b_1, \dots, b_N}^{G_0}] + [\omega_{b'_1, \dots, b'_N}^{G_0}] = [\omega_{b_1+b'_1, \dots, b_N+b'_N}^{G_0}]$. Applying the basic forgetful map for_B: $H_{G_0}^*(M_0) \rightarrow H_B^*(M_0)$ to the above, we have $[\omega_{b_1, \dots, b_N}] + [\omega_{b'_1, \dots, b'_N}] = [\omega_{b_1+b'_1, \dots, b_N+b'_N}]$, proving Part (1).

For (2), let $(b_1, \dots, b_N) \in \mathbb{R}^N$ be admissible and r a real positive number. Let $i = 1, \dots, N$. Since $\langle \tilde{x}, \mu_i \rangle \geq b_i$ if and only if $\langle r\tilde{x}, \mu_i \rangle \geq rb_i$, we have that (rb_1, \dots, rb_N) is admissible and $rP_{b_1, \dots, b_m} = P_{rb_1, \dots, rb_m}$. By Lemma 6.20, we have $\kappa_j^*([\omega_{rb_1, \dots, rb_N}^{G_0}]) = [q_0^* \circ \tilde{\Phi}_{rb_1, \dots, rb_N}(Z_j)]$. By Lemma 8.9 and the fact that $rP_{b_1, \dots, b_m} = P_{rb_1, \dots, rb_m}$, we have $\kappa_j^*([\omega_{rb_1, \dots, rb_N}^{G_0}]) = r\kappa_j^*([\omega_{b_1, \dots, b_N}^{G_0}])$. Thus by Corollary 6.19 we have $[\omega_{rb_1, \dots, rb_N}^{G_0}] = r[\omega_{b_1, \dots, b_N}^{G_0}]$. Applying the basic forgetful map for_B: $H_{G_0}^*(M_0) \rightarrow H_B^*(M_0)$, we have $r[\omega_{b_1, \dots, b_N}] = [\omega_{rb_1, \dots, rb_N}]$, proving Part (2). The lemma is proved. \square

Proposition 8.13. *Let M_0 be as above. Then we have*

$$H_B^{p,q}(M_0) = \begin{cases} 0 & \text{if } p \neq q, \\ H_B^{2p}(M_0) \otimes \mathbb{C} & \text{if } p = q. \end{cases}$$

Proof. By Corollary 7.4, it is enough to show that $H_B^{1,1}(M_0) = H_B^2(M_0) \otimes \mathbb{C}$. We show that $H_B^2(M_0)$ is generated by basic $(1, 1)$ -forms. Let $b = (b_1, \dots, b_N) \in \mathbb{R}^N$ be admissible. Since the set of all admissible elements in \mathbb{R}^N is nonempty and open, we have that for $i = 1, \dots, m$ there exists $\epsilon_i > 0$ such that $b + \epsilon_i e_i$ is admissible, where e_i denotes the i th standard basis vector of \mathbb{R}^N . For short, we denote by P_b the polytope P_{b_1, \dots, b_m} . We define the basic $(1, 1)$ -form τ_i on M_0 by

$$\tau_i := \frac{1}{\epsilon_i} (\omega_{b+\epsilon_i e_i} - \omega_b)$$

for $i = 1, \dots, m$. We define a linear map $L: \mathbb{R}^m \rightarrow H_B^2(M_0)$ by

$$L(a_1, \dots, a_m) := \sum_{i=1}^m a_i [\tau_i], \quad (a_1, \dots, a_m) \in \mathbb{R}^m.$$

Let $(a_1, \dots, a_m) \in \ker L$. We set

$$I_+ := \{i \mid a_i > 0\}, \quad I_- := \{i \mid a_i < 0\}.$$

Then we have

$$\sum_{i \in I_+} a_i [\tau_i] = \sum_{i \in I_-} -a_i [\tau_i].$$

By definition of τ_i , we have

$$(8.8) \quad \sum_{i \in I_+} \frac{a_i}{\epsilon_i} [\omega_{b+\epsilon_i e_i}] + \sum_{i \in I_-} -\frac{a_i}{\epsilon_i} [\omega_b] = \sum_{i \in I_+} \frac{a_i}{\epsilon_i} [\omega_b] + \sum_{i \in I_-} -\frac{a_i}{\epsilon_i} [\omega_{b+\epsilon_i e_i}].$$

All coefficients in (8.8) are nonnegative. By Lemma 8.12, we have that

$$b' := \sum_{i \in I_+} \frac{a_i}{\epsilon_i} (b + \epsilon_i e_i) + \sum_{i \in I_-} -\frac{a_i}{\epsilon_i} b$$

and

$$b'' := \sum_{i \in I_+} \frac{a_i}{\epsilon_i} b + \sum_{i \in I_-} -\frac{a_i}{\epsilon_i} (b + \epsilon_i e_i)$$

both are admissible. By Lemma 8.12 and (8.8), we have $[\omega_{b'}] = [\omega_{b''}]$. By Lemma 8.10, there exists $\tilde{y} \in V_0^*$ such that $P_{b'} = P_{b''} + \tilde{y}$. By Lemma 8.11, we have $\langle \tilde{y}, \mu_i \rangle = b'_i - b''_i$ for $i = 1, \dots, m$, where $b'_i \in \mathbb{R}$ and $b''_i \in \mathbb{R}$ are the i th entries of $b' \in \mathbb{R}^N$ and $b'' \in \mathbb{R}^N$, respectively. On the other hand, we have

$$b' - b'' = \sum_{i \in I_+} a_i e_i + \sum_{i \in I_-} a_i e_i = \sum_{i=1}^m a_i e_i.$$

Therefore we have $a_i = \langle \tilde{y}, \mu_i \rangle$ for $i = 1, \dots, m$. Thus we have

$$\ker L \subset \{(a_1, \dots, a_m) \mid \exists \tilde{y} \in \tilde{V}_0^* \text{ s.t. } \forall i, \langle \tilde{y}, \mu_i \rangle = a_i\}.$$

Since $\dim \tilde{V}_0^* = N - k$, we have $\dim \ker L \leq N - k$. Thus we have $\dim \operatorname{im} L \geq m - k + N$. On the other hand, it follows from Proposition 7.2 that $\dim H_B^2(M_0) = m - k + N$. Thus $L: \mathbb{R}^m \rightarrow H_B^2(M_0)$ is surjective. Since each element of $\operatorname{im} L$ is represented by a difference of 2 positive $(1, 1)$ -forms, we have that $H_B^2(M_0)$ is generated by $(1, 1)$ -forms, as required. \square

Proof of Theorem 8.1. By Lemma 7.1, there exists an α -equivariant holomorphic map $f: M_0 \rightarrow M$ such that

- (1) $\ker \alpha$ is connected and
- (2) $f: M_0 \rightarrow M$ is a principal $\ker \alpha$ -bundle.

By Proposition 3.2, f induces an isomorphism $f^*: H_B^*(M) \rightarrow H_B^*(M_0)$ of basic cohomologies. Since f is holomorphic, we have that f^* preserves the basic Hodge structures. This together with Proposition 8.13 yields that

$$H_B^{p,q}(M) = \begin{cases} 0 & \text{if } p \neq q, \\ H_B^{2p}(M) \otimes \mathbb{C} & \text{if } p = q, \end{cases}$$

as required. \square

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