Integrable derivations in the sense of Hasse-Schmidt for some binomial plane curves

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Abstract

We describe the module of integrable derivations in the sense of Hasse-Schmidt of the quotient of the polinomial ring in two variables over an ideal generated by the equation $x^n - y^q$.

Keywords: Hasse-Schmidt derivation, Integrability, Plane curve.

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INTRODUCTION

Let k be a commutative ring and A a commutative k-algebra. A Hasse-Schmidt derivation of A over k of length $m \in \mathbb{N}$ or $m = \infty$ is a sequence $D = (D_i)_{i\geq 0}^m$ such that:

$$D_0 = \mathrm{Id}_A, \quad D_i(xy) = \sum_{a+b=n} D_a(x)D_b(y)$$

for all $x, y \in A$. We denote by $\mathrm{HS}_k(A;m)$ the set of Hasse-Schmidt derivations of A of length m. The component D_i of a Hasse-Schmidt derivation is a differential operator of order $\leq i$, in particular D_1 is a k-derivation.

The Hasse-Schmidt derivations of length m, also called higher derivation of order m (see [Ma]), were introduced by H.Hasse and F.K. Schmidt ([H-S]) and they have been used by several authors in different contexts (see [Na1], [Se] or [Tr]). An important notion related with Hasse-Schmidt derivations is integrability. Let $m \in \mathbb{N}$ or $m = \infty$, then we say that $\delta \in \mathrm{Der}_k(A)$ is m-integrable if there exists $D \in \mathrm{HS}_k(A;m)$ such that $\delta = D_1$. The set of all m-integrable k-derivations is an A-submodule of $\mathrm{Der}_k(A)$ for all m, which is denoted by $\mathrm{IDer}_k(A;m)$.

If k has characteristic 0 or A is 0-smooth over k, then any k-derivation is ∞-integrable ([Ma]), that means that $\mathrm{Der}_k(A) = \mathrm{IDer}_k(A;\infty)$. If we consider k a ring of positive characteristic and A any commutative k-algebra, the modules $\mathrm{IDer}_k(A;m)$ have better properties than $\mathrm{Der}_k(A)$ (see [Mo]). So exploring these modules seems interesting to better understand singularities in positive characteristic.

The aim of this paper is to describe the modules of m-integrable derivations, for $m \geq 1$ and $m = \infty$, of the quotient of the polynomial ring in two variables over an ideal generated by an equation of type $x^n - y^q$.

This paper is organized as follows: In section 1 we recall the definition of Hasse-Schmidt derivations and give some known properties that will be useful in later sections. In section 2 we focus on the integrability of derivations in the sense of Hasse-Schmidt in quotients of polynomial rings in two variables over the ideal generated by the equation $x^n - y^q$. Namely, we calculate the module of integrable k-derivations when k is a reduced ring of characteristic $p > 0$ and n or q are not multiple of p. In section 2.1, we assume that k is a unique factorization domain and we see the relationship between integrable derivations of the quotient of a polynomial ring over $\langle f \rangle$ and over $\langle f^p \rangle$ where f is a polynomial. Thanks to this relationship, we can describe the integrable derivations of $k[x,y]/\langle x^n - y^q \rangle$ when n and q are both multiples of p. In section 3, we calculate the module of integrable derivations in some examples taken from [Gr].

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1 Hasse-Schmidt derivations

Let k be any commutative ring and A a commutative k-algebra. In this section we will define Hasse-Schmidt derivations and we will give some of their properties, ending with the case where A is a polynomial ring. We denote $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. For each integer $m \geq 1$, we will write $A[|\mu|]_m := A[|\mu|]/\langle \mu^{m+1} \rangle$ and $A[|\mu|]_\infty := A[|\mu|]$.

Definition 1.1 *A Hasse-Schmidt derivation (over k) of A of length $m \geq 1$ (resp. of length ∞) is a sequence $D := (D_0, D_1, \ldots, D_m)$ (or resp. $D = (D_0, D_1, \ldots)$) of k-linear maps $D_i : A \to A$, satisfying the conditions:*

$$D_0 = \mathrm{Id}_A, \quad D_i(xy) = \sum_{a+b=n} D_a(x) D_b(y)$$

for all $x, y \in A$ and for all i. We write $\mathrm{HS}_k(A; m)$ (resp. $\mathrm{HS}_k(A)$) for the set of Hasse-Schmidt derivations (over k) of A of length m (resp. ∞).

Remark 1.2 ([Ma]; cf. [Na2]) *1. Any Hasse-Schmidt derivation $D \in \mathrm{HS}_k(A; m)$ is determined by the k-algebra homomorphism*

$$\begin{array}{rccl} \varphi_D : & A & \to & A[|\mu|]_m \\ & a & \mapsto & \sum_{i \geq 0}^{m} D_i(a) \mu^i \end{array}$$

satisfying $\varphi_D(x) = x \mod \mu$. φ_D can be uniquely extended to a k-algebra automorphism $\widetilde{\varphi}_D : A[|\mu|]_m \to A[|\mu|]_m$ with $\widetilde{\varphi}_D(\mu) = \mu$. So, $\mathrm{HS}_k(A; m)$ has a canonical group structure. Namely, $D \circ D' = D'' \in \mathrm{HS}_k(A; m)$ with $D''_n = \sum_{i+j=n} D_i \circ D'_j$ for $n \leq m$. Moreover, the component D_1 is a k-derivation. So, the map $(\mathrm{Id}, D_1) \in \mathrm{HS}_k(A; 1) \mapsto D_1 \in \mathrm{Der}_k(A)$ is a group isomorphism.

2. *For any $a \in A$ and any $D \in \mathrm{HS}_k(A; m)$, the sequence $a \bullet D = (a^i D_i) \in \mathrm{HS}_k(A; m)$.*

3. *For any $1 \leq n \leq m$ and any $D \in \mathrm{HS}_k(A; m)$, we define the truncation map by $\tau_{mn}(D) = (\mathrm{Id}, D_1, \ldots, D_n) \in \mathrm{HS}_k(A; n)$.*

Definition 1.3 *Let $D \in \mathrm{HS}_k(A; m)$ where $m \in \overline{\mathbb{N}}$ and $n \geq m$. Let I be an ideal of A.*

- *We say that D is I-logarithmic if $D_i(I) \subseteq I$ for all i. The set of I-logarithmic Hasse-Schmidt derivations is denoted by $\mathrm{HS}_k(\log I; m)$, $\mathrm{HS}_k(\log I) := \mathrm{HS}_k(\log I; \infty)$ and $\mathrm{Der}_k(\log I) := \mathrm{HS}_k(\log I; 1)$.*
- *We say that D is n-integrable if there exists $E \in \mathrm{HS}_k(A, n)$ such that $\tau_{nm}(E) = D$. Any such E will be called a n-integral of D. If D is ∞-integrable we say that D is integrable. If $m = 1$, we write $\mathrm{IDer}_k(A; n)$ for the set of n-integrable derivations and $\mathrm{IDer}_k(A) := \mathrm{IDer}_k(A; \infty)$.*
- *We say that D is I-logarithmically n-integrable if there exists $E \in \mathrm{HS}_k(\log I; n)$ such that E is a n-integral of D. We put $\mathrm{IDer}_k(\log I; n)$ for the set of I-logarithmically n-integrable derivations when $m = 1$ and $\mathrm{IDer}_k(\log I) := \mathrm{IDer}_k(\log I, \infty)$.*

Remark 1.4 $\mathrm{IDer}_k(A; n)$ *is an A-submodule of* $\mathrm{Der}_k(A)$ *thanks to the group structure of* $\mathrm{HS}_k(A; n)$ *and operation 2.*

Definition 1.5 *A has a leap on $s > 1$ if the inclusion $\mathrm{IDer}_k(A; s-1) \supsetneq \mathrm{IDer}_k(A; s)$ is proper.*

Lemma 1.6 *Let k be a ring of characteristic $p > 0$ and $h \in A$. Consider $D \in \mathrm{HS}_k(A; m)$ with $m \in \overline{\mathbb{N}}$ and $\tau \geq 0$. Then, for all $i \leq m$,*

$$D_i\left(h^{p^\tau}\right) = \begin{cases} 0 & \text{if} \quad p^\tau \not| i \\ D_{i/p^\tau}(h)^{p^\tau} & \text{if} \quad p^\tau | i \end{cases}$$

Proof.
Let $\varphi : A \to A[|\mu|]_m$ be the k-algebra homomorphism determined by D. Then,

$$\sum_{i\geq 0}^{m} D_i\left(h^{p^\tau}\right)\mu^i = \varphi\left(h^{p^\tau}\right) = \varphi(h)^{p^\tau} = \sum_{j\geq 0}^{m} D_j(h)^{p^\tau}\mu^{jp^\tau} \mod \langle \mu^{m+1}\rangle$$

and we obtain the result by equating the coefficients in the above equation. □

Lemma 1.7 *Consider $g \in A$ and $D \in \mathrm{HS}_k(A;m)$. Suppose that $D_j(g) \in \langle g\rangle$ for all $0 \leq j < m$. Then, for all $r \geq 1$,*

$$D_m(g^r) \in rg^{r-1}D_m(g) + \langle g^r\rangle.$$

Proof.
We will prove that $D_j(g^r) \in \langle g^r\rangle$ for all $j < m$ and $r \geq 1$. We proceed by induction on j. For $j = 0$ the result is clear since $D_0 = \mathrm{Id}$. Let us assume that $D_a(g^r) \in \langle g^r\rangle$ for all $a < j$ and all r. We will show the result for j by induction on r. When $r = 1$, it's obvious from the hypothesis. Let us suppose that $D_j(g^{r-1}) \in \langle g^{r-1}\rangle$. From the definition of Hasse-Schmidt derivation,

$$D_j\left(g^r\right) = D_j\left(g^{r-1}\right)g + \sum_{\substack{a+b=j\\ a,b\neq 0}} D_a\left(g^{r-1}\right)D_b(g) + g^{r-1}D_j(g) \in \langle g^r\rangle.$$

Now, we will prove the lemma by induction on $r \geq 1$. It is obvious for $r = 1$, let us suppose that $D_m(g^{r-1}) \in (r-1)g^{r-2}D_m(g) + \langle g^{r-1}\rangle$. From the definition of Hasse-Schmidt derivation,

$$D_m\left(g^r\right) = D_m\left(g^{r-1}\right)g + D_m(g)g^{r-1} + \sum_{\substack{a+b=m\\ a,b\neq 0}} D_a\left(g^{r-1}\right)D_b(g) \in rg^{r-1}D_m(g) + \langle g^r\rangle$$

and the lemma is proved. □

1.1 Polynomial ring and integrability

Consider $R = k[x_1, \dots, x_d]$ the polynomial ring over a commutative ring k. In this section, we recall, for the ease of the reader, some results related with the integrability of k-derivation in a polynomial ring.

Theorem 1.8 [Ma, **Th. 27.1**] *Let $R = k[x_1, \dots, x_d]$ the polynomial ring over k, then $\mathrm{IDer}_k(R) = \mathrm{Der}_k(R)$.*

Corollary 1.9 *Any Hasse-Schmidt derivation of R over k of length $m \geq 1$ is integrable.*

Proof. This is consequence of Theorem 1.8 and Proposition 2.1.5 of [Na2]. □

Corollary 1.10 [Na2, **Corollary. 2.1.10**] *The map $\Pi : \mathrm{IDer}_k(\log I;m) \to \mathrm{IDer}_k(R/I;m)$ defined by $\Pi(D) = \overline{D}$ where $\overline{D_i}(a+I) = D_i(a) + I$ is a surjective group homomorphism.*

Corollary 1.11 *Let $I \subset R$ be an ideal and $A = R/I$. Then, A has a leap on $s \geq 1$ if and only if the inclusion $\mathrm{IDer}_k(\log I; s-1) \supsetneq \mathrm{IDer}_k(\log I; s)$ is proper.*

Proposition 1.12 [Na2, **Prop. 2.2.4**] *Let $f \in R$, $I = \langle f\rangle$, and $J^0 = \langle \partial_1(f), \dots, \partial_d(f)\rangle$ the gradient ideal. If $\delta : R \to R$ is an I-logarithmic k-derivation with $\delta \in J^0\,\mathrm{Der}_k(R)$, then δ admits an I-logarithmic integral $D \in \mathrm{HS}_k(\log I)$ with $D_i(f) = 0$ for all $i > 1$. In particular, if $\delta(f) = 0$, the integral D can be taken with $\varphi_D(f) = f$.*

2 Integrable derivations for $x^n - y^q$

Let $R = k[x, y]$ be the polynomial ring in two variables over a reduced ring k of characteristic $p > 0$ and $h = x^n - y^q \in R$. In this section we will study the modules of n-integrable derivations of $A = R/\langle h\rangle$ of length $n \in \overline{\mathbb{N}}$.

In this section we will follow the following notation: Let $\alpha := \mathrm{val}_p(n)$ be the p-adic valuation of n and $s = n/p^\alpha$. We will denote by m the remainder of the division of q by p and $\beta := \mathrm{val}_p(q - m)$. We write

$$\gamma := \min\{i | ip^\alpha \geq q - 1\} = \lceil (q - 1)/p^\alpha \rceil.$$

Proposition 2.1 *Let k be a commutative reduced ring of characteristic $p > 0$ and $R = k[x, y]$ the polynomial ring over k. We set $A = R/\langle h\rangle$ where $h = x^n - y^q$. For $\delta \in \mathrm{Der}_k(\log h)$, we denote $\overline{\delta} = \Pi(\delta)$ (Corollary 1.10).*

- *If $n, q \neq 0$, then*
$$\mathrm{IDer}_k(A) = \mathrm{Der}_k(A) = \langle \overline{\delta_1}, \overline{\delta_2}\rangle$$
where $\delta_1 = qx\partial_x + ny\partial_y$ and $\delta_2 = qy^{q-1}\partial_x + nx^{n-1}\partial_y$.

- *If $n = 0 \mod p$ and $q = 1$, then*
$$\mathrm{IDer}_k(A) = \mathrm{Der}_k(A) = \langle \overline{\partial_x}\rangle$$

- *If $\alpha, m \geq 1$ and $q \geq 2$, then*

$$\mathrm{IDer}_k(A; i) = \begin{cases} \begin{cases} \langle \overline{\partial_x}\rangle & 1 \leq i < p^\alpha \\ \langle \overline{x\partial_x}, \overline{y^\gamma\partial_x}\rangle & p^\alpha \leq i < p^{\alpha+\beta} \\ \left\langle \overline{x\partial_x}, \overline{y^{\gamma+1}\partial_x}\right\rangle & i \geq p^{\alpha+\beta} \text{ or } i = \infty \end{cases} & \text{if } s = 1,\ \alpha \leq \beta,\ m = 1 \\ \begin{cases} \langle \overline{\partial_x}\rangle & 1 \leq i < p^\alpha \\ \langle \overline{x\partial_x}, \overline{y^\gamma\partial_x}\rangle & i \geq p^\alpha \text{ or } i = \infty \end{cases} & \text{otherwise} \end{cases}$$

Proof.
Let $\delta = u\partial_x + v\partial_y$ be a k-derivation of R. To prove this result it is enough to show which derivations are h-logarithmically i-integrable for $i \in \overline{\mathbb{N}}$ (Corollary 1.10).

- $n, q \neq 0 \mod p$.

We have to find the pairs (u, v) such that $\delta(h) = nux^{n-1} - qvy^{q-1} \in \langle h\rangle$. It easy to see that $\mathrm{Der}_k(\log h) = \langle \delta_1, \delta_2\rangle$ where $\delta_1 = qx\partial_x + ny\partial_y$ and $\delta_2 = qy^{q-1}\partial_x + nx^{n-1}\partial_y$. Note that h is a quasi-homogenous polynomial with respect to the weights $w(x) = q$ and $w(y) = n$. By Theorem 1.2. of [Tr], the Euler vector field, δ_1, is h-logarithmically ∞-integrable. On the other hand, the gradient of h is $J^0 = \langle x^{n-1}, y^{q-1}\rangle$, so $\delta_2 \in J^0\,\mathrm{Der}_k(R)$ and from Proposition 1.12 we know that δ_2 is h-logarithmically ∞-integrable too. So, $\mathrm{IDer}_k(A) = \mathrm{Der}_k(A) = \langle \overline{\delta_1}, \overline{\delta_2}\rangle$.

- $n = 0 \mod p$ *and* $q = 1$.

The condition for δ to be h-logarithmic is that $v \in \langle h\rangle$, so $\mathrm{Der}_k(\log h) = \langle \partial_x, h\partial_y\rangle$. In this case $J^0 = \langle 1\rangle$, hence any $\langle h\rangle$-logarithmic derivation is integrable (Prop. 1.12). Then, $\mathrm{IDer}_k(A) = \mathrm{Der}_k(A) = \langle \overline{\partial}_x\rangle$.

- $\alpha, m \geq 1$ *and* $q \geq 2$.

Note that $n = sp^\alpha$. In order for δ to be h-logarithmic, $v \in \langle h\rangle$ so $\mathrm{Der}_k(\log h) = \langle \partial_x, h\partial_y\rangle$. Since $h\partial_y$ is the zero derivation on A, we can focus on the h-logarithmically integrability of $\delta = u\partial_x$ with $u \in R$. Let $u_x \in k[x, y]$ and $u_y \in k[y]$ such that

$$u = u_x(x, y)x + u_y(y) \Rightarrow \delta = u\partial_x = u_x x\partial_x + u_y\partial_x.$$

Since h is a quasi-homogeneous polynomial with respect to the weights $w(x) = q$ and $w(y) = sp^\alpha$, the Euler vector field, $\chi = qx\partial_x$, is h-logarithmically integrable, and hence also $u_x x\partial_x$ are. Since $\mathrm{IDer}_k(\log h; i)$ is a R-modules for all i,

$$\delta \in \mathrm{IDer}_k(\log h; i) \Leftrightarrow u_y\partial_x \in \mathrm{IDer}_k(\log h; i)$$

Let us consider $\delta = u\partial_x$ where $u \in k[y]$. Let $\varphi : R \to R[|\mu|]$ be a k-algebra homomorphism:

$$\begin{array}{rccl} \varphi : & R & \longrightarrow & R[|\mu|] \\ & x & \longmapsto & x + u\mu + u_2\mu^2 + \cdots \\ & y & \longmapsto & y \quad + \quad v_2\mu^2 + \cdots \end{array}$$

To show that δ is i-integrable it is enough to prove that there exist u_j, v_j for $2 \le j \le i$ such that $\varphi(h) \in \langle h \rangle$ mod μ^{i+1}, or, equivalently, the coefficients of μ^j in $\varphi(h)$ belong to $\langle h \rangle$ for all $j \le i$. We will denote by μ_j the coefficient of μ^j in the equation

$$\varphi(h) = \left(x^{p^\alpha} + u^{p^\alpha}\mu^{p^\alpha} + u_2^{p^\alpha}\mu^{2p^\alpha} + \cdots\right)^s - \left(y + v_2\mu^2 + v_3\mu^3 + \cdots\right)^q \tag{1}$$

Suppose that there exists i such that $2 \le i < p^\alpha$. Then, $\mu_2 = -qy^{q-1}v_2$ has to belong to $\langle h \rangle$. Hence, $v_2 \in \langle h \rangle$, so we can put $v_2 = 0$. Let us assume that $v_l = 0$ for all $2 \le l < i < p^\alpha$. In this case, $\mu_i = -qy^{q-1}v_i$ and, as the same before, we can put $v_i = 0$. Then,

$$\mathrm{Der}_k(A) = \mathrm{IDer}_k(A; i) = \left\langle \overline{\partial_x} \right\rangle \ \forall i < p^\alpha$$

and we can write the equation (1) as:

$$\left(x^{p^\alpha} + u^{p^\alpha}\mu^{p^\alpha} + u_2^{p^\alpha}\mu^{2p^\alpha} + \cdots\right)^s - \left(y + v_{p^\alpha}\mu^{p^\alpha} + v_{p^\alpha+1}\mu^{p^\alpha+1} + \cdots\right)^q \in \langle h \rangle \tag{2}$$

Now, we have to see that there are $u_{p^\alpha}, v_{p^\alpha} \in R$ such that

$$\mu_{p^\alpha} = sx^{p^\alpha(s-1)}u^{p^\alpha} - qy^{q-1}v_{p^\alpha} \in \langle h \rangle \tag{3}$$

Since $u \in k[y]$, the previous expression implies that $u^{p^\alpha} \in \langle y^{q-1} \rangle$. Therefore, if we write $u = \sum_{i \ge 0} u_i y^i$ with $u_i \in k$, then $u_i^{p^\alpha} = 0$ for all i such that $ip^\alpha < q - 1$, so $u_i = 0$ because k is reduced. Hence, we can write $u = w(y)y^\gamma$ where $\gamma = \min\{i | ip^\alpha \ge q - 1\}$ and $w(y) \in k[y]$. Substituting the expression of u on (3), we can deduce that

$$sx^{p^\alpha(s-1)}w^{p^\alpha}y^{\gamma p^\alpha-(q-1)} - qv_{p^\alpha} \in \langle h \rangle \ \Rightarrow \ v_{p^\alpha} \in (s/q)x^{p^\alpha(s-1)}w^{p^\alpha}y^{\gamma p^\alpha-(q-1)} + \langle h \rangle \tag{4}$$

Therefore, A has a leap on p^α and

$$\mathrm{IDer}_k(A; p^\alpha) = \langle \overline{x\partial_x}, \overline{y^\gamma\partial_x} \rangle \text{ where } \gamma = \min\{i| \ ip^\alpha \ge q - 1\}.$$

Let us write $q = tp^\beta + m$. Note that the only case where $\gamma p^\alpha = q - 1$ is $q = tp^\beta + 1$ and $\alpha \le \beta$. Let us focus on this case when $s = 1$.

- *Case $q = tp^\beta + 1$, $\alpha \le \beta$ and $s = 1$.* Observe that $t \ne 0$ because $q \ge 2$. It is easy to see that $\gamma = tp^{\alpha-\beta}$. We will study the integrability of $w(y)y^\gamma\partial_x$ in this particular case.

  Substituting the values of q and s in the equation (2) and (4) we obtain:

$$\left(x^{p^\alpha} + u^{p^\alpha}\mu^{p^\alpha} + u_2^{p^\alpha}\mu^{2p^\alpha} + \cdots\right) - \left(y^{p^\beta} + v_{p^\alpha}^{p^\beta}\mu^{p^{\alpha+\beta}} + v_{p^\alpha+1}^{p^\beta}\mu^{(p^\alpha+1)p^\beta} + \cdots\right)^t \left(y + v_{p^\alpha}\mu^{p^\alpha} + \cdots\right) \in \langle h \rangle$$

  and

$$v_{p^\alpha} = cw^{p^\alpha} + Fh$$

  for $c = 1/q$ and some $F \in k[x, y]$. Let us consider i such that $p^\alpha < i < p^{\alpha+\beta}$. If $i = jp^\alpha$ for some $j \ge 2$, then $\mu_i = u_j^{p^\alpha} - y^{tp^\beta}v_i$. Otherwise, $\mu_i = -y^{tp^\beta}v_i$. So, $wy^\gamma\partial_x$ is h-logarithmically i-integrable for all $i < p^{\alpha+\beta}$ (it's enough to put $u_j = v_i = 0$ so that $\mu_i \in \langle h \rangle$). Now,

$$\mu_{p^{\alpha+\beta}} = u_{p^\beta}^{p^\alpha} - ty^{(t-1)p^\beta+1}v_{p^\alpha}^{p^\beta} - y^{tp^\beta}v_{p^{\alpha+\beta}}$$

has to belong to $\langle h \rangle$. So, substituting the value of v_{p^α}, we have that

$$u_{p^\beta}^{p^\alpha} - ctw^{p^{\alpha+\beta}} y^{(t-1)p^\beta+1} - y^{tp^\beta} v_{p^{\alpha+\beta}} = G\left(x^{p^\alpha} - y^{tp^\beta+1}\right)$$

for some $G \in k[x,y]$. The coefficient of y^j with $j = (t-1)p^\beta + 1$ in this equality is $tcw_0^{p^\alpha} = 0$ where w_0 is the independent term of w. Since R is reduced, $w_0 = 0$. Hence, $y^\gamma \partial_x$ is not $p^{\alpha+\beta}$-integrable. However, if $w = w'y$ with $w' \in k[y]$, the previous equation is

$$u_{p^\beta}^{p^\alpha} - ctw'^{p^{\alpha+\beta}} y^{q+p^\beta(p^\alpha-1)} - y^{tp^\beta} v_{p^{\alpha+\beta}} = G\left(x^{p^\alpha} - y^{tp^\beta+1}\right)$$

Then, there exists a solution, for instance $u_{p^\beta} = 0$ and $v_{p^{\alpha+\beta}} = -ctw'^{p^{\alpha+\beta}} y^{p^\beta(p^\alpha-1)+1}$. In conclusion, in this case A has a leap in $p^{\alpha+\beta}$ and

$$\operatorname{IDer}_k\left(A; p^{\alpha+\beta}\right) = \left\langle \overline{x\partial_x}, \overline{y^{\gamma+1}\partial_x} \right\rangle$$

Until now we saw that, for all $q \geq 2$

$$\operatorname{IDer}_k\left(A; p^{\alpha}\right) = \left\langle \overline{x\partial_x}, \overline{y^{\gamma}\partial_x} \right\rangle \text{ where } \gamma = \min\{i |\ ip^\alpha \geq q-1\}$$

and moreover, when $q = tp^\beta + 1$, $1 \leq \alpha \leq \beta$ and $s = 1$, $y^\gamma \partial_x$ is not h-logarithmically integrable but

$$\operatorname{IDer}_k\left(A; p^{\alpha+\beta}\right) = \left\langle \overline{x\partial_x}, \overline{y^{\gamma+1}\partial_x} \right\rangle$$

Let us rewrite $\gamma := \gamma + 1$ in the latter case. We will see that $y^\gamma \partial_x$ is integrable on A for all $q \geq 2$. Consider

$$\begin{array}{rccl} \varphi: & A & \longrightarrow & A[|\mu|] \\ & x & \longmapsto & x + y^\gamma \mu \\ & y & \longmapsto & y + v_1 \mu^{p^\alpha} + v_2 \mu^{2p^\alpha} + \cdots \end{array}$$

where

$$v_i = C_i x^{p^\alpha(s-\sigma)} y^{i\gamma p^\alpha - (\tau+1)q+1} \text{ for } i = \tau s + \sigma \text{ with } \tau \geq 0 \text{ and } \sigma = 1, \ldots, s,$$

$$C_i = \frac{1}{q}\left[\binom{s}{i} - \sum_{j \in I_i} D_j\right] \text{ where } \binom{s}{i} = 0 \text{ if } i > s,$$

$$I_i = \left\{ j = (j_0, j_1, \ldots, j_{i-1}) \mid j_k \geq 0\ \forall k = 0, \ldots, i-1, |j| = q, \sum_{k=1}^{i-1} k j_k = i \right\}$$

and, for all $j = (j_0, j_1, \ldots, j_l)$ with $l \geq 1$,

$$D_j = \binom{q}{j} C_1^{j_1} \cdots C_l^{j_l} \text{ with } \binom{q}{j} = \frac{q!}{j_0! \cdots j_l!}.$$

We have to prove that φ is well defined. First we see that $i\gamma p^\alpha - (\tau+1)q + 1 \geq 0$, i.e., $(\tau s + \sigma)\gamma p^\alpha - \tau q \geq q - 1$.

- When $\gamma p^\alpha > q - 1$, then $\gamma p^\alpha \geq q$, but q is not multiple of p, so $\gamma p^\alpha \geq q + 1$ and therefore

$$(\tau s + \sigma)\gamma p^\alpha - \tau q \geq (\tau s + \sigma)(q+1) - \tau q = (\tau(s-1) + \sigma)q + \tau s + \sigma \geq q - 1$$

because $s - 1 \geq 0$ and $\sigma \geq 1$.

- Let us consider $\gamma p^\alpha = q-1$. As we have seen before, the previous equality only hold if $q = tp^\beta + 1$ and $\alpha \le \beta$. If $s = 1$, then we have considered $\gamma + 1$, so we are in the first point. Therefore, we have just considered $s \ge 2$. In this case, we have to prove that $(\tau s + \sigma)\gamma p^\alpha - \tau q = (\tau s+\sigma)(q-1) - \tau q \ge q-1$. Then
$$(\tau s+\sigma)(q-1) - \tau q \ge (2\tau+\sigma)(q-1) - \tau q = (\tau+\sigma)q - (2\tau+\sigma)$$
So,
$$(\tau+\sigma)q - (2\tau+\sigma) \ge q-1 \Leftrightarrow (\tau+\sigma-1)q \ge 2\tau+\sigma-1$$
and this is true because $q \ge 2$ and $\tau+\sigma-1 \ge 0$. Note that if $\tau+\sigma-1 = 0$ then $\tau = 0$ and $\sigma = 1$, so $2\tau+\sigma-1 = 0$ too.

Now, we have to show that $\varphi(h) = 0$ in $A[|\mu|]$. The equation is:

$$\varphi(h) = \left(x^{p^\alpha} + y^{\gamma p^\alpha}\mu^{p^\alpha}\right)^s - \left(y + v_1\mu^{p^\alpha} + v_2\mu^{2p^\alpha} + \cdots\right)^q$$

Since all degrees of the monomial which appeared in this equation are multiple of p^α, let us denote μ_i to the coefficient of degree ip^α. Then
$$\mu_i = \binom{s}{i} x^{p^\alpha(s-i)} y^{i\gamma p^\alpha} - \widetilde{\mu_i}$$
where $\widetilde{\mu_i}$ is the coefficient of μ^{ip^α} from $\left(y + v_1\mu^{p^\alpha} + v_2\mu^{2p^\alpha} + \cdots\right)^q$. This coefficient can be found on

$$\left(y + v_1\mu^{p^\alpha} + \cdots + v_i\mu^{ip^\alpha}\right)^q = \sum_{|j|=q} \binom{q}{j} y^{j_0} v_1^{j_1} \cdots v_i^{j_i} \mu^{p^\alpha(j_1+\ldots+ij_i)}$$

We just have to consider all j such that $j_1 + \ldots + ij_i = i$. Observe that there exists only one j holding this equation such that $j_i \neq 0$, This j is $(q-1, 0, \ldots, 0, 1)$ where 1 is in the position i. So, we can identify the set of all these j with $I_i \cup (q-1, 0, \ldots, 0, 1)$. Let us calculate a term of $\widetilde{\mu_i}$. Fixed j, we have

$$\binom{q}{j} y^{j_0} v_1^{j_1} \cdots v_i^{j_i} = \binom{q}{j} C_1^{j_1} \ldots C_i^{j_i} x^{ap^\alpha} y^b = D_j x^{ap^\alpha} y^b$$

where

$$a = \sum_{1\le \tau s+\sigma \le i} j_{\tau s+\sigma}(s-\sigma) \ge 0 \quad \text{and} \quad b = j_0 + \sum_{1\le \tau s+\sigma\le i} j_{\tau s+\sigma}\left(\gamma p^\alpha(\tau s+\sigma) - (\tau+1)q + 1\right) \ge 0$$

We are going to study these exponents.

$$a = s\sum_{1\le \tau s+\sigma\le i} j_{\tau s+\sigma} - \sum_{1\le \tau s+\sigma\le i} j_{\tau s+\sigma}\sigma = s(q-j_0) - \sum_{1\le \tau s+\sigma\le i} j_{\tau s+\sigma}\sigma$$

On the other side, we have

$$\sum_{1\le \tau s+\sigma\le i} j_{\tau s+\sigma}(\tau s+\sigma) = s\sum_{1\le \tau s+\sigma\le i} j_{\tau s+\sigma}\tau + \sum_{1\le \tau s+\sigma\le i} j_{\tau s+\sigma}\sigma = ls + r$$

where $i = ls + r$ (remember: $l \ge 0$ and $1 \le r \le s$). Then, if we denote $T = \sum_{1\le \tau s+\sigma\le i} j_{\tau s+\sigma}\tau$ and we substitute on a, we have
$$a = s(q-j_0) - ((l-T)s + r) = s(q - j_0 - l + T) - r \ge 0$$
If $q - j_0 - l + T < 1$, then $a < 0$ so $q - j_0 - l + T \ge 1$ and we can write

$$a = (q - j_0 - l + T - 1)s + s - r$$

Observe that $s-r\geq 0$ because $1\leq r\leq s$. Now,

$$\begin{aligned} b = &\; j_0+\gamma p^\alpha \sum_{1\leq \tau s+\sigma\leq i} j_{\tau s+\sigma}(\tau s+\sigma) - q\sum_{1\leq \tau s+\sigma\leq i} j_{\tau s+\sigma}\tau - q\sum_{1\leq \tau s+\sigma\leq i} j_{\tau s+\sigma} + \sum_{1\leq \tau s+\sigma\leq i} j_{\tau s+\sigma} \\ = &\; \gamma p^\alpha i - qT - q(q-j_0)+(q-j_0)+j_0 = i\gamma p^\alpha - q(T+q-j_0-1) \end{aligned}$$

So,

$$\binom{q}{j} y^{j_0} v_1^{j_1}\cdots v_i^{j_i} = D_j x^{(q-j_0-l+T-1)sp^\alpha+(s-r)p^\alpha} y^{i\gamma p^\alpha - q(T+q-j_0-1)}$$

Since $x^{sp^\alpha}=y^q$ in A,

$$\binom{q}{j} y^{j_0} v_1^{j_1}\cdots v_i^{j_i} = D_j x^{(s-r)p^\alpha} y^{i\gamma p^\alpha + q(q-j_0-l+T-1)-q(T+q-j_0-1)} = D_j x^{(s-r)p^\alpha} y^{i\gamma p^\alpha - lq}$$

Hence,

$$\begin{aligned} \widetilde{\mu_i} = & \sum_{\substack{|j|=q \\ j_1+\ldots+ij_i=i}} D_j x^{p^\alpha(s-r)} y^{i\gamma p^\alpha - lq} = \left(\sum_{j\in I_i} D_j + D_{(q-1,0,\ldots,0,1)}\right) x^{p^\alpha(s-r)} y^{i\gamma p^\alpha - lq} \\ = & \left(\sum_{j\in I_i} D_j + qC_i\right) x^{p^\alpha(s-r)} y^{i\gamma p^\alpha - lq} = \left(\sum_{j\in I_i} D_j + q(1/q)\left[\binom{s}{i} - \sum_{j\in I_i} D_j\right]\right) x^{p^\alpha(s-r)} y^{i\gamma p^\alpha - lq} \\ = & \binom{s}{i} x^{p^\alpha(s-r)} y^{i\gamma p^\alpha - lq} \end{aligned}$$

So,

$$\mu_i = \binom{s}{i} x^{p^\alpha(s-i)} y^{i\gamma p^\alpha} - \binom{s}{i} x^{p^\alpha(s-r)} y^{i\gamma p^\alpha - lq}$$

If $i>s$, then $\binom{s}{i}=0$, and hence $\mu_i=0$.
If $i\leq s$, then $i=0\cdot s+i$, i.e., $l=0$ and $r=i$, then

$$\mu_i = \binom{s}{i} x^{p^\alpha(s-i)} y^{i\gamma p^\alpha} - \binom{s}{i} x^{p^\alpha(s-i)} y^{i\gamma p^\alpha} = 0$$

so, φ is well defined and the proposition is proved.

□

Examples 2.2 *Let us consider k a reduced ring of characteristic $p=3$ and $h=x^3-y^4\in k[x,y]$, then $\gamma=1$ so, according with Proposition 2.1,*

$$\mathrm{IDer}_k(A;i) = \begin{cases} \langle\overline{\partial_x}\rangle & 1\leq i<3 \\ \langle\overline{x\partial_x}, \overline{y\partial_x}\rangle & 3\leq i<9 \\ \langle\overline{x\partial_x}, \overline{y^2\partial_x}\rangle & i\geq 9 \end{cases}$$

Now, if we consider $h=x^3-y^5$, then $\gamma=2$ and

$$\mathrm{IDer}_k(A;i) = \begin{cases} \langle\overline{\partial_x}\rangle & 1\leq i<3 \\ \langle\overline{x\partial_x}, \overline{y^2\partial_x}\rangle & i\geq 3 \end{cases}$$

Remark 2.3 *Note that if k is not reduced, Proposition 2.1 is not true. For example, if $k=\mathbb{F}_3[t]/\langle t^3\rangle$ and $h=x^3-y^5$, then $\overline{t\partial_x}\in\mathrm{IDer}_k(A)$ with the integral*

$$\begin{array}{ccc} A & \to & A[|\mu|] \\ x & \mapsto & x+t\mu \\ y & \mapsto & y \end{array}$$

Corollary 2.4 *Let k be a commutative reduced ring of characteristic $p>0$ and $A=k[x,y]/\langle h\rangle$ where $h=x^n-y^q$ such that $\alpha, m\geq 1$ and $q\geq 2$. We denote $B_i:=\operatorname{Ann}_A(\operatorname{IDer}_k(A;i-1)/\operatorname{IDer}_k(A;i))$ for $i>1$. Then,*

$$B_i=\begin{cases}\langle x,y^\gamma\rangle & \text{if } i=p^\alpha\\ \langle y\rangle & \text{if } i=p^{\alpha+\beta},\ s=1,\ \alpha\leq\beta \text{ and } m=1\end{cases}$$

Moreover, $B_i\supseteq J^0=\langle y^{q-1}\rangle$ where J^0 is the gradient ideal of h defined in Proposition 1.12.

Proof.

Let us start with $i=p^\alpha$. From Proposition 2.1, we can deduce that

$$\operatorname{IDer}_k(A;p^\alpha-1)/\operatorname{IDer}_k(A;p^\alpha)=\langle\partial_x\rangle/\langle x\partial_x,y^\gamma\partial_x\rangle$$

where $\partial_x\in\operatorname{Der}_k(A)$. By definition, $a\in B_i$ if $a\partial_x=0\mod\langle x\partial_x,y^\gamma\partial_x\rangle$, i.e, if there exist $F,G\in A$ such that $a\partial_x=Fx\partial_x+Gy^\gamma\partial_x$. Applying this derivation to x, we have that $a\in\langle x,y^\gamma\rangle$.

Now, when $\alpha\leq\beta$, $s=m=1$ and $i=p^{\alpha+\beta}$, from Proposition 2.1,

$$\operatorname{IDer}_k\left(A;p^{\alpha+\beta}-1\right)/\operatorname{IDer}_k\left(A;p^{\alpha+\beta}\right)=\langle x\partial_x,y^\gamma\partial_x\rangle/\langle x\partial_x,y^{\gamma+1}\partial_x\rangle=\langle y^\gamma\partial_x/y^{\gamma+1}\partial_x\rangle$$

In this case, $a\in B_{p^\alpha+\beta}$ if and only if $ay^\gamma\partial_x\in\langle y^{\gamma+1}\partial_x\rangle$, i.e, if $(a-Fy)y^\gamma\partial_x=0$ for some $F\in A$. This implies that $a\in\langle y\rangle$ and we have proved the corollary.

□

2.1 I^p-logarithmic derivations

In this section, we want to calculate the m-integrable derivations of $A=k[x,y]/\langle h\rangle$ where k is a unique factorization domain (UFD) of characteristic $p>0$ and $h=x^n-y^q$ with $n,q=0\mod p$. We start with some general results about the relationship between $\langle f\rangle$-logarithmic and $\langle f^p\rangle$-logarithmic derivations. In this section, we denote $R=k[x_1,\ldots,x_d]$.

Proposition 2.5 *If $f,g\in R=k[x_1,\ldots,x_d]$ are coprime, then, for all $n\in\overline{\mathbb{N}}$, we have:*

$$\operatorname{HS}_k(\log fg;n)=\operatorname{HS}_k(\log f;n)\cap\operatorname{HS}_k(\log g;n).$$

Proof.

- $\supseteq$. Let $D\in\operatorname{HS}_k(\log f;n)\cap\operatorname{HS}_k(\log g;n)$. By definition, $D_i(f)\in\langle f\rangle$ and $D_i(g)\in\langle g\rangle$ for all $i\leq n$. Then $D_i(fg)=\sum_{a+b=i}D_a(f)D_b(g)\in\langle fg\rangle$, so $D\in\operatorname{HS}_k(\log fg;n)$.
- $\subseteq$. Let $D\in\operatorname{HS}_k(\log fg;n)$. This implies that $D_i(fg)\in\langle fg\rangle$ for all $i\leq n$. We will prove the result by induction on i. When $i=1$, then $D_1(fg)=D_1(f)g+D_1(g)f\in\langle fg\rangle\subseteq\langle f\rangle,\langle g\rangle$. So, $D_1(f)g\in\langle f\rangle$. Since g and f are coprime, $D_1(f)\in\langle f\rangle$. For g is analogous.

  Now let us assume that $D_i(f)\in\langle f\rangle$ and $D_i(g)\in\langle g\rangle$ for all $i<n$. By definition,

  $$D_n(fg)=D_n(f)g+D_n(g)f+\sum_{\substack{a+b=n\\a,b\neq 0}}D_a(f)D_b(g)\in\langle fg\rangle\Rightarrow D_n(f)g+D_n(g)f\in\langle fg\rangle$$

  and we can proceed like case $i=1$. □

Corollary 2.6 *If $f,g\in R$ are coprime, then $\operatorname{IDer}_k(\log fg;n)\subseteq\operatorname{IDer}_k(\log f;n)\cap\operatorname{IDer}_k(\log g;n)$ for all $n\in\overline{\mathbb{N}}$.*

Proof. If $\delta\in\operatorname{IDer}_k(\log fg;n)$ then, there exists $D\in\operatorname{HS}_k(\log fg;n)$ a n-integral of δ. By Proposition 2.5, $D\in\operatorname{HS}_k(\log f;n)\cap\operatorname{HS}_k(\log g;n)$ so, $\delta\in\operatorname{IDer}_k(\log f;n)\cap\operatorname{IDer}_k(\log g;n)$.

□

Remark 2.7 *In general, equality in Proposition 2.5 does not hold. For example: Let $k = \mathbb{F}_2$ and $f = y^2$ and $g = x^2 - y$ two polynomial of $k[x, y]$. Then $\partial_x \in \mathrm{IDer}_k(\log f; 4) \cap \mathrm{IDer}_k(\log g; 4)$. However $\partial_x \notin \mathrm{IDer}_k(\log fg; 4)$.*

Corollary 2.8 *Let $f_1, \ldots, f_m \in R$. If f_i, f_j are coprime whenever $i \neq j$, then, for all $\overline{\mathbb{N}}$ we have:*

$$\mathrm{HS}_k(\log f_1 \cdots f_m; n) = \bigcap_i \mathrm{HS}_k(\log f_i; n) \quad \textit{and} \quad \mathrm{IDer}_k(\log f_1 \cdots f_m; n) \subseteq \bigcap_i \mathrm{IDer}_k(\log f_i; n)$$

Proof. The result is obtained thanks to Proposition 2.5 and Corollary 2.6 by induction on m.

□

Lemma 2.9 *Let f be an irreducible polynomial, $a \geq 1$ and $n \in \overline{\mathbb{N}}$. Consider $D \in \mathrm{HS}_k(R; n)$. Suppose that $D_i(f^a)^p \in \langle f^{ap} \rangle$ for all $i \leq n$. Then, $D \in \mathrm{HS}_k(\log f^a; n)$.*

Proof.

We write $a = sp^\alpha$ where $\alpha = \mathrm{val}_p(a) \geq 0$ and $s \geq 1$. By Lemma 1.6,

$$D_i\left(f^{sp^\alpha}\right) = \begin{cases} 0 & \text{if} \quad p^\alpha \not| i \\ D_{i/p^\alpha}(f^s)^{p^\alpha} & \text{if} \quad p^\alpha | i \end{cases}$$

Hence, we can focus on the case $n \geq p^\alpha$ and $i = jp^\alpha \leq n$. It's enough to show that $D_j(f) \in \langle f \rangle$ because, if this is true, we have that $D_j(f^s) \in \langle f^s \rangle$ by Lemma 1.7, and $D_i\left(f^{sp^\alpha}\right) = D_j(f^s)^{p^\alpha} \in \left\langle f^{sp^\alpha} \right\rangle$ so we would have the result.

Since $i = jp^\alpha \leq n$,

$$D_j\left(f^s\right)^{p^{\alpha+1}} = D_{jp^\alpha}\left(f^{sp^\alpha}\right)^p \in \left\langle f^{sp^{\alpha+1}} \right\rangle \tag{5}$$

When $j = 1$, $D_1(f^s) = sf^{s-1}D_1(f)$. Substituting in the previous expression, we have that

$$D_1\left(f^s\right)^{p^{\alpha+1}} = sf^{(s-1)p^{\alpha+1}}D_1(f)^{p^{\alpha+1}} \in \left\langle f^{sp^{\alpha+1}} \right\rangle \tag{6}$$

Since R is UFD and $f, s \neq 0$, $D_1(f)^{p^{\alpha+1}} \in \left\langle f^{p^{\alpha+1}} \right\rangle \subseteq \langle f \rangle$ and hence $D_1(f) \in \langle f \rangle$.

Let us assume that $D_l(f) \in \langle f \rangle$ for all $l < j$ with $jp^\alpha \leq n$. Thanks to the hypothesis, we can use Lemma 1.7, and we have

$$D_j\left(f^s\right) = sf^{s-1}D_j(f) + Ff^s$$

for some $F \in R$. Substituting this expression in (5),

$$sf^{(s-1)p^{\alpha+1}}D_j(f)^{p^{\alpha+1}} + F^{p^{\alpha+1}}f^{sp^{\alpha+1}} \in \left\langle f^{sp^{\alpha+1}} \right\rangle \Rightarrow sf^{(s-1)p^{\alpha+1}}D_j(f)^{p^{\alpha+1}} \in \left\langle f^{sp^{\alpha+1}} \right\rangle$$

Observe that it is the same condition that (6), so we can deduce that $D_j(f) \in \langle f \rangle$. □

Proposition 2.10 *Let k be an UFD of characteristic $p > 0$ and $R = k[x_1, \ldots, x_d]$ the polynomial ring over k. Let h be a polynomial of R. For all $n \in \overline{\mathbb{N}}$, we have:*

$$\mathrm{IDer}_k(\log h; n) = \mathrm{IDer}_k\left(\log h^p, np\right).$$

Proof.

$\subseteq$. Let $D_1 \in \mathrm{IDer}_k(\log h; n)$ and $D \in \mathrm{HS}_k(\log h; n)$ an integral. If $n < \infty$, from Corollary 1.9, D is np-integrable, so let D' be a np-integral of D. If $n = \infty$, we put $D' = D$. Observe that $D'_1 = D_1$ so, if $D' \in \mathrm{HS}_k\left(\log h^p; np\right)$ then $D_1 \in \mathrm{IDer}_k(\log h^p; np)$. We have to see that $D'_i(h^p) \in \langle h^p \rangle$ for all $i \leq np$.

By Lemma 1.6,

$$D'_i\left(h^p\right) = \begin{cases} 0 & \text{if} \quad p \not| i \\ D'_{i/p}(h)^p & \text{if} \quad p | i \end{cases}$$

Then, we can focus on $i = jp$ where $1 \le j \le n$. Note that $D'_j = D_j$ for all $1 \le j \le n$, so

$$D'_i(h^p) = D'_j(h)^p = D_j(h)^p \in \langle h^p \rangle.$$

Therefore, $D'_i(h^p) \in \langle h^p \rangle$ for all $i \le np$ and we have the inclusion.

$\supseteq$. Let $D_1 \in \mathrm{IDer}_k(\log h^p; np)$ and $D \in \mathrm{HS}_k(\log h^p; np)$ a np-integral of D_1. Let $h = h_1^{a_1} \cdots h_m^{a_m}$ be the factorization of h in irreducible factors, i.e, h_i is irreducible and $a_i \ge 1$ for all $i = 1, \dots, m$ and $h_i \ne h_j$ if $i \ne j$. Then $h_i^{a_i}$ and $h_j^{a_j}$ are coprime whenever $i \ne j$, and therefore, $h_1^{a_1 p}, \dots, h_m^{a_m p}$ are coprime too. By Corollary 2.8,

$$D \in \mathrm{HS}_k(\log h^p; np) = \bigcap_i \mathrm{HS}_k\left(\log h_i^{a_i p}; np\right).$$

Hence, $D_j(h_i^{a_i})^p = D_{jp}(h_i^{a_i p}) \in \langle h_i^{a_i p} \rangle$ for $j \le n$. By Lemma 2.9, $D_j(h_i^{a_i}) \in \langle h_i^{a_i} \rangle$ for all $i = 1, \dots, m$, and $j \le n$. So, $\tau_{np,n}(D) \in \cap \mathrm{HS}_k(\log h_i^{a_i}; n) = \mathrm{HS}_k(\log h; n) \Rightarrow D_1 \in \mathrm{IDer}_k(\log h; n)$. □

Corollary 2.11 *For all* $\tau \ge 0$ *and* $n \in \overline{\mathbb{N}}$, $\mathrm{IDer}_k(\log h; n) = \mathrm{IDer}_k\left(\log h^{p^\tau}; np^\tau\right)$.

Proof. By induction on τ using Proposition 2.10. □

Proposition 2.12 *Let* k *be a UFD of characteristic* $p > 0$, $R = k[x_1, \dots, x_d]$ *the polynomial ring over* k, $h \in R$ *and* $\tau \ge 1$. *Then the set of the leaps of* $A := R/\left\langle h^{p^\tau} \right\rangle$ *is*

$$\begin{cases} \{np^\tau |\ n \text{ leap of } R/\langle h \rangle\} & \text{if } \mathrm{Der}_k(\log h) = \mathrm{Der}_k(R) \\ \{np^\tau |\ n \text{ leap of } R/\langle h \rangle\} \cup p^\tau & \text{if } \mathrm{Der}_k(\log h) \ne \mathrm{Der}_k(R) \end{cases}$$

Proof.

By Corollary 1.11, A has a leap on $s > 1$ if and only if the inclusion $\mathrm{IDer}_k\left(\log h^{p^\tau}; s-1\right) \supsetneq \mathrm{IDer}_k\left(\log h^{p^\tau}; s\right)$ is proper. First of all, we will prove the next two equalities:

1. For $s < p^\tau$, $\mathrm{IDer}_k\left(\log h^{p^\tau}; s\right) = \mathrm{Der}_k(R)$.

$\subseteq$ is always true. Let $D_1 \in \mathrm{Der}_k(R) = \mathrm{IDer}_k(R)$ and $D \in \mathrm{HS}_k(R)$ an integral. Since $s < p^\tau$, for all $j \le s$, $p^\tau \nmid j$. By Lemma 1.6, $D_j\left(h^{p^\tau}\right) = 0 \in \left\langle h^{p^\tau} \right\rangle$ for all $j \le s$. Then, any derivation D_1 has a h^{p^τ}-logarithmic s-integral and the other inclusion holds. So, A does not have a leap on s.

2. Let s be an integer such that $np^\tau < s < (n+1)p^\tau$ for some $n \ge 1$. Then $\mathrm{IDer}_k\left(\log h^{p^\tau}; s\right) = \mathrm{IDer}_k\left(\log h^{p^\tau}; np^\tau\right)$.

Since $s > np^\tau$, the inclusion $\subseteq$ is true. Let $D_1 \in \mathrm{IDer}_k\left(\log h^{p^\tau}; np^\tau\right)$. By definition there exists an integral $D \in \mathrm{HS}_k\left(\log h^{p^\tau}; np^\tau\right)$. By Corollary 1.9, we can consider $D' \in \mathrm{HS}_k(R; s)$ an integral of D. Hence, for all j such that $np^\tau < j \le s < (n+1)p^\tau$, $p^\tau \not| j$ and, by Lemma 1.6, $D'_j\left(h^{p^\tau}\right) = 0 \in \left\langle h^{p^\tau} \right\rangle$. Since $D'_l = D_l$ for all $l \le np^\tau$, $D' \in \mathrm{HS}_k\left(\log h^{p^\tau}; s\right)$. Therefore, $D_1 \in \mathrm{IDer}_k\left(\log h^{p^\tau}; s\right)$ and A does not have a leap on s.

Thanks to these two equalities we know that the leaps are given on $s = np^\tau$ for some $n \ge 1$. Let us suppose that $s = p^\tau$. By Corollary 2.11 and the point 1.,

$$\mathrm{Der}_k(R) = \mathrm{IDer}_k\left(\log h^{p^\tau}; s-1\right) \supseteq \mathrm{IDer}_k\left(\log h^{p^\tau}; p^\tau = s\right) = \mathrm{Der}_k(\log h)$$

Hence, A has a leap on p^τ if and only if $\mathrm{Der}_k(\log h) \ne \mathrm{Der}_k(R)$. Now, let us consider $s = np^\tau$ for $n \ge 2$. By Corollary 2.11 and the point 2.

$$\begin{aligned} \mathrm{IDer}_k(\log h; n-1) = &\ \mathrm{IDer}_k\left(\log h^{p^\tau}; (n-1)p^\tau\right) = \mathrm{IDer}_k\left(\log h^{p^\tau}; np^\tau - 1\right) \\ \supseteq &\ \mathrm{IDer}_k\left(\log h^{p^\tau}; np^\tau\right) = \mathrm{IDer}_k(\log h; n) \end{aligned}$$

Then, A has a leap on $s = np^\tau$ if and only if n is a leap on $R/\langle h \rangle$ and we have proved the result. □

Proposition 2.13 *Let k be a UFD of characteristic $p > 0$ and $h = x^n - y^q \in k[x, y]$. Suppose $\alpha := \mathrm{val}_p(n) \geq 1$ and $\beta := \mathrm{val}_p(q) \geq 1$. We write $\tau = \min \alpha, \beta \geq 1$, $s = n/p^\tau$ and $t = q/p^\tau$. Then,*

$$\mathrm{IDer}_k(k[x,y]/\langle h\rangle; np) = \left\{\overline{\delta}\,|\ \delta \in \mathrm{IDer}_k(\log\langle x^s - y^t\rangle, n)\right\}$$

where the leaps occur in $\{np^\tau |\ n$ is a leap of $k[x, y]/\langle H\rangle\} \cup p^\tau$.

Proof. Using Corollary 2.11 and Proposition 2.1. □

3 Other examples

We are going to calculate the integrable derivations of the quotient of a polynomial ring over some non-binomial equations. These examples have been taken from the article [Gr].

Example 1.

Let k be a domain of characteristic $p > 0$ and $h = x^p + tx^{p+1} \in R = k[x]$ with $t \in k$. Let $A = R/\langle h\rangle$. The module of $\mathrm{Der}_k(\log h)$ is generated by $(1 + tx)\partial_x$. From Example (2.1.2) of [Na2], we have that $(1 + tx)\partial_x$ is h-logarithmically $(p - 1)$-integrable. So, let us consider $E \in \mathrm{HS}_k(\log h; p - 1)$ an integral of $u(1 + tx)\partial_x$ where $u \in R$. From Corollary 1.9, there exists $D \in \mathrm{HS}_k(R)$ an integral of E. In order for D to be h-logarithmic,

$$D_p(x^p + tx^{p+1}) = D_1(x)^p + t(xD_1(x)^p + D_p(x)x^p) = u^p(1 + tx)^{p+1} + tD_p(x)x^p \in \langle h\rangle$$

So, $u \in \langle x\rangle$ and $\mathrm{IDer}_k(\log h; p) = \langle x(1 + tx)\partial_x\rangle$. Observe that this generator is ∞-integrable, for example $x \in A \mapsto x + x(1 + tx)\mu \in A[|\mu|]$ is an integral. In conclusion,

$$\mathrm{IDer}_k(A; i) = \begin{cases} \langle\overline{(1 + tx)\partial_x}\rangle & \text{if } i \leq p - 1 \\ \langle\overline{x(1 + tx)\partial_x}\rangle & \text{if } i \geq p \end{cases}$$

Example 2.

Let k be a domain of characteristic $p = 2$ and $h = x^4 + y^6 + y^7 \in R = k[x, y]$. Let $A = R/\langle h\rangle$. In this case, the module of h-logarithmic derivations is generated by ∂_x and $h\partial_y$. Since $h\partial_y$ is h-logarithmically ∞-integrable, we can focus on the h-logarithmically integrability of $u\partial_x$ where $u \in k[x, y]$. Let $\varphi : R \to R[|\mu|]$ a k-algebra homomorphism:

$$\begin{array}{rccl} \varphi: & R & \longrightarrow & R[|\mu|] \\ & x & \longmapsto & x + u\mu + u_2\mu^2 + \cdots \\ & y & \longmapsto & y \quad + \quad v_2\mu^2 + \cdots \end{array}$$

We want to see that there exist $u_i, v_i \in R$ for $i \geq 2$ such that φ is h-logarithmic. The coefficient of μ^i for $i = 2, 3$ in $\varphi(h)$ is $y^6 v_i$. In order for φ to be h-logarithmic, $v_i \in \langle h\rangle$, so we can put $v_i = 0$. In fact, we can put $v_i = 0$ for all i such that $4 \not| i$. Thanks to this, we can write:

$$\varphi(h) = (x + u\mu + u_2\mu^2 + \ldots)^4 + (y + v_4\mu^4 + v_8\mu^8 \ldots)^6(1 + y + v_4\mu^4 + v_8\mu^8 \ldots) \tag{7}$$

The coefficient of μ^4 in (7) is $\mu_4 := u^4 + y^6 v_4$ and it has to belong to $\langle h\rangle$. Hence, $u \in \langle x, y^2\rangle$ and $\mathrm{IDer}_k(\log h; 4) = \langle x\partial_x, y^2\partial_x, h\partial_y\rangle$. It's easy to proof the following lemma through the calculation of a term in the equation (7):

Lemma 3.1 *Suppose that $u_j = 0$ for all $j \geq 2$ and $v_{4n} \in \langle y^2\rangle$ for all $n < i$, then there exists $v_{4i} \in \langle y^2\rangle$ such that the coefficient of μ^{4i} in (7) belongs to $\langle h\rangle$.*

Using this lemma repeatedly we deduce that $y^2\partial_x$ and $xy\partial_x$ are h-logarithmically integrable since a possible solution so that μ_4 is h-logarithmic is $v_4 = y^2$ and $v_4 = (1 + y)y^4$ respectively. Therefore, we need to see the h-logarithmically integrability of $ux\partial_x$ where $u \in k[x]$. In this case, $v_4 \in (1 + y)u^4 + \langle h\rangle$. Calculating the coefficient of μ^8 in (7), we obtain $\mu_8 := u_2^4 + y^6 v_8 + v_4^2(1 + y)y^4$. In order for μ_8 to be in $\langle h\rangle$, $u \in \langle x\rangle$. Hence, $v_4 \in \langle x^4, h\rangle$. We deduce that $x^2\partial_x$ is h-logarithmically integrable by the following lemma:

Lemma 3.2 *Suppose that $u_j = 0$ for all $j \geq 2$ and $v_{4n} \in \langle x^4 \rangle$ for all $n < i$, then there exists $v_{4i} \in \langle x^4 \rangle$ such that the coefficient of μ^{4i} in (7) belongs to $\langle h \rangle$.*

In conclusion,

$$\mathrm{IDer}_k(A; i) = \begin{cases} \langle \overline{\partial_x} \rangle & \text{if } 1 \leq i < 4 \\ \langle \overline{x\partial_x}, \overline{y^2\partial_x} \rangle & \text{if } 4 \leq i < 8 \\ \langle \overline{x^2\partial_x}, \overline{xy\partial_x}, \overline{y^2\partial_x} \rangle & \text{if } i \geq 8 \end{cases}$$

Example 3.

Let k be a domain of characteristic $p = 3$ and $h = x^3 + y^5 + x^2y^2 \in R = k[x, y]$. Let $A = R/\langle h \rangle$. The module of h-logarithmic derivation is generated by $\delta_1 := x^2\partial_x + y^3\partial_y$ and $\delta_2 := 2y^2\partial_x + (x + y^2)\partial_y$. These two derivations are h-logarithmically integrable. To verify this claim, let us consider $\varphi : R \to R[|\mu|]$ a homomorphism of k-algebras

$$\begin{array}{rccc} \varphi : & R & \longrightarrow & R[|\mu|] \\ & x & \longmapsto & x + u_1\mu + u_2\mu^2 + \cdots \\ & y & \longmapsto & y + v_1\mu + v_2\mu^2 + \cdots \end{array}$$

As in the previous example, we want to prove that there exist $u_i, v_i \in R$ for $i \geq 2$ such that $\varphi(h) \in \langle h \rangle$ where u_1 and v_1 are determined by δ_1 or δ_2. By calculating a generic term of $\varphi(h)$, we can show the following lemmas:

Lemma 3.3 *Let $u_1 = x^2$ and $v_1 = y^3$. Suppose that $v_j = 0$ for all $j \geq 2$ and $u_n \in \langle x^2 \rangle$ for all $n < i$. Then, there exists $u_i \in \langle x^2 \rangle$ such that the coefficient of μ^i in $\varphi(h)$ belongs to $\langle h \rangle$.*

Lemma 3.4 *Let $u_1 = 2y^2$ and $v_1 = x + y^2$. Suppose $u_n \in \langle xy, y^3 \rangle$ and $v_n \in \langle y^2 \rangle$ for all $2 \leq n < i$. Then, there exist $u_i \in \langle xy, y^3 \rangle$ and $v_i \in \langle y^2 \rangle$ such that the coefficient of μ^i in $\varphi(h)$ belongs to $\langle h \rangle$.*

Using Lemma 3.3 for the integrability of δ_1 and Lemma 3.4 for the integrability of δ_2, we can deduce that

$$\mathrm{IDer}_k(A) = \langle \overline{\delta_1}, \overline{\delta_2} \rangle.$$

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