

An infinitesimal approach to the study of Cycles on Abelian Varieties

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Abstract

This paper is a work in progress on Bloch’s conjecture asserting the vanishing of the Pontryagin product of a p codimensional cycle on an abelian variety by $p+1$ zero cycles of degree zero. We prove an infinitesimal version of the conjecture and we discuss, in particular, the case of 3 dimensional cycles.

Key Words: Abelian variety; Algebraic Cycles; Bloch’s Conjecture; Beauville’s Conjecture; Chow Group; Chow Ring; Rational equivalence.

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In this paper we investigate the geometry of the Chow ring of an abelian variety. Let A be a complex abelian variety of dimension n and denote by $CH_{\bullet}(A)$ its Chow group of algebraic cycles modulo rational equivalence, graded by dimension. Recall that $CH_{\bullet}(A)$ has two ring structures: the first one is given by the intersection product, which we shall denote by “ \cdot ”, the second one is given by the Pontryagin product, which we shall always denote by “ \star ”. Observe that the subgroup of zero cycles $CH_0(A)$ is a subring of $CH_{\bullet}(A)$ under the Pontryagin ring structure; we shall denote by I the augmentation ideal, namely the ideal of zero cycles of degree zero. In [Bl], Bloch conjectures a natural generalization to higher codimension of the celebrated square theorem: if $Y \in CH_k(A)$, then $I^{\star n-k+1} \star Y = 0$ ($I^{\star d}$ denotes the d^{th} -power of I under Pontryagin product). Observe that for $k = n-1$, the cycle Y is a divisor and the formula above particularizes to the cited square theorem: $Y_{a+b} - Y_a - Y_b + Y = 0$, $\forall a, b \in A$, where Y_x denotes the translate of Y by x . For $k = n$ the formula above is trivial. Bloch proves his conjecture for $k = 0, 1, 2$. After Bloch’s results, a great improvement in the understanding of the structure of the Chow ring has been given by Beauville ([Be1], [Be2]): working with rational coefficients, by the use of Fourier theory he gave a decomposition $CH_{\bullet}(A) \otimes \mathbb{Q} = \bigoplus_{d,s} [CH_d(A)]_s$ and conjectured that negative pieces (those one with $s < 0$) should be trivial, where cycles of Beauville’s degree s are defined by putting $[CH_d(A)]_s = \{\alpha \in CH_d(A) \otimes \mathbb{Q} \mid \text{mult}(m)^* \alpha = m^{2(n-d)-s} \alpha\}$. In spite of the great achievement and clarification about the structure of the Chow ring

he obtained, he only proved that negative pieces of his decomposition are trivial in the range of dimension ≤ 2 and codimension ≤ 1 , the same range of Bloch's results, outside of this range the main problems are still open. It seems there are no approaches that can manage the higher dimension/codimension situation (namely dimension ≥ 3 and codimension ≥ 2). We may stress that the first unknown result concerns 3 dimensional cycles of a 5 dimensional abelian variety.

This paper is a work in progress on Bloch's conjecture: we prove an infinitesimal version of the conjecture and we discuss, in particular, the case where $k = 3$; we also show some further results and connections with the Fourier transform associated with the Poincaré bundle. More in detail, considering the formula

$$(1) \quad \text{Pic}^0(A)^k \cdot CH_k(A) \subseteq I^{\star k},$$

we shall prove the following:

- at least for $k = 3$, formula (1) implies Bloch's conjecture (see Proposition 16);
- an infinitesimal version of formula (1) holds (see Proposition 11).

We may note that formula (1) is trivial for $k \leq 2$. In fact, the intersection of a divisor in $\text{Pic}^0(A)$ with a 1-cycle is a 0-cycle of degree zero (case $k = 1$); furthermore, $I^{\star 2}$ is the kernel of the abelian sum map $\text{Sum} : I \rightarrow A$ and, by rigidity, the intersection between a 2-cycle and two divisors in $\text{Pic}^0(A)$ is a zero cycle of degree zero whose abelian sum must be zero (this take care of the case $k = 2$).

We also generalize a Mumford's result concerning the countability of components of the rational orbits of the difference map ([Mu]), cfr. Lemma (12), that may help in our situation, cfr. Lemma (14).

§1 An infinitesimal version of formula (1).

Let A be as above, fix an identification $A = \mathbb{C}^n/\Lambda$ and denote by $\alpha : \mathbb{C}^n \rightarrow A$ the natural projection. Let W denote an irreducible and reduced subvariety of dimension k . Then consider a very ample linear system $|D|$ and choose divisors $D_1, \dots, D_k \in |D|$ such that the intersection $D_1 \cap \dots \cap D_k \cap W$ is transverse, in particular it consists of distinct points. Let r denote the degree of this intersection, set

$$\{p_1, \dots, p_r\} := D_1 \cap \dots \cap D_k \cap W$$

and fix once for all the ordering of the p_i 's. For a divisor D on A and $a \in A$, we denote with $t_a D$ the translate of D by a , i.e. $t_a D = D \star \{a\}$. It is convenient translating by points in the universal covering \mathbb{C}^n rather than in A , so we also put $t_z D = t_{\alpha(z)} D$ for $z \in \mathbb{C}^n$. Now let $U := \{z \in \mathbb{C}^n \mid \|z\| < \epsilon_0\}$ and fix ϵ_0 such that the intersection $t_{u_1} D_1 \cap \dots \cap t_{u_k} D_k \cap W$ is transverse for all $u_1, \dots, u_k \in U$, shrinking U if necessary we also assume that the set of all the intersections above consists of r connected components. Each of such components contains exactly one of the p_i 's, this fact allow us to define the holomorphic map

$$U^k \longrightarrow A^r$$

$$\underline{u} := (u_1, \dots, u_k) \mapsto (p_1(\underline{u}), \dots, p_r(\underline{u}))$$

where the $p_i(\underline{u})$'s are the points of the intersection $t_{u_1}D_1 \cap \dots \cap t_{u_k}D_k \cap W$, ordered in such a way to match the previous choice concerning the ordering of the p_i 's.

Local results.

Fix one of the p_i 's above and denote it by p . As the intersection $D_1 \cap \dots \cap D_k \cap W$ is 0-dimensional and transverse at p , the tangent space $T_p(A)$ decomposes as

$$(2) \quad T_p(A) = L_1 \oplus \dots \oplus L_k \oplus N ,$$

where

$$L_j := \bigcap_{i \neq j} T_p(D_i) \cap T_p(W) \quad \text{and} \quad N := \bigcap_i T_p(D_i) .$$

We may note that $\dim L_j = 1, \forall j$ and $\dim N = n - k$. Furthermore,

$$T_p(D_j) = \bigoplus_{i \neq j} L_i \oplus N , \quad T_p(W) = \bigoplus_j L_j ,$$

thus L_j is a complement of $T_p(D_j)$ and N is a complement of $T_p(W)$.

Now let $\tilde{\pi}_j$ denote the projections as indicated in the commutative diagram

$$\begin{array}{ccccccc} T_p(A) & \longrightarrow & L_j & \subseteq & T_p(A) & & \\ \phi \downarrow & & & & \downarrow \phi & & \\ \mathbb{C}^n & \xrightarrow{\tilde{\pi}_j} & \mathbb{C}^n & \xrightarrow{\alpha} & \frac{\mathbb{C}^n}{\Lambda} = A & & \end{array}$$

where the map at the first row is the projection associated with the decomposition (2) and ϕ is the natural isomorphism defined by the canonical identifications of vector spaces $T_p(A) = T_0(A) = \mathbb{C}^n$. Similarly, we define $\tilde{\pi}_N : \mathbb{C}^n \rightarrow \mathbb{C}^n$ as the morphism induced by the projection $T_p(A) \rightarrow N$. Furthermore, we define

$$\pi_j = \alpha \circ \tilde{\pi}_j , \quad \pi_N = \alpha \circ \tilde{\pi}_N .$$

Let $\underline{u} = (u_1, \dots, u_k) \in U^k$ and $p(\underline{u})$ be as previously defined. Also, consider a series expansion of $p(\underline{u})$ and define $p(\underline{u})_i := i$ -order-terms of $p(\underline{u})$. Then clearly $p(\underline{u})_0 = p$ does not depend on \underline{u} .

Lemma 3. *We have $p(\underline{u})_1 = \sum_{j=1}^k \pi_j(u_j)$, i.e.*

$$p(\underline{u}) = p + \sum_{j=1}^k \pi_j(u_j) + \text{higher-order-terms}$$

(in particular, only moving D_i in the L_i -direction affects the local contribution near $p = p_i$ of the first order approximation of the intersection $t_{u_1}D_1 \cap \dots \cap t_{u_k}D_k \cap W$).

Proof: As we are interested to the first-order-approximation of $p(\underline{u})$, we are free to work with tangent spaces:

$$\begin{aligned} p(\underline{u}) &= p + \alpha \left((\phi(T_p(D_1)) + u_1) \cap \dots \cap (\phi(T_p(D_k)) + u_k) \cap \phi(T_p(W)) \right) \\ &\quad + \text{higher-order-terms.} \end{aligned}$$

Since by definition $T_p(D_j) = \oplus_{i \neq j} L_i \oplus N$ and $T_p(W) = \oplus_i L_i$, then

$$(3.1) \quad \begin{aligned} \tilde{\pi}_i(u_j) &\in \phi(T_p(D_j)), \quad \forall i \neq j; \\ \tilde{\pi}_N(u_j) &\in \phi(T_p(D_j)), \quad \forall j. \end{aligned}$$

By the decomposition $T_p(A) = L_1 \oplus \dots \oplus L_k \oplus N$ and by (3.1) we obtain $\phi(T_p(D_j)) + u_j = \phi(T_p(D_j)) + \tilde{\pi}_1(u_j) + \dots + \tilde{\pi}_k(u_j) + \tilde{\pi}_N(u_j) = \phi(T_p(D_j)) + \tilde{\pi}_j(u_j)$. Therefore,

$$\begin{aligned} &(\phi(T_p(D_1)) + u_1) \cap \dots \cap (\phi(T_p(D_k)) + u_k) \cap \phi(T_p(W)) \\ &= (\phi(T_p(D_1)) + \tilde{\pi}_1(u_1)) \cap \dots \cap (\phi(T_p(D_k)) + \tilde{\pi}_1(u_k)) \cap \phi(T_p(W)) \\ &= \sum_{j=1}^k \tilde{\pi}_j(u_j) \end{aligned}$$

where the last equality follows again by (3.1). This concludes our proof. \square

We now introduce some further notation. Denote the k -cartesian product of the set $\{0, 1\}$ by Φ , for $\sigma = (\sigma_1, \dots, \sigma_k) \in \Phi$ and $\underline{a} = (a_1, \dots, a_k) \in A^k$ we define $\underline{a}\sigma := (\dots, a_j\sigma_j, \dots)$, namely multiplication by σ replaces a_j with 0 whenever $\sigma_j = 0$. We also define $|\sigma|$ as the number of zeros appearing in σ . As usual, we denote by $Z_0(A)$ the free abelian group generated by points of A and, for $x \in A$, we denote by $\{x\}$ the corresponding cycle in $Z_0(A)$. We denote by \tilde{I} the ideal of $Z_0(A)$ of 0-cycles of degree zero and by $\tilde{I}^{\star k}$ its k -Pontryagin-power (as far it concerns 0-cycles, the Pontryagin product is defined even if one does not quotient by rational equivalence). Note that $\tilde{I}^{\star k}$ is the ideal of $Z_0(A)$ generated by the image of A^k , via the natural map $(a_1, \dots, a_k) \mapsto (\{a_1\} - \{0\}) \star \dots \star (\{a_k\} - \{0\})$.

By a slight abuse of notation we use the same term ‘‘cycle’’ to indicate an element in $Z_\bullet(A)$ as well as to indicate its associated class in $CH_\bullet(A)$. No confusion will occur since we shall always specify where to consider our ‘‘cycle’’.

Remark 4. For $\underline{u} = (u_1, \dots, u_k) \in U^k$, the contribution near p of the intersection

$$(t_{u_1} D_1 - D_1) \cap \dots \cap (t_{u_k} D_k - D_k) \cap W$$

is just the zero cycle

$$\zeta_p(\underline{u}) := \sum_{\sigma \in \Phi} (-1)^{|\sigma|} \{p(\underline{u}\sigma)\} \in Z_0(A)$$

(The word ‘‘near’’ means ‘‘in the connected component of containing p ’’, see the definition of U given at the beginning of this section).

Proof: Straightforward. \square

It is convenient to our purposes to see our contribution as an element in $\text{Hom}(\Phi, A)$, namely to factorize ζ_p as

$$(5) \quad \zeta_p = \delta' \circ \nu_p$$

where the maps ν_p and δ' are defined as follows

$$\begin{array}{ccccc} U^k & \xrightarrow{\nu_p} & \text{Hom}(\Phi, A) & \xrightarrow{\delta'} & Z_0(A) \\ \underline{u} & \mapsto & (\sigma \mapsto p(\underline{u}\sigma)) & & \\ & & f & \mapsto & \sum_{\sigma \in \Phi} (-1)^{|\sigma|} \{f(\sigma)\} \end{array}$$

The class under rational equivalence of the local contribution $\zeta_p(\underline{u})$ might not belong to $I^{\star k}$, nonetheless we shall see (proposition (6) below) that the first-order approximation of $\zeta_p(\underline{u})$ even belongs to $\tilde{I}^{\star k}$ (as far it concern the approximation to the first order we need not to take the quotient under rational equivalence).

We now come to an elementary proposition, or rather a trivial consequence of lemma (3):

Proposition 6. *Let $\underline{u} = (u_1, \dots, u_k) \in U^k$ and let $p(\underline{u})_{\leq 1} := p(\underline{u})_0 + p(\underline{u})_1$ be the truncation of $p(\underline{u})$ to the first order. Then*

$$\sum_{\sigma \in \Phi} (-1)^{|\sigma|} \{p(\underline{u}\sigma)_{\leq 1}\} \in \tilde{I}^{\star k}$$

where, as usual, \star denotes the Pontryagin product.

Proof: First of all we may note that by lemma (3) and by the definition of the multiplication by σ the following holds

$$(6.1) \quad p(\underline{u}\sigma)_1 = \sum_{j=1}^k \pi_j(u_j \sigma_j) = \sum_{j=1}^k \pi_j(u_j) \sigma_j$$

We have

$$\begin{aligned} \sum_{\sigma \in \Phi} (-1)^{|\sigma|} \{p + p(\underline{u}\sigma)_1\} &= \{p\} \star \sum_{\sigma \in \Phi} (-1)^{|\sigma|} \{p(\underline{u}\sigma)_1\} \\ &= \{p\} \star \sum_{\sigma \in \Phi} (-1)^{|\sigma|} \left\{ \sum_{j=1}^k \pi_j(u_j) \sigma_j \right\} \\ &= \{p\} \star \left((\{\pi_1(u_1)\} - \{0\}) \star \dots \star (\{\pi_k(u_k)\} - \{0\}) \right) \\ &\in \{p\} \star \tilde{I}^{\star k} = \tilde{I}^{\star k} \end{aligned}$$

where the second equality follows by (6.1) and the third equality follows by the straightforward identity

$$\sum_{\sigma \in \Phi} (-1)^{|\sigma|} \{a_j \sigma_j\} = (\{a_1\} - \{0\}) \star \dots \star (\{a_k\} - \{0\}),$$

for all $(a_1, \dots, a_k) \in A^k$.

□

By the previous proof, introducing the map

$$\begin{aligned} \lambda_p : T^k &\longrightarrow \text{Hom}(\Phi, A) \\ \underline{a} &\mapsto \left(\sigma \mapsto p + \sum_{j=1}^k \pi_j(a_j) \sigma_j \right) \end{aligned}$$

where $T = \mathbb{C}^n$ denotes the universal covering of A and $\underline{a} = (a_1, \dots, a_k)$, we obtain

$$\delta' \circ \lambda_p(\underline{a}) = \{p\} \star \star_{j=1}^k \left(\{\pi_j(a_j)\} - \{0\} \right).$$

We want to stress that the various π_j appearing above depend on decomposition (2), thus they depend on our $p = p_\iota$.

Global results.

So far we have considered contributions near some point p of the intersection at remark (4). Now we put these contributions all together, namely we define

$$\begin{aligned} \nu : U^k &\longrightarrow \text{Hom}(\Phi, A)^r, \quad \nu(\underline{u}) := (\nu_{p_1}(\underline{u}), \dots, \nu_{p_r}(\underline{u})) \\ \zeta : U^k &\longrightarrow Z_0(A), \quad \zeta(\underline{u}) := \sum_{\iota=1}^r \zeta_{p_\iota}(\underline{u}) \\ \lambda : T^k &\longrightarrow \text{Hom}(\Phi, A)^r, \quad \lambda(\underline{a}) := (\lambda_{p_1}(\underline{a}), \dots, \lambda_{p_r}(\underline{a})) \\ \delta : \text{Hom}(\Phi, A)^r &\longrightarrow Z_0(A), \quad \delta(f_1, \dots, f_r) := \sum_{\iota=1}^r \delta'(f_\iota). \end{aligned}$$

We want to stress that by the definition given in remark (4) we have

$$\zeta(\underline{u}) = (t_{u_1} D_1 - D_1) \cap \dots \cap (t_{u_k} D_k - D_k) \cap W$$

and formula (5) becomes

$$(7) \quad \zeta = \delta \circ \nu,$$

namely we have a commutative diagram

$$(7') \quad \begin{array}{ccc} U^k & \xrightarrow{\zeta} & Z_0(A) \\ \nu \downarrow & & \nearrow \delta \\ \text{Hom}(\Phi, A)^r & & \end{array}$$

We may note that $\text{Hom}(\Phi, A)^r$ is an abelian variety (in fact its elements are sets of $r \cdot 2^k$ points in A). Furthermore, the vertical map is holomorphic and, in the sense it

will be clarified at notation (8) and remark (9) below (that we will introduce also for other purposes) the map δ is Mumford's difference map.

Observe that, by remark (4), the composition of ζ with the natural projection $cl : Z_0(A) \rightarrow CH_0(A)$ is the restriction to U^k of the intersection map

$$\begin{aligned} \xi : \quad A^k &\longrightarrow CH_0(A) \\ (a_1, \dots, a_k) &\mapsto (t_{a_1}D - D) \bullet \cdots \bullet (t_{a_k}D - D) \bullet W \end{aligned}$$

(here it is not necessary to distinguish the various $D_j \in |D|$, cfr. our former setting) which is a group-morphism on each factor by the square theorem.

Notation 8. Let $B := \text{Hom}(\Phi, A)$, we identify

$$B^r \cong (B^+)^r \times (B^-)^r \cong A^{r \cdot 2^{k-1}} \times A^{r \cdot 2^{k-1}}$$

where $B^+ := \text{Hom}(\Phi^+, A)$, $B^- := \text{Hom}(\Phi^-, A)$, $\Phi^+ := \{ \sigma \in \Phi \mid (-1)^{|\sigma|} = 1 \}$, $\Phi^- := \{ \sigma \in \Phi \mid (-1)^{|\sigma|} = -1 \}$ (we may note that $\Phi = \Phi^+ \cup \Phi^-$).

Remark 9. The map δ takes (f_1, \dots, f_r) to $\sum_i \sum_{\sigma \in \Phi} (-1)^{|\sigma|} \{ f_i(\sigma) \}$, thus under the previous identifications it is the difference map

$$\begin{aligned} d : \quad A^m \times A^m &\longrightarrow Z_0(A) \\ (\underline{z}', \underline{z}'') &\mapsto \sum_{j=1}^m \{ z'_j \} - \{ z''_j \} \end{aligned}$$

(here $m = r2^{k-1}$ and the notation is the obvious one: $\underline{z}' = (z'_1, \dots, z'_m)$ etcetera).

Recalling our identification $A^m \times A^m \cong \text{Hom}(\Phi^+ \cup \Phi^-, A)^r$, $m = r2^{k-1}$, let us look at the diagram

$$(10) \quad \begin{array}{ccccc} U^k & \hookrightarrow & A^k & \xrightarrow{\xi} & CH_0(A) & \xrightarrow{\mu} & \frac{CH_0(A)}{I^{\star k}} \\ & & \nu \downarrow & & \uparrow cl & & \\ T^k & \xrightarrow{\lambda} & A^m \times A^m & \xrightarrow{\delta} & Z_0(A) & & \end{array}$$

where $\mu : CH_0(A) \rightarrow \frac{CH_0(A)}{I^{\star k}}$ is the natural projection. In the following proposition we collect a few properties of the diagram (10):

Proposition 11. For $a_1, \dots, a_k \in A$ we have

$$(11.1) \quad \xi(a_1, \dots, a_k) = (t_{a_1}D - D) \bullet \cdots \bullet (t_{a_k}D - D) \bullet W,$$

elements of this type generate $\text{Pic}^0(A)^k \bullet CH_k(A)$. Furthermore, the following properties hold:

a) the diagram (10) commutes: $\xi|_{U^k} = cl \circ \delta \circ \nu$;

- b) ξ is a group-morphism on each factor of $A^k = A \times \dots \times A$;
c) ν is holomorphic, it is a concrete realization of the intersection at the r.h.s. of (11.1), the first order approximation of $\nu(a_1, \dots, a_k)$ is $\lambda(a_1, \dots, a_k)$, in particular we have an inclusion of tangent spaces $\nu_* T_0(U^k) \subseteq \lambda(T^k)$;
d) $\mu \circ cl \circ \delta \circ \lambda = 0$.

Proof: Property (a) has already been proved; property (b) trivially follows by the square theorem; properties (c) and (d) follow by lemma (3) and proposition (6). \square

We want to stress that on one hand we have maps

$$\begin{array}{ccccc} & & U^k & & \\ & & \downarrow \nu & & \\ T^k & \xrightarrow{\lambda} & B^r & \xrightarrow{\mu \circ cl \circ \delta} & \frac{CH_0(A)}{I^{\star k}} \end{array}$$

where the bottom row vanishes, on the other hand the composition $\mu \circ cl \circ \delta \circ \nu$ is (the restriction of) an additive map on each factor. As a consequence, by property c), if $CH_0(A)$ had a good topology, or better if its quotient by $I^{\star k}$ had a good topology, formula (1) would easily follow (and, in turn, Bloch's conjecture would follow).

We now recall a fundamental result concerning the chow group $CH_0(A)$ due to Mumford, then we generalize it to the quotient $\frac{CH_0(A)}{I^{\star k}}$.

Lemma 12. See [Mu]. Let A be an abelian variety, m a positive integer and consider the difference map (the notation is the usual one)

$$\begin{aligned} d : A^m \times A^m &\longrightarrow CH_0(A) \\ (\underline{z}', \underline{z}'') &\mapsto \sum_{j=1}^m \{z'_j\} - \{z''_j\} \end{aligned}$$

Then each fibre of d is a countable union of algebraic subsets.

By observing that $cl \circ \delta$ from diagram (10) is the difference map d up to the identification

$$B^r \cong A^{r \cdot 2^{k-1}} \times A^{r \cdot 2^{k-1}}$$

(cfr. notation 8 and remark 9), we obtain the following immediate corollary:

Corollary 13. Each fibre of the composition $cl \circ \delta$ from diagram (10) is a countable union of algebraic subsets.

We now come to a generalization of Mumford's lemma.

Lemma 14. Let A be an abelian variety and let m, k be integers. Then each fibre of the composition

$$A^m \times A^m \xrightarrow{d} CH_0(A) \xrightarrow{\mu} \frac{CH_0(A)}{I^{\star k}}$$

is a countable union of algebraic subsets (c.u.a.s.). Here, as usual, d is the difference map and μ is the natural projection.

Proof: The idea of proof is elementary: calling “generator” any Pontryagin product of k elements in I , we show that for $z_0 \in CH_0(A)$ and $t \geq 1$, the set of elements in $A^m \times A^m$ whose image in $CH_0(A)$ belongs to the same class of z_0 up to the sum of t generators is a c.u.a.s..

Keeping the notation (8), let $z_0 \in CH_0(A)$, let $t \geq 1$ be an integer and consider

$$\begin{aligned} \varphi : A^m \times A^m \times (A^k)^t &\longrightarrow A^m \times (B^+)^t \times A^m \times (B^-)^t \\ (z_1 ; z_2 ; \dots, \underline{x}_i, \dots) &\mapsto (z_1 ; \dots, \mathcal{F}_k^+(\underline{x}_i), \dots ; z_2 ; \dots, \mathcal{F}_k^-(\underline{x}_i), \dots) \end{aligned}$$

where $\underline{x}_1, \dots, \underline{x}_t \in A^k$, for $\underline{x} \in A^k$ one defines $\mathcal{F}_k^+(\underline{x})$ to be the map defined on Φ^+ that takes σ to $\underline{x}\sigma$ and $\mathcal{F}_k^-(\underline{x})$ is defined in the same way (but now the σ 's run over a different set, namely Φ^-). Furthermore, denote by π the natural projection $A^m \times (B^+)^t \times A^m \times (B^-)^t \longrightarrow A^m \times A^m$. Now set

$$W_t := \{ (z_1, \omega_1; z_2, \omega_2) \in A^m \times (B^+)^t \times A^m \times (B^-)^t \mid z_1 + \omega_1 = z_2 + \omega_2 + z_0 \in CH_0(A) \}$$

where ω_1 and ω_2 are considered as effective cycles under the isomorphisms $(B^+)^t \cong (A^{2k-1})^t \cong (B^-)^t$. Also set

$$V := \pi \left(\bigcup_{t=1}^{\infty} W_t \cap \text{image } \varphi \right).$$

As W_t is a c.u.a.s. by Mumford's lemma, then V is a c.u.a.s.. Furthermore, V is the fibre $V = (\mu \circ d)^{-1}([z_0])$. In fact, on one hand the left inclusion “ \subseteq ” is clear, on the other hand, if $z_1 - z_2 \equiv z_0$ modulo $I^{\star k}$, then we can write

$$\begin{aligned} z_1 - z_2 - z_0 &= cl \left(\sum_{i=1}^t (\{a_{i,1}\} - \{0\}) \star \dots \star (\{a_{i,k}\} - \{0\}) \right) \\ &= cl \left(\sum_{i=1}^t \left(\sum_{\sigma \in \Phi^+} \{x_i \sigma\} - \sum_{\sigma \in \Phi^-} \{x_i \sigma\} \right) \right) \end{aligned}$$

for some t and some $\underline{x}_i \in A^k$. □

As Mumford's lemma tell us that the fibres of $cl \circ \delta$ are c.u.a.s. (corollary 13), lemma (14) tell us the following:

Corollary 15. *Each fibre of the composition $\mu \circ cl \circ \delta$ from diagram (10) is a countable union of algebraic subsets.*

§2 Reduction of Bloch's conjecture to formula (1).

We now consider Bloch's conjecture for 3 dimensional cycles, namely formula

$$(BC) \quad CH_3(A) \star I^{\star n-2} = 0, \quad \text{where } n := \dim A.$$

We prove that formula (1) implies Bloch's conjecture (BC), namely it is the particular case where $k = r = 0$ of Proposition (16) below. We remained that, unfortunately, we were only able to prove that formula (1) holds at the level of first order approximation.

Notation. For F and G subgroups of $CH_\bullet(A)$ we denote by $F \bullet G$ the subgroup generated by intersections YZ , where $Y \in F$, $Z \in G$; we also define F^r to be $F^r := F \cdots F$ (r -times). Furthermore, for a cycle $W \in CH_\bullet(A)$, we let $W \bullet F$ (as well as $F \bullet W$) denote the group of intersections $W \bullet Y$, $Y \in F$. The groups $F \star G$, $F \star^r$ and $W \star F$ are defined in a similar way.

For the sequel, we fix an ample symmetric divisor D , we put $\mathcal{C} = D^{n-1}$ (namely \mathcal{C} is an ample 1-cycle), and for non-negative integers j, k, r we put

$$\mathcal{N}_{j,k,r} := I \star^j \star (\text{Pic}^0(A)^k \bullet CH_3(A)) \star \mathcal{C} \star^r,$$

where $\text{Pic}^0(A)^0 = [A]$ (the unit for the intersection product) and $I \star^0 = \mathcal{C} \star^0 = \{o\}$ (the unit for the Pontryagin product).

We recall that by Bloch's proposition [Bl, 4.2] we have $\mathcal{C} \star I \star^n = 0$. In fact, more in general:

$$(\spadesuit) \quad \mathcal{C} \star^{n-j+1} \star I \star^j = 0, \quad j = 0, 1, 2, \dots, n+1.$$

Proof: This result is elementary and well-known, however we remind its proof. As intersection with divisors in $\text{Pic}^0(A)$ is a derivation for the Pontryagin product one has

$$0 = E_1 \bullet \dots \bullet E_j \bullet (\mathcal{C} \star^{n+1}) = \binom{n+1}{j} \sigma_1 \star \dots \star \sigma_j \star \mathcal{C} \star^{n+1-j},$$

where $E_i \in \text{Pic}^0(A)$, $\sigma_i = E_i \bullet \mathcal{C} \in I$. As $\text{Pic}^0(A)$ is divisible, such expressions generate $\mathcal{C} \star^{n-j+1} \star I \star^j$ and we are done. \square

Proposition 16. *Let A be an abelian variety of dimension n . Assuming*

$$\text{Pic}^0(A)^3 \bullet CH_3(A) \subseteq I \star^3$$

we then have

$$(\star) \quad \mathcal{N}_{j,k,r} = 0, \quad \forall j+r \geq n-2 \quad (j, k, r \text{ non-negative integers}).$$

As already mentioned, for $k = r = 0$ (and $j = n-2$) we obtain Bloch's vanishing

$$CH_3(A) \star I \star^{n-2} = 0.$$

Defining $\mathcal{N}_{j,k,r,d}$ by replacing $CH_3(A)$ with $CH_d(A)$, namely

$$\mathcal{N}_{j,k,r,d} := I \star^j \star (\text{Pic}^0(A)^k \bullet CH_d(A)) \star \mathcal{C} \star^r = 0, \quad \forall j+r \geq n-d+1,$$

proposition (16) is the first not-known case of the following:

Conjecture 17. Let A be an abelian variety of dimension n , then formula (1) implies

$$\mathcal{N}_{j,k,r,d} = 0, \quad \forall j+r \geq n-d+1.$$

It is interesting (after considering rational coefficients) to look at things in the setting of Beauville's theory. First, (1) is equivalent to the vanishing of

$$\mathrm{Pic}^0(A)^k \bullet [CH_k(A)_{\mathbb{Q}}]_s = 0, \quad \text{for } s \leq -1,$$

vanishing that would trivially follow by Beauville's conjecture stating that

$$[CH_d(A)_{\mathbb{Q}}]_s = 0, \quad \text{for } s \leq -1$$

(s denotes Beauville's degree, cfr. [Be1]). Indeed, Beauville's conjecture would directly imply Bloch conjecture as well the vanishing (\star) from Proposition (16) and, more in general, the vanishing of $\mathcal{N}_{j,k,r,d}$ (in the correct range, cfr. Conjecture 17).

In fact, on one hand, it is not restrictive working with rational coefficients: as $\mathrm{Pic}^0(A)$ is divisible, also I is divisible, being $I/I^{\star 2}$ divisible and being $I^{\star n+1} = 0$. On the other hand, since $\mathrm{Pic}^0(A) \otimes \mathbb{Q} = [CH_{n-1}(A)_{\mathbb{Q}}]_1$, $I \otimes \mathbb{Q} = [CH_0(A)_{\mathbb{Q}}]_{\geq 1}$ and $C \otimes 1 \in [CH_1(A)_{\mathbb{Q}}]_0$, our set $\mathcal{N}_{j,k,r}$ from (\star) , after tensoring with \mathbb{Q} , is contained in $[CH_{3-k+r}(A)_{\mathbb{Q}}]_{\geq j+k}$. Finally, since $j+r \geq n-2$, i.e. $j+k > n-(3+r-k)$, assuming that Beauville's conjecture hold we get $[CH_{3-k+r}(A, \mathbb{Q})]_{\geq j+k} = 0$.

We want to stress that Beauville proved his conjecture for $d = 0, 1, 2$, again the case where $d = 3$ is the first case where the main question is open.

Proof: (of prop. 16). We shall perform an induction that uses r and k . First, we need to deal with the "boundary" cases. We want to prove the following

$$(16_a) \quad (\star) \text{ holds provided that } k \geq 3;$$

$$(16_b) \quad (\star) \text{ holds provided that } 3-k+r \geq n-1.$$

Property (16_a) holds by trivial reasons of dimension in case $k > 3$. In the remaining case, namely the case $k = 3$, as by the hypothesis $\mathrm{Pic}^0(A)^3 \bullet CH_3(A) \subseteq I^{\star 3}$, for $j+r \geq n-2$ we then have $\mathcal{N}_{j,k,r} \subseteq I^{\star 3+j} \star C^{\star r} = 0$ (by \spadesuit).

As for property (16_b) we first observe that, for trivial reasons of dimension, it holds for $3-k+r > n$. So we distinguish the two cases $3-k+r = n$ and $3-k+r = n-1$. Consider $\sigma \in \mathcal{N}_{j,k,r}$, where $j+r \geq n-2$. The first case is trivial: as $j+r \geq n-2$ and $3-k+r = n$, then $j+k \geq 1$ and therefore σ is algebraically trivial. On the other hand $\dim \sigma = 3-k+r = n$, so σ must be trivial. As for the case where $3-k+r = n-1$ we first observe that our cycle σ is a divisor and that we must have $j+k = j+3+r-n+1 \geq 2$ (because $j+r \geq n-2$), then putting together these two observation the thesis follows: we have a divisor $\sigma \in I^{\star j} \star Y$ where $Y \in (\mathrm{Pic}^0(A)^k \bullet CH_3(A)) \star C^{\star r}$ and $j+k \geq 2$, if either $j \geq 2$, or $j = 1 \leq k$ the result follows by the square theorem, on the other hand, if $k \geq 2$, then Y is trivial: for D being an ample divisor, $R \in \mathrm{Pic}^0(A)^{k-2}$ and $W \in CH_3(A)$, the map

$$\begin{aligned} \Psi : A \times A &\longrightarrow \mathrm{Pic}^0(A) \\ (a, b) &\mapsto ((D_a - D) \bullet (D_b - D) \bullet R \bullet W) \star C^{\star r}, \end{aligned}$$

is trivial on $A \times \{0\} \cup \{0\} \times A$ and therefore it is the zero map. On the other hand Y is clearly a sum of divisors in the image of Ψ , so it must be trivial.

So far we are done with the “boundary cases” (16_a) and (16_b). We are now ready to prove (\star) . We proceed by descending induction on k and r , more precisely we shall prove that

$$\mathcal{N}_{j,k,r} = 0, \quad \forall j+r \geq n-2,$$

under the hypothesis that (\star) holds for $k' > k$ and that (\star) hold provided that $k' = k$ and $r' > r$.

By (16_a) and (16_b) we may assume $k \leq 2$ and $r \leq n-5+k$. In particular, as $j+r \geq n-2$ by hypotheses, we obtain $j \geq 3-k \geq 1$.

Let $\zeta := s_1 \star \dots \star s_j$ where $s_i = F_i \bullet \mathcal{C}$ and $F_i \in \text{Pic}^0(A)$, and let $W := E_1 \bullet \dots \bullet E_k$, $E_i \in \text{Pic}^0(A)$. Choose \widetilde{F}_j such that $(r+1)\widetilde{F}_j = F_j$. We have

$$\begin{aligned} 0 &= \widetilde{F}_j \bullet (s_1 \star \dots \star s_{j-1} \star (W \bullet Y) \star \mathcal{C}^{r+1}) \\ &= s_1 \star \dots \star s_{j-1} \star (\widetilde{F}_j \bullet W \bullet Y) \star \mathcal{C}^{r+1} + \zeta \star (W \bullet Y) \star \mathcal{C}^r \\ &= \zeta \star (W \bullet Y) \star \mathcal{C}^r \end{aligned}$$

where the inductive hypotheses take care of the first and the last equality, and the second equality holds since intersection with a divisor in $\text{Pic}^0(A)$ is a derivation for the Pontryagin product. □

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