ON THE KÄHLER–YANG–MILLS–HIGGS EQUATIONS

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To Simon Donaldson on his 60th birthday

Abstract. In this paper we introduce a set of equations on a principal bundle over a compact complex manifold coupling a connection on the principal bundle, a section of an associated bundle with Kähler fibre, and a Kähler structure on the base. These equations are a generalization of the Kähler–Yang–Mills equations introduced by the authors. They also generalize the constant scalar curvature for a Kähler metric studied by Donaldson and others, as well as the Yang–Mills–Higgs equations studied by Mundet i Riera. We provide a moment map interpretation of the equations, construct some first examples, and study obstructions to the existence of solutions.

Contents

1. Introduction

In the 1990s, Donaldson and Fujiki observed independently that moment maps play a central role in Kähler geometry [11, 15]. Since then, they have been fruitfully applied in the problem of finding constant scalar curvature Kähler metrics, acting as a guiding principle for many advances in this topic such as the recent solution of the Kähler–Einstein problem [10]. As noticed in [1], the moment map picture for Kähler metrics extends to the study of equations coupling a Kähler metric on a compact complex manifold and a connection on a principal bundle over it, known as the Kähler–Yang–Mills equations. Alike the constant scalar curvature Kähler metrics can be used to understand the moduli space of polarised manifolds, these equations are natural in the study of the algebro-geometric moduli problem for bundles and varieties, suggested in [36].

Motivated by the search of the simplest non-trivial solutions of the Kähler–Yang–Mills equations, the authors studied [2] the dimensional reduction of the equations on the product

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of a Riemann surface with the complex projective line. This approach to the Kähler–Yang–Mills equations provided a new theory for abelian vortices on the Riemann surface [7, 28, 18, 19] with back-reaction of the metric, described by solutions of the 'gravitating vortex equations', and showed an unexpected relation with the physics of cosmic strings [2, 3]. The further coupling of a Kähler metric and a connection with a 'Higgs field' considered in these works also reveils newly emergent phenomena, not observed in the theory originally introduced in [1].

Building on [1, 2, 3], this paper develops some basic pieces of a general moment-map theory for the coupling of a Kähler metric on a compact complex manifold X, a connection on a principal bundle E over X, and a Higgs field ϕ, given by a section of a Kähler fibration associated to E. Our treatment of the Higgs field ϕ is inspired by, on the one hand, work on the Yang–Mills–Higgs equations by Mundet i Riera [27], and, other hand, Donaldson's study of actions of diffeomorphism groups on spaces of sections of a bundle [14] (see Section 2). As we will see, the Kähler–Yang–Mills–Higgs equations introduced in this paper lead to a very rich theory (see Section 3), which comprises a large class of interesting examples of moment-map equations (see Sections 4 and 5). In addition, we expect that these equations may provide a natural framework for the interaction of Kähler geometry and a certain class of unified field theories in physics [34] (see Section 5.2).

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2. Hamiltonian actions of the extended gauge group

2.1. The space of connections.

Details for this section can be found in [1].

Let X be a compact symplectic manifold of dimension $2n$, with symplectic form ω. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ and E be a smooth principal G-bundle on X, with projection map $\pi\colon E \to X$. Let $\mathcal{H}$ be the group of Hamiltonian symplectomorphisms of (X,ω) and $\operatorname{Aut} E$ be the group of automorphisms of the bundle E. Recall that an *automorphism* of E is a G-equivariant diffeomorphism $g\colon E \to E$. Any such automorphism covers a unique diffeomorphism $\check{g}\colon X \to X$, i.e. a unique $\check{g}$ such that $\pi \circ g = \check{g} \circ \pi$. We define the *Hamiltonian extended gauge group* (to which we will simply refer as extended gauge group) of E,

$$\widetilde{\mathcal{G}} \subset \operatorname{Aut} E,$$

as the group of automorphisms which cover elements of $\mathcal{H}$. Then the gauge group of E is the normal subgroup $\mathcal{G} \subset \widetilde{\mathcal{G}}$ of automorphisms covering the identity.

The map $\widetilde{\mathcal{G}} \xrightarrow{p} \mathcal{H}$ assigning to each automorphism g the Hamiltonian symplectomorphism $\check{g}$ that it covers is surjective. We thus have an exact sequence of Lie groups

$$1 \to \mathcal{G} \xrightarrow{\iota} \widetilde{\mathcal{G}} \xrightarrow{p} \mathcal{H} \to 1, \tag{2.1}$$

where ι is the inclusion map.

The spaces of smooth k-forms on X and smooth k-forms with values in any given vector bundle F on X are denoted by Ω^k and $\Omega^k(F)$, respectively. Fix a positive definite inner product on $\mathfrak{g}$, invariant under the adjoint action, denoted

$$(\cdot,\cdot)\colon \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{R}. \tag{2.2}$$

This product induces a metric on the adjoint bundle $\operatorname{ad} E = E \times_G \mathfrak{g}$, which extends to a bilinear map on $(\operatorname{ad} E)$-valued differential forms (we use the same notation as in [6, §3])

$$\Omega^p(\operatorname{ad} E) \times \Omega^q(\operatorname{ad} E) \longrightarrow \Omega^{p+q} \colon (a_p, a_q) \longmapsto a_p \wedge a_q. \tag{2.3}$$

We consider the operator

$$\Lambda = \Lambda_\omega : \Omega^k \longrightarrow \Omega^{k-2} \colon \psi \longmapsto \omega^\sharp \lrcorner \psi, \tag{2.4}$$

where $\sharp$ is the operator acting on k-forms induced by the symplectic duality $\sharp\colon T^*X \to TX$ and $\lrcorner$ denotes the contraction operator. Its linear extension to $\Omega^k(\operatorname{ad} E)$ is also denoted $\Lambda : \Omega^k(\operatorname{ad} E) \to \Omega^{k-2}(\operatorname{ad} E)$ (we use the same notation as, e.g., in [12]).

Let $\mathcal{A}$ be the set of connections on E. This is an affine space modelled on $\Omega^1(\operatorname{ad} E)$. The 2-form on $\mathcal{A}$ defined by

$$\omega_{\mathcal{A}}(a, b) = \int_X a \wedge b \wedge \frac{\omega^{n-1}}{n-1!} \tag{2.5}$$

for $a, b \in T_A\mathcal{A} = \Omega^1(\operatorname{ad} E)$, $A \in \mathcal{A}$, is a symplectic form.

There is an action of $\operatorname{Aut} E$, and hence of the extended gauge group, on the space $\mathcal{A}$ of connections on E. To define this action, we view the elements of $\mathcal{A}$ as G-equivariant splittings $A\colon TE \to VE$ of the short exact sequence

$$0 \to VE \longrightarrow TE \longrightarrow \pi^*TX \to 0, \tag{2.6}$$

where $VE = \ker d\pi$ is the vertical bundle. Using the action of $g \in \operatorname{Aut} E$ on TE, its action on $\mathcal{A}$ is given by $g \cdot A := g \circ A \circ g^{-1}$. Any such splitting A induces a vector space splitting of the Atiyah short exact sequence

$$0 \to \operatorname{Lie} \mathcal{G} \xrightarrow{\iota} \operatorname{Lie}(\operatorname{Aut} E) \xrightarrow{p} \operatorname{Lie}(\operatorname{Diff} X) \to 0 \tag{2.7}$$

(cf. [6, equation (3.4)]), where $\operatorname{Lie}(\operatorname{Diff} X)$ is the Lie algebra of vector fields on X and $\operatorname{Lie}(\operatorname{Aut} E)$ is the Lie algebra of G-invariant vector fields on E. Abusing of the notation, this splitting is given by maps

$$A\colon \operatorname{Lie}(\operatorname{Aut} E) \longrightarrow \operatorname{Lie} \mathcal{G}, \quad A^\perp\colon \operatorname{Lie}(\operatorname{Diff} X) \longrightarrow \operatorname{Lie}(\operatorname{Aut} E) \tag{2.8}$$

such that $\iota \circ A + A^\perp \circ p = \operatorname{Id}$, where A is the vertical projection and $A^\perp$ the horizontal lift of vector fields on X to vector fields on E, given by the connection.

It is easy to see that the $\widetilde{\mathcal{G}}$-action on $\mathcal{A}$ is symplectic. An equivariant moment map for this action was calculated in [1]. To give an explicit formula, we use that the splitting (2.8) restricts to a splitting of the exact sequence

$$0 \to \operatorname{Lie} \mathcal{G} \xrightarrow{\iota} \operatorname{Lie} \widetilde{\mathcal{G}} \xrightarrow{p} \operatorname{Lie} \mathcal{H} \to 0 \tag{2.9}$$

induced by (2.1). Consider the isomorphism of Lie algebras

$$\operatorname{Lie} \mathcal{H} \cong C_0^\infty(X), \tag{2.10}$$

where $\operatorname{Lie} \mathcal{H}$ is the Lie algebra of Hamiltonian vector fields on X and $C_0^\infty(X)$ is the Lie algebra of smooth real functions on X with zero integral over X with respect to ω^n, with the Poisson bracket. This isomorphism is induced by the map $C^\infty(X) \to \operatorname{Lie} \mathcal{H}\colon f \mapsto \eta_f$, which to each function f assigns its Hamiltonian vector field η_f, defined by

$$df = \eta_f \lrcorner \omega. \tag{2.11}$$

Let $F_A \in \Omega^2(\operatorname{ad} E)$ be the curvature of $A \in \mathcal{A}$ and z be an element of the space

$$\mathfrak{z} = \mathfrak{g}^G \tag{2.12}$$

of elements of $\mathfrak{g}$ which are invariant under the adjoint G-action, that we identify with sections of $\operatorname{ad} E$. We have the following.

Proposition 2.1. *The $\widetilde{\mathcal{G}}$-action on $\mathcal{A}$ is Hamiltonian, with equivariant moment map $\mu_{\widetilde{\mathcal{G}}}\colon \mathcal{A} \to (\operatorname{Lie}\widetilde{\mathcal{G}})^*$ given by*

$$\langle \mu_{\widetilde{\mathcal{G}}}(A), \zeta\rangle = \int_X A\zeta \wedge (\Lambda F_A - z)\frac{\omega^n}{n!} - \frac{1}{4}\int_X f\left(\Lambda^2(F_A \wedge F_A) - 4\Lambda F_A \wedge z\right)\frac{\omega^n}{n!} \tag{2.13}$$

for all $\zeta \in \operatorname{Lie}\widetilde{\mathcal{G}}$, $A \in \mathcal{A}$, where $f \in C_0^\infty(X)$ corresponds to $p(\zeta)$ via (2.9) *and* (2.10)*.*

2.2. **Sections of a Kähler fibration.** Let $(F, \hat{J}, \hat{\omega})$ be a (possibly non-compact) Kähler manifold, with complex structure $\hat{J}$ and Kähler form $\hat{\omega}$. Following the notation of the previous section, we assume that G acts on F by Hamiltonian isometries, and fix a G-equivariant moment map

$$\hat{\mu}\colon F \to \mathfrak{g}^*.$$

Consider the associated fibre bundle $\mathcal{F} = E \times_G F$ with fibre F. We will denote by $V\mathcal{F} \subset T\mathcal{F}$ the vertical bundle of the fibration.

Let $\mathcal{S} := \Omega^0(X, \mathcal{F})$ the space of C^∞ global sections of the fibre bundle $\mathcal{F}$. Using the Kähler structure on the fibres of $\mathcal{F}$, we endow the infinite-dimensional space $\mathcal{S}$ with a Kähler structure. Given $\phi \in \mathcal{S}$, the symplectic form is given explicitly by

$$\omega_{\mathcal{S}}(\dot{\phi}_1, \dot{\phi}_2) = \int_X \hat{\omega}(\dot{\phi}_1, \dot{\phi}_2)\frac{\omega^n}{n!}$$

where $\dot{\phi}_i \in T_\phi\mathcal{S}$ are identified with elements in $\Omega^0(\phi^* V\mathcal{F})$.

An equivariant moment map for the action of the gauge group $\mathcal{G}$ of E on $(\mathcal{S}, \omega_{\mathcal{S}})$ was calculated in [27]. Here we are interested in a generalization of this result, where the gauge group is extended by the group of hamiltonian symplectomorphisms $\mathcal{H}$ of (X, ω). The action of the extended group $\widetilde{\mathcal{G}}$ on E induces an action on $\mathcal{S}$. This can be seen, for example, by regarding a section of $\mathcal{F}$ as a G-equivariant map $\phi\colon E \to F$. Furthermore, it is easy to see that $\widetilde{\mathcal{G}}$-action on $\mathcal{S}$ preserves the Kähler structure.

To compute the moment map, let us assume for a moment that the symplectic form $\hat{\omega}$ is exact (this is, e.g., the situation considered in [3]), that is, there exists $\hat{\sigma} \in \Omega^1(F)$ such that

$$d\hat{\sigma} = \hat{\omega}.$$

By averaging over G, we can assume that $\hat{\sigma}$ is invariant under the action of G, and it follows that $\omega_{\mathcal{S}} = d\sigma_{\mathcal{S}}$, with

$$\sigma_{\mathcal{S}}(\dot{\phi}) = \int_X \hat{\sigma}(\dot{\phi})\frac{\omega^n}{n!}.$$

Then, a $\widetilde{\mathcal{G}}$-equivariant moment map $\mu_{\widetilde{\mathcal{G}}}\colon \mathcal{S} \longrightarrow (\operatorname{Lie}\widetilde{\mathcal{G}})^*$ is given by

$$\langle \mu_{\widetilde{\mathcal{G}}}, \zeta\rangle = -\sigma_{\mathcal{S}}(Y_\zeta) = \int_X \hat{\sigma}(d\phi(\zeta))\frac{\omega^n}{n!}, \tag{2.14}$$

where Y_ζ denotes the infinitesimal action

$$Y_{\zeta|\phi} = -d\phi(\zeta)$$

of $\zeta \in \operatorname{Lie}\widetilde{\mathcal{G}}$ on $\phi \in \mathcal{S}$, where ϕ is regarded as a map $\phi\colon E \to F$ and we use the identification $E \times_G TF \cong \phi^* V\mathcal{F}$.

We want to obtain an equivalent formula for the moment map (2.14) which is independent of the choice of 1-form $\hat{\sigma}$. For this, choosing a connection $A\colon TE \to VE$ on E, we can write

$$d\phi(\zeta) = d\phi(A^\perp \zeta) + d\phi(A\zeta) = \check{\zeta} \lrcorner d_A\phi - A\zeta \cdot \phi,$$

where $\check{\zeta} := p(\zeta)$, $A\zeta \cdot \phi$ denotes the infinitesimal action of $A\zeta \in \Omega^0(VE)$ along the image of ϕ and $d_A\phi = d\phi(A^\perp \cdot) \in \Omega^1(\phi^* V\mathcal{F})$ is the covariant derivative induced by A. Using that $\hat{\sigma}$ induces a moment map for the G-action on F (that we can assume to be $\hat{\mu}$) it follows that

$$\hat{\sigma}(A\zeta \cdot \phi) = -\langle \phi^*\hat{\mu}, A\zeta\rangle$$

where $\phi^*\hat{\mu} \in \Omega^0(E \times_G \mathfrak{g}^*)$. We use now that $\check{\zeta} \in \mathcal{H}$, that is, $\check{\zeta} \lrcorner \omega = df$ for a smooth function $f \in C_0^\infty(X)$:

$$\begin{aligned}\int_X \hat{\sigma}(\check{\zeta} \lrcorner d_A\phi)\frac{\omega^n}{n!} &= \int_X \hat{\sigma}(d_A\phi) \wedge df \wedge \frac{\omega^{n-1}}{(n-1)!}\\ &= \int_X f d(\hat{\sigma}(d_A\phi)) \wedge \frac{\omega^{n-1}}{(n-1)!}.\end{aligned}$$

Finally, our desired formula follows from

$$d(\hat{\sigma}(d_A\phi)) = \frac{1}{2}\hat{\omega}(d_A\phi, d_A\phi) + \hat{\sigma}(F_A \cdot \phi) = \frac{1}{2}\hat{\omega}(d_A\phi, d_A\phi) - \langle \phi^*\hat{\mu}, F_A\rangle.$$

The next result is independent of the existence of the 1-form $\hat{\sigma}$ on F.

Proposition 2.2. *The $\widetilde{\mathcal{G}}$-action on $\mathcal{S}$ is Hamiltonian, with equivariant moment map*

$$\mu_{\widetilde{\mathcal{G}}}\colon \mathcal{S} \longrightarrow (\operatorname{Lie}\widetilde{\mathcal{G}})^*.$$

For any choice of unitary connection A on E, the moment map is given explicitly by

$$\langle \mu(\phi), \zeta\rangle = \int_X \langle \phi^*\hat{\mu}, A\zeta\rangle \frac{\omega^n}{n!} + \frac{1}{2}\int_X f(\hat{\omega}(d_A\phi, d_A\phi) - 2\langle \phi^*\hat{\mu}, F_A\rangle) \wedge \frac{\omega^{n-1}}{(n-1)!} \tag{2.15}$$

for all $\phi \in \mathcal{S}$ and $\zeta \in \operatorname{Lie}\widetilde{\mathcal{G}}$ covering $\check{\zeta} \in \mathcal{H}$, such that $df = \check{\zeta} \lrcorner \omega$ with $f \in C_0^\infty(X)$.

Proof. The variation of $\langle \phi^*\hat{\mu}, A\zeta\rangle$ with respect to ϕ is

$$\langle d\hat{\mu}(\dot{\phi}), A\zeta\rangle = \hat{\omega}(d\phi(A\zeta), \dot{\phi}).$$

In addition, we have

$$\begin{aligned}-\hat{\omega}(d_A\phi(\check{\zeta}), \dot{\phi})\frac{\omega^n}{n} &= -\hat{\omega}(d_A\phi, \dot{\phi}) \wedge df \wedge \omega^{n-1}\\ &= d(f\hat{\omega}(d_A\phi, \dot{\phi}) \wedge \omega^{n-1}) - fd(\hat{\omega}(d_A\phi, \dot{\phi})) \wedge \omega^{n-1},\end{aligned}$$

while the variation of $\hat{\omega}(d_A\phi, d_A\phi) - 2\langle \phi^*\hat{\mu}, F_A\rangle$ in the second integral is

$$\hat{\omega}(d_A\phi, d_A\dot{\phi}) + \hat{\omega}(d_A\dot{\phi}, d_A\phi) - 2\hat{\omega}(d\phi(F_A), \dot{\phi}) = -2d(\hat{\omega}(d_A\phi, \dot{\phi})).$$

Formula (2.15) follows now integrating by parts. □

2.3. **The Hermitian scalar curvature as a moment map.** Via its projection into the group of Hamiltonian symplectomorphisms $\mathcal{H}$ (see (2.1)), the extended gauge group acts on the space $\mathcal{J}$ of compatible almost complex structures on the symplectic manifold (X, ω). As proved by Donaldson [11], the $\mathcal{H}$-action on $\mathcal{J}$ is Hamiltonian, with moment map given by the Hermitian scalar curvature of the almost Kähler manifold. The moment map interpretation of the scalar curvature was first given by Quillen in the case of Riemann surfaces and Fujiki [15] for the Riemannian scalar curvature of Kähler manifolds, and generalized independently in [11].

First we recall the notion of Hermitian scalar curvature of an almost Kähler manifold, we follow closely Donaldson's approach. Fix a compact symplectic manifold X of dimension $2n$, with symplectic form ω. An almost complex structure J on X is called compatible with ω if the bilinear form $g_J(\cdot,\cdot) := \omega(\cdot, J\cdot)$ is a Riemannian metric on X. Any almost complex structure J on X which is compatible with ω defines a Hermitian metric on T^*X and there is a unique unitary connection on T^*X whose (0,1) component is the operator $\bar{\partial}_J \colon \Omega^{1,0}_J \to \Omega^{1,1}_J$ induced by J. The real 2-form ρ_J is defined as $-i$ times the curvature of the induced connection on the canonical line bundle $K_X = \Lambda^n_{\mathbb{C}} T^*X$, where i is the imaginary unit $\sqrt{-1}$. The Hermitian scalar curvature S_J is the real function on X defined by

$$S_J \omega^n = 2n \rho_J \wedge \omega^{n-1}. \tag{2.16}$$

The normalization is chosen so that S_J coincides with the Riemannian scalar curvature when J is integrable. The space $\mathcal{J}$ of almost complex structures J on X which are compatible with ω is an infinite dimensional Kähler manifold, with complex structure $\mathbf{J} \colon T_J\mathcal{J} \to T_J\mathcal{J}$ and Kähler form $\omega_{\mathcal{J}}$ given by

$$\mathbf{J}\Phi := J\Phi \text{ and } \omega_{\mathcal{J}}(\Psi, \Phi) := \frac{1}{2n!} \int_X \operatorname{tr}(J\Psi\Phi)\omega^n, \tag{2.17}$$

for Φ, $\Psi \in T_J\mathcal{J}$, respectively. Here we identify $T_J\mathcal{J}$ with the space of endomorphisms $\Phi \colon TX \to TX$ such that Φ is symmetric with respect to the induced metric $\omega(\cdot, J\cdot)$ and satisfies $\Phi J = -J\Phi$.

The group $\mathcal{H}$ of Hamiltonian symplectomorphisms $h \colon X \to X$ acts on $\mathcal{J}$ by push-forward, i.e. $h \cdot J := h_* \circ J \circ h_*^{-1}$, preserving the Kähler form. As proved by Donaldson [11, Proposition 9], the $\mathcal{H}$-action on $\mathcal{J}$ is Hamiltonian with equivariant moment map $\mu_{\mathcal{H}} \colon \mathcal{J} \to (\operatorname{Lie}\mathcal{H})^*$ given by

$$\langle \mu_{\mathcal{H}}(J), \eta_f \rangle = -\int_X f S_J \frac{\omega^n}{n!}, \tag{2.18}$$

for $f \in C_0^\infty(X)$, identified with an element η_f in $\operatorname{Lie}\mathcal{H}$ by (2.10) and (2.11).

As a warm up for our discussion in Section 3, we note that the $\mathcal{H}$-invariant subspace $\mathcal{J}^i \subset \mathcal{J}$ of integrable almost complex structures is a complex submanifold (away from its singularities), and therefore inherits a Kähler structure. Over $\mathcal{J}^i$, the Hermitian scalar curvature S_J is the Riemannian scalar curvature of the Kähler metric determined by J and ω. Hence the quotient

$$\mu_{\mathcal{H}}^{-1}(0)/\mathcal{H}, \tag{2.19}$$

where $\mu_{\mathcal{H}}$ is now the restriction of the moment map to $\mathcal{J}^i$, is the moduli space of Kähler metrics with fixed Kähler form ω and constant scalar curvature. Away from singularities, this moduli space can thus be constructed as a Kähler reduction (see [15] and references therein for details).

3. The Kähler–Yang–Mills–Higgs equations

3.1. **The equations as a moment map condition.** Fix a compact symplectic manifold X of dimension $2n$ with symplectic form ω, a compact Lie group G and a smooth principal G-bundle E on X. We fix an Ad-invariant inner product $(\cdot,\cdot)\colon \mathfrak{g}\otimes\mathfrak{g}\to\mathbb{R}$ on the Lie algebra $\mathfrak{g}$ of G. Let $\mathcal{J}$ be the space of almost complex structures on X compatible with ω and $\mathcal{A}$ the space of connections on E. Consider the space of triples

$$\mathcal{J}\times\mathcal{A}\times\mathcal{S}, \tag{3.1}$$

endowed with the symplectic structure

$$\omega_{\mathcal{J}}+4\alpha\omega_{\mathcal{A}}+4\beta\omega_{\mathcal{S}}, \tag{3.2}$$

(for a choice of non-zero real coupling constants α,β). Similarly as in [1, Proposition 2.2], the space (3.1) has a formally integrable almost complex structure, which is compatible with (3.2) when $\alpha>0$ and $\beta>0$, thus inducing a Kähler structure in this case.

By Proposition 2.2 combined with Proposition 2.1 and 2.18, the diagonal action of $\widetilde{\mathcal{G}}$ on this space is Hamiltonian (here the action of $\widetilde{\mathcal{G}}$ on $\mathcal{J}$ is given by projecting to $\mathcal{H}$), with equivariant moment map $\mu_{\alpha,\beta}\colon \mathcal{J}\times\mathcal{A}\times\mathcal{S}\to(\operatorname{Lie}\widetilde{\mathcal{G}})^*$ given by

$$\begin{aligned}\langle\mu_{\alpha,\beta}(J,A,\phi),\zeta\rangle &= 4\int_X(A\zeta,\alpha\Lambda F_A+\beta\phi^*\hat{\mu}-z)\frac{\omega^n}{n!}\\ &\quad-\int_X f(S_J-2\beta\Lambda\hat{\omega}(d_A\phi,d_A\phi)+\alpha\Lambda^2(F_A\wedge F_A)+4(\Lambda F_A,\beta\phi^*\hat{\mu}-\alpha z))\frac{\omega^n}{n!},\end{aligned} \tag{3.3}$$

for any choice of central element z in the Lie algebra $\mathfrak{g}$.

Suppose now that X has Kähler structures with Kähler form ω. This means that the subspace $\mathcal{J}^i\subset\mathcal{J}$ of integrable almost complex structures compatible with ω is not empty. Define

$$\mathcal{T}\subset\mathcal{J}\times\mathcal{A}\times\mathcal{S} \tag{3.4}$$

by the conditions

$$J\in\mathcal{J}^i,\qquad A\in\mathcal{A}_J^{1,1},\qquad \bar\partial_{J,A}\phi=0,$$

where $\bar\partial_{J,A}\phi$ denotes the $(0,1)$-part of $d_A\phi$ with respect to J and $\mathcal{A}_J^{1,1}\subset\mathcal{A}$ consists of connections A with $F_A\in\Omega_J^{1,1}(\operatorname{ad}E)$, or equivalently satisfying

$$F_A^{0,2_J}=0.$$

Here $\Omega_J^{p,q}(\operatorname{ad}E)$ denotes the space of $(\operatorname{ad}E)$-valued smooth (p,q)-forms with respect to J and $F_A^{0,2_J}$ is the projection of F_A into $\Omega_J^{0,2}(\operatorname{ad}E)$. This space is in bijection with the space of holomorphic structures on the principal G^c-bundle E^c over (X,J) (see [30]).

By definition, $\mathcal{T}$ is a complex subspace of (3.1) (away from its singularities) preserved by the $\widetilde{\mathcal{G}}$-action, and hence it inherits a Hamiltonian $\widetilde{\mathcal{G}}$-action.

Proposition 3.1. *The $\widetilde{\mathcal{G}}$-action on $\mathcal{T}$ is Hamiltonian with $\widetilde{\mathcal{G}}$-equivariant moment map $\mu_{\alpha,\beta}\colon\mathcal{T}\to(\operatorname{Lie}\widetilde{\mathcal{G}})^*$ given by*

$$\begin{aligned}\langle\mu_{\alpha,\beta}(J,A,\phi),\zeta\rangle &= 4\int_X(A\zeta,\alpha\Lambda F_A+\beta\phi^*\hat{\mu}-z)\frac{\omega^n}{n!}\\ &\quad-\int_X f(S_J+\beta\Delta_g|\phi^*\hat{\mu}|^2+\alpha\Lambda^2(F_A\wedge F_A)-4\alpha(\Lambda F_A,z))\frac{\omega^n}{n!},\end{aligned} \tag{3.5}$$

for all $(J,A,\phi) \in \mathcal{T}$ *and* $\zeta \in \operatorname{Lie}\widetilde{\mathcal{G}}$*, where* Δ_g *denotes the Laplacian of* $g = \omega(\cdot, J\cdot)$*.*

Proof. Since $(J,A,\phi) \in \mathcal{T}$, we have $\bar{\partial}_A \phi = 0$, and hence

$$\Delta_g |\phi^*\hat{\mu}|^2 = 2i\Lambda\bar{\partial}\partial|\phi^*\hat{\mu}|^2 = -2\Lambda\hat{\omega}(d_A\phi, d_A\phi) + 4\langle \phi^*\hat{\mu}, \Lambda F_A\rangle.$$

The statement follows now from (3.3). □

The zeros of the moment map $\mu_{\alpha,\beta}$, restricted to the space of integrable pairs $\mathcal{T}$, correspond to a coupled system of partial differential equations which is the object of our next definition.

Definition 3.2. We say that a triple $(J,A,\phi) \in \mathcal{T}$ satisfies the *Kähler–Yang–Mills–Higgs equations* with coupling constants $\alpha, \beta \in \mathbb{R}$ if

$$\begin{aligned} \alpha\Lambda F_A + \beta\phi^*\hat{\mu} &= z, \\ S_J + \beta\Delta_g|\phi^*\hat{\mu}|^2 + \alpha\Lambda^2(F_A \wedge F_A) - 4\alpha(\Lambda F_A, z) &= c, \end{aligned} \tag{3.6}$$

where S_J is the scalar curvature of the metric $g_J = \omega(\cdot, J\cdot)$ on X, z is an element in the center of $\mathfrak{g}$ and $c \in \mathbb{R}$.

The constant $c \in \mathbb{R}$ in (3.6) is explicitly defined by the identity

$$c[\omega]^n = 2\pi n c_1(X) \cup [\omega]^{n-1} + 2\alpha n(n-1)p_1(E) \cup [\omega]^{n-2} - 4nc(E) \cup [\omega]^{n-1} \tag{3.7}$$

where $p_1(E) := [F_A \wedge F_A] \in H^4(X,\mathbb{R})$ and $c(E) \in H^2(X,\mathbb{R})$ are the Chern–Weil classes associated to the G-invariant symmetric forms $(\cdot,\cdot)$ and $(\cdot, z)$ on $\mathfrak{g}$ respectively, and so c only depends on $[\omega]$ and the topology of E.

The set of solutions of (3.6) is invariant under the action of $\widetilde{\mathcal{G}}$ and we define the moduli space of solutions as the set of all solutions modulo the action of $\widetilde{\mathcal{G}}$. We can identify this moduli space with the quotient

$$\mu_{\alpha,\beta}^{-1}(0)/\widetilde{\mathcal{G}}, \tag{3.8}$$

where $\mu_{\alpha,\beta}$ denotes now the restriction of the moment map to $\mathcal{T}$. Away from singularities, this is a Kähler quotient for the action of $\widetilde{\mathcal{G}}$ on the smooth part of $\mathcal{T}$ equiped with the Kähler form obtained by the restriction of (3.2).

3.2. **Futaki invariant and geodesic stability.** In this section, we explain briefly some general obstructions to the existence of solutions of the Kähler–Yang–Mills–Higgs equations (3.6), which follow the general method developed in [1, §3]. To describe them, it is helpful to adopt a dual view point, based on complex differential geometry.

We fix a compact complex manifold X of dimension n, a Kähler class $\Omega \in H^{1,1}(X)$ and a holomorphic principal bundle E^c over X. We assume that the structure group of E^c is a complex reductive Lie group G^c, and that the Lie algebra $\mathfrak{g}^c$ of G^c is endowed with an Ad-invariant symmetric bilinear form. Let $(F, \hat{J}, \hat{\omega})$ be a (possibly non-compact) Kähler manifold, with complex structure $\hat{J}$ and Kähler form $\hat{\omega}$. We assume that a maximal compact subgroup $G \subset G^c$ acts on F by Hamiltonian isometries, and fix a G-equivariant moment map

$$\hat{\mu} \colon F \to \mathfrak{g}^*.$$

Consider the associated fibre bundle $\mathcal{F} = E^c \times_{G^c} F$ with fibre F, and assume that there exists a holomorphic section

$$\phi \in H^0(X, \mathcal{F}).$$

Then, the Kähler–Yang–Mills–Higgs equations on (X, E^c, ϕ), for fixed coupling constants $\alpha, \beta \in \mathbb{R}$, are

$$\begin{aligned} \alpha\Lambda_\omega F_H + \beta\phi^*\hat{\mu} &= z, \\ S_\omega + \beta\Delta_\omega|\phi^*\hat{\mu}|^2 + \alpha\Lambda_\omega^2(F_H \wedge F_H) - 4\alpha(\Lambda_\omega F_H, z) &= c, \end{aligned} \tag{3.9}$$

where the unknowns are a Kähler metric on X with Kähler form ω in Ω, and a reduction $H\colon X \to E^c/G$ to G. In this case, F_H is the curvature of the Chern connection A_H of H on E^c, and S_ω is the scalar curvature of the Kähler metric. Note that the constant $c \in \mathbb{R}$ depends on α, Ω and the topology of X and E^c. In the rest of this section, we will assume $\alpha > 0$ and $\beta > 0$ in the definition of (3.9).

Our first obstruction builds on the general method in [1, §3] and classical work of Futaki [16]. Consider the complex Lie group $\operatorname{Aut}(X, E^c)$ of automorphisms of (X, E^c) and the complex Lie subgroup fixing the section ϕ

$$\operatorname{Aut}(X, E^c, \phi) \subset \operatorname{Aut}(X, E^c).$$

We define a map

$$\mathcal{F}_{\alpha,\beta}\colon \operatorname{Lie}\operatorname{Aut}(X, E^c, \phi) \longrightarrow \mathbb{C}$$

given by the formula

$$\begin{aligned} \langle \mathcal{F}_{\alpha,\beta}, \zeta\rangle = 4\int_X (A_H\zeta, \alpha\Lambda_\omega F_H + \beta\phi^*\hat{\mu} - \alpha z)\frac{\omega^n}{n!} \\ - \int_X \varphi(S_\omega + \beta\Delta_\omega|\phi^*\hat{\mu}|^2 + \alpha\Lambda_\omega^2(F_H \wedge F_H) - 4(\Lambda_\omega F_H, z)))\frac{\omega^n}{n!}, \end{aligned} \tag{3.10}$$

for a choice a Kähler form $\omega \in \Omega$ and hermitian metric H on E. To explain this formula, we note that $\operatorname{Lie}\operatorname{Aut}(X, E^c)$ is the space of G^c-invariant holomorphic vector fields ζ on the total space of E^c. Any such ζ covers a real-holomorphic vector field $\check{\zeta}$ on X, and decomposes, in terms of the connection A_H, as

$$\zeta = A_H\zeta + A_H^\perp\check{\zeta},$$

where $A_H\zeta$ and $A_H^\perp\check{\zeta}$ are its vertical and horizontal parts. The complex-valued function

$$\varphi := \varphi_1 + i\varphi_2,$$

with $\varphi_1, \varphi_2 \in C_0^\infty(X, \omega)$, is determined by the unique decomposition

$$\check{\zeta} = \eta_{\varphi_1} + J\eta_{\varphi_2} + \gamma,$$

valid precisely because $\check{\zeta}$ is a real-holomorphic vector field, where J is the (integrable) almost complex structure of X, η_{φ_j} (for $j = 1, 2$) is the Hamiltonian vector field of φ_j, and γ is the dual of a 1-form that is harmonic with respect to the Kähler metric.

This *Futaki character* provides the following obstruction to the existence of solutions of the Kähler–Yang–Mills–Higgs equations equations (cf. [1, Theorem 3.9]). Let B be the space of pairs (ω, H) consisting of a Kähler form ω in the cohomology class Ω and a reduction H of E^c to $G \subset G^c$.

Proposition 3.3. *The map* (3.10) *is independent of the choice of element* (ω, H) *in* B. *It defines a character of* $\operatorname{Lie}\operatorname{Aut}(X, E^c, \phi)$, *which vanishes identically if there exists a solution of the Kähler–Yang–Mills–Higgs equations* (3.9) *with Kähler class* Ω.

Further obstructions to the existence of solutions of the Kähler–Yang–Mills equations are intimately related to the geometry of the infinite-dimensional space B. It is interesting to notice that this geometry is independent of the choice of holomorphic section ϕ on $\mathcal{F}$. The space B has a structure of symmetric space [1, Theorem 3.6], that is, it has a torsion-free affine connection ∇, with holonomy group contained in the extended gauge group (each point of B determines one such group) and covariantly constant curvature. The partial differential equations that define the geodesics (ω_t, H_t) on B, with respect to the connection ∇, are

$$\begin{aligned} dd^c(\ddot{\varphi}_t - (d\dot{\varphi}_t, d\dot{\varphi}_t)_{\omega_t}) &= 0, \\ \ddot{H}_t - 2J\eta_{\dot{\varphi}_t} \lrcorner d_{H_t}\dot{H}_t + iF_{H_t}(\eta_{\dot{\varphi}_t}, J\eta_{\dot{\varphi}_t}) &= 0, \end{aligned} \tag{3.11}$$

where $\eta_{\dot{\varphi}_t}$ is the Hamiltonian vector field of $\dot{\varphi}_t$ with respect to ω_t, i.e. $d\dot{\varphi}_t = \eta_{\dot{\varphi}_t} \lrcorner \omega_t$. Assuming existence of smooth geodesic rays, that is, smooth solutions (ω_t, H_t) of (3.11) defined on an infinite interval $0 \leq t < \infty$, with prescribed boundary condition at $t = 0$, one can define a stability condition for (X, E^c, ϕ). Define a 1-form $\sigma_{\alpha,\beta}$ on B by

$$\begin{aligned} \sigma_{\alpha,\beta}(\dot{\omega}, \dot{H}) = & - 4i \int_X (H^{-1}\dot{H}, \alpha\Lambda_\omega F_H + \beta\phi^*\hat{\mu} - \alpha z)\frac{\omega^n}{n!} \\ & - \int_X \dot{\varphi}(S_\omega + \beta\Delta_\omega|\phi^*\hat{\mu}|^2 + \alpha\Lambda_\omega^2(F_H \wedge F_H) - 4(\Lambda_\omega F_H, z)))\frac{\omega^n}{n!}, \end{aligned}$$

where $(\dot{\omega}, \dot{H})$ is a tangent vector to B at (ω, H) and $\dot{\omega} = dd^c\dot{\varphi}$ for $\varphi \in C_0^\infty(X, \omega)$.

Definition 3.4. The triple (X, E^c, ϕ) is *geodesically semi-stable* if for every smooth geodesic ray b_t on B, the following holds

$$\lim_{t\to+\infty} \sigma_{\alpha,\beta}(\dot{b}_t) \geq 0.$$

Under the assumption that B is geodesically convex, that is, that any two points in B can be joined by a smooth geodesic segment, *geodesic semi-stability* provides an obstruction to the existence of solutions of (3.9).

The proof of the next proposition follows from the fact that the quantity $\sigma_{\alpha,\beta}(\dot{b}_t)$ is increasing along geodesics in B, with speed controlled by the infinitesimal action on the space $\mathcal{T}$ in 3 (see the proof of [1, Proposition 3.14]).

Proposition 3.5. *Assume that B is geodesically convex. If there exists a solution of the Kähler–Yang–Mills–Higgs equations in B, then (X, P^c, ϕ) is geodesically semi-stable. Furthermore, such a solution is unique modulo the action of* $\mathrm{Aut}(X, E^c, \phi)$.

The space B defines a geodesic submersion over the symmetric space of Kähler metrics on the class Ω [13, 24, 29]. In particular, this implies that in general one cannot expect existence of smooth geodesic segments on B with arbitrary boundary conditions.

3.3. **Matsushima–Lichnerowicz for the Kähler–Yang–Mills–Higgs equations.** In this section we introduce a new obstruction to the existence of solutions of the Kähler–Yang–Mills–Higgs equations. This is based on an analogue of Matsushima–Lichnerowicz Theorem [23, 25] for (3.9), which relates the existence of a solution on (X, E^c, ϕ) with the reductivity of $\mathrm{Lie}\,\mathrm{Aut}(X, E^c, \phi)$. Our proof relies on the moment-map interpretation of the equations (3.9), following closely Donaldson–Wang's abstract proof [11, 32] of the Matsushima–Lichnerowicz Theorem.

For simplicity, we will assume that X has vanishing first Betti number, even though we expect that our analysis goes through with minor modifications to the general case.

Theorem 3.6. *Assume* $H^1(X,\mathbb{R}) = 0$*. If* (X, E^c, ϕ) *admits a solution of the Kähler–Yang–Mills–Higgs equations* (3.9) *with* $\alpha > 0$ *and* $\beta > 0$*, then the Lie algebra of* $\operatorname{Aut}(X, E^c, \phi)$ *is reductive.*

To prove our theorem we need some preliminary results. Let ω be a Kähler form on X and a reduction H of E^c to $G \subset G^c$. The following lemma gives a convenient formula for the elements of $\operatorname{Lie}\operatorname{Aut}(X, E^c, \phi)$ adapted to the pair (ω, H), and is reminiscent of the Hodge-theoretic description of holomorphic vector fields on compact Kähler manifolds (see, e.g., [20, Ch. 2]). As in (2), $\operatorname{Lie}\widetilde{\mathcal{G}}$ will denote the Lie algebra of the extended gauge group associated to the symplectic structure ω and the reduction E_H. For the proof, we will not assume that (ω, H) is a solution of (3.9). We denote by I the almost complex structure on the total space of E^c.

Lemma 3.7. *Assume* $H^1(X,\mathbb{R}) = 0$*. Then, for any* $y \in \operatorname{Lie}\operatorname{Aut}(X, E^c)$ *there exist* $\zeta_1, \zeta_2 \in \operatorname{Lie}\widetilde{\mathcal{G}}$ *such that*

$$y = \zeta_1 + I\zeta_2. \tag{3.12}$$

Proof. Let A be the Chern connection of H on E^c. We will use the decomposition of

$$y = Ay + A^\perp \check{y} \tag{3.13}$$

into its vertical and horizontal components Ay, $A^\perp \check{y}$, where $\check{y}$ is the unique holomorphic vector field on X covered by y. Using the anti-holomorphic involution on the Lie algebra $\mathfrak{g}^c$ determined by $G \subset G^c$, we decompose

$$Ay = \xi_1 + i\xi_2,$$

for $\xi_j \in \Omega^0(\operatorname{ad} E_H)$. Furthermore, as $H^1(X,\mathbb{R}) = 0$, we have

$$\check{y} = \check{y}_1 + J\check{y}_2,$$

where $\check{y}_1$ and $\check{y}_2$ are Hamiltonian vector fields for the symplectic form ω. Hence, defining the vector fields

$$\zeta_j = \xi_j + A^\perp \check{y}_j,$$

for $j = 1, 2$, we obtain the result. □

We will now apply Lemma 3.7 to the elements of $\operatorname{Lie}\operatorname{Aut}(X, E^c, \phi) \subset \operatorname{Lie}\operatorname{Aut}(X, E^c)$.

Lemma 3.8. *Assume* $H^1(X,\mathbb{R}) = 0$ *and that* (X, E^c, ϕ) *admits a solution* (ω, h) *of the Kähler–Yang–Mills–Higgs equations with* $\alpha > 0$ *and* $\beta > 0$*. Then, for any* $y \in \operatorname{Lie}\operatorname{Aut}(X, E^c, \phi)$*, the vector fields* ζ_1, ζ_2 *in* (3.12) *satisfy* $\zeta_1, \zeta_2 \in \operatorname{Lie}\operatorname{Aut}(X, E^c, \phi)$*.*

Proof. By the results of Section 3.1, if (ω, h) is a solution of (3.9), then the triple $t := (J, A, \phi)$ is a zero of a moment map

$$\mu_{\alpha,\beta} \colon \mathcal{T} \to \operatorname{Lie}\widetilde{\mathcal{G}}^*$$

for the action of $\widetilde{\mathcal{G}}$ on the space of 'integrable triples' $\mathcal{T}$ defined in (3.4). Recall that $\mathcal{T}$ is endowed with a (formally) integrable almost complex structure $\mathbf{I}$, and Kähler metric

$$g_{\alpha,\beta} = \omega_{\alpha,\beta}(\cdot, \mathbf{I}\cdot)$$

(as we are assuming $\alpha > 0$ and $\beta > 0$), where the compatible symplectic structure $\omega_{\alpha,\beta}$ is as in (3.2). Given $y \in \operatorname{Lie}\operatorname{Aut}(X, E^c)$, we denote by $Y_{y|t}$ the infinitesimal action of y on t). Then the proof reduces to show that $Y_{\zeta_1|t} = Y_{\zeta_2|t} = 0$ for $y \in \operatorname{Lie}\operatorname{Aut}(X, E^c, \phi)$, where ζ_1, ζ_2 as in (3.12). To prove this, we note that since the almost complex structure I on E^c is integrable, we have (see [1, Section 3.2])

$$0 = Y_{y|t} = Y_{\zeta_1 + I\zeta_2|t} = Y_{\zeta_1|t} + \mathbf{I}Y_{\zeta_2|t}.$$

Considering now the norm $\|\cdot\|$ on $T_t\mathcal{T}$ induced by the metric $g_{\alpha,\beta}$, we obtain

$$0 = \|Y_{y|t}\|^2 = \|Y_{\zeta_1|t}\|^2 + \|Y_{\zeta_2|t}\|^2 - 2\omega_{\alpha,\beta}(Y_{\zeta_1|t}, Y_{\zeta_1|t}).$$

Now, $\mu_{\alpha,\beta}(t) = 0$ and the moment map μ_α is equivariant, so

$$\omega_\alpha(Y_{\zeta_1|t}, Y_{\zeta_1|t}) = d\langle \mu_\alpha, \zeta_1 \rangle (Y_{\zeta_2|t}) = \langle \mu_\alpha(t), [\zeta_1, \zeta_2] \rangle = 0,$$

and therefore

$$\|Y_{\zeta_1|t}\|^2 = \|Y_{\zeta_2|t}\|^2 = 0,$$

so we conclude that $\zeta_1, \zeta_2 \in \operatorname{Lie}\operatorname{Aut}(X, E^c, \phi)$, as required. □

Theorem 3.6 is now a formal consequence of Lemma 3.8.

Proof of Theorem 3.6. Considering the $\widetilde{\mathcal{G}}$-action on $\mathcal{T}$, we note that the Lie algebra $\mathfrak{k} = \operatorname{Lie}\widetilde{\mathcal{G}}_t$ of the isotropy group $\widetilde{\mathcal{G}}_t$ of the triple $t = (J, A, \phi) \in \mathcal{T}$ satisfies

$$\mathfrak{k} \oplus I\mathfrak{k} \subset \operatorname{Lie}\operatorname{Aut}(X, E^c, \phi).$$

Furthermore, the Lie group $\widetilde{\mathcal{G}}_t$ is compact, because it can be regarded as a closed subgroup of the isometry group of a Riemannian metric on the total space of E_H (see [1, Section 2.3]). Now, Lemma 3.8 implies that

$$\operatorname{Lie}\operatorname{Aut}(X, E^c, \phi) = \mathfrak{k} \oplus I\mathfrak{k},$$

so $\operatorname{Lie}\operatorname{Aut}(X, E^c, \phi)$ is the complexification of the Lie algebra $\mathfrak{k}$ of a compact Lie group, and hence a reductive complex Lie algebra. □

4. Gravitating vortices and dimensional reduction

4.1. **Gravitating quiver vortex equations.** Here we consider in more detail the Kähler–Yang–Mills–Higgs equations when the Higgs field is a section of a special type of vector bundles, defining a quiver bundle. To fix notation, we recall the notions of quiver and quiver bundle (see, e.g., [4] for details). A *quiver* Q is a pair of sets (Q_0, Q_1), together with two maps $t, h\colon Q_1 \to Q_0$. The elements of Q_0 and Q_1 are called the vertices and arrows of the quiver, respectively. An arrow $a \in Q_1$ is represented pictorially as $a\colon i \to j$, where $i = ta$ and $j = ha$ are called the tail and the head of a. Suppose for simplicity that the quiver is finite, that is, both Q_0 and Q_1 are finite sets (this condition will be weakened in Section 4.2). Fix a compact complex manifold X of dimension n. A *holomorphic Q-bundle over X* is a pair (E, ϕ) consisting of a set E of holomorphic vector bundles E_i on X, indexed by the vertices $i \in Q_0$, and a set ϕ of holomorphic vector-bundle homomorphisms $\phi_a\colon E_{ta} \to E_{ha}$, indexed by the arrows $a \in Q_1$. Note that it is often useful to consider a category of *twisted* quiver bundles (see [4]), but they will not be needed for the application given in Corollary 5.3.

A *Hermitian metric* on (E, ϕ) is a set H of Hermitian metrics H_i on E_i, indexed by the vertices $i \in Q_0$. Any such Hermitian metric determines a C^∞ adjoint vector-bundle morphism $\phi_a^{*H_a}\colon E_{ha}\to E_{ta}$ of $\phi_a\colon E_{ta}\to E_{ha}$ with respect to the Hermitian metrics H_{ta} and H_{ha}, for each $a\in Q_1$, and we can construct a (H-self-adjoint) 'commutator'

$$[\phi,\phi^{*H}]=\bigoplus_{i\in Q_0}[\phi,\phi^{*H}]_i\colon \bigoplus_{i\in Q_0}E_i\longrightarrow\bigoplus_{i\in Q_0}E_i,$$

with components

$$[\phi,\phi^{*H}]_i:=\sum_{a\in h^{-1}(i)}\phi_a\circ\phi_a^{*H_a}-\sum_{a\in t^{-1}(i)}\phi_a^{*H_a}\circ\phi_a\colon E_i\longrightarrow E_i,$$

for all $i\in Q_0$. In the following, $\mathbb{R}_{>0}\subset\mathbb{R}$ is the set of positive real numbers, and for any two sets I and S, S^I is the set of maps $\sigma\colon I\to S$, $i\mapsto\sigma_i$; to avoid confusion with the symbols used to denote quiver vertices, $\mathbf{i}=\sqrt{-1}$ is the imaginary unit.

Definition 4.1. Fix constants $\rho\in\mathbb{R}_{>0}$, $\sigma\in\mathbb{R}_{>0}^{Q_0}$ and $\tau\in\mathbb{R}^{Q_0}$. The *gravitating quiver* (ρ,σ,τ)*-vortex equations* for a pair (ω,H), consisting of a Kähler metric ω on the complex manifold X and a Hermitian metric H on a holomorphic Q-bundle (E,ϕ), are

$$\sigma_i\,\mathbf{i}\,\Lambda_\omega F_{H_i}+[\phi,\phi^{*H}]_i=\tau_i\,\mathrm{Id}_{E_i}, \tag{4.1a}$$

$$S_\omega-\rho\sum_{i\in Q_0}\sigma_i\Lambda_\omega^2\operatorname{Tr}F_{H_i}^2+2\rho\sum_{a\in Q_1}\left(\Delta_\omega+2\left(\frac{\tau_{ha}}{\sigma_{ha}}-\frac{\tau_{ta}}{\sigma_{ta}}\right)\right)|\phi_a|^2_{H_a}=c. \tag{4.1b}$$

Here, $|\phi_a|^2_{H_a}:=\operatorname{Tr}(\phi_a\circ\phi_a^{*H_a})\in C^\infty(X)$ is the pointwise squared norm, and c is a constant, determined by the parameters ρ,σ,τ, the cohomology class of ω, and the characteristic classes of the manifold X and the vector bundles E_i. More precisely,

$$\begin{aligned}c\operatorname{Vol}_\omega(X)=2\int_X\rho_\omega\wedge\frac{\omega^{n-1}}{(n-1)!}-4\rho\sum_{i\in Q_0}\sigma_i\int_X\operatorname{Tr}F_{H_i}^2\wedge\frac{\omega^{n-2}}{(n-2)!}\\+4\rho\operatorname{Vol}_\omega(X)\sum_{i\in Q_0}\left(\frac{\tau_i}{\sigma_i}-\mu_\omega(E_i)\right)\tau_i r_i,\end{aligned}$$

where $\operatorname{Vol}_\omega(X)=\int_X\omega^n/n!$, r_i is the rank of E_i, its normalized ω-slope is

$$\mu_\omega(E_i):=\frac{1}{\operatorname{Vol}_\omega(X)}\frac{1}{r_i}\int_X\operatorname{Tr}(\mathbf{i}\,F_{A_i})\wedge\frac{\omega^{n-1}}{(n-1)!}, \tag{4.2}$$

and ρ_ω is the Ricci form. To see this, we integrate (4.1b), use (2.16), and also integrate the following identity (that follows from (4.1a))

$$\sum_{a\in Q_1}\left(\frac{\tau_{ha}}{\sigma_{ha}}-\frac{\tau_{ta}}{\sigma_{ta}}\right)|\phi_a|^2_{H_a}=\sum_{i\in Q_0}\frac{\tau_i}{\sigma_i}\operatorname{Tr}[\phi,\phi^{*H}]_i=\sum_{i\in Q_0}\left(\frac{\tau_i^2 r_i}{\sigma_i}-\tau_i\operatorname{Tr}(\mathbf{i}\,\Lambda_\omega F_{H_i})\right). \tag{4.3}$$

Given a fixed Kähler form ω on X, the first set of equations (4.1a), involving a Hermitian metric H on (E,ϕ), were called the (σ,τ)*-vortex equations* on (E,ϕ) over the Kähler manifold (X,ω) in [5], where their symplectic interpretation and their relation with a (σ,τ)-*polystability condition* were provided. To explain how the larger set of equations (4.1) fit in the general moment-map picture of Section 3, we now fix the metrics and consider the holomorphic data as the unknowns. More precisely, we fix a compact real manifold X of dimension $2n$, with a symplectic form ω, and a pair (E,H) consisting of a set of C^∞ (complex) vector bundles E_i of ranks r_i, and a set of Hermitian metrics H_i on E_i, indexed by the vertices $i\in Q_0$. Let P_i be the frame G_i-bundle of the Hermitian vector bundle E_i, where $G_i = \mathrm{U}(r_i)$, for all $i \in Q_0$, and $\widetilde{\mathcal{G}}_i$ the extended gauge group of P_i over (X,ω). Let $P \to X$ be the fibre product of the principal bundles $P_i \to X$, for all $i \in Q_0$, and $\widetilde{\mathcal{G}}$ the extended gauge group of P over (X,ω). Then P is a principal G-bundle, where G is the direct product of the groups G_i, for all $i \in Q_0$, and we have short exact sequences

$$1 \to \mathcal{G}_i \longrightarrow \widetilde{\mathcal{G}}_i \xrightarrow{p} \mathcal{H} \to 1, \qquad 1 \to \mathcal{G} \longrightarrow \widetilde{\mathcal{G}} \xrightarrow{p} \mathcal{H} \to 1,$$

where $\mathcal{G}_i$ is the gauge group of P_i, the gauge group of P is the direct product

$$\mathcal{G} = \prod_{i\in Q_0} \mathcal{G}_i, \tag{4.4}$$

and $p\colon \widetilde{\mathcal{G}} \to \mathcal{H}$ is the fibre product of the group morphisms $p\colon \widetilde{\mathcal{G}}_i \to \mathcal{H}$, for all $i \in Q_0$. Let $\mathcal{A}_i$ is the space of connections on P_i. Consider the space of connections on P, denoted

$$\mathcal{A} = \prod_{i\in Q_0} \mathcal{A}_i.$$

To specify a symplectic structure on $\mathcal{A}$, we fix a vector $\alpha \in \mathbb{R}^{Q_0}_{>0}$, and define an Ad-invariant positive definite inner product (2.2) on the Lie algebra $\mathfrak{g}$ of G, given for all $a, b \in \mathfrak{g}$ by

$$(a,b) = -\sum_{i\in Q_0} \alpha_i \operatorname{Tr}(a_i \circ b_i), \tag{4.5}$$

where a_i, b_i are in the Lie algebra $\mathfrak{g}_i$ of G_i. Then the symplectic form (2.5) on $\mathcal{A}$ becomes

$$\omega_{\mathcal{A}}(a,b) = -\sum_{i\in Q_0} \alpha_i \int_X \operatorname{Tr}(a_i \wedge b_i) \wedge \frac{\omega^{n-1}}{n-1!}, \tag{4.6}$$

for $A \in \mathcal{A}$, $a, b \in T_A\mathcal{A} = \Omega^1(\operatorname{ad} E)$. Consider the element z of the centre of $\mathfrak{g}$ given by $z_i = -\mathbf{i}\, c_i \operatorname{Id}_{E_i}$, for all $i \in Q_0$, for fixed $c_i \in \mathbb{R}$. By Proposition 2.1, the $\widetilde{\mathcal{G}}$-action on $\mathcal{A}$ has equivariant moment map $\mu_{\widetilde{\mathcal{G}}}\colon \mathcal{A} \to (\operatorname{Lie}\widetilde{\mathcal{G}})^*$ given for all $A \in \mathcal{A}$, $\zeta \in \operatorname{Lie}\widetilde{\mathcal{G}}$ by

$$\begin{aligned} \langle \mu_{\widetilde{\mathcal{G}}}(A), \zeta\rangle &= \mathbf{i} \sum_{i\in Q_0} \alpha_i \int_X \operatorname{Tr}\left(\xi_i(\mathbf{i}\,\Lambda_\omega F_{A_i} - c_i \operatorname{Id}_{E_i})\right) \frac{\omega^n}{n!} \\ &\quad + \frac{1}{4}\int_X f \sum_{i\in Q_0} \left(\alpha_i \Lambda_\omega^2 \operatorname{Tr} F_{A_i}^2 + 4 c_i \alpha_i \operatorname{Tr}(\mathbf{i}\,\Lambda_\omega F_{A_i})\right) \frac{\omega^n}{n!}, \end{aligned} \tag{4.7}$$

where $\xi := A\zeta \in \operatorname{Lie}\mathcal{G}$ (so $\xi_i = A_i\zeta \in \operatorname{Lie}\mathcal{G}_i$), and $p(\zeta) = \eta_f$ with $f \in C^\infty_0(X)$ (see (2.11)). Define a Hermitian vector bundle over X by

$$\mathcal{R} = \bigoplus_{a\in Q_0} \mathcal{R}_a, \text{ with } \mathcal{R}_a = \operatorname{Hom}(E_{ta}, E_{ha}),$$

where the Hermitian metric is the orthogonal direct sum of the Hermitian metrics H_a on the vector bundles $\mathcal{R}_a$, given by the formulae $(\phi_a, \psi_a)_{H_a} := \operatorname{Tr}(\phi_a \psi_a^{*H_a})$, for all ϕ_a, ψ_a in the same fibre of $\mathcal{R}_a$. Consider now the space of C^∞ global sections of $\mathcal{R}$,

$$\mathcal{S} = \bigoplus_{a\in Q_0} \mathcal{S}_a, \text{ with } \mathcal{S}_a = \Gamma(X, \mathcal{R}_a).$$

Then $\mathcal{S}$ has a symplectic form $\omega_{\mathcal{S}}$ defined for all $\phi \in \mathcal{S}$, $\dot\phi, \dot\psi \in T_\phi\mathcal{S} \cong \mathcal{S}$ by

$$\omega_{\mathcal{S}}(\dot\phi, \dot\psi) = \mathbf{i} \sum_{a\in Q_1} \int_X \operatorname{Tr}(\dot\phi_a \dot\psi_a^* - \dot\psi_a \dot\phi_a^*) \frac{\omega^n}{n!}.$$

Since $\omega_{\mathcal{S}} = d\sigma$, for the 1-form σ on $\mathcal{S}$ given for all $\phi \in \mathcal{S}, \dot{\phi} \in T_\phi \mathcal{S}$ by

$$\sigma(\dot{\phi}) = -\frac{\mathbf{i}}{2} \sum_{a \in Q_1} \int_X (\dot{\phi}_a \phi_a^* - \phi_a \dot{\phi}_a^*) \frac{\omega^n}{n!},$$

the canonical $\widetilde{\mathcal{G}}$-action is Hamiltonian, with equivariant moment map $\mu_{\mathcal{S}} \colon \mathcal{S} \to (\operatorname{Lie} \widetilde{\mathcal{G}})^*$ given by $\langle \mu_{\mathcal{S}}(\phi), \zeta \rangle = -\sigma(Y_\zeta(\phi))$, where the infinitesimal action of $\zeta \in \operatorname{Lie} \widetilde{\mathcal{G}}$ on $\mathcal{S}$ is the vector field on $\mathcal{S}$ with value $Y_\zeta(\phi) = \xi \cdot \phi - p(\zeta) \lrcorner d_A \phi$ on $\phi \in \mathcal{S}$. Here, $\xi = A\zeta \in \operatorname{Lie} \mathcal{G}$ (so $\xi_i = A_i(\zeta_i) \in \operatorname{Lie} \mathcal{G}_i$), the action of ξ on ϕ is given by $(\xi \cdot \phi)_a = \xi_{ha} \phi_a - \phi_a \xi_{ta}$, and $d_A \phi$ is the covariant derivative with respect to the connection induced by A on $\mathcal{R}$. More explicitly,

$$\langle \mu_{\mathcal{S}}(\phi), \zeta \rangle = \mathbf{i} \sum_{i \in Q_0} \int_X \operatorname{Tr}([\phi, \phi^*]_i \xi_i) \frac{\omega^n}{n!} - \mathbf{i} \int_X f \sum_{a \in Q_1} \Lambda_\omega d(d_{A_a} \phi_a, \phi_a)_{H_a} \frac{\omega^n}{n!}. \tag{4.8}$$

Fix $\rho \in \mathbb{R}_{>0}$. Then we consider the space of triples $\mathcal{J} \times \mathcal{A} \times \mathcal{S}$, with the symplectic form

$$\omega_{\alpha,\rho} = \omega_{\mathcal{J}} + 4\omega_{\mathcal{A}} + 4\rho\omega_{\mathcal{S}}, \tag{4.9}$$

with $\mathcal{J}$ and $\omega_{\mathcal{J}}$ as in Section 2.3. Adding (2.18), (4.7) and (4.8), we see that the diagonal $\widetilde{\mathcal{G}}$-action on $\mathcal{J} \times \mathcal{A} \times \mathcal{S}$ has equivariant moment map $\mu_{\alpha,\rho} \colon \mathcal{J} \times \mathcal{A} \times \mathcal{S} \to (\operatorname{Lie} \widetilde{\mathcal{G}})^*$ given by

$$\begin{aligned} \langle \mu_{\alpha,\rho}(J, A, \phi), \zeta \rangle &= 4\,\mathbf{i} \sum_{i \in Q_0} \int_X \operatorname{Tr}\left(\xi_i(\alpha_i \,\mathbf{i}\, \Lambda_\omega F_{A_i} + \rho[\phi, \phi^*]_i - \alpha_i c_i \operatorname{Id}_{E_i})\right) \frac{\omega^n}{n!} \\ &- \int_X f \left(S_J - \sum_{i \in Q_0} \left(\alpha_i \Lambda_\omega^2 \operatorname{Tr} F_{A_i}^2 + 4 c_i \alpha_i \operatorname{Tr}(\mathbf{i}\, \Lambda_\omega F_{A_i}) \right) + 4\rho \sum_{a \in Q_1} \mathbf{i}\, \Lambda_\omega d(d_{A_a} \phi_a, \phi_a)_{H_a} \right) \frac{\omega^n}{n!}, \end{aligned} \tag{4.10}$$

for all $(J, A, \phi) \in \mathcal{J} \times \mathcal{A} \times \mathcal{S}$, $\zeta \in \operatorname{Lie} \widetilde{\mathcal{G}}$, with $\xi := A\zeta \in \operatorname{Lie} \mathcal{G}$, $p(\zeta) = \eta_f$, $f \in C_0^\infty(X)$.

Consider the $\widetilde{\mathcal{G}}$-invariant subspace $\mathcal{T} \subset \mathcal{J} \times \mathcal{A} \times \mathcal{S}$ of 'integrable triples' (J, A, ϕ), defined by the conditions $J \in \mathcal{J}^i, A_i \in (\mathcal{A}_i)_J^{1,1}, \bar{\partial}_{J, A_a} \phi_a = 0$, for all $i \in Q_0, a \in Q_1$ (cf. (3.4)). Since

$$\Delta_\omega |\phi_a|_{H_a}^2 = 2\,\mathbf{i}\, \Lambda_\omega \bar{\partial}\partial |\phi_a|_{H_a}^2 = 2\,\mathbf{i}\, \Lambda_\omega d(d_{A_a} \phi_a, \phi_a)_{H_a}$$

when $\bar{\partial}_{A_a} \phi_a = 0$, the $\widetilde{\mathcal{G}}$-action has equivariant moment map $\mu_{\alpha,\rho} \colon \mathcal{T} \to (\operatorname{Lie} \widetilde{\mathcal{G}})^*$ given by

$$\begin{aligned} \langle \mu_{\alpha,\rho}(J, A, \phi), \zeta \rangle &= 4\,\mathbf{i} \sum_{i \in Q_0} \int_X \operatorname{Tr}\left(\xi_i(\alpha_i \,\mathbf{i}\, \Lambda_\omega F_{A_i} + \rho[\phi, \phi^*]_i - \alpha_i c_i \operatorname{Id}_{E_i})\right) \frac{\omega^n}{n!} \\ &- \int_X f \left(S_J + 2\rho \Delta_\omega \sum_{a \in Q_1} |\phi_a|_{H_a}^2 - \sum_{i \in Q_0} \left(\alpha_i \Lambda_\omega^2 \operatorname{Tr} F_{A_i}^2 + 4 c_i \alpha_i \operatorname{Tr}(\mathbf{i}\, \Lambda_\omega F_{A_i}) \right) \right) \frac{\omega^n}{n!}, \end{aligned} \tag{4.11}$$

for all $(J, A, \phi) \in \mathcal{T}$. Defining now $\sigma_i = \alpha_i / \rho$ and $\tau_i = \alpha_i c_i / \rho$, we see that the vanishing condition $\mu_{\alpha,\rho}(J, A, \phi) = 0$ for a triple $(J, A, \phi) \in \mathcal{T}$ is equivalent to the equations

$$\begin{aligned} \sigma_i \,\mathbf{i}\, \Lambda_\omega F_{H_i} + [\phi, \phi^{*H}]_i &= \tau_i \operatorname{Id}_{E_i}, \\ S_\omega + 2\rho \Delta_\omega \sum_{a \in Q_1} |\phi_a|_{H_a}^2 - \rho \sum_{i \in Q_0} (\sigma_i \Lambda_\omega^2 \operatorname{Tr} F_{H_i}^2 + 4\tau_i \operatorname{Tr}(\mathbf{i}\, \Lambda_\omega F_{H_i})) &= c', \end{aligned} \tag{4.12}$$

expressed in terms of the metrics ω and H, where $c' \in \mathbb{R}$. By (4.3), these equations are equivalent to the gravitating vortex equations (4.1), with c' replaced by another $c \in \mathbb{R}$.

4.2. **Dimensional reduction.** We will now consider the invariant solutions of the Kähler–Yang–Mills equations on an equivariant vector bundle over $M = X \times K^c/P$. Here, X is a compact complex manifold, K^c is a connected simply connected semisimple complex Lie group, and $P \subset K^c$ is a parabolic subgroup, so the quotient K^c/P for the P-action by right multiplication on K^c is a flag manifold. The group K^c acts trivially on the first factor X and in the standard way on K^c/P. The Kähler–Yang–Mills equations for the compact complex manifold M, a holomorphic vector bundle $\widetilde{E} \to M$, and a fixed real parameter $\alpha > 0$, are

$$\mathbf{i}\,\Lambda_{\widetilde{\omega}} F_{\widetilde{H}} = \mu_{\widetilde{\omega}}(\widetilde{E})\,\mathrm{Id}_{\widetilde{E}}, \tag{4.13a}$$

$$S_{\widetilde{\omega}} - \alpha\Lambda^2_{\widetilde{\omega}}\,\mathrm{Tr}\,F^2_{\widetilde{H}} = C. \tag{4.13b}$$

They involve a pair consisting of a Kähler form $\widetilde{\omega}$ on M and a Hermitian metric $\widetilde{H}$ on $\widetilde{E}$, with normalized slope $\mu_{\widetilde{\omega}}(\widetilde{E})$ defined by (4.2).

Let $L \subset P$ be a (reductive) Levi subgroup, and $K \subset K^c$ a maximal compact Lie subgroup. Then the K-invariant Kähler 2-forms ω_ε on the complex K^c-manifold $K/(K \cap P) \cong K^c/P$ are parametrized by elements $\varepsilon \in \mathbb{R}^\Sigma_{>0}$ (see [4, p. 38, Lemma 4.8]), where Σ is a fixed set of 'non-parabolic simple roots', as defined in [4, §1.5.1]. For a fixed $\varepsilon \in \mathbb{R}^\Sigma_{>0}$ and each choice of Kähler form ω on X, we consider the K-invariant Kähler form on M defined by

$$\widetilde{\omega} = \omega + \omega_\varepsilon \tag{4.14}$$

(hereafter we omit the symbols for the pullbacks by the canonical projections $M \to X$, $M \to K^c/P$).

In [4], the first and the third authors proved that there exist an infinite quiver Q and a set of relations $\mathcal{K}$ of Q, such that a K^c-equivariant holomorphic vector bundle $\widetilde{E}$ over M is equivalent to a holomorphic Q-bundle (E, ϕ) over X that satisfies the relations in $\mathcal{K}$ (see [4, p. 19, Theorem 2.5]). The vertex set Q_0 consists of the isomorphism classes of (finite-dimensional complex) irreducible representations of L. Under this equivalence, the K-invariant Hermitian metrics $\widetilde{H}$ on the vector bundle $\widetilde{E}$ over M are in bijection with the Hermitian metrics H on the quiver bundle (E, ϕ) over X (see [4, §4.2.4]). Furthermore, for each choice of Kähler form ω on X, a K-invariant Hermitian metric $\widetilde{H}$ satisfies the Hermitian–Yang–Mills equation (4.13a) on $\widetilde{E}$ over $(M, \widetilde{\omega})$ if and only if the corresponding Hermitian metric H on (E, ϕ) over (X, ω) satisfies the quiver (σ, τ)-vortex equations (4.1a) (see [4, §4.2.2, Theorem 4.13]). Here, the parameters $\sigma \in \mathbb{R}^{Q_0}_{>0}$ and $\tau \in \mathbb{R}^{Q_0}$ are given by

$$\sigma_\lambda = \dim_{\mathbb{C}} M_\lambda, \quad \tau_\lambda = \sigma_\lambda(\mu_{\widetilde{\omega}}(\widetilde{E}) - \mu_\varepsilon(\mathcal{O}_\lambda)), \tag{4.15}$$

for all $\lambda \in Q_0$, where M_λ is an irreducible representation of L (or P) in the isomorphism class λ, $\mathcal{O}_\lambda = K^c \times_P M_\lambda$ is the homogeneous vector bundle over K^c/P associated to M_λ, and the normalized slopes $\mu_\varepsilon(\mathcal{O}_\lambda) := \mu_{\omega_\varepsilon}(\mathcal{O}_\lambda)$, defined by (4.2), are explicitly given by [4, (4.16), §4.2.3]. Note that the vortex equations (4.1a), and the symplectic interpretation in Section 4.1, make sense for the infinite quiver Q, as $E_\lambda \neq 0$ only for finitely many $\lambda \in Q_0$, and the quiver Q is locally finite, that is, $t^{-1}(a)$ and $h^{-1}(a)$ are finite sets for all $a \in Q_1$.

The following correspondence extends these bijections to the Kähler–Yang–Mills equations. It includes [2, Proposition 3.4] for a particular class of equivariant bundles when $K^c/P = \mathbb{P}^1$. As above, $\widetilde{E}$ is a K^c-equivariant holomorphic vector bundle over M, and (E, ϕ) is the corresponding holomorphic Q-bundle over X.

Theorem 4.2. *Let ω be a Kähler form on X and $\widetilde{\omega}$ the K-invariant Kähler form on M defined by* (4.14)*. Let $\widetilde{H}$ be a K-invariant Hermitian metric on $\widetilde{E}$, and H the corresponding Hermitian metric on (E,ϕ). Then the pair $(\widetilde{\omega},\widetilde{H})$ satisfies the Kähler–Yang–Mills equations* (4.13) *if and only if (ω,H) satisfies the quiver (ρ,σ,τ)-vortex equations* (4.1)*, where $\rho:=\alpha$, and $\sigma\in\mathbb{R}^{Q_0}_{>0}$ and $\tau\in\mathbb{R}^{Q_0}$ are given by* (4.15).

Proof. Let $Q'\subset Q$ be the finite full subquiver with vertex set Q'_0 consisting of the vertices $\lambda\in Q_0$ such that $E_\lambda\neq 0$, so (E,ϕ) is a Q'-bundle over X. Let $\widetilde{A}$ and A_λ be the Chern connections of $\widetilde{H}$ and H_λ on the holomorphic vector bundles $\widetilde{E}$ and E_λ, respectively, for $\lambda\in Q'_0$. The vector bundles E_λ and the Hermitian metrics H_λ on E_λ, for $\lambda\in Q'_0$, specify the K-action on $\widetilde{E}$ and its Hermitian metric $\widetilde{H}$, respectively, via the identification

$$\widetilde{E}=\bigoplus_{\lambda\in Q'_0}E_\lambda\otimes\mathcal{O}_\lambda \tag{4.16}$$

between K-equivariant C^∞ Hermitian vector bundles, where the homogeneous vector bundles $\mathcal{O}_\lambda$ are endowed with their unique (up to scale) K-invariant Hermitian metrics. Furthermore, the Higgs fields ϕ_a and the unitary connections A_λ, for $a\in Q_1,\lambda\in Q'_0$, specify the unitary connection $\widetilde{A}$ on $\widetilde{E}$, given by $d_{\widetilde{A}}=d_{A^\circ}+\theta$, with $\theta=\beta-\beta^*\in\Omega^1(\operatorname{ad}\widetilde{E})$ and

$$d_{A^\circ}=\sum_{\lambda\in Q'_0}(d_{A_\lambda}\otimes\mathrm{Id}_{\mathcal{O}_\lambda}+\mathrm{Id}_{E_\lambda}\otimes d_{A'_\lambda})\circ\pi_\lambda,\quad \beta=\sum_{a\in Q'_1}\phi_a\otimes\eta_a, \tag{4.17}$$

where A'_λ is the unique K-invariant unitary connection on $\mathcal{O}_\lambda$, $\pi_\lambda\colon\widetilde{E}\to E_\lambda\otimes\mathcal{O}_\lambda$ are the canonical projections, and $\{\eta_a\mid a\in t^{-1}(\lambda)\cap h^{-1}(\mu)\}$ is a basis of the space of K-invariant $\mathrm{Hom}(\mathcal{O}_\lambda,\mathcal{O}_\mu)$-valued $(0,1)$-forms on K^c/P, for all $\lambda,\mu\in Q'_0$ (see [4, §3.4.5]).

We will use the moment-map interpretations of the Kähler–Yang–Mills equations and the quiver gravitating vortex equations. Let $\widetilde{\mathcal{J}}$, $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{G}}_M$ be the space of almost complex structures $\widetilde{J}$ on $(M,\widetilde{\omega})$, the space of unitary connections on $(\widetilde{E},\widehat{H})$, and the extended gauge group of the symplectic manifold $(M,\widetilde{\omega})$ and the Hermitian vector bundle $(\widetilde{E},\widetilde{H})$, respectively. By [1, Proposition 2.1], the $\widetilde{\mathcal{G}}_M$-action on $\widetilde{\mathcal{J}}\times\widetilde{\mathcal{A}}$, with symplectic form $\omega_{\widetilde{\mathcal{J}}}+4\alpha\omega_{\widetilde{\mathcal{A}}}$, has equivariant moment map $\widetilde{\mu}_{\widetilde{\mathcal{G}}_M}\colon\widetilde{\mathcal{J}}\times\widetilde{\mathcal{A}}\to(\operatorname{Lie}\widetilde{\mathcal{G}}_M)^*$ given by

$$\langle\widetilde{\mu}_{\widetilde{\mathcal{G}}_M}(\widetilde{J},\widetilde{A}),\widetilde{\zeta}\rangle=4\alpha\langle\widetilde{\mu}_{\mathcal{G}_M}(\widetilde{A}),\widetilde{\xi}\rangle+\int_M\widetilde{f}S_\alpha(\widetilde{J},\widetilde{A})\frac{\widetilde{\omega}^m}{m!}, \tag{4.18}$$

for all $\widetilde{\zeta}\in\operatorname{Lie}\widetilde{\mathcal{G}}_M$, where $\widetilde{\xi}=\widetilde{A}\widetilde{\zeta}\in\operatorname{Lie}\mathcal{G}_M$, $\widetilde{f}\in C^\infty_0(M)$ is such that $d\widetilde{f}=p(\widetilde{\zeta})\lrcorner\widetilde{\omega}$, $m=\dim_{\mathbb{C}}M$, and $\widetilde{\mu}_{\mathcal{G}_M}\colon\widetilde{\mathcal{A}}\to(\operatorname{Lie}\mathcal{G}_M)^*$, $S_\alpha(\widetilde{J},\widetilde{A})\in C^\infty(M)$ (cf. [1, (3.78)]) are given by

$$\langle\widetilde{\mu}_{\mathcal{G}_M}(\widetilde{A}),\widetilde{\xi}\rangle=\mathbf{i}\int_M\mathrm{Tr}((\mathbf{i}\,\Lambda_{\widetilde{\omega}}F_{\widetilde{A}}-\mu_{\widetilde{\omega}}(\widetilde{E})\,\mathrm{Id}_{\widetilde{E}})\widetilde{\xi})\frac{\widetilde{\omega}^m}{m!}, \tag{4.19a}$$

$$S_\alpha(\widetilde{J},\widetilde{A})=-S_{\widetilde{J}}+4\alpha\mu_{\widetilde{\omega}}(\widetilde{E})\,\mathrm{Tr}(\mathbf{i}\,\Lambda_{\widetilde{\omega}}F_{\widetilde{A}})+\alpha\Lambda^2_{\widetilde{\omega}}\,\mathrm{Tr}\,F^2_{\widetilde{A}}. \tag{4.19b}$$

By construction, $\Lambda_{\widetilde{\omega}}F_{\widetilde{A}}+\mathbf{i}\,\mu_{\widetilde{\omega}}(\widetilde{E})\,\mathrm{Id}_{\widetilde{E}}\in\operatorname{Lie}\widetilde{\mathcal{G}}^K_M$, so $\widetilde{\mu}_{\mathcal{G}_M}(\widetilde{A})=0$ if and only if $\langle\widetilde{\mu}_{\mathcal{G}_M}(\widetilde{A}),\widetilde{\xi}\rangle=0$ for all $\widetilde{\xi}\in\operatorname{Lie}\mathcal{G}^K_M$ (where $(-)^K$ means the fixed-point subspace for the K-action). Using the last displayed formula for $\mathbf{i}\,\Lambda_{\widetilde{\omega}}F_{\widetilde{A}}$ in [4, §4.2.4]), we see that

$$\langle\widetilde{\mu}_{\mathcal{G}_M}(\widetilde{A}),\widetilde{\xi}\rangle=\mathrm{Vol}_\varepsilon(K^c/P)\sum_{\lambda\in Q'_0}\mathbf{i}\int_X\mathrm{Tr}\,((\sigma_\lambda\,\mathbf{i}\,\Lambda_\omega F_{A_\lambda}+[\phi,\phi^*]_\lambda-\tau_\lambda\,\mathrm{Id}_{E_\lambda})\xi_\lambda)\,\frac{\omega^n}{n!}, \tag{4.20}$$

where $\xi \in \operatorname{Lie}\mathcal{G}^K$ corresponds to $(\xi_\lambda)_{\lambda\in Q_0'}$, with $\xi_\lambda \in \operatorname{Lie}\mathcal{G}_\lambda$, by [4, Proposition 3.4], $\mathcal{G}_\lambda$ being the unitary gauge group of E_λ, and $\operatorname{Vol}_\varepsilon(K^c/P) = \int_{K^c/P} \omega_\varepsilon^l/l!$, with $l = \dim_{\mathbb{C}}(K^c/P)$. This gives the correspondence for the vortex equations (4.1a) and the Hermitian–Yang–Mills equation (4.13a). To compare (4.1b) and (4.13b), we calculate separately the terms involved in (4.19b), namely,

$$-S_{\widetilde{J}} = -S_J + \text{const.}, \tag{4.21a}$$

$$4\alpha\mu_{\widetilde{\omega}}(\widetilde{E}) \operatorname{Tr}(\mathbf{i}\,\Lambda_{\widetilde{\omega}}F_{\widetilde{A}}) = 4\rho\mu_{\widetilde{\omega}}(\widetilde{E}) \sum_{\lambda\in Q_0'} \sigma_\lambda \operatorname{Tr}(\mathbf{i}\,\Lambda_\omega F_{A_\lambda}\xi_\lambda) + \text{const.}, \tag{4.21b}$$

$$\begin{aligned}\alpha\Lambda_{\widetilde{\omega}}^2 \operatorname{Tr} F_{\widetilde{A}}^2 = &\sum_{\lambda\in Q_0'} \big(\rho\sigma_\lambda\Lambda_\omega^2 \operatorname{Tr} F_A^2 - 4\rho\sigma_\lambda\mu_\varepsilon(\mathcal{O}_\lambda)\operatorname{Tr}(\mathbf{i}\,\Lambda_\omega F_{A_\lambda})\big)\\ &- \sum_{a\in Q_1'} 4\rho\,\mathbf{i}\,\Lambda_\omega d\operatorname{Tr}(d_{A_a}\phi_a\circ\phi_a^*)\\ &- \rho\sum_{a,b\in Q_1'} \operatorname{Tr}(\phi_a\circ\phi_b^*)\Lambda_{\omega_\varepsilon}^2 d\operatorname{Tr}(\eta_a\wedge d_{A_b'}\eta_b^* + d_{A_a'}\eta_a\wedge\eta_b^*) + \text{const.},\end{aligned} \tag{4.21c}$$

where A_a (resp. A_a') is the connection induced by A_{ta} and A_{ha} (resp. A_{ta}' and A_{ha}') on the vector bundle $\operatorname{Hom}(E_{ta}, E_{ha})$ (resp. $\operatorname{Hom}(\mathcal{O}_{ta}, \mathcal{O}_{ha})$), and the sums in $a, b \in Q_1'$ in (4.21b) are constrained to the condition $ta = tb, ha = hb$ (so that the traces are well defined). Formula (4.21a) follows because the scalar curvature of ω_ε on K^c/P is K-invariant by construction, and hence it is constant, as K acts effectively on K^c/P. Formula (4.21b) is obtained taking traces in the last displayed formula for $\mathbf{i}\,\Lambda_{\widetilde{\omega}}F_{\widetilde{A}}$ in [4, §4.2.4]). We prove (4.21c) making the substitution $F_{\widetilde{A}} = F_{A^\circ} + d_{A^\circ}\theta + \theta^2$, obtaining

$$\begin{aligned}\Lambda_{\widetilde{\omega}}^2 \operatorname{Tr} F_{\widetilde{A}}^2 = &\Lambda_{\widetilde{\omega}}^2 \operatorname{Tr} F_{A^\circ}^2 + 2\Lambda_{\widetilde{\omega}}^2 \operatorname{Tr}(F_{A^\circ}\wedge d_{A^\circ}\theta) + \Lambda_{\widetilde{\omega}}^2 \operatorname{Tr}(\theta^4)\\ &+ 2\Lambda_{\widetilde{\omega}}^2 \operatorname{Tr}(d_{A^\circ}\theta\wedge\theta^2) + \Lambda_{\widetilde{\omega}}^2 \operatorname{Tr}((d_{A^\circ}\theta)^2) + 2\Lambda_{\widetilde{\omega}}^2 \operatorname{Tr}(F_{A^\circ}\wedge\theta^2),\end{aligned} \tag{4.22}$$

and calculating the six terms in the right-hand side:

$$\Lambda_{\widetilde{\omega}}^2 \operatorname{Tr} F_{A^\circ}^2 = \sum_{\lambda\in Q_0'} \big(\sigma_\lambda\Lambda_\omega^2 \operatorname{Tr} F_{A_\lambda}^2 - 4\sigma_\lambda\mu_\varepsilon(\mathcal{O}_\lambda)\operatorname{Tr}(\mathbf{i}\,\Lambda_\omega F_{A_\lambda})\big) + \text{ const.}, \tag{4.23a}$$

$$\Lambda_{\widetilde{\omega}}^2 \operatorname{Tr}(F_{A^\circ}\wedge d_{A^\circ}\theta) = 0,\ \Lambda_{\widetilde{\omega}}^2 \operatorname{Tr}(\theta^4) = 0,\ \Lambda_{\widetilde{\omega}}^2 \operatorname{Tr}(d_{A^\circ}\theta\wedge\theta^2) = 0, \tag{4.23b}$$

$$\Lambda_{\widetilde{\omega}}^2 \operatorname{Tr}((d_{A^\circ}\theta)^2) = \sum_{a,b\in Q_1'} \big(4\operatorname{Tr}\mathbf{i}\,\Lambda_\omega(d_{A_a}\phi_a\wedge d_{A_b}\phi_b^*) - 2\operatorname{Tr}(\phi_a\circ\phi_b^*)\Lambda_{\omega_\varepsilon}^2 \operatorname{Tr}(d_{A_a'}\eta_a\wedge d_{A_b'}\eta_b^*)\big)\,, \tag{4.23c}$$

$$\begin{aligned}\Lambda_{\widetilde{\omega}}^2 \operatorname{Tr}(F_{A^\circ}\wedge\theta^2) = &-2\sum_{a\in Q_1'} \operatorname{Tr}\left(\mathbf{i}\,\Lambda_\omega F_{A_{ha}}\circ\phi_a\circ\phi_a^* - \mathbf{i}\,\Lambda_\omega F_{A_{ta}}\circ\phi_a^*\circ\phi_a\right)\\ &+ \sum_{a,b\in Q_1'} \operatorname{Tr}(\phi_a\circ\phi_b^*)\Lambda_{\omega_\varepsilon}^2 \operatorname{Tr}(F_{A_{ha}'}\wedge\eta_a\wedge\eta_b^* + F_{A_{ta}'}\wedge\eta_a^*\wedge\eta_b).\end{aligned} \tag{4.23d}$$

Formula (4.23a) follows from the definition of d_{A° in (4.17), the identities $\mathbf{i}\,\Lambda_{\omega_\varepsilon}F_{A_\lambda'} = \mu_\varepsilon(\mathcal{O}_\lambda)\operatorname{Id}_{\mathcal{O}_\lambda}$ (see [4, Lemma 4.15, §4.2.3]), and the fact that $\Lambda_{\omega_\varepsilon}^2 \operatorname{Tr} F_{A_\lambda'}^2 \in C^\infty(K^c/P)$ is K-invariant, and hence constant. The first identity in (4.23b) follows by using (4.17) and observing the quiver Q' has no oriented cycles [4, Lemma 1.15]. The second identity in (4.23b) follows from $\operatorname{Tr}(\theta\wedge\theta^3) = -\operatorname{Tr}(\theta^3\wedge\theta)$ (as θ is a 1-form). Using (4.17) and the orthogonal direct-sum decomposition

$$TM = TX \oplus T(K^c/P) \tag{4.24}$$

(with respect to the metric $\widetilde{\omega}$), it is not difficult to derive the identity

$$\Lambda^2_{\widetilde{\omega}} \operatorname{Tr}(d_{A^\circ}\theta \wedge \theta^2) = -\sum_{a,b,c\in Q'_1} \operatorname{Tr}(\phi_a \circ \phi_b \circ \phi_c^*) \otimes \Lambda^2_{\omega_\varepsilon} d \operatorname{Tr}(\eta_a \wedge \eta_b \wedge \eta_c^*) + \text{c. c.}, \tag{4.25}$$

where "c.c." means complex conjugate. The third identity in (4.23b) now follows because the function $\Lambda^2_{\widetilde{\omega}} \operatorname{Tr}(d_{A^\circ}\theta \wedge \theta^2)$ is K-invariant (by construction), so it equals its average by fibre integration along the canonical projection $M \to X$, that vanishes because in (4.25),

$$\int_{K^c/P} \Lambda^2_{\omega_\varepsilon} d \operatorname{Tr}(\eta_a \wedge \eta_b \wedge \eta_c^*)\omega_\varepsilon^l = 0.$$

To prove (4.23c), we use (4.17) with $\theta = \beta - \beta^*$, and the facts that the quiver Q' has no oriented cycles and the direct-sum decomposition (4.24) is orthogonal, obtaining

$$\Lambda^2_{\widetilde{\omega}} \operatorname{Tr}((d_{A^\circ}\theta)^2) = -2\Lambda^2_{\widetilde{\omega}} \operatorname{Tr}(d_{A^\circ}\beta \wedge d_{A^\circ}\beta^*).$$

To prove that this equals the right-hand side of (4.23c), one needs to make another calculation using (4.17), choosing the basis $\{\eta_a\}$ as in [4, §4.2.4], so the pointwise inner product

$$-\operatorname{Tr} \mathbf{i}\, \Lambda_{\omega_\varepsilon}(\eta_a \wedge \eta_b^*) = \delta_{ab} \tag{4.26}$$

is the Kronecker delta for $ta = tb, ha = hb$, and the orthogonal decomposition (4.24).

Finally, using (4.17) and the decomposition $\theta = \beta - \beta^*$, one can prove that

$$\Lambda^2_{\widetilde{\omega}} \operatorname{Tr}(F_{A^\circ} \wedge \theta^2) = -\Lambda^2_{\widetilde{\omega}} \operatorname{Tr}(F_{A^\circ} \wedge [\beta, \beta^*]),$$

where $[\beta, \beta^*] = \beta \wedge \beta^* + \beta^* \wedge \beta$, because $\Lambda_{\omega_\varepsilon}$ and $\Lambda^2_{\omega_\varepsilon}$ respectively vanish when applied to $(2,0)$ and $(0,2)$-forms, and to $(1,3)$ and $(3,1)$-forms. To show that this is equal to the right-hand side of (4.23d), one has to use (4.17) once again, and (4.26).

Formula (4.21c) follows from (4.22), (4.23), and the fact that the connections A_λ and A'_λ are unitary, and so putting together the right-hand sides of (4.23c) and (4.23d), we obtain

$$\begin{aligned}\Lambda^2_{\widetilde{\omega}} \operatorname{Tr}((d_{A^\circ}\theta)^2) + 2\Lambda^2_{\widetilde{\omega}} \operatorname{Tr}(F_{A^\circ} \wedge \theta^2) = &- 4 \sum_{a\in Q'_0} \mathbf{i}\, \Lambda_\omega d \operatorname{Tr}(d_{A_a}\phi_a \circ \phi_a^*) \\ &- \sum_{a,b\in Q'_0} \operatorname{Tr}(\phi_a \circ \phi_b^*)\Lambda^2_{\omega_\varepsilon} d \operatorname{Tr}(\eta_a \wedge d_{A'_b}\eta_b^* + d_{A'_a}\eta_a \wedge \eta_b^*).\end{aligned}$$

We can now compare (4.1b) and (4.13b). By construction, $S_\alpha(\widetilde{J}, \widetilde{A}) \in C^\infty(M)^K$, so $S_\alpha(\widetilde{J}, \widetilde{A}) = \text{const.}$ if and only if the last term in (4.19b) vanishes for all $\widetilde{f} \in C_0^\infty(M)^K$, i.e. $\widetilde{f} = f \circ p_X$ with $f \in C_0^\infty(X)$, $p_X \colon M \to X$ being the canonical projection. In this case,

$$\int_M \widetilde{f} S_\alpha(\widetilde{J}, \widetilde{A}) \frac{\widetilde{\omega}^m}{m!} = \operatorname{Vol}_\varepsilon(K^c/P) \int_X f S_{\rho,\sigma,\tau}(J, A, \phi)\frac{\omega^n}{n!},$$

where, adding the three identities in (4.21), we have

$$S_{\rho,\sigma,\tau}(J, A, \phi) := -S_J + \rho \sum_{\lambda\in Q'_0} (\sigma_\lambda \Lambda^2_\omega \operatorname{Tr} F^2_{A_\lambda} + 4\tau_\lambda \operatorname{Tr}(\mathbf{i}\, \Lambda_\omega F_{A_\lambda})) - 4\rho \sum_{a\in Q'_1} \mathbf{i}\, \Lambda_\omega d \operatorname{Tr}(d_{A_a}\phi_a \circ \phi_a^*).$$

Combining this and (4.20) in (4.18), we see that $\langle \widetilde{\mu}_{\widetilde{\mathcal{G}}_M}(\widetilde{J}, \widetilde{A}), \widetilde{\zeta}\rangle$ equals (4.10), up to a factor $\operatorname{Vol}_\varepsilon(K^c/P)$. This implies the correspondence for (4.1b) and (4.13b), as required. □

Note that the relations in the set $\mathcal{K}$ have not played any role in the proof of Theorem 4.2.

5. Examples

5.1. **Solutions in the weak coupling limit.** In this section we consider the Kähler–Yang–Mills–Higgs equations with coupling constants $\alpha = \beta$. Assuming $\alpha > 0$ and normalizing the first equation in (3.9), we obtain the system of equations

$$\begin{aligned} \Lambda_\omega F_H + \phi^*\hat{\mu} &= z, \\ S_\omega + \alpha\Delta_\omega|\phi^*\hat{\mu}|^2 + \alpha\Lambda^2_\omega(F_H \wedge F_H) - 4\alpha(\Lambda_\omega F_H, z) &= c. \end{aligned} \tag{5.1}$$

Following [1], this section is concerned with the existence of solutions of (5.1) in 'weak coupling limit' $0 < |\alpha| \ll 1$ by deforming solutions (ω, H) with coupling constants $\alpha = 0$.

Note that for $\alpha = 0$, the coupled equations (5.1) are the condition that ω is a constant scalar curvature Kähler (cscK) metric on X and H is a solution of the Yang–Mills–Higgs equation, as studied in [27]. If $\phi = 0$, then the existence of solutions of the first equation in (5.1) is equivalent, by the Theorem of Donaldson, Uhlenbeck and Yau [12, 31], to the polystability of the holomorphic bundle E^c with respect to the Kähler class $[\omega] \in H^2(X, \mathbb{R})$. For $\phi \neq 0$, Mundet i Riera [27] gave the following characterization of the existence of solutions.

Theorem 5.1 ([27]). *Assume that $\phi \neq 0$ and that (E^c, ϕ) is a simple pair. Then, for every fixed Kähler form ω, there exists a solution H of the Yang–Mills–Higgs equation if and only if (E^c, ϕ) is z-stable, in which case the solution is unique.*

The conditions of simplicity and z-stability in the previous theorem are rather technical, and we refer the reader to [27] for a detailed definition. To give an idea in the language of Section 3.2, a sufficient condition for (E^c, ϕ) to be a simple pair (see [27, Definition 2.17]) is that the Lie algebra $\operatorname{Lie}\operatorname{Aut}(E^c, \phi)$ of infinitesimal automorphisms of (E^c, ϕ) vanishes, where $\operatorname{Lie}\operatorname{Aut}(E^c, \phi)$ is given by the Kernel of

$$\operatorname{Lie}\operatorname{Aut}(X, E^c, \phi) \to H^0(X, TX).$$

The z-stability of the pair (E^c, ϕ) can regarded as a version of the geodesic stability in Definition 3.4, for (weak) geodesic rays (ω_t, H_t) with $\omega_t = \omega$ constant (see [27, Definition 2.16]).

Our next result is a consequence of the implicit function theorem in Banach spaces, combined with Theorem 5.1 and the moment map interpretation of the constant scalar curvature Kähler metric equation. The proof follows along the lines of [1, Theorem 4.18].

Theorem 5.2. *Assume that $\phi \neq 0$ and that (E^c, ϕ) is a simple pair. Assume that there is a cscK metric ω_0 on X with cohomology class $[\omega_0] = \Omega_0$ and that there are no non-zero Hamiltonian Killing vector fields on X. If (E^c, ϕ) is z-stable with respect to ω_0, then there exists an open neighbourhood $U \subset \mathbb{R} \times H^{1,1}(X, \mathbb{R})$ of $(0, \Omega_0)$ such that for all $(\alpha, \Omega) \in U$ there exists a solution of (5.1) with coupling constant α such that $[\omega] = \Omega$.*

We next provide an application of the previous theorem to the Kähler–Yang–Mills equations. Using the notation of Section 4.2, we fix a Kähler form ω_0 on X, a K-invariant Kähler form ω_ε on K^c/P (with $\varepsilon \in \mathbb{R}^\Sigma_{>0}$), and the product Kähler form $\widetilde{\omega}_0 = \omega_0 + \omega_\varepsilon$ on $M = X \times K^c/P$. Let $\Omega_0 = [\omega_0]$, $\Omega_\varepsilon = [\omega_\varepsilon]$ and $\widetilde{\Omega}_0 = [\widetilde{\omega}_0] = \Omega_0 + \Omega_\varepsilon$ be their cohomology classes on X, K^c/P and M, respectively. We also fix a K^c-equivariant holomorphic vector bundle $\widetilde{E}$ on M, and say $\widetilde{E}$ is *K^c-invariantly stable* (with respect to $\widetilde{\Omega}_0$)

if for all K^c-invariant proper subsheaves $\widetilde{E}' \subset \widetilde{E}$, their slopes with respect to $\widetilde{\Omega}_0$ satisfy $\mu_{\widetilde{\Omega}_0}(\widetilde{E}') < \mu_{\widetilde{\Omega}_0}(\widetilde{E})$ (cf. [4, Definition 4.6, §4.1.2]).

Corollary 5.3. *Assume that ω_0 is a constant scalar curvature Kähler metric on X, there are no non-zero Hamiltonian Killing vector fields on X, and $\widetilde{E}$ is K^c-invariantly stable with respect to Ω_0. Then there exists an open neighbourhood $U \subset \mathbb{R} \times H^{1,1}(X, \mathbb{R})$ of $(0, \Omega_0)$ such that for all $(\alpha, \Omega) \in U$, there exists a K-invariant solution $(\widetilde{\omega}, \widetilde{H})$ of the Kähler–Yang–Mills equations* (4.13) *on M with coupling constant α such that $[\widetilde{\omega}] = \Omega + \Omega_\varepsilon$.*

Proof. This follows from Theorems 5.2 and 4.2, and the correspondences of [4, §4]. To apply Theorem 5.2, we consider the holomorphic Q-bundle (E, ϕ) over X corresponding to $\widetilde{E}$, and the symplectic form (3.2) given by $\omega_{\mathcal{J}} + 4\rho\omega_{\mathcal{A}} + 4\rho\omega_{\mathcal{S}}$, i.e., with $\alpha = \beta$ both equal to ρ, where $\omega_{\mathcal{A}}$ is now defined using the invariant inner product (4.6) with $\alpha_\lambda = \dim_{\mathbb{C}} M_\lambda$. □

Note that this result is not covered by [1, Theorem 4.18], since the infinitesimal action by any non-zero element of $\mathfrak{k}^c/\mathfrak{p} \cong T_P(K^c/P)$ induces a nowhere-vanishing real holomorphic vector field over $X \times K^c/P$ (where $\mathfrak{p} \subset \mathfrak{k}^c$ are the Lie algebras of $P \subset K^c$, respectively).

To illustrate further the scope of application of Theorem 5.2, consider now a compact Riemann surface Σ with genus $g(\Sigma) > 1$, endowed with a Kähler metric ω_0 with constant curvature -1. We fix a holomorphic principal G^c-bundle over Σ and consider a unitary representation $\rho\colon G \to \mathrm{U}(W)$, for a hermitian vector space W. We take $F = \mathbb{P}(W)$, endowed with the Fubini–Study metric, rescaled by a real constant $\tau > 0$. Consider the associated ruled manifold

$$\mathcal{F} = E^c \times_{G^c} F = \mathbb{P}(E^c \times_{G^c} W).$$

Denote by $\mathbb{P}(W)^s \subset \mathbb{P}(W)$ the locus of stable points for the linearized G^c-action, and set

$$\mathcal{F}^s = E^c \times_{G^c} \mathbb{P}(W)^s \subset \mathcal{F}.$$

Then, if E^c is semistable with respect the Kähler class $[\omega_0]$ and $\phi \in H^0(\Sigma, \mathcal{F})$ is such that $\phi(\Sigma) \subset \mathcal{F}^s$, then (E^c, ϕ) is z-stable, for any z and any value of τ (see [26, p. 74]). Furthermore, we can also choose ϕ such that the pair is simple, by taking its image outside any proper G^c-invariant subspace $W' \subset W$.

For the sake of concreteness, consider the case that $G^c = \mathrm{GL}(r, \mathbb{C})$ and ρ is the standard representation in $W = \mathbb{C}^r$. Then $V = E^c \times_{G^c} W$ is a holomorphic vector bundle and $\phi \in H^0(\Sigma, \mathcal{F})$ can be identified with the inclusion

$$L \subset V$$

for a holomorphic subbundle L of rank one, as considered by Bradlow and the third author in [9]. Then, the pair (E^c, ϕ) is not simple if and only if one can find a holomorphic splitting

$$V = V' \oplus V''$$

such that L is contained in V'. Identifying $z = -i\lambda \operatorname{Id} \in \mathfrak{u}(r)$ for a real constant $\lambda \in \mathbb{R}$, the pair (E^c, ϕ) is z-stable if and only if for any non-zero proper subbundle $V' \subset V$ we have

$$\frac{\deg(V') + \tau \operatorname{rk}(L \cap V')}{\operatorname{rk}(V')} < \frac{\deg(V) + \tau}{r}.$$

In this simple situation, the equations (5.1) are for a Kähler metric ω on Σ and a hermitian metric H on V, and reduce to

$$i\Lambda_\omega F_H + \tau\pi_L^H = \lambda \,\mathrm{Id},$$
$$S_\omega + \alpha\tau^2\Delta_\omega|\pi_L^H|^2 = c',$$

for a suitable real constant $c' \in \mathbb{R}$, where $\pi_L^H \colon V \to L$ denotes the H-orthogonal projection.

5.2. **Gravitating vortices and Yang's Conjecture.** In this section we apply Theorem 3.6 to find an obstruction to the *gravitating vortex equations* on the Riemann sphere, as introduced in [2]. As an application, we give an alternative affirmative answer to Yang's Conjecture for the Einstein–Bogomol'nyi equations [33] (see also [34, p. 437]), that shall be compared with the original proof in [3, Corollary 4.7].

Consider $X = \mathbb{P}^1$, with $G^c = \mathbb{C}^*$ and $F = \mathbb{C}$, endowed with the standard hermitian structure. A $\mathbb{C}^*$-principal bundle on $\mathbb{P}^1$ is equivalent to a line bundle $\mathcal{O}_{\mathbb{P}^1}(N)$ of degree N, while the Higgs field is $\phi \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(N))$. Here we are concerned with the case $\phi \neq 0$, so we assume $N > 0$. Choose a real constant $\tau > 0$, and consider $z = -i\alpha\tau/2$. Then, the Kähler–Yang–Mills–Higgs equations (3.9) with coupling constants $\alpha = \beta > 0$ are equivalent to the gravitating vortex equations [2]

$$\begin{aligned} i\Lambda_\omega F_H + \frac{1}{2}(|\phi|_H^2 - \tau) &= 0, \\ S_\omega + \alpha(\Delta_\omega + \tau)(|\phi|_H^2 - \tau) &= c, \end{aligned} \tag{5.2}$$

where ω is a Kähler metric on $\mathbb{P}^1$ and H is a hermitian metric on $\mathcal{O}_{\mathbb{P}^1}(N)$. The constant $c \in \mathbb{R}$ is topological, and is explicitly given by

$$c = 2 - 2\alpha\tau N, \tag{5.3}$$

where have assumed the normalization $\int_{\mathbb{P}^1} \omega = 2\pi$.

The first equation in (5.2) is the *abelian vortex equation*. A theorem by Noguchi [28], Bradlow [7] and the third author [18, 19] implies that, upon a choice of Kähler metric with volume 2π, the equation

$$i\Lambda_\omega F_H + \frac{1}{2}(|\phi|_H^2 - \tau) = 0$$

admits a (unique) solution provided that $N < \tau/2$. As we will show next, this numerical condition is not enough to ensure the existence of solutions of the coupled system (5.2).

Theorem 5.4. *If ϕ has only one zero, then there are no solutions of the gravitating vortex equations for $(\mathbb{P}^1, L, \phi)$.*

Proof. Choose homogeneous coordinates $[x_0, x_1]$ on $\mathbb{P}^1$ such that ϕ is identified with the polynomial

$$\phi \cong x_0^N.$$

Here we use the natural identification $H^0(\mathbb{P}^1, L) \cong S^N(\mathbb{C}^2)^*$, where the right hand side is the space of degree N homogeneous polynomials in the coordinates x_0, x_1. By [3, Lemma 4.3], it follows that

$$\mathrm{Aut}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(N), \phi) \cong \mathbb{C}^* \rtimes \mathbb{C},$$

which is non-reductive. Consequently, the proof follows from Theorem 3.6. □

When the constant c in (5.3) is zero, the gravitating vortex equations (5.2) turn out to be a system of partial differential equations that have been extensively studied in the physics literature, known as the *Einstein–Bogomol'nyi equations*. Based on partial results in [33], Yang posed a conjecture about non-existence of solutions of the Einstein–Bogomol'nyi equations with ϕ having exactely one zero. This conjecture has been recently settled in the affirmative in [3]. As an application of Theorem 5.4, we provide here an alternative proof.

Corollary 5.5 (Yang's conjecture). *There is no solution of the Einstein–Bogomol'nyi equations for ϕ having exactly one zero.*

5.3. **Non-abelian vortices on $\mathbb{P}^1$.** We consider now the case of non-abelian rank-two vortices on the Riemann sphere (corresponding to $G = \mathrm{U}(2)$).

Let $X = \mathbb{P}^1$, with $G^c = \mathrm{GL}(2,\mathbb{C})$ and $F = \mathbb{C}^2$, endowed with the standard hermitian structure. A G^c-principal bundle on $\mathbb{P}^1$ is equivalent to a split rank-two bundle

$$V = \mathcal{O}_{\mathbb{P}^1}(N_1) \oplus \mathcal{O}_{\mathbb{P}^1}(N_2),$$

while the Higgs field is

$$\phi = (\phi_1, \phi_2) \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(N_1)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(N_2)).$$

We will assume $0 < N_1 \leqslant N_2$. Choose a real constant $\tau > 0$, and consider the central element $z = -i(\alpha\tau/2)\,\mathrm{Id}$. Then, the Kähler–Yang–Mills–Higgs equations (3.9) with coupling constants $\alpha = \beta > 0$ are equivalent to

$$\begin{aligned} i\Lambda_\omega F_H + \frac{1}{2}\phi \otimes \phi^{*H} &= \frac{\tau}{2}\,\mathrm{Id}, \\ S_\omega + \alpha(\Delta_\omega + \tau)(|\phi|_H^2 - 2\tau) &= c, \end{aligned} \tag{5.4}$$

where ω is a Kähler metric on $\mathbb{P}^1$ and H is a hermitian metric on V. The constant $c \in \mathbb{R}$ is topological, and is explicitly given by

$$c = 2 - 2\alpha\tau(N_1 + N_2), \tag{5.5}$$

where have assumed the normalization $\int_{\mathbb{P}^1} \omega = 2\pi$.

The first equation in (5.4) is the *non-abelian vortex equation*, as studied in [8]. Applying [8, Theorem 2.1.6] we obtain that this equation admits a solution provided that

$$N_2 < \frac{\tau}{2} < N_1 + N_2 - \deg([\phi]), \tag{5.6}$$

where $[\phi]$ denotes the line bundle given by the saturation of the image of $\phi\colon \mathcal{O}_{\mathbb{P}^1} \to V$. We want to show next that condition (5.6) is not sufficient to solve the full system of equations (5.4). For this, we will apply the Futaki invariant in Proposition 3.3. Fix homogeneous coordinates $[x_0, x_1]$, so that $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(N_j)) \cong S^{N_j}(\mathbb{C}^2)^*$ is the space of degree N_j homogeneous polynomials in x_0, x_1. Following [3], consider

$$\phi_j = x_0^{N-\ell_j} x_1^{\ell_j}, \tag{5.7}$$

with $0 \leq \ell_j < N_j$ (the case $\ell_1 = \ell_2 = 0$ corresponds to a Higgs field ϕ that has only one zero). In this case, it can be easily checked that the numerical condition (5.6) reduces to

$$N_2 < \frac{\tau}{2} < N_1 + N_2 - \min\{\ell_1, \ell_2\} - \min\{N_1 - \ell_1, N_2 - \ell_2\}, \tag{5.8}$$

and, by choosing suitable values of the parameters τ, N_j, and ℓ_j, the non-abelian vortex equation admits a solution. To evaluate the Futaki invariant, note that the Lie algebra element

$$y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{gl}(2, \mathbb{C}) \tag{5.9}$$

can be identified with an element in $\operatorname{Lie}\operatorname{Aut}(\mathbb{P}^1, V, \phi)$ for any choice of ℓ_j as before.

Lemma 5.6.

$$\langle \mathcal{F}_{\alpha,\alpha}, y \rangle = 2\pi i\alpha(2N_1 - \tau)(2\ell_1 - N_1) + 2\pi i\alpha(2N_2 - \tau)(2\ell_2 - N_2) \tag{5.10}$$

The proof follows along the lines of [3, Lemma 4.6], by direct evaluation of the Futaki invariant using the Fubini–Study metric on $\mathbb{P}^1$ and the product ansatz $H = H_1 \oplus H_2$, with H_j the Fubini–Study hermitian metric on the line bundle $\mathcal{O}_{\mathbb{P}^1}(N_j)$.

As a direct consequence of Proposition 3.3 and the previous lemma, we obtain the following.

Theorem 5.7. *Let* (V, ϕ) *as before, and assume that* (5.8) *is satisfied. Then, there is no solution of the equations* (5.4) *on* $(\mathbb{P}^1, V, \phi)$*, unless the following balancing condition holds*

$$\frac{2\ell_1 - N_1}{2N_2 - \tau} + \frac{2\ell_2 - N_2}{2N_1 - \tau} = 0.$$

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