SEMI-ORTHOGONAL DECOMPOSITION OF SYMMETRIC PRODUCTS OF CURVES AND CANONICAL SYSTEM

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Abstract. Let C be an irreducible smooth complex projective curve of genus $g \geq 2$ and C_d its d-fold symmetric product. In this paper, we study the question of semi-orthogonal decompositions of the derived category of C_d. This entails investigations of the canonical system on C_d, in particular its base locus.

1. Introduction

Let C be an irreducible smooth complex projective curve of genus $g \geq 2$. Let $C^d = C\times \overset{d}{\cdots} \times C$ be its Cartesian product, and let C_d be the d-fold symmetric product, meaning the quotient of C^d by the action of the symmetric group of d letters. We address the question whether the bounded derived category of coherent sheaves

$$D(C_d) := D^b_{coh}(C_d)$$

admits a non-trivial semi-orthogonal decomposition (Definition 4.1).

Semi-orthogonal decomposition is one of the basic notions in the theory of derived categories of coherent sheaves on algebraic varieties. When the derived category of an algebraic variety admits a semi-orthogonal decomposition, we can divide the category into smaller pieces and try to understand the whole triangulated category via its components. It turns out that semi-orthogonal decompositions of derived categories of algebraic varieties are closely related to birational geometry, Hochschild homology and cohomology, K-theory, mirror symmetry, moduli theory, motives, etc. See [Ku14] and [AB] and references therein for an overview of the role of semi-orthogonal decompositions in algebraic geometry. As a concrete example, we can mention the result of Bernardara and Bolognesi [BB] giving a criterion for the rationality of a conic bundle on a minimal rational surface in terms of the existence of certain semi-orthogonal decomposition of its derived category. Also, there is a conjecture of Kuznetsov [Ku10, Conjecture 1.1] about the rationality of a smooth cubic fourfold in terms of one of the components of certain semi-orthogonal decomposition. Last but not least, semi-orthogonal decomposition can be used to investigate geometry of moduli spaces of instanton or ACM bundles on some Fano varieties [Ku12].

In general, it is very hard to classify all possible semi-orthogonal decompositions of the derived category of a given variety. Even the question of which algebraic variety can have a non-trivial semi-orthogonal decomposition is still widely open.

There are several known classes of varieties who do not admit non-trivial semi-orthogonal decomposition (cf. [KO, Ok]). From the earlier works, we can see that

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the existence of non-trivial semi-orthogonal decomposition of the derived category of an algebraic variety is closely related to the base locus of the canonical bundle of the variety. This motivated us to study base locus of the canonical line bundle of C_d.

This study of the base locus of the canonical bundle and semi-orthogonal decomposition of the derived category of C_d was inspired by a conjecture of M. S. Narasimhan. Recently, Narasimhan proved in [Na1, Na2] that the derived category of C can be embedded into the derived category of the moduli space $\mathcal{SU}_C(2, L)$ of stable bundles of rank 2 and fixed determinant L of degree 1. A similar result was obtained by Fonarev and Kuznetsov for general curves via different method (cf. [FK]). Narasimhan conjectured that the derived category of the moduli space admits a semi-orthogonal decomposition as follows (Belmans, Galkin and Mukhopadhyay have independently stated the same conjecture in [BGM]).

Conjecture 1.1. *The derived category of $\mathcal{SU}_C(2, L)$ has the following semi-orthogonal decomposition*

$$D(\mathcal{SU}_C(2, L)) = \langle D(pt), D(pt), D(C), D(C), \cdots, D(C_{g-2}), D(C_{g-2}), D(C_{g-1})\rangle.$$

(two copies of $D(C_i)$ for $i < g-1$ and one copy for $i = g-1$).

It turns out that there is a motivic decomposition of $\mathcal{SU}_C(2, L)$ which is compatible with the above conjecture (cf. [Lee]). From the point of view of this conjecture it is of interest whether the derived categories of symmetric powers of curves can be further decomposed. Okawa proved that the derived category of a curve of genus $g \geq 2$ cannot have a non-trivial semi-orthogonal decomposition [Ok]. On the other hand, Toda gives an explicit semi-orthogonal decomposition of C_d for large d in [To, Corollary 5.11]. Let J denotes the Jacobian of C. Toda proves that, if $d > g-1$, then:

$$D(C_d) = \langle \overbrace{D(J), \ldots, D(J)}^{d-g+1}, D(C_{-d+2g-2})\rangle.$$

Recall that the gonality $\mathrm{gon}(C)$ of a curve C is the lowest degree among all nonconstant morphisms from C to the projective line $\mathbb{P}^1$. Equivalently, it is the lowest degree of a line bundle L on C with $h^0(L) \geq 2$. In this paper, we prove the following theorem.

Theorem 1.2 (Corollary 4.3)**.** *Let C be a smooth complex projective curve of genus $g \geq 3$, and let d be a positive integer with $d < \mathrm{gon}(C)$. Then there is no non-trivial semi-orthogonal decomposition of $D(C_d)$.*

We note that, for a generic curve C of genus g, the gonality satisfies

$$\mathrm{gon}(C) = \left\lfloor \frac{g+3}{2} \right\rfloor.$$

If $d = 2$ we prove the following result for all curves (no condition on gonality) with genus at least 3:

Theorem 1.3 (Theorem 4.9)**.** *Let C be a smooth projective curve of genus $g \geq 3$. Then there is no non-trivial semi-orthogonal decomposition on $D(C_2)$.*

This result is sharp because, if $g = 2$ then $D(C_2)$ admits a semi-orthogonal decomposition (recall that, for $g = 2$, the Albanese map $C_2 \longrightarrow J$ is the blow-up of the Jacobian at a point, and then apply the semi-orthogonal decomposition formula for the blow-up [Or]). We conjecture the following (this has independently been stated in [BGM]).

Conjecture 1.4. *Let C be a projective smooth curve of genus $g \geq 2$. Then there is no non-trivial semi-orthogonal decomposition on $D(C_d)$ for $1 \leq d \leq g-1$.*

In this direction, we prove some results on the base locus of the canonical divisor K_{C_d} of the symmetric product C_d.

Proposition 1.5 (Proposition 3.4)**.** *Let $1 \leq d \leq g-1$. The base locus of the canonical divisor K_{C_d} is the set of points $(x_1, \cdots, x_d)$ in C_d such that $h^0(\mathcal{O}_C(x_1 + \cdots + x_d)) > 1$.*

Equivalently, the base locus is the set of points in C_d where the Albanese map is not injective.

The following conjecture is known to the experts.

Conjecture 1.6. *Let X be a smooth projective variety. If the canonical bundle K_X is nef and $h^0(K_X) > 0$, then X admits no non-trivial semi-orthogonal decomposition.*

In Lemma 2.2 we prove that Conjecture 1.6 implies Conjecture 1.4

2. Nefness of the canonical divisor of C_d

Let Θ be the theta divisor on the Jacobian $J(C)$. Fixing a point $p \in C$, the Albanese map of the symmetric product C_d is constructed as follows

$$u\,:\, C_d \longrightarrow J(C)\,, \quad D \longmapsto \mathcal{O}_C(D - dp)\,.$$

Note that the fiber of the Albanese map is

$$u^{-1}(u(D)) \,=\, \mathbb{P}H^0(\mathcal{O}_C(D))\,. \tag{2.1}$$

We also define

$$i\,:\, C_{d-1} \longrightarrow C_d\,, \quad D \longmapsto D + p\,.$$

Let $\theta := u^*\Theta$. The class of the divisor $i(C_{d-1})$ of C_d will be denoted by x.

Lemma 2.1. *The canonical class of the symmetric product C_d is given by the formula*

$$K_{C_d} \,=\, (g-d-1)x + \theta\,. \tag{2.2}$$

Proof. Let $\Delta \subset C^d$ be the big diagonal where at least two points coincide. The image of Δ under the quotient map $\pi : C^d \longrightarrow C_d$, for the action of the symmetric group, will be denoted by Δ', so we have the commutative diagram

$$\begin{array}{ccc} \Delta & \longrightarrow & C^d \\ {\scriptstyle \pi|_\Delta}\downarrow & & \downarrow{\scriptstyle \pi} \\ \Delta' & \longrightarrow & C_d \end{array}$$

Note that $\pi^*(\Delta') \,=\, 2\Delta$, and hence Δ is the ramification divisor. The divisor Δ' is divisible by 2; in fact,

$$K_{C_d} \,=\, (2g-2)x - \Delta'/2$$

[K2, Proposition 2.6]. On the other hand,

$$-\Delta'/2 \,=\, \theta - (d+g-1)x$$

[K1, Lemma 7]. The lemma follows from these two facts. □

Recall the formula of Macdonald [Ma, § 11]

$$H^0(C_d,\, K_{C_d}) \,=\, \bigwedge^d H^0(C,\, K_C) \tag{2.3}$$

Lemma 2.2. *If Conjecture 1.6 holds, then for any curve C with $g \geq 3$ and $1 < d < g$, the symmetric product C_d admits no non-trivial semi-orthogonal decomposition.*

Proof. The class θ is nef, being the pullback of an ample class, while the class x is ample, hence $K_{C_d} = (g-d-1)x + \theta$ is nef under the given conditions on d. Furthermore $H^0(C_d,\, K_{C_d}) = \bigwedge^d H^0(C,\, K_C) \neq 0$ (2.3), so all the conditions in Conjecture 1.6 are satisfied. □

3. Base locus of canonical divisor of C_d

Let C be a smooth projective curve of genus g. Take any positive integer $d \leq g-1$. In this section we prove that the base locus of the canonical line bundle K_{C_d} of the symmetric product C_d coincides with the locus where the Albanese map is not injective (Proposition 3.4). We give two independent proofs. The first proof is a combination of Proposition 3.1 and Lemma 3.2. The second proof is given after the statement of Proposition 3.4.

When we write a point of C_d as $z = (z_1, \cdots, z_d)$, the points z_i of C need not be distinct. We also denote by z the subscheme of C defined by $\sum z_i$.

Proposition 3.1. *Let $1 \leq d \leq g-1$. Let $z = (z_1, \cdots, z_d) \in C_d$ be a point of the base locus of the complete linear system $|K_{C_d}| = \mathbb{P}(H^0(C_d,\, K_{C_d}))$. Then the dimension of*

$$H^0(C,\, \mathcal{O}_C(z)) \,=\, H^0(C,\, \mathcal{O}_C(\sum_{i=1}^d z_i))$$

is at least two.

Proof. We shall first describe a subset of $\mathbb{P}(H^0(C_d,\, K_{C_d}))$ whose linear span is whole $\mathbb{P}(H^0(C_d,\, K_{C_d}))$.

Let $S \subset H^0(C,\, K_C)$ be a linear subspace of dimension d. Now define

$$D_S \,:=\, \{(y_1, \cdots, y_d) \,\in\, C_d \mid \operatorname{div}(\omega) - \sum_{i=1}^d y_i \ \text{ is effective for some } \ \omega \,\in\, S \setminus \{0\}\}\,.$$

Note that $\operatorname{div}(\omega) - \sum_{i=1}^d y_i$ is effective if and only if ω vanishes on the subscheme of C defined by $\sum_{i=1}^d y_i$.

We claim that D_S is a divisor on C_d linearly equivalent to K_{C_d} and moreover the collection $\{D_S\}_{S\in \mathrm{Gr}(d,H^0(C,K_C))}$ spans $H^0(C_d,\, K_{C_d})$.

To prove this, using (2.3) the above divisor D_S corresponds to the line

$$\bigwedge^d S \subset \bigwedge^d H^0(C,\, K_C) \,=\, H^0(C_d,\, K_{C_d})\,.$$

The collection of all such lines with S running over $\mathrm{Gr}(d, H^0(C,\, K_C))$ evidently spans $\bigwedge^d H^0(C,\, K_C)$, proving the claim (for more details, see [Ba, Lemma 1.5.4]).

As in the statement of the proposition, take a point $z = (z_1, \cdots, z_d) \in C_d$ of the base locus of $\mathbb{P}(H^0(C_d,\, K_{C_d}))$. Note that this means that for every linear subspace $S \subset H^0(C,\, K_C)$ of dimension d, there is a nonzero $\omega \in S$ such that

$\mathrm{div}(\omega) - \sum_{i=1}^{d} z_i$ is effective. We shall now interpret this condition in order to be able to use it. Consider the short exact sequence of sheaves

$$0 \longrightarrow K_C \otimes \mathcal{O}_C(-z) \longrightarrow K_C \longrightarrow K_C|_z \longrightarrow 0 \tag{3.1}$$

on C. Let

$$0 \longrightarrow H^0(C,\, K_C \otimes \mathcal{O}_C(-z)) \stackrel{\beta}{\longrightarrow} H^0(C,\, K_C) \stackrel{\gamma}{\longrightarrow} H^0(K_C|_z) \tag{3.2}$$

be the long exact sequence of cohomologies associated to (3.1). This implies that

$$\dim \beta(H^0(C,\, K_C \otimes \mathcal{O}_C(-z))) \,\geq\, g-d\,,$$

because $\dim H^0(K_C|_z) \,=\, d$.

We shall show that

$$\dim \beta(H^0(C,\, K_C \otimes \mathcal{O}_C(-z))) \,\geq\, g-d+1\,. \tag{3.3}$$

To prove this, if $\dim \beta(H^0(C,\, K_C \otimes \mathcal{O}_C(-z))) \,=\, g-d$, then take a subspace of dimension d

$$S \,\subset\, H^0(C,\, K_C)$$

which is complementary to the subspace $\beta(H^0(C,\, K_C \otimes \mathcal{O}_C(-z)))$ of $H^0(C,\, K_C)$. Then the restriction $\gamma|_S$, where γ is the homomorphism in (3.2), is injective. Therefore, there is no nonzero $\omega \,\in\, S$ such that $\mathrm{div}(\omega) - \sum_{i=1}^{d} z_i$ is effective, because such an element ω has to be in the kernel of $\gamma|_S$. This proves (3.3).

From (3.3) and (3.2) it follows immediately that the homomorphism γ in (3.2) is *not* surjective.

Now consider the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C(z) \longrightarrow \mathcal{O}_C(z)|_z \longrightarrow 0$$

on C. Let

$$0 \longrightarrow H^0(C,\, \mathcal{O}_C) \longrightarrow H^0(C,\, \mathcal{O}_C(z)) \stackrel{\eta}{\longrightarrow} H^0(\mathcal{O}_C(z)|_z) \stackrel{\phi}{\longrightarrow} H^1(C,\, \mathcal{O}_C) \tag{3.4}$$

be the corresponding long exact sequence of cohomologies. By Serre duality,

$$H^1(C,\, \mathcal{O}_C) \,=\, H^0(C,\, K_C)^*\,.$$

Using this duality, the homomorphism ϕ in (3.4) is the dual of the homomorphism γ in (3.2). We proved earlier that γ is not surjective. Consequently, ϕ is not injective. Hence from (3.4) it follows that $\dim H^0(C,\, \mathcal{O}_C(z)) \,\geq\, 2$. This completes the proof. □

In view of the above proof, the following converse of Proposition 3.1 is now rather straightforward.

Lemma 3.2. *Let $1 \,\leq\, d \,\leq\, g-1$. Let $z \,=\, (z_1,\, \cdots,\, z_d) \,\in\, C_d$ be a point such that the dimension of*

$$H^0(C,\, \mathcal{O}_C(z)) \,=\, H^0(C,\, \mathcal{O}_C(\sum_{i=1}^{d} z_i))$$

is at least two. Then z lies on the base locus of the complete linear system $|K_{C_d}| \,=\, \mathbb{P}(H^0(C_d,\, K_{C_d}))$.

Proof. Since $\dim H^0(C, \mathcal{O}_C(z)) \geq 2$, the homomorphism η in (3.4) is nonzero. Hence ϕ in (3.4) is not injective. Consequently, the dual homomorphism γ in (3.2) is not surjective. Therefore,

$$\dim \gamma(H^0(C, K_C)) < \dim H^0(K_C|_z) = d\,.$$

This implies that for any linear subspace $S \subset H^0(C, K_C)$ of dimension d, the restriction $\gamma|_S$ is not injective. Now for any nonzero $\omega \in \mathrm{kernel}(\gamma|_S)$ the divisor $\mathrm{div}(\omega) - \sum_{i=1}^d y_i$ is effective. Consequently, z lies on the base locus of the complete linear system $|K_{C_d}|$. □

Let $f : C \longrightarrow \mathbb{P}^1$ be a surjective map of degree d. For any $b \in \mathbb{P}^1$, we have $f^{-1}(b) \in C_d$, where $f^{-1}(b)$ is the scheme theoretic inverse image. Therefore, we have morphism

$$\widehat{f} : \mathbb{P}^1 \longrightarrow C_d\,, \quad b \longmapsto f^{-1}(b)\,.$$

This is a morphism, and not just a set theoretic map, because C_d is the Hilbert scheme $\mathrm{Hilb}^d(C)$ of subschemes of C of dimension 0 and length d, the graph of f gives a closed subscheme of $\mathbb{P}^1 \times C$, flat over $\mathbb{P}^1$, and the morphism $\mathbb{P}^1 \longrightarrow C_d$ associated to this subscheme by the universal property of the Hilbert scheme is precisely $\widehat{f}$.

Corollary 3.3. *The image of the above map $\widehat{f}$ is contained in the base locus of the complete linear system $|K_{C_d}|$.*

Proof. For any $b \in \mathbb{P}^1$, we have

$$\dim H^0(C, \mathcal{O}_C(f^{-1}(b))) \geq \dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b)) = 2\,.$$

So Lemma 3.2 completes the proof. □

Proposition 3.4. *Let $1 \leq d \leq g-1$. The base locus of the canonical divisor K_{C_d} is the set of points $(x_1, \cdots, x_d)$ in C_d such that $h^0(\mathcal{O}_C(x_1 + \cdots + x_d)) > 1$.*

Equivalently, the base locus is the set of points in C_d where the Albanese map is not injective.

Proof. The first part follows from the combination of Proposition 3.1 and Lemma 3.2. The second part follows from the observation that the fiber of the Albanese map $u : C_d \longrightarrow J(C)$ is $u^{-1}(u(z)) = \mathbb{P}(H^0(\mathcal{O}_C(z)))$.

□

We shall now give the second proof of Proposition 3.4. Consider the Albanese map

$$u : C_d \longrightarrow J(C)$$

Let $z = (z_1, \cdots, z_d) \in C_d$ be a point, which can also be thought as a subscheme in C. The tangent space $T_z C_d$ of C_d at z is

$$T_z C_d = \mathrm{Hom}(\mathcal{O}_C(-z), \mathcal{O}_z) = H^0(C, \mathcal{O}_C(z)|_z)\,. \tag{3.5}$$

Therefore, the differential of the Albanese map gives a linear map

$$\phi : H^0(C, \mathcal{O}_C(z)|_z) = T_z C_d \stackrel{du_z}{\longrightarrow} T_{u(z)} J(C) = H^1(C, \mathcal{O}_C)\,.$$

This map ϕ is the connecting homomorphism in the long exact sequence of cohomologies associated to the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C(z) \longrightarrow \mathcal{O}_C(z)|_z \longrightarrow 0 \tag{3.6}$$

on C. The dual to the map ϕ is the homomorphism

$$\gamma\,:\, H^0(C,K_C)\;\longrightarrow\; H^0(C,K_C|_z)$$

in the long exact sequence of cohomologies associated to the short exact sequence

$$0\longrightarrow K_C(-z)\longrightarrow K_C\longrightarrow K_C|_z\longrightarrow 0$$

obtained by applying the functor $\mathcal{H}om(\cdot,K_C)$ to the short exact sequence (3.6).

Using (3.5) and Serre duality, we obtain the following canonical isomorphism for the fiber of K_{C_d} over a point $z=(z_1,\cdots,z_d)\in C_d$

$$K_{C_d}|_z \;=\; \bigwedge^d \mathrm{Hom}(\mathcal{O}_C(-z),\mathcal{O}_z)^\vee \;=\; \bigwedge^d \mathrm{Ext}^1(\mathcal{O}_z,K_C(-z)) \;=\; \bigwedge^d H^0(K_C|_z) \tag{3.7}$$

Using the identifications (2.3), and (3.7) and taking the d-fold exterior product, we get that

$$e_z: H^0(C_d,\,K_{C_d}) \;=\; \bigwedge^d H^0(C,\,K_C)\stackrel{\wedge^d\gamma}{\longrightarrow}\bigwedge^d H^0(K_C|_z)=K_{C_d}|_z\,.$$

The above map e_z is the evaluation map at z. We note that a point $z\in C_d$ is in the base locus of K_{C_d} if and only if e_z is zero. This is equivalent to the map γ being non-surjective, which in turn is equivalent to the assertion that the map ϕ is not injective. We have identified the map ϕ with the differential du_z of the Albanese map at z. Therefore, a point $z\in C_d$ is in the base locus of K_{C_d} if and only the Albanese map is not injective at z.

4. Semi-orthogonal decompositions of $D(C_d)$

Definition 4.1. A triangulated category $\mathcal{T}$ admits a nontrivial semi-orthogonal decomposition if there are two full non-trivial triangulated subcategories $\mathcal{A},\mathcal{B}$ of $\mathcal{T}$ such that

(1) $\mathrm{Hom}_{\mathcal{T}}(b,a)=0$ for every $b\in\mathcal{B}$, $a\in\mathcal{A}$ and
(2) $\mathcal{A},\mathcal{B}$ generate $\mathcal{T}$.

In this section, using our results on the base locus of the canonical bundle, and applying the work of Kawatani and Okawa [KO], we will obtain restrictions to the existence of semi-orthogonal decompositions of the triangulated category $D(C_d)$.

Theorem 4.2 ([KO, Corollary 1.3])**.** *Let X be a smooth projective variety such that the base locus of the canonical divisor is a finite set. Then there is no non-trivial semi-orthogonal decomposition of $D(X)$.*

Corollary 4.3. *Let C be a smooth complex projective curve of genus $g\geq 3$ and let d be a integer with $d<\mathrm{gon}(C)$. Then there is no non-trivial semi-orthogonal decomposition of $D(C_d)$.*

Proof. Note that $h^0(\mathcal{O}_C(\sum z_i))=1$ for any $z=(z_1,\cdots,z_d)$, because $d<\mathrm{gon}(C)$. So Proposition 3.1 implies that z is not a base point of K_{C_d}, and hence the canonical divisor K_{C_d} is base-point free. Now from Theorem 4.2 it follows that $D(C_d)$ has no non-trivial semi-orthogonal decomposition. □

When $d=2$ we are able to prove a stronger result which disposes of condition on the gonality of C. We will use the following result:

Theorem 4.4 ([KO, Theorem 1.8])**.** *Let S be a minimal smooth projective surface of general type with $h^0(K_S) > 1$ and satisfying the condition that for any one-dimensional connected component $Z \subset \mathrm{Bs}\,|K_S|$, its intersection matrix is negative definite. Then there is no non-trivial semi-orthogonal decomposition of $D(S)$.*

We start with some results about the geometry of C_2.

Lemma 4.5. *Let $g \geq 3$. The surface C_2 is minimal. It has an embedded rational curve if and only if C is hyperelliptic, and in this case*

- *the rational curve is $\Gamma = \{x + \sigma(x)\}$, where σ is the hyperelliptic involution, and*
- *$\Gamma^2 = 1 - g$, i.e., Γ is a $(1-g)$-curve.*

Proof. For all points in the image of the Albanese map $u : C_2 \longrightarrow J(C)$, the fiber is a projective space (2.1). Since C_2 is a surface, the fiber of the Albanese has at most dimension 2. But the fiber cannot be $\mathbb{P}^2$, because this would mean that there is a degree 2 line bundle A on C with $h^0(A) = 3$, and this would imply that the genus of C is $\mathbb{P}^1$, contradicting the hypothesis $g \geq 3$. In particular, the Albanese map is not constant.

If C is hyperelliptic then C_2 has no rational curve. Indeed, a rational curve has to map to a point, so the fiber over this point would be exactly $\mathbb{P}^1$ (by the previous argument), and we would have a line bundle A on C with $h^0(A) = 2$, so A would be the hyperelliptic divisor.

Let us now suppose that C is hyperelliptic. A rational curve in C_2 has to be in a fiber of the Albanese map, but the only positive dimensional fiber of this map is one dimensional (by the argument in the first paragraph of this proof), and it is the fiber over the hyperelliptic line bundle.

We denote the above mentioned fiber of u by Γ, so Γ is isomorphic to $\mathbb{P}^1$. We now calculate its self-intersection. The self-intersection of the diagonal $\Delta \subset C \times C$ is

$$\Delta^2 = 2 - 2g$$

by Poincaré–Hopf theorem. The automorphism of $C \times C$, which is identity on the first factor and the hyperelliptic involution on the second, sends the diagonal Δ to the graph $\widetilde{\Gamma} = \{x, \sigma(x)\}$ of the hyperelliptic involution, hence also $\widetilde{\Gamma}^2 = 2 - 2g$. Consider the diagram

$$\begin{array}{ccc} \widetilde{\Gamma} & \hookrightarrow & C \times C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi} \\ \Gamma & \longrightarrow & C_2 \end{array}$$

Observe that $\widetilde{\Gamma} \cong C$ and the morphism f is just the quotient by the hyperelliptic involution. Therefore we have $\pi_*\widetilde{\Gamma} = 2\Gamma$ and $\pi^*\Gamma = \widetilde{\Gamma}$ as cycles, and the projection formula for intersection gives

$$\pi_*(\widetilde{\Gamma} \cdot \widetilde{\Gamma}) = \pi_*(\widetilde{\Gamma} \cdot \pi^*\Gamma) = \pi_*(\widetilde{\Gamma}) \cdot \Gamma = 2\Gamma \cdot \Gamma$$

and hence $\Gamma^2 = 1 - g$. □

Now we prove that C_2 is a surface of general type.

Lemma 4.6. *If $g \geq 3$, then the symmetric product C_2 is of general type.*

Proof. In view of [Be, Proposition X.1] it suffices to show that

- the self-intersection of the canonical divisor of C_2 is positive, and C_2 is irrational surface.

From [Be, Proposition I.8], we can compute the self-intersection on C^2. We know that the pull back of the canonical divisor on C_2 to C^2 is $K_C \boxtimes K_C(-\Delta)$. The self-intersection of $K_C \boxtimes K_C(-\Delta)$ is

$$2(2g-2)^2 - (2g-2) - 4(2g-2) = (2g-2)(4g-9)$$

and it is positive when $g \geq 3$.

To prove that C_2 is not rational by contradiction, assume that C_2 is rational. Then C_2 can be covered by rational curves, which implies that the Albanese map $u : C_2 \to J(C)$ is constant, but we know that the Albanese map is not constant by the argument in the first paragraph of the proof of 4.5. Hence C_2 is not rational.

Therefore C_2 is of general type when $g \geq 3$. □

Remark 4.7. When $g = 2$, we know that C_2 is the blow-up of the $J(C)$ at a point. This has two consequences: C_2 is not a surface of general type, and $D(C_2)$ admits a nontrivial semi-orthogonal decomposition (using the blow-up formula in [Or]).

Finally we check that $p_g > 1$ for C_2.

Lemma 4.8. *The canonical bundle K_{C_2} has $h^0(K_{C_2}) = \binom{g}{2} > 1$ (recall that $g \geq 3$).*

Proof. Macdonald, [Ma], proves that $H^0(C_d, K_{C_d}) = \bigwedge^d H^0(C, K_C)$. □

Theorem 4.9. *Let C be a smooth projective curve of genus $g \geq 3$. Then there is no non-trivial semi-orthogonal decomposition on $D(C_2)$.*

Proof. If C is hyperelliptic, by Lemma 4.5 the Albanese map fails to be injective exactly on $\Gamma \subset J(C)$. Therefore, Proposition 3.4 implies that

$$\mathrm{Bs}\,|K_{C_2}| = \Gamma$$

and the only connected component of the base locus is Γ, which is irreducible. Hence the intersection matrix is just $\Gamma^2 = 1 - g < 0$, so it is negative definite.

If C is not hyperelliptic, then the Albanese map is injective. Proposition 3.4 implies that K_{C_d} has no base locus, and hence there is no non-trivial semi-orthogonal decomposition by Theorem 4.2.

In view of Lemmas 4.5, 4.6, and 4.8, the hypothesis of Theorem 4.4 are satisfied, so there is no non-trivial semi-orthogonal decomposition. □

Remark 4.10. We thank an anonymous referee for the following alternative argument to check the assumptions of Theorem 4.4. By Lemma 2.1 and the assumption $g \geq 3$, we know that the canonical divisor K_{C_2} is nef and big. Hence C_2 is minimal surface of general type. Since C_2 is a minimal model, the Albanese map should be birational onto its image (otherwise, C_2 is either $\mathbb{P}^2$ or a ruled surface, contradiction). In other words, the morphism of C_2 onto its image is a resolution of singularities of a surface $u(C_2)$. Now the base locus of K_{C_2} coincides with the u-exceptional curve (Proposition 3.4), hence the intersection matrix of each connected component is negative definite.

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