THE WEIL CONJECTURES I

By Pierre Deligne (translated by Evgeny Goncharov)
eg555@cam.ac.uk[*], eagoncharov@edu.hse.ru[†]

I attempted to write the full translation of this article to make the remarkable proof of Pierre Deligne available to a greater number of people. Overviews of the proofs can be found elsewhere. I especially recommend the notes of James Milne on Etale Cohomology that also contain a justification for the theory underlying this article and proofs of the results used by Deligne. The footnotes are mostly claims that some details appear in Milne, clarifications of some of the terminology or my personal struggles. I have also made a thorough overview of the proof together with more detailed explanations - `https://arxiv.org/abs/1807.10812`*. Enjoy!*

Abstract

In this article I prove the Weil conjecture about the eigenvalues of Frobenius endomorphisms. The precise formulation is given in (1.6). I tried to make the demonstration as geometric and elementary as possible and included reminders: only the results of paragraphs 3, 6, 7 and 8 are original.

In the article following this one I will give various refinements of the intermediate results and the applications, including the hard Lefschetz theorem (on the iterated cup products by the cohomology class of a hyperplane section).

The text faithfully follows from the six lectures given at Cambridge in July 1973. I thank N.Katz for allowing me to use his notes.

Contents

[*]Cambridge University, Center for Mathematical Sciences, Wilberforce Road, Cambridge, UK
[†]National Research University Higher School of Economics (NRU HSE), Usacheva 6, Moscow, Russia.

1 The theory of Grothendieck: a cohomological interpretation of L-functions

(1.1) Let X be a scheme of finite type over $\mathbb{Z}$, $|X|$ be the set of closed points of X and for $x \in |X|$ we denote by $N(x)$ the number of elements in the residue field $k(x)$ of X at x. The Hasse-Weil zeta function of X is

$$\zeta_X(s) = \prod_{x \in |X|} (1 - N(x)^{-s})^{-1} \quad \textbf{(1.1.1)}$$

(this product converges absolutely for $Re(s)$ large enough). For $X = Spec(\mathbb{Z})$, $\zeta_X(s)$ is the Riemann zeta function.

We will consider exclusively the case when X is a scheme over a finite field $\mathbb{F}_q$.

For $x \in |X|$ we will write g_x instead of $N(x)$. Denoting $\deg(x) = [k(x) : \mathbb{F}_q]$ we have $q_x = q^{\deg(x)}$. It makes sense to introduce a new variable $t = q^{-s}$. Let

$$Z(X;t) = \prod_{x \in |X|} (1 - t^{\deg(x)})^{-1} \quad \textbf{(1.1.2)};$$

this product converges for $|t|$ small enough and we have

$$\zeta_X(s) = Z(X; q^{-s}) \quad \textbf{(1.1.3)}$$

(1.2) Dwork (On the rationality of the zeta function of an algebraic variety, Amer. J. Math., 82, 1960, p. 631-648) and Grothendieck ([1] and SGA5) have demonstrated that $Z(X;t)$ is a rational function of t.

For Grothendieck, this is a corollary of general results in l-adic cohomology (where l is a prime number not equal to the characteristic p of $\mathbb{F}_q$). These provide a cohomological interpretation of the zeros and poles of $Z(X;t)$, and a functional equation when X is proper and smooth. The methods of Dwork are p-adic. For X a non-singular hypersurface in a projective space they also provided him with a cohomological interpretation of zeros and poles, and the functional equation. They inspired the crystalline theory of Grothendieck and Berthelot, which for X proper and smooth provides a p-adic cohomological interpretation of zeros and poles, and the functional equation. Based on Washnitzer ideas, Lubkin created a variant of this theory, valid only for X proper, smooth and liftable to characteristic 0 (A p-adic proof of Weil's conjectures, Ann of Math, 87, 1968, pp. 125-255).

We will make essential use of Grothendieck's results and recall them below.

(1.3) Let X be an algebraic variety over an algebraically closed field k of characteristic p, i.e. a separated scheme of finite type over k. We do not exclude the case $p = 0$. For any prime number $l \neq p$, Grothendieck defined l-adic cohomology groups $H^i(X, \mathbb{Q}_l)$. He also defined cohomology groups with compact support $H^i_c(X, \mathbb{Q}_l)$. For X proper the two coincide. $H^i_c(X, \mathbb{Q}_l)$ are vector spaces of finite dimension over $\mathbb{Q}_l$, zero for $i > 2dim(X)$.

(1.4) Let X_0 be an algebraic variety over $\mathbb{F}_q$, $\bar{\mathbb{F}}_q$ the algebraic closure of $\mathbb{F}_q$ and X the algebraic variety over $\bar{\mathbb{F}}_q$ obtained from X_0 by extension of scalars of $\mathbb{F}_q$ to $\bar{\mathbb{F}}_q$. In the language of Weil and Shimura we would express this situation by: "Let X be an algebraic variety defined over $\mathbb{F}_q$". Let $F : X \to X$ be the Frobenius morphism; it sends a point with coordinates x to the point with coordinates x^q; in other words, for U_0 a Zariski open subset of X_0, defining an open subset U of X, we have $F^{-1}(U) = U$; for $x \in H^0(U_0, \mathcal{O})$ we have $F^*x = x^q$. Let us identify the set $|X|$ of closed points of X with $X_0(\bar{\mathbb{F}}_q)$ (all the points $Hom_{\mathbb{F}_q}(Spec(\bar{\mathbb{F}}_q), X_0)$ of X_0 with coefficients in $\bar{\mathbb{F}}_q$) and let $\varphi \in Gal(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ be the substitution of Frobenius: $\varphi(x) = x^q$. The action of F on $|X|$ identifies with the action of φ on $X_0(\bar{\mathbb{F}}_q)$. Then:

a) The set X^F of closed points of X fixed under F is identified with the set $X_0(\mathbb{F}_q) \subset X_0(\bar{\mathbb{F}}_q)$ of points of X defined over $\mathbb{F}_q$. This simply expresses the fact that for $x \in \bar{\mathbb{F}}_q$ we have $x \in \mathbb{F}_q \Leftrightarrow x^q = x$.

b) Similarly, the set X^{F^n} of closed points of X fixed under the n-th iteration of F is identified with $X_0(\mathbb{F}_{q^n})$.

c) The set $|X|$ of closed points of X is identified with the set $|X|_F$ of orbits of F (or φ) on $|X|$. The degree $\deg(x)$ of $x \in |X_0|$ is the number of elements in the corresponding orbit.

d) From b) and c) we see that

$$\#X^{F^n} = \#X_0(\mathbb{F}_{q^n}) = \sum_{\deg(x)|n} \deg(x) \quad \textbf{(1.4.1)}$$

(for $x \in |X_0|$ and $\deg(x)|n$, x defines $\deg(x)$ points with coordinates in $\mathbb{F}_{q^n}$ all conjugate over $\mathbb{F}_q$).

(1.5) The morphism F is finite, in particular, proper. Therefore, it induces morphisms

$$F^* : H^i_c(X, \mathbb{Q}_l) \to H^i_c(X, \mathbb{Q}_l).$$

Grothendieck proved the formula of Lefschetz

$$\#X^F = \sum_i (-1)^i Tr(F^*, H^i_c(X, \mathbb{Q}_l));$$

the right side, that is a priori an l-adic number is an integer and is equal to the left side. We should note that such a formulation is only reasonable because $dF = 0$, even at infinity (X is not assumed to be proper); the relation $dF = 0$ implies that fixed points of F have multiplicity one.

We have a similar formula for the iterations of F:

$$\#X^{F^n} = X_0(\mathbb{F}_{q^n}) = \sum_i (-1)^i Tr(F^{*n}, H^i_c(X, \mathbb{Q}_l)) \quad \textbf{(1.5.1)}$$

We take the logarithmic derivative of (1.1.2):

$$t\frac{d}{dt} \log Z(X_0, t) = \frac{t\frac{d}{dt}Z(X_0,t)}{Z(X_0,t)} = \sum_{x\in|X_0|} -\frac{-\deg(x)t^{\deg(x)}}{1-t^{\deg(x)}} =$$

$$= \sum_{x\in|X_0|}\sum_{n>0} \deg(x) t^{n\deg(x)} \overset{(1.4.1)}{=} \sum_n X_0(\mathbb{F}_{q^n})t^n \quad \textbf{(1.5.2)}$$

For F an endomorphism of a vector space V we have a formal series identity

$$t\frac{d}{dt}\log(\det(1-Ft,V)^{-1}) = \sum_{n>0} Tr(F^n, V)t^n \quad \textbf{(1.5.3)}$$

(check for $\dim V = 1$ and observe that both sides are additive in V when we take short exact sequences). By substituting (1.5.1) into (1.5.2) and applying (1.5.3) one finds

$$t\frac{d}{dt}\log Z(X_0,t) = \sum_i (-1)^i t\frac{d}{dt}\log\det(1-F^*t, H^i_c(X,\mathbb{Q}_l))^{-1},$$

or

$$Z(X,t) = \prod_i \det(1 - F^*t, H^i_c(X, \mathbb{Q}_l))^{(-1)^{i+1}} \quad \textbf{(1.5.4)}$$

The right side is in $\mathbb{Q}_l(t)$. The formula implies that its Taylor expansion at $t = 0$, a priori a formal series in $\mathbb{Q}_l[[t]]$ with constant coefficient one, is in $\mathbb{Z}[[t]]$ and is equal to the left side, also considered as a formal series in t. This formula is the Grothendiek's cohomological interpretation of the Z-function.

Our main result is the following:

Theorem (1.6). *Let X_0 be a projective nonsingular (= smooth) variety over $\mathbb{F}_q$. For each i, the characteristic polynomial $\det(1 - F^*t, H^i(X, \mathbb{Q}_l))$ has integer coefficients independent of l ($l \neq p$). The complex roots α of this polynomial (complex conjugates of the eigenvalues of F^*) are of absolute value $|\alpha| = q^{\frac{i}{2}}$.*

We show that (1.6) is a consequence of the following apparently weaker statement:

Lemma (1.7). *For each i and each $l \neq p$ the eigenvalues of the Frobenius endomorphism F^* on $H^i(X, \mathbb{Q}_l)$ are algebraic numbers of absolute value $|\alpha| = q^{\frac{i}{2}}$.*

Proof of (1.7) ⇒ (1.6): Let's look at $Z(X_0, t)$ as a formal series with constant term 1, an element of $\mathbb{Z}[[t]] : Z(X_0, t) = \sum_n a_n t^n$. From (1.5.3), the image of $Z(X_0, t)$ in $\mathbb{Q}_l[[t]]$ is a Taylor expansion of a rational function. This means that for N and M large enough ($\geqslant$ the degrees of numerator and denominator) the Hankel determinants

$$H_k = \det((a_{i+j+k})_{0 \leqslant i,j \leqslant M}) \ (k > N)$$

are zero. They vanish in $\mathbb{Q}_l$ if and only if they vanish in $\mathbb{Q}$; $Z(X_0, t)$ is a Taylor expansion of an element in $\mathbb{Q}(t)$. In other words,

$$Z(X_0, t) \in \mathbb{Z}[[t]] \cap \mathbb{Q}_l(t) \subset \mathbb{Q}(t).$$

Let $Z(X_0, t) = \frac{P}{Q}$, with $P, Q \in \mathbb{Z}[t]$ coprime and with positive constant terms. According to a lemma of Fatou, since $Z(X_0, t)$ lies in $\mathbb{Z}[[t]]$ and has constant term 1, the constant terms of P and Q are 1[1]. Let

$$P_i(t) = \det(1 - F^*t, H^i(X, \mathbb{Q}_l)).$$

(1.7) implies that P_i are coprime. The right hand side of (1.5.4) is therefore an irreducible fraction and

$$P(t) = \prod_{i \ odd} P_i(t)$$

$$Q(t) = \prod_{i \ even} P_i(t).$$

Let K be the subfield of the algebraic closure $\bar{\mathbb{Q}}_l$ of $\mathbb{Q}_l$ generated over $\mathbb{Q}$ by the roots of $R(t) = P(t)Q(t)$. The roots of $P_i(t)$ are the roots of $R(t)$ such that all their complex conjugates have absolute value $q^{-\frac{i}{2}}$. This set is stable under $Gal(K/\mathbb{Q})$. Therefore, $P_i(t)$ has rational coefficients. According to a lemma of Gauss (or because roots of P_i, being roots of R, are inverses of algebraic

[1] See the proof in James Milne's lectures on Etale Cohomology (Milne). Here and below - footnotes of the translator.

integers), it even has integer coefficients. The above description of the roots of $P_i(t)$ is independent of l, therefore, the polynomial $P_i(t)$ is also independent of l.

The rest of the article is dedicated to the demonstration of (1.7).

(1.8) The theory of Grothendieck provides cohomological interpretation not only of zeta functions but also of L-functions. The results are as follows.

(1.9) Let X be an algebraic variety over a field k. For the definition of a constructible $\mathbb{Q}_l$-sheaf on X consult SGA 5 VI[2]. It suffices to say that:

a) If $\mathcal{F}$ is a constructible $\mathbb{Q}_l$-sheaf, there exists a finite partition of X into locally closed parts such that $\mathcal{F}|X_i$ is locally constant.

b) Assume that X is connected and let $\bar{x}$ be a geometric point of X. For $\mathcal{F}$ locally constant, $\pi_1(X, \bar{x})$ acts on the stalks $\mathcal{F}_{\bar{x}}$; the map $\mathcal{F} \to \mathcal{F}_{\bar{x}}$ defines an equivalence of categories (locally constant $\mathbb{Q}_l$-sheaves on X) $\to$ (continuous representations of $\pi_1(X, \bar{x})$ on $\mathbb{Q}_l$ vector spaces of finite dimension). Such a representation in general does not factor through a finite quotient.

c) If $k = \mathbb{C}$, the constructible $\mathbb{Q}_l$-sheaves over X are identified with the sheaves of $\mathbb{Q}_l$ vector spaces $\mathcal{F}$ on X^{an} and their exists a finite partition of X into Zariski locally closed parts and for each i a local system[3] of free of finite type $\mathbb{Z}_l$-modules $\mathcal{F}_i$ on X_i such that

$$\mathcal{F}|X_i = \mathcal{F}_i \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

We will only consider constructible $\mathbb{Q}_l$-sheaves and call them just $\mathbb{Q}_l$-sheaves.

(1.10) Assume that k is algebraically closed and let $\mathcal{F}$ be a $\mathbb{Q}_l$-sheaf on X. Grothendieck defined the l-adic cohomology groups $H^i(X, \mathcal{F})$ and $H^i_c(X, \mathcal{F})$. $H^i_c(X, \mathcal{F})$ are vector spaces of finite dimension over $\mathbb{Q}_l$, zero for $i > 2\dim(X)$. For $k = \mathbb{C}$, $H^i(X, \mathcal{F})$ and $H^i_c(X, \mathcal{F})$ are the usual cohomology groups (resp. groups with compact support) of X^{an} with coefficients in $\mathcal{F}$.

(1.11) Let X_0 be an algebraic variety over $\mathbb{F}_q$, X the corresponding variety over $\bar{\mathbb{F}}_q$ and $\mathcal{F}_0$ a sheaf of sets on X_0 (for the etale topology). We denote by $\mathcal{F}$ its inverse image on X. In addition to the Frobenius isomorphism $F : X \to X$, we have a canonical isomorphism $F^* : F^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$. Here is a description. We regard $\mathcal{F}_0$ as an etale space over X_0, i.e. we identify $\mathcal{F}_0$ with an algebraic space $[\mathcal{F}_0]$, equipped with an etale morphism $f : [\mathcal{F}_0] \to X_0$ such that $\mathcal{F}_0$ is the sheaf of local sections of $[\mathcal{F}_0]$. The similar etale space $[\mathcal{F}]$ over X is obtained from $[\mathcal{F}_0]$ by extension of scalars. So we have a commutative diagram

$$\begin{array}{ccc} [\mathcal{F}] & \xrightarrow{F} & [\mathcal{F}] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{F} & X \end{array}$$

and a morphism $[\mathcal{F}] \to X \times_{(F,X,f)} [\mathcal{F}] = [F^*\mathcal{F}]$, that is an isomorphism because f is etale. The inverse of this isomorphism defines the isomorphism $F^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$ that we seek.

This construction is generalized to $\mathbb{Q}_l$-sheaves.

(1.12) Let X_0 be an algebraic variety over $\mathbb{F}_q$, $\mathcal{F}_0$ a $\mathbb{Q}_l$-sheaf on X_0, $(X, \mathcal{F})$ is obtained by extension of scalars of $\mathbb{F}_q$ to $\bar{\mathbb{F}}_q$, $F : X \to X$ and $F^* : F^*\mathcal{F} \to \mathcal{F}$. Finite morphisms F and F^* define an endomorphism

$$F^* : H^i_c(X, \mathcal{F}) \to H^i_c(X, F^*\mathcal{F}) \to H^i_c(X, \mathcal{F}).$$

[2] Or Milne.

[3] We will also call it a locally constant sheaf (french. constant tordu). This is an abuse of terminology. A sheaf $\mathcal{M} = (\mathcal{M}_n)$ of $\mathbb{Z}_l$-modules is called locally constant if each $\mathcal{M}_n$ is locally constant. It is not, in general, locally constant in the classical sense. Similar remarks apply elsewhere to $\mathbb{Z}_l$ and $\mathbb{Q}_l$-sheaves.

For $x \in |X|$, F^* defines a morphism $F^*_x : \mathcal{F}_{F(x)} \to \mathcal{F}_x$. For $x \in X^F$ it is an endomorphism of $\mathcal{F}_x$. Grothendieck proved the formula of Lefschetz

$$\sum_{x \in X^F} Tr(F^*_x, \mathcal{F}_x) = \sum_i (-1)^i Tr(F^*, H^i_c(X, \mathcal{F})).$$

A similar formula holds for the iterations of F: n-th iteration of F^* defines morphisms $F^{*n}_x : \mathcal{F}_{F^n(x)} \to \mathcal{F}_x$; for x fixed under F^n, F^{*n}_x is an endomorphism and

$$\sum_{x \in X^{F^n}} Tr(F^{*n}_x, \mathcal{F}_x) = \sum_i (-1)^i Tr(F^{*n}, H^i_c(X, \mathcal{F})) \quad \textbf{(1.12.1)}$$

(1.13) Let $x_0 \in |X|$, Z be the orbit corresponding to F in $|X|$ and $x \in Z$. The orbit Z has $\deg(x_0)$ elements (1.4). We denote by $F^*_{x_0}$ the endomorphism $F^{*\deg(x_0)}_x$ of $\mathcal{F}_x$ and let

$$\det(1 - F^*_{x_0} t, \mathcal{F}_0) = \det(1 - F^*_{x_0} t, \mathcal{F}_x).$$

Because of the local isomorphism[4], $(\mathcal{F}_x, F^*_{x_0})$ does not depend on the choice of X. This justifies omitting x in the notation. We will use a similar notation for other functions of $(\mathcal{F}_x, F^*_{x_0})$.

(1.14) Define $Z(X_0, \mathcal{F}_0, t) \in \mathbb{Q}_l[[t]]$ by the product

$$Z(X_0, \mathcal{F}_0, t) = \prod_{x \in |X_0|} \det(1 - F^*_x t^{\deg(x)}, \mathcal{F}_0)^{-1} \quad \textbf{(1.14.1)}$$

For the constant sheaf $\mathbb{Q}_l$ we recover (1.1.2). According to (1.5.3), the logarithmic derivative of Z is

$$t\frac{d}{dt} \log Z(X_0, \mathcal{F}_0, t) \overset{def}{=} \frac{t\frac{d}{dt} Z(X_0, \mathcal{F}_0, t)}{Z(X_0, \mathcal{F}_0, t)} = \sum_n \sum_{x \in X^{F^n} = X_0(\mathbb{F}_{q^n})} Tr(F^{*n}_x, \mathcal{F}_0) t^n \quad \textbf{(1.14.2)}$$

Substituting (1.12.1) in (1.14.2) we find by a calculation similar to (1.5) the generalization of (1.5.4)

$$Z(X_0, \mathcal{F}_0, t) = \prod_i \det(1 - F^* t, H^i_c(X, \mathcal{F}))^{(-1)^{i+1}} \quad \textbf{(1.14.3)}$$

This formula is an identity in $\mathbb{Q}_l[[t]]$.

(1.15) It is sometimes convenient to use Galois language instead of the geometric one. Here is the dictionary.

If $\bar{\mathbb{F}}^1_q$ and $\bar{\mathbb{F}}^2_q$ are two algebraic closures of $\mathbb{F}_q$, $(X_0, \mathcal{F}_0)$ over $\mathbb{F}_q$ defines by extension of scalars $(X_1, \mathcal{F}_1)$ over $\bar{\mathbb{F}}^1_q$ and $(X_2, \mathcal{F}_2)$ over $\bar{\mathbb{F}}^2_q$. All $\mathbb{F}_q$-isomorphisms $\sigma : \bar{\mathbb{F}}^1_q \xrightarrow{\sim} \bar{\mathbb{F}}^2_q$ define isomorphisms

$$H^*_c(X_1, \mathcal{F}_1) \xrightarrow{\sim} H^*_c(X_2, \mathcal{F}_2).$$

In particular, for $\bar{\mathbb{F}}^1_q = \bar{\mathbb{F}}^2_q$ (denote by $\bar{\mathbb{F}}_q$), we find that $Gal(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ acts on $H^*_c(X, \mathcal{F})$ (action by transport of structure [5]). Let $\varphi \in Gal(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ be the substitution of Frobenius. We verify that

$$F^* = \varphi^{-1} \quad (in\ End(H^*_c(X, \mathcal{F}))).$$

[4] The Frobenius F is a local isomorphism and hence an isomorphism on stalks.

[5] This phrase is commonly used to state the principle that any isomorphism $Y_1 \to Y_2$ extends canonically to an isomorphism of objects constructed from Y_1 and Y_2 (cohomology groups, sheaves, etc). When $Y_1 = Y_2$, automorphisms also extend.

This leads to the definition of the *geometric Frobenius* $F \in Gal(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ as φ^{-1}. We have

$$F^* = F \quad \mathbf{(1.15.1)}$$

Let x be a geometric point of X_0, localized to $x_0 \in |X_0|$. By transport of structure, the group $Gal(k(x)/k(x_0))$ acts on the stalk $(\mathcal{F}_0)_x$ of $\mathcal{F}_0$ at x; in particular, we have a geometric Frobenius relative to $k(x_0)$: $F_{x_0} \in Gal(k(x)/k(x_0))$. For x defined by a closed point, still denoted by x, in X we have $\mathcal{F}_x = (\mathcal{F}_0)_x$ and

$$F^*_{x_0} \stackrel{def}{=} F_x^{*\deg(x_0)} = F_{x_0} \quad (in\ End(\mathcal{F}_x)) \quad \mathbf{(1.15.2)}$$

In the Galois notation, (1.14.3) looks like

$$\prod_{x \in |X_0|} \det(1 - F_x t^{\deg(x)}, \mathcal{F}_0)^{-1} = \prod_i \det(1 - Ft, H^i_c(X, \mathcal{F}))^{(-1)^{i+1}}.$$

2 The theory of Grothendieck: Poincare duality

(2.1) To explain the relationship between the roots of unity and orientations I will first repeat the two classical cases in a wacky language.

a) *Differentiable manifolds.* - Let X be a differentiable manifold purely of dimension n. The orientation sheaf $\mathbb{Z}'$ on X is the sheaf locally isomorphic to the constant sheaf $\mathbb{Z}$, whose invertible sections on an open U in X correspond to the orientations of U. An *orientation* of X is an isomorphism of $\mathbb{Z}'$ with the constant sheaf $\mathbb{Z}$. The *fundamental class* of X is a morphism Tr : $H^n_c(X, \mathbb{Z}') \to \mathbb{Z}$; if X is orientable, it is identified with a morphism $Tr : H^n_c(X, \mathbb{Z}) \to \mathbb{Z}$. The Poincare duality is expressed using the fundamental class.

b) *Complex varieties.* - Let $\mathbb{C}$ be the closure of $\mathbb{R}$. A smooth complex algebraic variety or rather the underlying differentiable variety is always orientable. To justify this it suffices to orient $\mathbb{C}$ itself. This amounts to a choice:

a) choosing one of the two roots of the equation $X^2 = -1$; we call it $+i$;

b) choosing an isomorphism from $\mathbb{R}/\mathbb{Z}$ to $U^1 = \{z \in \mathbb{C} |\ |z| = 1\}$; $+i$ is the image of $\frac{1}{4}$;

c) choosing one of the two isomorphisms $x \to \exp(\pm 2\pi i x)$ from $\mathbb{Q}/\mathbb{Z}$ to the group of the roots of unity of $\mathbb{C}$, which extends continuously to an isomorphism from $\mathbb{R}/\mathbb{Z}$ to U^1.

We denote by $\mathbb{Z}(1)$ a free $\mathbb{Z}$-module of rank one whose set of generators has two elements canonically corresponding to one of the two-element sets *a), b), c)*. The simplest is to take $\mathbb{Z}(1) = Ker(\exp : \mathbb{C} \to \mathbb{C}^*)$. The generator $y = \pm 2\pi i$ corresponds to the isomorphism *c)*: $x \to \exp(xy)$. Let $\mathbb{Z}(r)$ be the r-th tensor power of $\mathbb{Z}(1)$. If X is a smooth complex algebraic variety purely of complex dimension r, the orientation sheaf on X is the constant sheaf of value $\mathbb{Z}(r)$.

(2.2) To "orient" an algebraic variety over an algebraically closed k of characteristic zero, we must choose an isomorphism from $\mathbb{Q}/\mathbb{Z}$ to the group of the roots of unity of k. The set of such isomorphisms is the principal homogeneous space for $\hat{\mathbb{Z}}^*$ (no longer for $\mathbb{Z}^*$). When one is only interested in the l-adic cohomology, it suffices to consider the roots of unity of order a power of l, and to assume that the characteristic p of k differs from l. We denote by $\mathbb{Z}/l^n(1)$ the group of the roots of unity of k of order dividing l^n. For various n, $\mathbb{Z}/l^n(1)$ form a projective system with transition maps

$$\sigma_{m,n} : \mathbb{Z}/l^m(1) \to \mathbb{Z}/l^n(1) : x \to x^{l^{m-n}}.$$

We let $\mathbb{Z}_l = \lim proj \mathbb{Z}/l^n(1)$ and $\mathbb{Q}_l(1) = \mathbb{Z}_l(1) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Denote by $\mathbb{Q}_l(r)$ the r-th tensor power of $\mathbb{Q}_l(1)$; for $r \in \mathbb{Z}$ negative we put $\mathbb{Q}_l(r) = \mathbb{Q}_l(-r)\check{}$.

As a vector space over $\mathbb{Q}_l$, $\mathbb{Q}_l(1)$ is isomorphic to $\mathbb{Q}_l$. However, the automorphism group of k acts non-trivially on $\mathbb{Q}_l(1)$: it acts via the character with values in $\mathbb{Z}_l^*$, which gives its action on the roots of unity. In particular, for $k = \bar{\mathbb{F}}_q$, the substitution of Frobenius $\varphi : x \to x^q$ acts by multiplication by q.

Let X be an algebraic variety purely of dimension n over k. The *orientation sheaf* of X for the l-adic cohomology is the constant $\mathbb{Q}_l$-sheaf $\mathbb{Q}_l(n)$. The *fundamental class* is a morphism

$$Tr : H_c^{2n}(X, \mathbb{Q}_l(n)) \to \mathbb{Q}_l,$$

or rather even

$$Tr : H_c^{2n}(X, \mathbb{Q}_l) \to \mathbb{Q}_l(-n).$$

Theorem (2.3) (Poincare duality). *For X proper and smooth, purely of dimension n, the bilinear form*

$$Tr(x \cup y) : H^i(X, \mathbb{Q}_l) \otimes H^{2n-i}(X, \mathbb{Q}_l) \to \mathbb{Q}_l(-n)$$

is a perfect paring (it identifies $H^i(X, \mathbb{Q}_l)$ with the dual of $H^{2n-i}(X, \mathbb{Q}_l(n))$).

(2.4) Let X_0 be a proper and smooth algebraic variety over $\mathbb{F}_q$, purely of dimension n and we obtain X over $\bar{\mathbb{F}}_q$ from X_0 by extension of scalars. The morphism (2.3) is compatible with the action of $Gal(\bar{\mathbb{F}}_q/\mathbb{F}_q)$. If (α_j) are the eigenvalues of the geometric Frobenius acting on $H^i(X, \mathbb{Q}_l)$, the eigenvalues of F acting on $H^{2n-i}(X, \mathbb{Q})$ are $(q^n \alpha_j^{-1})$.

(2.5) Assume for simplicity that X is connected. The proof of (2.4) goes as follows, once we transpose to the geometric language instead of the Galois one (see (1.15)).

a) The cup-product puts $H^i(X, \mathbb{Q}_l)$ and $H^{2n-i}(X, \mathbb{Q}_l)$ into perfect duality with values in $H^{2n}(X, \mathbb{Q}_l)$ that has dimension one.

b) The cup product commutes with the inverse image of F^* by the Frobenius morphism $F : X \to X$.

c) The morphism F is finite of degree q^n: on $H^{2n}(X, \mathbb{Q}_l)$ F^* is multiplication by q^n.

d) Therefore, the eigenvalues of F^* satisfy the property (2.4).

(2.6) We let $\chi(X) = \sum_i (-1)^i \dim H^i(X, \mathbb{Q}_l)$. For n odd, the form $Tr(x \cup y)$ on $H^n(X, \mathbb{Q}_l)$ is skew-symmetric; the integer $n\chi(X)$ is always even. It is easy to deduce from (1.5.4) and (2.3), (2.4) that

$$Z(X_0, t) = \varepsilon q^{\frac{-n\chi(X)}{2}} t^{-\chi(X)} Z(X_0, q^{-n} t^{-n})$$

for $\varepsilon = \pm 1$. If n is even, let N denote the multiplicity of the eigenvalue $q^{\frac{n}{2}}$ of F^* acting on $H^n(X, \mathbb{Q}_l)$ (i.e the dimension of the corresponding invariant subspace). We have

$$\varepsilon = \begin{cases} 1, & \text{if } n \text{ is odd} \\ (-1)^N, & \text{if } n \text{ is even} \end{cases}$$

This is the Grothendieck's formulation of the functional equation for Z-functions.

(2.7) We will need other forms of the duality theorem. The case of curves will be enough for our purposes. If $\mathcal{F}$ is a $\mathbb{Q}_l$-sheaf on an algebraic variety X over an algebraically closed k, we denote by $\mathcal{F}(r)$ the sheaf $\mathcal{F} \otimes \mathbb{Q}_l(r)$. This sheaf is (not canonically) isomorphic to $\mathcal{F}$.

Theorem (2.8). *Let X be smooth purely of dimension n and $\mathcal{F}$ be a locally constant sheaf. We denote by $\check{\mathcal{F}}$ the dual of $\mathcal{F}$. The bilinear form*

$$Tr(x \cup y) : H^i(X, \mathcal{F}) \otimes H_c^{2n-i}(X, \check{\mathcal{F}}(n)) \to H_c^{2n}(X, \mathcal{F} \otimes \check{\mathcal{F}}(n)) \to H_c^{2n}(X, \mathbb{Q}_l(n)) \to \mathbb{Q}_l$$

is a perfect pairing.

(2.9) Assume that X is connected and that x is a closed point of X. The functor $\mathcal{F} \to \mathcal{F}_x$ is an equivalence of the category of locally constant $\mathbb{Q}_l$-sheaves with that of l-adic representations of $\pi_1(X, x)$. Via this equivalence, $H^0(X, \mathcal{F})$ is identified with the invariants of $\pi_1(X, x)$ acting on $\mathcal{F}_x$:

$$H^0(X, \mathcal{F}) \xrightarrow{\sim} \mathcal{F}_x^{\pi_1(X,x)}.$$

According to (2.8), for X smooth and connected of dimension n we have

$$H_c^{2n}(X, \mathcal{F}) = H^0(X, \check{\mathcal{F}}(n))\check{} = ((\check{\mathcal{F}}_x(n))^{\pi_1(X,x)})\check{}.$$

The duality exchanges invariants (the largest invariant subspaces) with coinvariants (the largest invariant quotients)[6]. The formula takes form

$$H_c^{2n}(X, \mathcal{F}) = (\mathcal{F}_x)_{\pi_1(X,x)}(-n).$$

We will only use it for $n = 1$.

Statement (2.10). *Let X be a connected smooth curve over an algebraically closed field k, x a closed point of X and $\mathcal{F}$ a locally constant $\mathbb{Q}_l$-sheaf. We have*
(i)$H_c^0(X, \mathcal{F}) = 0$ if X is affine.
(ii)$H_c^2(X, \mathcal{F}) = (\mathcal{F}_x)_{\pi_1(X,x)}(-1)$.

Assertion (i) simply states that $\mathcal{F}$ does not have sections with finite support.

(2.11) Let X be a connected smooth projective curve over an algebraically closed k, U an open set in X, the complement of the finite set S of closed points of X, j the inclusion $U \hookrightarrow X$ and $\mathcal{F}$ a locally constant $\mathbb{Q}_l$-sheaf on U. Let $j_*\mathcal{F}$ be the constructible $\mathbb{Q}_l$-sheaf - the direct image of $\mathcal{F}$. Its stalk at $x \in S$ has rank less or equal to the rank of the stalk at a general point; it is the space of invariants of the local monodromy group.

Theorem (2.12). *The bilinear form*

$$Tr(x \cup y) : H^i(X, j_*\mathcal{F}) \otimes H^{2-i}(X, j_*\check{\mathcal{F}}(1)) \to H^2(X, j_*\mathcal{F} \otimes j_*\check{\mathcal{F}}(1)) \to$$

$$\to H^2(X, j_*(\mathcal{F} \otimes \check{\mathcal{F}})(1)) \to H_c^2(X, \mathbb{Q}_l(1)) \to \mathbb{Q}_l$$

is a perfect pairing.

(2.13) It will be convenient to have $\mathbb{Q}_l$-sheaves $\mathbb{Q}_l(r)$ on any scheme X where l is invertible. The point is to define $\mathbb{Z}/l^n(1)$. By definition, $\mathbb{Z}/l^n(1)$ is the etale sheaf of the l^n-th roots of unity.

(2.14) *Bibliographical notes on paragraphs 1 and 2.*

A) All the important results in etale cohomology are first proved for torsion sheaves. The extension to $\mathbb{Q}_l$-sheaves is done by passing to formal limits. In what follows, for each theorem

[6]I haven't seen the terms "invariant" and "coinvariant" used in this setting. For a group G acting on M the "invariants" (resp. "coinvariants") are denoted M^G (resp M_G).

mentioned I will not refer to the reference where it is proved, but to the reference where a similar statement for a torsion sheaf is proved.

B) With the exception of the Lefschetz formula and (2.12), results in etale cohomology used in this article are all proved in SGA 4. For those already stated, the references are: definition of H^i: VII; definition of H^I_c: XVII 5.1; finiteness theorem: XIV 1, completed in XVII 5.3; cohomological dimension: X; Poincare duality: XVIII.

C) The relation between the various Frobeniuses ((1.4), (1.11), (1.15)) is explained in detail in SGA 5, XV, par. 1, 2.

D) The cohomological interpretation of the Z-functions is clearly exposed in [1]; however, Lefschetz formula (1.12) for X a smooth projective curve is used, but not proved. For the proof, one has to consult SGA 5.

E) The form (2.12) of the Poincare duality theorem follows from the general result SGA 4, XVIII (3.2.5) (for $S = Spec(k)$, $X = X$, $K = j_*\mathcal{F}$, $L = \mathbb{Q}_l$) by a local calculation that is not difficult. The statement will be explicitly included in the final version of SGA 5. For the case interesting to us (tame ramification of $\mathcal{F}$), we could obtain it transcendentially[7] by lifting X and $\mathcal{F}$ to characteristic 0.

3 The main lemma (La majoration fondamentale)

The result of this paragraph was catalyzed by reading the lecture of Rankin [3][8].

(3.1) Let U_0 be a curve on $\mathbb{F}_q$, complement in $\mathbb{P}^1$ to a finite set of closed points, U be the curve over $\bar{\mathbb{F}}_q$ corresponding to it, u a closed point of U, $\mathcal{F}_0$ a locally constant sheaf on U_0 and $\mathcal{F}$ its inverse image on U.

Let $\beta \in \mathbb{Q}$. We say that $\mathcal{F}_0$ is of *weight* β if for all $x \in |U_0|$, the eigenvalues of F_x acting on $\mathcal{F}_0$ (1.13) are algebraic numbers all of which complex conjugates are of absolute value $q_x^{\beta/2}$. For example, $\mathbb{Q}_l(r)$ is of weight $-2r$.

Theorem (3.2). *Let's make the following hypotheses:*

(i) $\mathcal{F}_0$ is equipped with a bilinear skew-symmetric nondegenerate form

$$\psi : \mathcal{F}_0 \otimes \mathcal{F}_0 \to \mathbb{Q}_l(-\beta) \quad (\beta \in \mathbb{Z}).$$

(ii) The image of $\pi_1(U, u)$ in $GL(\mathcal{F}_u)$ is an open subgroup of the symplectic group $Sp(\mathcal{F}_u, \psi_u)$.

(iii) For all $x \in |U_0|$, the polynomial $\det(1 - F_x t, \mathcal{F}_0)$ *has rational coefficients.*

Then $\mathcal{F}$ is of weight β.

We may and do assume that U is affine and that $\mathcal{F} \neq 0$.

Lemma (3.3). *Let $2k$ be an even integer and denote by $\overset{2k}{\otimes}\mathcal{F}_0$ the $2k$-th tensor power of $\mathcal{F}_0$. For $x \in |U_0|$ the logarithmic derivative*

$$t\frac{d}{dt}\log(\det(1 - F_x t^{\deg(x)}, \overset{2k}{\otimes}\mathcal{F}_0)^{-1})$$

is a formal series with positive rational coefficients.

[7]Transcendential algebraic geometry deals with varieties defined over $\mathbb{C}$ and concentrates on their structure of holomorphic manifolds, that allows one to use powerful techniques of topology, analysis, differential equations, etc.

[8]For a slightly different exposition of the main lemma using the equivalence of (1.9) b) consult Milne.

Hypothesis (iii) ensures that for all n $Tr(F_x^n, \mathcal{F}_0) \in \mathbb{Q}$. The number

$$Tr(F_x^n, \overset{2k}{\otimes}\mathcal{F}_0) = Tr(F_x^n, \mathcal{F}_0)^{2k}$$

is a positive rational and we apply (1.5.3).

Lemma (3.4). *The local factors* $\det(1 - F_x t^{\deg(x)}, \overset{2k}{\otimes}\mathcal{F}_0)^{-1}$ *are formal series with positive rational coefficients.*

The formal series $\log(\det(1 - F_x t^{\deg(x)}, \overset{2k}{\otimes}\mathcal{F}_0)^{-1})$ has constant term zero, from (3.3) all the coefficients are $\geqslant 0$; the coefficients of the exponentiation are therefore also positive.

Lemma (3.5). *Let* $f_i = \sum_n a_{i,n} t^n$ *be a sequence of formal series with positive real coefficients. We assume that the order of* $f_i - 1$ *tends to infinity with* i*; and we denote* $f = \prod_i f_i$*. Then the radius of convergence of* f_i *is greater or equal to that of* f*.*

If $f = \sum_n a_n t^n$, we have $a_{i,n} \leqslant a_n$.

Lemma (3.6). *Under the assumptions of (3.5), if* f *and the* f_i *are Taylor expansions of meromorphic functions, then*

$$inf\{|z| \, |f(z) = \infty\} \leqslant inf\{|z| \, |f_i(z) = \infty\}$$

Indeed, those numbers are the radii of convergence.

(3.7) For each partition P of $[1, 2k]$ [9] into the two element sets $\{i_\alpha, j_\alpha\}(i_\alpha \leqslant j_\alpha)$, we define

$$\psi_P : \overset{2k}{\otimes}\mathcal{F}_0 \to \mathbb{Q}_l(-k\beta) : x_1 \otimes \cdots \otimes x_{2k} \to \prod_\alpha \psi(x_{i_\alpha}, x_{j_\alpha}).$$

Let x be a closed point of X [10]. Hypothesis (ii) ensures that the coivariants of $\pi_1(U, u)$ on $\overset{2k}{\otimes}\mathcal{F}_u$ are the coinvariants on $\overset{2k}{\otimes}\mathcal{F}_u$ of the entire symplectic group (π_1 is Zariski-dense in Sp). Let $\mathcal{P}$ be the set of partitions P. From H.Weil (*The classical groups*, Princeton University Press, chap. VI, par. 1), for an appropriate $\mathcal{P}' \subset \mathcal{P}$, depending on $\dim(\mathcal{F}_u)$, ψ_P (for $P \in \mathcal{P}'$) defines an isomorphism

$$(\overset{2k}{\otimes}\mathcal{F}_u)_{\pi_1} = (\overset{2k}{\otimes}\mathcal{F}_u)_{Sp} \xrightarrow{\sim} \mathbb{Q}_l(-k\beta)^{\mathcal{P}'}.$$

Let N be the number of elements in $\mathcal{P}'$. According to (2.10) the formula above gives

$$H_c^2(U, \overset{2k}{\otimes}\mathcal{F}) \simeq \mathbb{Q}_l(-k\beta - 1)^N.$$

Since $H_c^0(U, \overset{2k}{\otimes}\mathcal{F}) = 0$ [11], the formula (1.14.3) reduces to

$$Z(U_0, \overset{2k}{\otimes}\mathcal{F}_0, t) = \frac{\det(1 - F^* t, H^1(U, \overset{2k}{\otimes}\mathcal{F}))}{(1 - q^{k\beta+1} t)^N}.$$

[9] of $\{1, \cdots, 2k\}$
[10] Let u be a closed point of U.
[11] Again, see (2.10).

This Z-function is therefore the Taylor series expansion of a rational function having only one pole at $t = 1/q^{k\beta+1}$. We will only use the fact that the poles are of modulus $t = 1/q^{k\beta+1}$ in $\mathbb{C}$. This could be concluded from the general arguments about reductive groups. If α is an eigenvalue of F_x on $\mathcal{F}_0$, then α^{2k} is an eigenvalue of F_x on $\overset{2k}{\otimes}\mathcal{F}_0$. We now let α be any complex conjugate of [12] α. The inverse power $1/\alpha^{2k/\deg(x)}$ is a pole of $\det(1 - F_x t^{\deg(x)}, \overset{2k}{\otimes}\mathcal{F})^{-1}$. According to (3.4) and (3.6) we have

$$|1/q^{k\beta+1}| \leqslant |1/\alpha^{2k/\deg(x)}|,$$

or

$$|\alpha| \leqslant q_x^{\frac{\beta}{2}+\frac{1}{2k}}.$$

Letting k go to infinity we find that

$$|\alpha| \leqslant q_x^{\frac{\beta}{2}}.$$

On the other hand, the existence of ψ ensures that $q_x^\beta \alpha^{-1}$ is an eigenvalue, so we have the inequality

$$|q_x^\beta \alpha^{-1}| \leqslant q_x^{\beta/2},$$

or

$$q_x^{\beta/2} \leqslant |\alpha|.$$

This completes the proof.

Corollary (3.8). *Let α be an eigenvalue of F^* acting on $H_c^1(U, \mathcal{F})$. Then α is an algebraic number all of which complex conjugates satisfy*

$$|\alpha| \leqslant q^{\frac{\beta+1}{2}+\frac{1}{2}}.$$

The formula (1.14.3) for $\mathcal{F}_0$ reduces to

$$Z(U_0, \mathcal{F}, t) = \det(1 - F^* t, H_c^1(U, \mathcal{F})).$$

The left hand side is a formal series with rational coefficients, given its representation as a product and hypothesis (iii). The right hand side is therefore a polynomial with rational coefficients, $1/\alpha$ is its root. This already implies that α is algebraic. To complete the proof, it suffices to show that the infinite product that defines $Z(U_0, \mathcal{F}_0, t)$ converges absolutely (thus it is nonzero) for $|t| < q^{\frac{-\beta}{2}-1}$.

Let N be the rank of $\mathcal{F}$ and let

$$\det(1 - F_x t, \mathcal{F}) = \prod_{i=1}^{N} (1 - \alpha_{i,x} t).$$

According to (3.2), $|\alpha_{i,x}| = q_x^{\beta/2}$. The convergence of the infinite product for Z follows from that of the series

$$\sum_{i,x} |\alpha_{i,x} t^{\deg(x)}|.$$

For $|t| = q^{\frac{-\beta}{2}-1-\varepsilon}$ ($\varepsilon > 0$) we have

$$\sum_{i,x} |\alpha_{i,x} t^{\deg(x)}| = N \sum_x q_x^{-1-\varepsilon}.$$

[12]the original

On the affine line there are q^n points with coordinate in $\mathbb{F}_{q^n}$, so there are at most q^n closed points of degree n. So we have

$$\sum_x q_x^{-1-\varepsilon} \leqslant \sum_n q^n q^{n(-1-\varepsilon)} = \sum_n q^{-n\varepsilon} < \infty,$$

which completes the proof.

Corollary (3.9). *Let j_0 be the inclusion of U_0 in $\mathbb{P}^1_{\mathbb{F}_q}$, j that of U into $\mathbb{P}^1$ and α an eigenvalue of F^* acting on $H^1(\mathcal{P}^1, j_*\mathcal{F})$. Then α is an algebraic number all of which complex conjugates satisfy*

$$q^{\frac{\beta+1}{2}-\frac{1}{2}} \leqslant |\alpha| \leqslant q^{\frac{\beta+1}{2}+\frac{1}{2}}.$$

The segment of the long exact sequence in cohomology defined by the short exact sequence

$$0 \to j_!\mathcal{F} \to j_*\mathcal{F} \to j_*\mathcal{F}/j_!\mathcal{F} \to 0$$

($j_!$ is the extension by 0) is

$$H^1_c(U, \mathcal{F}) \to H^1(\mathbb{P}, j_*\mathcal{F}) \to 0.$$

So the eigenvalue α already appears in $H^1_c(U, \mathcal{F})$ [13] and so by (3.8) we have:

$$|\alpha| \leqslant q^{\frac{\beta+1}{2}+\frac{1}{2}}.$$

The Poincare duality (2.12) implies that $q^{\beta+1}\alpha^{-1}$ is an eigenvalue, so we have the inequality

$$|q^{\beta+1}\alpha^{-1}| \leqslant q^{\frac{\beta+1}{2}+\frac{1}{2}}$$

and the corollary is proved.

4 Lefschetz theory: local theory

(4.1) On $\mathbb{C}$ Lefshietz local results are as follows. Let $D = \{z | \, |z| < 1\}$ be the unit disk, $D^* = D - \{0\}$ and $f : X \to D$ be a morphism of analytic spaces. We assume that

a) X is nonsingular and purely of dimension $n + 1$;

b) f is proper;

c) f is smooth outside the point x of the special fiber $X_0 = f^{-1}(0)$;

d) At x f has a nondegenerate double point.

Let $t \neq 0$ in D and $X_t = f^{-1}(t)$ "the" general fiber. To the previous data we associate:

α) Specialization morphisms $sp : H^i(X_0, \mathbb{Z}) \to H^i(X_t, \mathbb{Z})$: X_0 is a deformation retract of X and sp is the composition arrow

$$H^i(X_0, \mathbb{Z}) \xleftarrow{\sim} H^i(X, \mathbb{Z}) \to H^i(X_t, \mathbb{Z})$$

β) The monodromy transformations $T : H^i(X_t, \mathbb{Z}) \to H^i(X_t, \mathbb{Z})$, which describe the effect on the singular cycles of X_t of "rotating t around 0". This is even an action on $H^i(X_t, \mathbb{Z})$, the stalk at t of the local system $R^i f_* \mathbb{Z}|D^*$, of the positive generator of $\pi_1(D^*, t)$.

[13] We will repeatedly use the following facts:
A) If $H^i(X') \to H^i(X)$ is a surjection, then (1.7) for $H^i(X') \Rightarrow$ (1.7) for $H^i(X)$.
B) If $H^i(X) \hookrightarrow H^i(X'')$ is an embedding, then (1.7) for $H^i(X'') \Rightarrow$ (1.7) for $H^i(X)$.

Lefschetz theory describes α) and β) in terms of the *vanishing cycle*[14] $\delta \in H^n(X_t, \mathbb{Z})$. This cycle is well-defined up to sign. For $i \neq n,\ n+1$ we have

$$H^i(X_0, \mathbb{Z}) \xrightarrow{\sim} H^i(X_t, \mathbb{Z})\ (i \neq n,\ n+1).$$

For $i = n,\ n+1$ we have an exact sequence

$$0 \to H^n(X_0, \mathbb{Z}) \to H^n(X_t, \mathbb{Z}) \overset{x \to (x,\delta)}{\to} \mathbb{Z} \to H^{n+1}(X_0, \mathbb{Z}) \to H^{n+1}(X_t, \mathbb{Z}) \to 0.$$

For $i \neq n$, the monodromy T is the identity. For $i = n$ we have

$$Tx = x \pm (x, \delta)\delta.$$

The values of $\pm$, $T\delta$ and (δ, δ) are as follows:

$n \mod 4$	0	1	2	3
$Tx = x \pm (x, \delta)\delta$	-	-	+	+
(δ, δ)	2	0	-2	0
$T\delta$	$-\delta$	δ	$-\delta$	δ

The monodromy transformation preserves the intersection form $Tr(x \cup y)$ on $H^n(X_t, \mathbb{Z})$. For n odd, it is the symplectic transvection. For n even, it is the symmetric orthogonal.

(4.2) There is an analog of (4.1) in abstract algebraic geometry. The disk D is replaced by the spectrum of a henselian discrete valuation ring A with an algebraically closed residue field. Let S be the spectrum, η its generic point (spectrum of the field of fractions of A), s the closed point (spectrum of the residue field). The role of t is played by the geometric generic point $\bar{\eta}$ (spectrum of the closure of the field of fractions of A).

Let $f : X \to S$ be a proper morphism, with X regular purely of dimension $n+1$. We assume that f is smooth except for an ordinary double point x of the special fiber X_s. Let l be a prime number different from the residual characteristic[15] p of S. Denoting by $X_{\bar{\eta}}$ the generic geometric fiber, we have the specialization morphism

$$sp : H^i(X_s, \mathbb{Q}_l) \xleftarrow{\sim} H^i(X, \mathbb{Q}_l) \to H^i(X_{\bar{\eta}}, \mathbb{Q}_l) \quad \textbf{(4.2.1)}$$

The role of T is played by the action of the inertia group $I = Gal(\bar{\eta}/\eta)$ on $H^i(X_{\bar{\eta}}, \mathbb{Q}_l)$ by transport of structure (see (1.15)):

$$I = Gal(\bar{\eta}/\eta) \to GL(H^i(X_{\bar{\eta}}, \mathbb{Q}_l)) \quad \textbf{(4.2.2)}$$

The data (4.2.1), (4.2.2) fully determines the sheaf $R^i f_* \mathbb{Q}_l$ on S.

(4.3) Let $n = 2m$ for n even and $n = 2m+1$ for n odd. (4.2.1) and (4.2.2) can still be described in terms of the vanishing cycle

$$\delta \in H^n(X_{\bar{\eta}}, \mathbb{Q}_l)(m) \quad \textbf{(4.3.1)}$$

This cycle is well-defined up to sign.

[14] We can define δ to be the unique (up to sign) generator of $H^n(X_0, \mathbb{Z})^\perp \subset H^n(X_t, \mathbb{Z})$ (under the pairing induced by Poincare duality).

[15] Characteristic of the residue field of A.

For $i \neq n,\ n+1$ we have

$$H^i(X_s,\mathbb{Q}_l) \xrightarrow{\sim} H^i(X_{\bar\eta},\mathbb{Q}_l)\ (i \neq n,\ n+1) \quad \textbf{(4.3.2)}$$

For $i = n,\ n+1$ we have an exact sequence

$$0 \to H^n(X_s,\mathbb{Q}_l) \to H^n(X_{\bar\eta},\mathbb{Q}_l) \overset{x \to Tr(x \cup \delta)}{\longrightarrow} \mathbb{Q}_l(m-n) \to H^{n+1}(X_s,\mathbb{Q}_l) \to H^{n+1}(X_{\bar\eta},\mathbb{Q}_l) \to 0 \quad \textbf{(4.3.3)}$$

The action (4.2.2) (local monodromy) is trivial for $i \neq n$. For $i = n$, it is described as follows.

A) *n odd.* - We have a canonical homomorphism

$$t_l : I \to \mathbb{Z}_l(1),$$

and the action of $\sigma \in I$ is

$$x \to x \pm t_l(\sigma)(x,\delta)\delta.$$

B) *n even.* - We will not need this case. Let's just say that if $p \neq 2$, there exists a unique character of order two

$$\varepsilon : I \to \{\pm 1\},$$

and we have

$$\begin{array}{lll} \sigma x = x & \text{if} & \varepsilon(\sigma) = 1 \\ \sigma x = x \pm (x,\delta)\delta & \text{if} & \varepsilon(\sigma) = -1 \end{array}$$

The signs $\pm$ in A) and B) are the same as in (4.1).

(4.4) These results imply the following information[16] about $R^i f_* \mathbb{Q}_l$.

a) If $\delta \neq 0$:

1) For $i \neq n$ the sheaf $R^i f_* \mathbb{Q}_l$ is constant.

2) Let j be the inclusion of η in S. We have

$$R^i f_* \mathbb{Q}_l = j_* j^* R^i f_* \mathbb{Q}_l.$$

b) If $\delta = 0$: (This is an exceptional case. Since $(\delta,\delta) = \pm 2$ for n even, it can only happen for n odd.)

1) For $i \neq n+1$ the sheaf $R^i f_* \mathbb{Q}_l$ is constant.

2) Let $\mathbb{Q}_l(m-n)_s$ be the sheaf $\mathbb{Q}_l(m-n)$ on $\{s\}$, extended by zero on S. Then we have an exact sequence

$$0 \to \mathbb{Q}_l(m-n)_s \to R^{n+1} f_* \mathbb{Q}_l \to j_* j^* R^{n+1} f_* \mathbb{Q}_l \to 0,$$

where $j_* j^* R^{n+1} f_* \mathbb{Q}_l$ is a constant sheaf.

5 Lefschetz theory: global theory

(5.1) On $\mathbb{C}$ the results of Lefshietz are as follows. Let $\mathbb{P}$ be a projective space of dimension $\geqslant 1$ and $\check{\mathbb{P}}$ the dual projective space; its points parameterize the hyperplanes of $\mathbb{P}$ and we denote by H_t the hyperplane defined by $t \in \check{\mathbb{P}}$. If A is a linear subspace of codimension 2 in $\mathbb{P}$, the hyperplanes containing A are parameterized by points of the line $D \subset \check{\mathbb{P}}$, the *dual* of A. These hyperplanes $(H_t)_{t \in D}$ form the *pencil with axis A*.

[16] Derivation of some of the results can be found in Milne.

Let $X \subset \mathbb{P}$ be a connected nonsingular projective variety of dimension $n+1$. Let $\tilde{X} \subset X \times D$ be the set of pairs (x, t) such that $x \in H_t$. Projections to the first and second coordinates form a diagram[17]

$$\begin{array}{ccc} X & \xleftarrow{\pi} & \tilde{X} \\ & & \big\downarrow f \\ & & D \end{array} \quad \textbf{(5.1.1)}$$

The fiber of f at $t \in D$ is the hyperplane section $X_t = X \cap H_t$ of X.

Fix X and take a general enough A. Then:

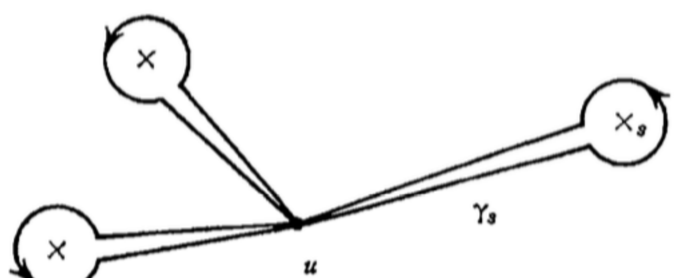
A) A is transverse to X and $\tilde{X}$ is the blowing up of X along $A \cap X$. In particular, $\tilde{X}$ is nonsingular.

B) There is a finite subset S of D and for each $s \in S$ a point $x_s \in X_s$ such that f is smooth outside x_s.

C) The x_s are critical nondegenerate points of f.

Therefore, for each $s \in S$ local Lefschetz theory (4.1) applies to a small disk D_s around s and $f^{-1}(D_s)$.

(5.2) Let $U = D - S$. Let $u \in U$ and choose disjoint loops $(\gamma_s)_{s \in S}$ starting from u, with γ_s turning once around s:



These loops generate the fundamental group $\pi_1(U, u)$. This group acts on $H^i(X_u, \mathbb{Z})$, the stalk at u of the local system $R^i f_* \mathbb{Z}|U$. According to the local theory (4.1), to each $s \in S$ corresponds a vanishing cycle $\delta_s \in H^n(X_u, \mathbb{Z})$; these cycles depend on the choice of γ_s. For $i \neq n$, the action of $\pi_1(U, u)$ on $H^i(X_u, \mathbb{Z})$ is trivial. For $i = n$ we have

$$\gamma_s x = x \pm (x, \delta_s)\delta_s \quad \textbf{(5.2.1)}$$

Let E be the subspace of $H^n(X_u, \mathbb{Q})$ generated by the δ_s (*vanishing* part of the cohomology).

Proposition (5.3). *E is stable under the action of the monodromy group $\pi_1(U, u)$. The orthogonal $E^\perp$ of E (for the intersection form $Tr(x \cup y)$) is the space of the invariants of the monodromy in $H^n(X_u, \mathbb{Q})$.*

The γ_s generate the monodromy group, so this is clear from (5.2.1).

Theorem (5.4). *The vanishing cycles $\pm\delta_s$ are conjugate (up to sign)*[18] *under the action of $\pi_1(U, u)$.*

[17] In the original paper the letter f is absent.

[18] That is, given $s,\ s' \in S$, there exists a $\sigma \in \pi_1$ such that $\sigma\delta_s = \pm\delta_s$.

Let $\check{X} \subset \check{\mathbb{P}}$ be the dual variety of X; it is the set of $t \in \check{\mathbb{P}}$ such that H_t is tangent to X, i.e. such that X_t is singular or $X \subset H_t$. The variety $\check{X}$ is irreducible. Let $Y \subset X \times \check{\mathbb{P}}$ be the space of pairs (x, t) such that $x \in H_t$. We have a diagram

$$\begin{array}{ccc} X & \longleftarrow & Y \\ & & \big\downarrow g \\ & & \check{\mathbb{P}} \end{array}$$

The fiber of g at $t \in \check{\mathbb{P}}$ is the hyperplane section $X_t = X \cap H_t$ of X and g is smooth on the complement of the inverse image of $\check{X}$.

We retrieve the situation of (5.1) by replacing $\check{\mathbb{P}}$ by the line $D \subset \check{\mathbb{P}}$ and Y by $g^{-1}(D)$. We have $S = D \cap \check{X}$. According to a theorem of Lefschetz, for D general enough, the map

$$\pi_1(D - S, u) \to \pi_1(\check{\mathbb{P}} - \check{X}, u)$$

is surjective. It suffices to show that $\pm\delta_s$ are conjugate under $\pi_1(\check{\mathbb{P}} - \check{X})$.

For x in the smooth locus of codimension 1 of $\check{X}$, let ch be the path from t to x in $\check{\mathbb{P}} - \check{X}$ and γ_x the loop that follows ch until the neighborhood of $\check{X}$, turns once around $\check{X}$ and then returns to t by ch. The loops γ_x (for various ch) are mutually conjugate. Since $\check{X}$ is irreducible, two points in the smooth locus of $\check{X}$ can always be joined, in $\check{X}$, by a path that does not leave the smooth locus. It follows that the conjugation class of γ_x does not depend on x. In particular, γ_s are mutually conjugate. We see from (5.2.1) that this implies the conjugacy of $\pm\delta_s$.

Corollary (5.5). *The action of $\pi_1(U, u)$ on $E/(E \cap E^\perp)$ is absolutely irreducible*[19].

Let $F \subset E \otimes \mathbb{C}$ be the subspace stable under the monodromy. If $F \not\subset (E \cap E^\perp) \otimes \mathbb{C}$, there exists an $x \in F$ and $s \in S$ such that $(x, \delta_s) \neq 0$. We then have

$$\gamma_s x - x = \pm(x, \delta_s)\delta_s \in F$$

and $\delta_s \in F$. According to (5.4), all the δ_s are then in F and $F = E$. This proves (5.5).

(5.6) These results transpose as follows into abstract algebraic geometry. Let $\mathbb{P}$ be a projective space of dimension > 1 over an algebraically closed field k of characteristic p and $X \subset \mathbb{P}$ a connected projective nonsingular variety of dimension $n + 1$. For A a linear subspace of codimension 2 we define D, the pencil $(H_t)_{t \in D}$, $\tilde{X}$ and the diagram (5.1.1) as in (5.1). We say that $(H_t)_{t \in D}$ form a *Lefschetz pencil* of hyperplane sections if the following conditions are satisfied:

A) The axis A is transverse to X. $\tilde{X}$ is obtained by blowing up X along $A \cap X$ and is smooth.

B) There is a finite subset S of D and for each $s \in S$ a point $x_s \in X_s$ such that f is smooth outside x_s.

C) The x_s are ordinary double singular points of f.

For each $s \in S$ the local Lefschetz theory of par. 4 applies to the spectrum D_s of the henselization of the local ring of D at s and to $\tilde{X}_{D_s} = \tilde{X} \times_D D_s$.

(5.7) Let N be the dimension of $\mathbb{P}$, r an integer $\geqslant 1$ and $\iota_{(r)}$ the embedding of $\mathbb{P}$ into the projective space of dimension $\binom{N+r}{N} - 1$, the homogeneous coordinates of which are monomials

[19] The action on the k-vector space V is called absolutely irreducible if the corresponding action on $V \otimes_k \bar{k}$ is irreducible.

of degree r in the homogeneous coordinates of $\mathbb{P}$ [20]. The hyperplane sections of $\iota_{(r)}(\mathbb{P})$ are the hypersurfaces of degree r of $\mathbb{P}$.

For $p \neq 0$ it might happen that there is no such pencil of hyperplane sections of X that is Lefschetz. However, if $r \geqslant 2$ and we replace the projective embedding $\iota_1 : X \hookrightarrow \mathbb{P}$ by $\iota_r = \iota_{(r)} \circ \iota_1$, then, in this new embedding, any general enough pencil of hypersurface sections of degree r on X is still Lefschetz.

(5.8) For the rest of this discussion, we are studying the Lefschetz pencil of hyperplane sections of X, *excluding the case $p = 2$, n even.* The case of n odd will suffice for our purposes. We put $U = D - S$. Take $u \in U$ and l a prime number $\neq p$. The local results of par. 4 show that $R^n f_* \mathbb{Q}_l$ is *tamely* ramified at each $s \in S$. The tame fundamental group of U is a quotient of the profinite completion of the corresponding transcendential[21] fundamental groups (lifting to characteristic 0 of the tame coverings and the Riemann existence theorem). The algebraic situation is therefore similar to the transcendential situation and the transfer of Lefschetz's results is done by standard arguments. In the proof of (5.4) the theorem of Lefschetz for π_1 is replaced by a theorem of Bertiny and we have to invoke Abhyankar's lemma to control the ramification of $R^* g_* \mathbb{Q}_l$ along the smooth locus of codimension one in $\check{X}$ [22].

The results are as follows[23]:

a) *If the vanishing cycles are nonzero*:

1) For $i \neq n$ the sheaf $R^i f_* \mathbb{Q}_l$ is constant.

2) Let j be the inclusion of U in D. We have

$$R^n f_* \mathbb{Q}_l = j_* j^* R^n f_* \mathbb{Q}_l.$$

3) Let $E \subset H^n(X_u, \mathbb{Q}_l)$ be the subspace of the cohomology generated by the vanishing cycles. This subspace is stable under $\pi_1(U, u)$ and

$$E^{\perp} = H^n(X_u, \mathbb{Q}_l)^{\pi(U,u)}.$$

The representation of $\pi(U, u)$ on $E/(E \cap E^{\perp})$ is absolutely irreducible and the image of π_1 in $GL(E/(E \cap E^{\perp}))$ is generated (topologically[24]) by the $x \to x \pm (x, \delta_s)\delta_s$ $(s \in S)$ (the $\pm$ sign is determined as in (4.1)).

b) *If the vanishing cycles are zero*: (This is an exceptional case. Since $(\delta, \delta) = \pm 2$ for n even, it can only happen for n odd: $n = 2m + 1$. Note that if one vanishing cycle is zero, they all are because of conjugacy.)

1) For $i \neq n + 1$ the sheaf $R^i f_* \mathbb{Q}_l$ is constant.

2) We have an exact sequence

$$0 \to \bigoplus_{s \in S} \mathbb{Q}_l(m - n)_s \to R^{n+1} f_* \mathbb{Q}_l \to \mathcal{F} \to 0$$

with $\mathcal{F}$ constant.

3) $E = 0$.

(5.9) The subspace $E \cap E^{\perp}$ of E is the kernel of the restriction to E of the intersection form $Tr(x \cup y)$. Therefore, this form induces a bilinear nondegenerate form

$$\psi : E/(E \cap E^{\perp}) \otimes E/(E \cap E^{\perp}) \to \mathbb{Q}_l(-n),$$

[20] Deligne describes the Veronese mapping.

[21] That is, taken over $\mathbb{C}$, see footnote 7.

[22] See the sketch of this proof in Milne.

[23] Again, proofs of some of the results appear in Milne.

[24] They generate a dense subgroup.

skew-symmetric for n odd and symmetric for n even. This form is preserved by the monodromy; for n odd, therefore, the monodromy representation induces

$$\rho : \pi_1(U,u) \to Sp(E/(E \cap E^\perp), \psi).$$

Theorem (5.10) (Kajdan-Margulis). [25] *The image of ρ is open.*

The image of ρ is a compact, therefore, analytic l-adic[26] subgroup of $Sp(E/(E \cap E^\perp), \psi)$. It suffices to show that its Lie algebra $\mathfrak{L}$ equals $\mathfrak{sp}(E/(E \cap E^\perp), \psi)$. The transcendential analog of this Lie algebra is the Lie algebra of the Zariski closure of the monodromy group.

We deduce from (5.8) that $\mathfrak{L}$ is generated by transformations with zero square

$$N_s : x \to (x, \delta_s)\delta_s \quad (s \in S)$$

and that $E/(E \cap E^\perp)$ is an absolutely irreducible representation of $\mathfrak{L}$. The theorem comes from the following lemma.

Lemma (5.11). *Let V be a finite dimensional vector space over the field k of characteristic 0, ψ a nondegenerate skew-symmetric form on a Lie subalgebra $\mathfrak{L}$ of the Lie algebra $\mathfrak{sp}(V, \eta)$. We assume that:*

(i) V is a simple representation of $\mathfrak{L}$.

(ii) $\mathfrak{L}$ is generated by the family of endomorphisms of V of the form $x \to \psi(x, \delta)\delta$.

Then $\mathfrak{L} = \mathfrak{sp}(V, \psi)$.

We may and do assume that V, and thus $\mathfrak{L}$ are nonzero. Let $W \subset V$ be the set of $\delta \in V$ such that $N(\delta) : x \to \psi(x, \delta)\delta$ is in $\mathfrak{L}$.

a) W is stable under homotheties (since $\mathfrak{L}$ is a vector subspace of $\mathfrak{gl}(V)$).

b) If $\delta \in W$, $\exp(\lambda N(\delta))$ is an automorphism of $(V, \psi, \mathfrak{L})$, therefore, transforms W to itself. If δ', $\delta'' \in W$, we have $\exp(\lambda N(\delta'))\delta'' = \delta'' + \lambda\psi(\delta'', \delta')\delta' \in W$ [27]; if $\psi(\delta', \delta'') \neq 0$, then the vector subspace spanned by δ' and δ'' lies in W.

c) It follows that W is the union of its maximal linear subspaces W_α and that those are pairwise orthogonal. Each W_α is therefore stable under the $N(\delta)$ $(\delta \in W)$, so it is stable under $\mathfrak{L}$. By hypothesis (i), $W_\alpha = V$ and $\mathfrak{L}$ contains all $N(\delta)$ for $\delta \in V$. We conclude by noting that Lie algebra $\mathfrak{sp}(V, \psi)$ is generated by the $N(\delta)$ $(\delta \in V)$.

Remark **(5.12)** (not necessary for the exposition). - It is now easy to prove (1.6) for a hypersurface of odd dimension n in $\mathbb{P}^{n+1}_{\mathbb{F}_q}$.

Let X_0 be such a hypersurface and $\bar{X}_0$ the hypersurface over $\bar{\mathbb{F}}_q$, which is obtained by extension of scalars. We have

$$H^i(\bar{X}_0, \mathbb{Q}_l) = \mathbb{Q}_l(-i) \quad (0 \leqslant i \leqslant n);$$

$H^i(\bar{X}_0, \mathbb{Q}_l(i))$ is generated by the i-th cup power of η, the cohomology class $c_1(\mathcal{O}(1))$ of a hyperplane section. Therefore, we have

$$Z(X_0, t) = \det(1 - F^*t, H^n(\bar{X}_0, \mathbb{Q}_l) / \prod_{i=0}^{n}(1 - q^i t)$$

and $\det(1 - F^*t, H^n(\bar{X}_0, \mathbb{Q}_l)$ is a polynomial with integer coefficients independent of l.

[25]See a somewhat more detailed exposition of this part in Milne.

[26]A Lie group over $\mathbb{Q}_l$.

[27]We use the nilpotence of $N(\delta)$ first to define $\exp(\lambda N(\delta))$ and then to obtain the formula.

Let's vary X_0 within the Lefschetz pencil of hypersurfaces that is defined over $\mathbb{F}_q$ (see (5.7) for $X = \mathbb{P}^{n+1}$; the existence of such a pencil is not clear; if we wanted to complete the argument sketched here, we would have to use the arguments that will be given in (7.1). One verifies that E coincides here with the whole H^n and (3.2) provides the Weil conjecture for all the hypersurfaces of the pencil, in particular for X_0.

(5.13) *Bibliographical notes on paragraphs 3 and 4.*

A) The results of Lefschetz (4.1) and (5.1) to (5.5) are contained in this book [2]. For the local theory (4.1), it may be more handy to consult SGA 7, XIV (3.2).

B) The results of paragraph 4 are proved in parts XIII, XIV and XV of SGA 7.

C) (5.7) is proved in SGA 7, XVII[28].

D) (5.8) is proved in SGA 7, XVIII. The irreducibility theorem is proved there for E but only under the hypothesis that $E \cap E^{\perp} = \{0\}$. Proof of the general case (for $E/(E \cap E^{\perp})$) is similar.

6 The rationality theorem

(6.1) Let $\mathbb{P}_0$ be a projective space of dimension $\geqslant 1$ over $\mathbb{F}_q$, $X_0 \subset \mathbb{P}_0$ a projective nonsingular variety, $A_0 \subset \mathbb{P}_0$ a linear subspace of codimension two, $D_0 \subset \check{\mathbb{P}}_0$ the dual line, $\bar{\mathbb{F}}_q$ the algebraic closure of $\mathbb{F}_q$ and $\mathbb{P}, X, A, D$ over $\bar{\mathbb{F}}_q$ obtained from $\mathbb{P}_0, X_0, A_0, D_0$ by extension of scalars. The diagram (5.1.1) from (5.6) comes from a similar diagram over $\mathbb{F}_q$:

$$\begin{array}{ccc} X_0 & \xleftarrow{\pi_0} & \tilde{X}_0 \\ & & \downarrow{\scriptstyle f_0} \\ & & D_0 \end{array} \qquad \textbf{(6.1.1)}$$

We assume that X is connected of *even* dimension $n+1 = 2m+2$ and that the pencil of hyperplane sections of X defined by D is a *Lefschetz pencil*. The set S of $t \in D$ such that X_t is singular and defined over $\mathbb{F}_q$ comes from $S_0 \subset D_0$. We denote $U_0 = D_0 - S_0$ and $U = D - S$.

Let $u \in U$. The vanishing part of the cohomology $E \subset H^n(X_u, \mathbb{Q}_l)$ is stable under $\pi_1(U, u)$, so it is defined over U by a local subsystem $\mathcal{E}$ of $R^n f_* \mathbb{Q}_l$. The latter is defined over $\mathbb{F}_q$: $R^i f_* \mathbb{Q}_l$ is the inverse image of the $\mathbb{Q}_l$ sheaf $R^i f_{0*} \mathbb{Q}_l$ on D_0 and, on U, $\mathcal{E}$ is the inverse image of a local subsystem

$$\mathcal{E}_0 \subset R^n f_{0*} \mathbb{Q}_l.$$

The cup product is a skew-symmetric form

$$\psi : R^n f_{0*} \mathbb{Q}_l \otimes R^n f_{0*} \mathbb{Q}_l \to \mathbb{Q}_l(-n).$$

Denoting by $\mathcal{E}_0^{\perp}$ the orthogonal of $\mathcal{E}_0$ relative to ψ, on $R^n f_{0*} \mathbb{Q}_l | U_0$ we see that ψ induces a a perfect pairing

$$\psi : \mathcal{E}_0/(\mathcal{E}_0 \cap \mathcal{E}_0^{\perp}) \otimes \mathcal{E}_0/(\mathcal{E}_0 \cap \mathcal{E}_0^{\perp}) \to \mathbb{Q}_l(-n).$$

Theorem (6.2). *For all $x \in |U_0|$ the polynomial* $\det(1 - F_x^* t, \mathcal{E}_0/(\mathcal{E}_0 \cap \mathcal{E}_0^{\perp}))$ *has rational coefficients.*

Corollary (6.3). *Let j_0 be the inclusion of U_0 in D_0 and j that of U in D. The eigenvalues of F^* acting on $H^1(D, j_* \mathcal{E}_0/(\mathcal{E}_0 \cap \mathcal{E}_0^{\perp}))$ are algebraic numbers all of which complex conjugates α satisfy*

$$q^{\frac{n+1}{2} - \frac{1}{2}} \leqslant |\alpha| \leqslant q^{\frac{n+1}{2} + \frac{1}{2}}.$$

[28] Also see the sketch of the proof presented in Milne.

According to (5.10) and (6.2), the hypotheses of (3.2) are in fact verified for $(U_0, \mathcal{E}_0/(\mathcal{E}_0 \cap \mathcal{E}_0^{\perp}), \psi)$ for $\beta = n$ and we apply (3.9).

Lemma (6.4). *Let $\mathcal{G}_0$ be a locally constant $\mathbb{Q}_l$-sheaf on U_0 such that its inverse image $\mathcal{G}$ on U is a constant sheaf. Then there exist units α_i in $\bar{\mathbb{Q}}_l$ such that for each $x \in |U_0|$ we have*

$$\det(1 - F_x^* t, \mathcal{G}_0) = \prod_i (1 - \alpha_i^{\deg(x)} t).$$

The lemma expresses the fact that $\mathcal{G}_0$ is the inverse image of a sheaf on $Spec(\mathbb{F}_q)$, namely, its direct image on $Spec(\mathbb{F}_q)$ [29]. The latter identifies with an l-adic representation G_0 of $Gal(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ and we have[30]

$$\det(1 - Ft, G_0) = \prod_i (1 - \alpha_i t).$$

Lemma (6.4) applies to $R^i f_{0*}\mathbb{Q}_l$ $(i \neq n)$, to $R^n f_{0*}\mathbb{Q}_l/\mathcal{E}_0$ and to $\mathcal{E}_0 \cap \mathcal{E}_0^{\perp}$[31] .

For $x \in |U_0|$ the fiber $X_x = f_0^{-1}(x)$ is a variety over the finite field $k(x)$. If $\bar{x}$ is a point of U above x, $X_{\bar{x}}$ is obtained from X_x by extension of scalars of $k(x)$ to the algebraic closure $k(\bar{x}) = \bar{\mathbb{F}}_q$ and $H^i(X_{\bar{x}}, \mathbb{Q}_l)$ is the stalk of $R^i f_* \mathbb{Q}_l$ at $\bar{x}$. The formula (1.5.4) for the variety X_x over $k(x)$ gives

$$Z(X_x, t) = \prod_i \det(1 - F_x^* t, R^i f_{0*}\mathbb{Q}_l)^{(-1)^{i+1}}$$

and $Z(X_x, t)$ is a product of

$$Z^f = \det(1 - F_x^* t, R^n f_{0*}\mathbb{Q}_l/\mathcal{E}_0) \det(1 - F_x^* t, \mathcal{E}_0 \cap \mathcal{E}_0^{\perp}) \prod_{i \neq n} \det(1 - F_x^* t, R^i f_{0*}\mathbb{Q}_l)^{(-1)^{i+1}}$$

and

$$Z^m = \det(1 - F_x^* t, \mathcal{E}_0/(\mathcal{E}_0 \cap \mathcal{E}_0^{\perp}).$$

Put $\mathcal{F}_0 = \mathcal{E}_0/(\mathcal{E}_0 \cap \mathcal{E}_0^{\perp})$, $\mathcal{F} = \mathcal{E}/(\mathcal{E} \cap \mathcal{E}^{\perp})$ and apply (6.4) to the factors of Z^f. We find that there exist l-adic units α_i $(1 \leqslant i \leqslant N)$ and β_j $(1 \leqslant j \leqslant M)$ in $\bar{\mathbb{Q}}_l$ such that for all $x \in |U_0|$

$$Z(X_x, t) = \frac{\prod_i (1 - \alpha_i^{\deg(x)} t)}{\prod_j (1 - \beta_j^{\deg(x)} t)} \det(1 - F_x^* t, \mathcal{F}_0)$$

and in particular the right side is in $\mathbb{Q}(t)$. If some α_i coincides with a β_j, we can simultaneously delete α_i from the family of α and β_j from the family of β. Therefore, we may and do assume that $\alpha_i \neq \beta_j$ for all i and all j.

(6.5) It suffices to prove that polynomials $\prod_i (1 - \alpha_i t)$ and $\prod_j (1 - \beta_j t)$ have rational coefficients, i.e. the family of α_i (resp. the family of β_j) is defined over $\mathbb{Q}$. We will deduce that from the following propositions.

[29] Indeed, if $\varepsilon : U_o \to Spec\mathbb{F}_q$ is the canonical morphism, the assumption implies that $\varepsilon^* \varepsilon_* \mathcal{G}_0 \to \mathcal{G}_0$ becomes an isomorphism once we base change to $\bar{\mathbb{F}}_q$.

[30] To establish the lemma, we also use the relationship between the stalks of the sheaf and those of its inverse/direct image (see Milne).

[31] To see that the inverse images of the latter two are constant, use the equivalence of categories of locally constant sheaves and continuous representations of π_1 and the fact that the monodromy action acts by deforming the cohomology by the vanishing cycles.

Proposition (6.6). *Let* (γ_i) $(1 \leqslant i \leqslant P)$ *and* (δ_j) $(1 \leqslant j \leqslant Q)$ *be two families of* l*-adic units in* $\bar{\mathbb{Q}}_l$*. Assume that* $\gamma_i \neq \delta_j$*. If* K *is a large enough set of integers* $\neq 1$*, and* L *is a large enough nowhere dense subset of* $|U_0|$*, then, if* $x \in |U_0|$ *satisfies* $k \nmid \deg(x)$ *(for all* $k \in K$*) and* $x \notin L$*, the denominator of*

$$\det(1 - F_x^* t, \mathcal{F}_0) \prod_i (1 - \gamma_i^{\deg(x)} t) / \prod_j (1 - \delta_j^{\deg(x)} t) \quad \textbf{\textit{(6.6.1)}}$$

written in irreducible form, is $\prod_j (1 - \delta_j^{\deg(x)} t)$*.*

The proof will be given in (6.10-13). According to (6.7) below, (6.6) provides an intrinsic description of the family of δ_j in terms of the family of rational fractions (6.6.1) for $x \in |U_0|$.

Lemma (6.7). *Let* K *be a finite set of integers* $\neq 1$ *and* (δ_j) $(1 \leqslant j \leqslant Q)$ *and* (ε_j) $(1 \leqslant j \leqslant Q)$ *be two families of elements of a field. If, for all* n *large enough, not divisible by any of the* $k \in K$*, the family of* δ_j^n *coincides with that of* ε_j^n *(up to order), then the family of* δ_j *coincides with that of* ε_j *(up to order).*

We proceed by induction on Q. The set of integers n such that $\delta_Q^n = \varepsilon_j^n$ is an ideal (n_j). Let's prove that there exists a j_0 such that $\delta_Q = \varepsilon_{j_0}$. Otherwise the n_j would be distinct from 1 and there would be arbitrarily large integers n, not divisible by any of the n_j nor by any of the $k \in K$. We would have $\delta_Q^n \neq \varepsilon_j^n$ and this contradicts the hypothesis. So there exists a j_0 such that $\delta_Q = \varepsilon_{j_0}$.We conclude by applying the induction hypothesis to the families (δ_j) $(j \neq Q)$ and (ε_j) $(j \neq j_0)$.

Proposition (6.8). *Let* (γ_i) $(1 \leqslant i \leqslant P)$ *and* (δ_j) $(1 \leqslant j \leqslant Q)$ *be two families of* p*-adic units in* $\bar{\mathbb{Q}}_l$*,* $R(t) = \prod_i (1 - \gamma_i t)$ *and* $S(t) = \prod_j (1 - \delta_j t)$*. Assume that for all* $x \in |U_0|$ $\prod_j (1 - \delta_j^{\deg(x)} t)$ [32] *divides*

$$\prod_i (1 - \gamma_i^{\deg(x)} t) \det(1 - F_x^* t, \mathcal{F}_0).$$

Then $S(t)$ *divides* $R(t)$*.*

Remove from the families (γ_i) and (δ_j) pairs of common elements until they verify the hypothesis of (6.6). Apply (6.6). By hypothesis, the rational fractions (6.6.1) are polynomials. Therefore, no δ survives, which means that[33] $S(t)$ divides $R(t)$.

(6.9) We prove (6.5) and (6.2) (modulo (6.6)). Let's put $(\gamma_i) = (\alpha_i)$ and $(\delta_i) = (\beta_i)$ in (6.6). We get an intrinsic characterization of the family of β_j in terms of the family of rational functions $Z(X_x, t)$ $(x \in |U_0|)$. These being in $\mathbb{Q}(t)$, the family of β_j is defined over $\mathbb{Q}$.

Polynomials[34] $\prod_i (1 - \alpha_j^{\deg(x)} t) \det(1 - F_x^* t, \mathcal{F}_0)$ are therefore in $\mathbb{Q}[t]$. Proposition (6.8) provides an intrinsic description of the family of α_i in terms of this family of polynomials[35]. The family of α_i is thus defined over $\mathbb{Q}$.

(6.10) Let $u \in U$ and $\mathcal{F}_u$ the stalk of $\mathcal{F}$ at u. The arithmetic fundamental group $\pi_1(U_0, u)$, the extension of $\hat{\mathbb{Z}} = Gal(\bar{\mathbb{F}}/\mathbb{F})$ (generator: φ) by the geometric fundamental group[36] $\pi_1(U, u)$, acts on

[32]In the original paper i appears instead of j.
[33]By the remarks after Proposition (6.6) and Lemma (6.7).
[34]for various x
[35]Since a polynomial is uniquely characterized by its quotients.
[36]Both of the aforementioned groups are also called etale fundamental groups (of U_0 and U respectively).

$\mathcal{F}_u$ by symplectic similitudes[37]

$$\rho : \pi_1(U_0, u) \to CSp(\mathcal{F}_u, \psi).$$

We denote by $\mu(g)$ the multiplier of the symplectic similitude g. Let

$$H \subset \hat{\mathbb{Z}} \times CSp(\mathcal{F}_u, \psi)$$

be the subgroup defined by the equation

$$q^{-n} = \mu(g)$$

(q being an l-adic unit, $q^n \in \mathbb{Q}_l^*$ is defined for all $n \in \hat{\mathbb{Z}}$). The fact that ψ has values in $\mathbb{Q}_l(-n)$ implies that the map from π_1 to $\hat{\mathbb{Z}} \times CSp$, the coordinates of the canonical projection to $\hat{\mathbb{Z}}$ and ρ factor through

$$\rho_1 : \pi_1(U_0, u) \to H.$$

Lemma (6.11). *The image H_1 of ρ_1 is open in H.*

Indeed, $\pi_1(U_0, u)$ projects *onto* $\hat{\mathbb{Z}}$ and the image of $\pi_1(U, u) = Ker(\pi_1(U_0, u) \to \hat{\mathbb{Z}})$ in $Sp(\mathcal{F}_u, \psi) = Ker(H \to \hat{\mathbb{Z}})$ is open (5.10).

Lemma (6.12). *For $\delta \in \bar{\mathbb{Q}}_l$ an l-adic unit, the set Z of $(n, g) \in H_1$ such that δ^n is an eigenvalue of g is closed of measure 0* [38].

It is clear that Z is closed. For each $n \in \hat{\mathbb{Z}}$ let CSp_n be the set of $g \in CSp(\mathcal{F}_u, \psi)$ such that $\mu(g) = q^{-n}$ and let Z_n be the set of $g \in CSp_n$ such that δ^n is an eigenvalue of g. Then CSp_n is a homogeneous space for Sp and we check that Z_n is a proper algebraic subspace, thus, of measure 0. According to (6.11), $H^1 \cap (\{n\} \times Z_n)$ is of measure 0 in the inverse image in H_1 of n and we apply Fubini to the projection $H_1 \to \hat{\mathbb{Z}}$.

(6.13) Let us prove (6.6). For each i and j, the set of integers n such that $\gamma_i^n = \delta_j^n$ is the set of multiples of a fixed integer n_{ij} (we do not exclude $n_{ij} = 0$). By hypothesis, $n_{ij} \neq 1$.

According to (6.12) and the Chebotarev's density theorem, the set of $x \in |U_0|$ such that $\beta_j^{\deg(x)}$ is an eigenvalue of F_x^* acting on $\mathcal{F}_0$ is nowhere dense. We take for K the set of n_{ij} and for L the set of x as above.

7 Completion of the proof of (1.7)

Lemma (7.1). *Let X_0 be a nonsingular absolutely irreducible*[39] *projective variety of even dimension over $\mathbb{F}_q$. Let X over $\bar{\mathbb{F}}_q$ be obtained from X_0 by extension of scalars and α an eigenvalue of F^* acting on $H^d(X, \mathbb{Q}_l)$. Then α is an algebraic number all of which complex conjugates, still denoted α, satisfy*

$$q^{\frac{d}{2}-\frac{1}{2}} \leqslant |\alpha| \leqslant q^{\frac{d}{2}+\frac{1}{2}} \quad \textbf{\textit{(7.1.1)}}$$

[37] i.e. a group $A(V)$ of linear transformations g of a vector space V equipped with a nondegenerate bilinear form $<,>: V \times V \to k$ such that $< gv, gw >= \mu(g) < v, w >$ for a multiplicative character $\mu : A(V) \to k^\times$ called the similitude multiplier

[38] With respect to the Haar measure on $\hat{\mathbb{Z}} \times CSp$.

[39] or geometrically irreducible

We proceed by induction on d (always assumed even). The case $d = 0$ is trivial even without assuming that X_0 is absolutely irreducible; we assume from now on $d \geqslant 2$. We put $d = n + 1 = 2m + 2$.

If $\mathbb{F}_{q^r}$ is an extension of degree r of $\mathbb{F}_q$ and $X'_0/\mathbb{F}_{q^r}$ is obtained from $X_0/\mathbb{F}_q$ by extension of scalars, the statement (7.1) for $X_0/\mathbb{F}_q$ is equivalent to (7.1) for $X'_0/\mathbb{F}_{q^r}$; in the same way as q is replaced by q^r, the eigenvalues of F^* are replaced by their r-th powers.

According to (5.7), in a suitable projective embedding $i : X \to \mathbb{P}$, X admits a Lefschetz pencil of hyperplane sections. The previous remark allows us to assume that the pencil is defined over $\mathbb{F}_q$ (once we replace $\mathbb{F}_q$ by a finite extension).

Therefore, assume that there exists a projective embedding $X_0 \to \mathbb{P}_0$ and a subspace $A_0 \subset \mathbb{P}_0$ of codimension two that defines the Lefschetz pencil. We recall the notations of (6.1) and (6.3). A new extension of scalars allows us to assume that:

a) The points of S are defined over $\mathbb{F}_q$.

b) The vanishing cycles for x_s ($s \in S$) are defined over $\mathbb{F}_q$ (since only $\pm\delta$ is intrinsic, they can only be defined over quadratic extensions).

c) There exists a rational point $u_0 \in U_0$. We take the corresponding point u of U as the base point.

d) $X_{u_0} = f_0^{-1}(u_0)$ admits a smooth hyperplane section Y_0 defined over $\mathbb{F}_q$. We let $Y = Y_0 \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$.

Since $\tilde{X}$ is obtained from X by blowing up along a smooth subvariety $A \cap X$ of dimension two, we have

$$H^i(X, \mathbb{Q}_l) \hookrightarrow H^i(\tilde{X}, \mathbb{Q}_l)$$

(in fact, $H^i(\tilde{X}, \mathbb{Q}_l) = H^i(X, \mathbb{Q}_l) \oplus H^{i-2}(A \cap X, \mathbb{Q}_l)(-1)$)[40]. It suffices to prove (7.1.1) for the eigenvalues α of F^* acting on $H^d(\tilde{X}, \mathbb{Q}_l)$.

The Leray spectral sequence for f is

$$E_2^{pq} = H^p(D, R^q f_* \mathbb{Q}_l) \Rightarrow H^{p+q}(\tilde{X}, \mathbb{Q}_l).$$

It suffices to prove (7.1.1) for the eigenvalues of F^* acting on E_2^{pq} for $p + q = d = n + 1$. Those are[41]:

A) $E_2^{2,n-1}$. According to (5.8), $R^{n-1} f_* \mathbb{Q}_l$ is constant. From (2.10) we have

$$E_2^{2,n-1} = H^{n-1}(X_u, \mathbb{Q}_l)(-1).$$

Applying the weak Lefschetz theorem (corollary of SGA 4, XIV (3.2)) and the Poincare duality (SGA 4, XVIII), we have

$$H^{n-1}(X_u, \mathbb{Q}_l)(-1) \hookrightarrow H^{n-1}(Y, \mathbb{Q}_l)(-1)$$

and we apply the induction hypothesis to Y_0.

B) $E_2^{0,n+1}$. If the vanishing cycles are nonzero, $R^{n+1} f_* \mathbb{Q}_l$ is constant and

$$E_2^{0,n+1} = H^{n+1}(X_u, \mathbb{Q}_l).$$

The Gysin map

$$H^{n-1}(Y, \mathbb{Q}_l)(-1) \to H^{n+1}(X_u, \mathbb{Q}_l)$$

is surjective (by an argument dual to that of A)) and we apply the induction hypothesis to Y_0.

[40] This is by the Thom isomorphism theorem. See a (rather technical) proof in Milne.

[41] I highly recommend consulting Milne's book for the explanations of the steps in A), B), C).

If the vanishing cycles are zero, the exact sequence of (5.8) b) gives the following exact sequence

$$\bigoplus_{s\in S}\mathbb{Q}_l(m-n) \to E_2^{0,n+1} \to H^{n+1}(X_u,\mathbb{Q}_l).$$

The eigenvalues of F acting on $\mathbb{Q}_l(m-n)$ are $q^{d/2}$ and for H^{n+1} everything is as above.

C) $E_2^{1,n}$. If we had the hard Lefschietz theorem, we would know that $\mathcal{E}\cap\mathcal{E}^\perp$ is zero and that $R^nf_*\mathbb{Q}_l$ is the direct sum of $j_*\mathcal{E}$ and a constant sheaf. The H^1 of a constant sheaf on $\mathbb{P}^1$ is zero and it would suffice to apply (6.3).

Since we have not proved the hard Lefshetz theorem yet, we will have to figure a way out. If the vanishing cycles are zero, $R^nf_*\mathbb{Q}_l$ is constant ((5.8) b)) and $E_2^{1,n}=0$. Therefore we may and do assume that the vanishing cycles are nonzero. Filter $R^nf_*\mathbb{Q}_l = j_*j^*R^nf_*\mathbb{Q}_l$ (5.8) by the subsheafs $j_*\mathcal{E}$ and $j_*(\mathcal{E}\cap\mathcal{E}^\perp)$. If the vanishing cycles δ are not in $\mathcal{E}\cap\mathcal{E}^\perp$ [42] we have exact sequences[43]:

$$0\to j_*\mathcal{E}\to R^nf_*\mathbb{Q}_l \to \textit{constant sheaf} \to 0 \quad \textbf{(7.1.2)}$$

$$0\to \textit{ constant sheaf } j_*(\mathcal{E}\cap\mathcal{E}^\perp))\to j_*\mathcal{E}\to j_*(\mathcal{E}/(\mathcal{E}\cap\mathcal{E}^\perp))\to 0 \quad \textbf{(7.1.3)}$$

If, God forbid, the δ are in $\mathcal{E}\cap\mathcal{E}^\perp$, we have $\mathcal{E}\subset\mathcal{E}^\perp$ and exact sequences[44]:

$$0\to \textit{ the constant sheaf } j_*\mathcal{E}^\perp \to R^nf_*\mathbb{Q}_l \to \textit{ a sheaf } \mathcal{F}\to 0 \quad \textbf{(7.1.4)}$$

$$0\to\mathcal{F}\to \textit{ the constant sheaf } j_*j^*\mathcal{F}\to \bigoplus_{s\in S}\mathbb{Q}_l(n-m)_s\to 0 \quad \textbf{(7.1.5)}$$

In the first case the long exact sequences in cohomology give

$$H^1(D,j_*\mathcal{E})\to H^1(D,R^nf_*\mathbb{Q}_l)\to 0 \quad \textbf{(7.1.2')}$$

$$0\to H^1(D,j_*\mathcal{E})\to H^1(D,j_*(\mathcal{E}/(\mathcal{E}\cap\mathcal{E}^\perp))) \quad \textbf{(7.1.3')}$$

and we apply (6.3).

In the second case, they give

$$0\to H^1(D,R^nf_*\mathbb{Q}_l)\to H^1(D,\mathcal{F}) \quad \textbf{(7.1.4')}$$

$$\bigoplus_{s\in S}\mathbb{Q}_l(n-m)\to H^1(D,\mathcal{F})\to 0 \quad \textbf{(7.1.5')}$$

and we remark that F acts on $\mathbb{Q}_l(n-m)$ by multiplication by $q^{d/2}$.

Lemma (7.2). *Let X_0 be a nonsingular projective absolutely irreducible variety of dimension d over $\mathbb{F}_q$. Let X over $\bar{\mathbb{F}}_q$ be obtained from X_0 by extension of scalars and α be an eigenvalue of F^* acting on $H^d(X,\mathbb{Q}_l)$. Then α is an algebraic number all of which complex conjugates, still denoted α, satisfy*

$$|\alpha| = q^{\frac{d}{2}}.$$

[42] More precisely, not in $E\cap E^\perp$.

[43] The constant sheaf in (7.1.2) is $j_*(R^nf_*\mathbb{Q}_l/\mathcal{E})$. To see that this (and the next few) sheaves are constant, reason as in footnote 31 and use that the inverse/direct images of constant sheaves via j are constant (the second claim is false in general).

[44] For $\mathcal{F}=R^nf_*\mathbb{Q}_l/j_*\mathcal{E}^\perp$.

We first prove that (7.2) ⇒ (1.7). For X_0 projective nonsingular over $\bar{\mathbb{F}}_q$ we have to prove the following statements:

$W(X_0, i)$. *Let X be obtained from X_0 by extension of scalars of $\mathbb{F}_q$ to $\bar{\mathbb{F}}_q$. If α is an eigenvalue of F^* acting on $H^i(X, \mathbb{Q}_l)$, then α is an algebraic number all of which complex conjugates, still denoted α, satisfy $|\alpha| = q^{i/2}$.*

a) If $\mathbb{F}_{q^n}$ is an an extension of degree n of $\mathbb{F}_q$ and $X_0'/\mathbb{F}_{q^n}$ is obtained from $X_0/\mathbb{F}_q$ by extension of scalars, then $W(X_0, i)$ is equivalent to $W(X_0', i)$: the extension of scalars replaces α by α^n and q by q^n.

b) If X_0 is purely of dimension n, $W(X_0, i)$ is equivalent to $W(X_0, 2n - i)$; this follows from Poincare duality[45].

c) If X_0 is a an union of irreducible X_0^α, $W(X_0, i)$ is equivalent to the collection of $W(X_0^\alpha, i)$.

d) If X_0 is purely of dimension n, Y_0 is a smooth hyperplane section of X_0 and $i < n$, then $W(Y_0, i) \Rightarrow W(X_0, i)$: this follows from the weak Lefschetz theorem[46].

To prove the statements $W(X_0, i)$ we move in succession:

-by c) we assume that X_0 is purely of dimension n;

-by b) we also assume that $0 \leqslant i \leqslant n$;

-by a) and d) we also assume $i = n$;

-by a) and c) we also assume that X_0 is absolutely irreducible.

Now the case satisfies the conditions of (7.2).

(7.3) We prove (7.2). For every integer k, α^k is an eigenvalue of F^* acting on $H^{kd}(X^k, \mathbb{Q}_l)$ (Kunneth's formula). For k even, X^k satisfies the conditions of (7.1), so we have

$$q^{\frac{kd}{2} - \frac{1}{2}} \leqslant |\alpha^k| \leqslant q^{\frac{kd}{2} + \frac{1}{2}}$$

and

$$q^{\frac{d}{2} - \frac{1}{2k}} \leqslant |\alpha| \leqslant q^{\frac{d}{2} + \frac{1}{2k}}.$$

Letting k go to infinity, we establish (7.2).

8 First applications

Theorem (8.1). *Let $X_0 \subset \mathbb{P}_0^{n+r}$ be a nonsingular complete intersection over $\mathbb{F}_q$ of dimension n and of multidegree $(d_1, \cdots d_r)$. Let b' be the n-th Betti number of the complex nonsingular complete intersection with the same dimension and multidegree. Put $b = b'$ for n odd and $b = b' - 1$ for n even. Then*

$$|\#X_0(\mathbb{F}_q) - \#\mathbb{P}^n(\mathbb{F}_q)| \leqslant bq^{n/2}.$$

Let $X/\bar{\mathbb{F}}_q$ be obtained from X_0 and $\mathbb{Q}_l\eta^i$ be the line in $H^{2n}(X, \mathbb{Q}_l)$ generated by the i-th cup power of the cohomology class of the hyperplane section. On this line F^* acts by multiplication by q^i. The cohomology of X is the direct sum of the $\mathbb{Q}_l\eta^i$ $(0 \leqslant i \leqslant n)$ and the primitive part of

[45] If α is an eigenvalue of F^* acting on $H^i(X, \mathbb{Q}_l)$, then q^n/α is an eigenvalue of F^* acting on $H^{2n-i}(X, \mathbb{Q}_l)$. See Milne.

[46] Depending on what one takes for the weak Lefschetz theorem, this follows either directly from it and the properties of Frobenius (if we assume that the weak Lefschetz theorem provides us with surjectivity of the Gysin map in cohomology) or one has to also invoke the Gysin sequence to establish surjectivity (if we only assume $H^i(X, \mathbb{Q}_l) = 0$ for $i > d$ and X affine). One can also avoid using the weak Lefschetz theorem and apply a Kunneth formula argument instead (due to A. Mellit). See Milne for clarifications.

$H^n(X, \mathbb{Q}_l)$ of dimension b. According to (1.5), therefore, there exist b algebraic numbers α_j, the eigenvalues of F^* acting on this primitive cohomology, such that

$$\#X_0(\mathbb{F}_q) = \sum_{i=0}^{n} q^i + (-1)^n \sum_j \alpha_j.$$

According to (1.7), $|\alpha_j| = q^{n/2}$ and

$$|\#X_0(\mathbb{F}_q) - \#\mathbb{P}^n(\mathbb{F}_q)| = |\#X_0(\mathbb{F}_q) - \sum_{i=0}^{n} q^i| = |\sum_j \alpha_j| \leqslant \sum_j |\alpha_j| = bq^{n/2}.$$

Theorem (8.2). *Let N be an integer $\geqslant 1$, $\varepsilon : (\mathbb{Z}/N)^* \to \mathbb{C}^*$ a character, k an integer $\geqslant 2$ and f a holomorphic modular form on $\Gamma_0(N)$ of weight k and with character ε : f is a holomorphic function on the the Poincare half-plane X such that for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, with $c \equiv 0$ (N) we have*

$$f\left(\frac{az+b}{cz+d}\right) = \varepsilon(a)^{-1}(cz+d)^k f(z).$$

We assume that f is cuspidal and primitive ("new" in the sence of Arkin-Lehner and Miyake), in particular an eigenvector of the Hecke operators T_p $(p \nmid N)$. Let $f = \sum_{n=1}^{\infty} a_n q^n$ with $q = e^{2\pi iz}$ (and $a_1 = 1$). Then for p prime not dividing N

$$|a_p| \leqslant 2p^{\frac{k-1}{2}}.$$

In other words, the roots of the equation

$$T^2 - a_p T + \varepsilon(p)p^{k-1}$$

are of absolute value $p^{\frac{k-1}{2}}$.

These roots are indeed the eigenvalues of the Frobenius acting on H^{k-1} of a nonsingular projective variety of dimension $k-1$ defined over $\mathbb{F}_p$.

Under restrictive assumptions, this fact is proved in my Bourbaki expose (Formes modulaires et representations l-adiques, expose 355, February 1969, in: *Lecture Notes in Mathematics, 179).* The general case is not much more difficult.

Remark **(8.3)** J.P.Serre and myself have recently proved that (8.2) remains true for $k = 1$. The proof is quite different.

The following application was suggested to me by E.Bombiery.

Theorem (8.4). *Let Q be a polynomial in n variables and of degree d over $\mathbb{F}_q$, Q_d a homogeneous part of degree d of Q and $\psi : \mathbb{F}_q \to \mathbb{C}^*$ an additive nontrivial character on $\mathbb{F}_q$. We assume that:*

(i) d is coprime to p [47]

(ii) The hypersurface H_0 in $\mathbb{P}^{n-1}_{\mathbb{F}_q}$ defined by Q_d is smooth.

Then

$$|\sum_{x_1,\cdots,x_n \in \mathbb{F}_q} \psi(Q(x_1,\cdots,x_n))| \leqslant (d-1)^n q^{n/2}.$$

[47](the characteristic of $\mathbb{F}_q$)

After replacing Q by a scalar multiple, we may (and do) assume that[48]

$$\psi(x) = \exp(2\pi i Tr_{\mathbb{F}_q/\mathbb{F}_p}(x)/p) \quad \textbf{(8.4.1)}$$

Let X_0 be an etale covering of the affine space $\mathbb{A}_0$ of dimension n over $\mathbb{F}_q$ with equation $T^p - T = Q$ and let σ be the projection of X_0 to $\mathbb{A}_0$:

$$\sigma : X_0 \to \mathbb{A}_0$$

$$X_0 = Spec(\mathbb{F}_q[x_1, \cdots, x_n, T]/(T^p - T - Q)).$$

The covering X_0 is Galois with Galois group $\mathbb{Z}/p$; $i \in \mathbb{Z}/p = \mathbb{F}_p$ acts by $T \to T + i$.

We let $x \in \mathbb{A}_0(\mathbb{F}_q)$ and compute the Frobenius endomorphism on the fiber of $X_0/\mathbb{A}_0$ at x. Let $q = p^f$ and let $\bar{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$. For $(x, T) \in X_0(\bar{\mathbb{F}})$ above x we have $F((x, T)) = (x, T^q)$ and

$$T^q = T + \sum_{i=1}^{f}(T^{p^i} - T^{p^{i-1}}) = T + \sum Q(x)^{p^{i-1}} = T + Tr_{\mathbb{F}_q/\mathbb{F}_p}(Q(x)).$$

This is the action of the element $Tr_{\mathbb{F}_q/\mathbb{F}_p}(Q(x))$ of the Galois group.

Let E be the field of the p-th roots of unity and λ a finite place of E coprime to p. We will work in λ-adic cohomology. For $j \in \mathbb{Z}/p$, let $\mathcal{F}_{j,0}$ be a E_λ local system of rank one on $\mathbb{A}_0$ defined by X_0 and $\psi(-jx) : \mathbb{Z}/p \to E^* \to E_\lambda^*$: we have $\iota : X_0 \to \mathcal{F}_{j,0}$ and $\iota(i \star x) = \psi(-ij)\iota(x)$. Denote without $_0$ objects obtained from $\mathbb{A}_0, X_0, \mathcal{F}_{j,0}$ by extension of scalars to $\bar{\mathbb{F}}_q$. The trace formula (1.12.1) for $\mathcal{F}_{j,0}$ gives:

$$\sum_{x_1, \cdots, x_n \in \mathbb{F}_q} \psi(Q(x_1, \cdots, x_n)) = \sum_i Tr(F^*, H^i_c(\mathbb{A}, \mathcal{F}_1)) \quad \textbf{(8.4.2)}$$

We have $\sigma_* E_\lambda = \bigoplus_j \mathcal{F}_j$ and so

$$H^*_c(X, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} E_\lambda = \bigoplus_j H^*_c(\mathbb{A}, \mathcal{F}_j) \quad \textbf{(8.4.3)}$$

For $j = 0$, $\mathcal{F}_j$ is the constant sheaf E_λ; this factor corresponds to inclusion, by taking the inverse image, of the cohomology of $\mathbb{A}$ in that of X.

Lemma (8.5). *(i) For $j \neq 0$, $H^i_c(\mathbb{A}, \mathcal{F}_j)$ is zero for $i \neq n$, for $i = n$, the cohomology space has dimension $(d-1)^n$.*

(ii) For $j \neq 0$, the cup product

$$H^n_c(\mathbb{A}, \mathcal{F}_j) \otimes H^n_c(\mathbb{A}, \mathcal{F}_{-j}) \to H^{2n}(\mathbb{A}, E_\lambda) \xrightarrow{Tr} E_\lambda(-n)$$

is a perfect pairing.

(iii) X_0 is open in a nonsinqular projective variety Z_0.

Let's deduce (8.4) from (8.5). Let $j_0 : X_0 \hookrightarrow Z_0$ and $j : X \hookrightarrow Z$ be obtained by extension of scalars of $\bar{\mathbb{F}}_q$. According to (8.4.2), (i) and (1.7) for Z_0, it suffices to prove the injectivity of

$$H^n_c(\mathbb{A}, \mathcal{F}_1) \xrightarrow{\sigma^*} H^n_c(X, \mathcal{F}_1) = H^n_c(X, E_\lambda) \xrightarrow{j_!} H^n(Z, E_\lambda).$$

[48]For $q = p^f$ we have $Tr_{\mathbb{F}_q/\mathbb{F}_p}(x) = x + x^p + \cdots + x^{p^{f-1}}$.

We have $Tr(a \cup b) = \frac{1}{p} Tr(j_! \sigma^* a \cap j_! \sigma^* b)$, so injectivity follows from (ii)[49].

(8.6) We prove (8.5) (iii). Let $\mathbb{P}_0$ be the projective space over $\mathbb{F}_q$, obtained from $\mathbb{A}_0$ by adding a hyperplane at infinity $\mathbb{P}_0^\infty$, $H_0 \subset \mathbb{P}_0^\infty$ with equation $Q_d = 0$ and Y_0 the covering of $\mathbb{P}_0$ normalizing $\mathbb{P}_0$ along X_0.

$$\begin{array}{ccccccc} X_0 & \hookrightarrow & Y_0 & & & & \\ \downarrow{\scriptstyle\sigma} & & \downarrow & & & & \\ \mathbb{A}_0 & \hookrightarrow & \mathbb{P}_0 & \hookleftarrow & \mathbb{P}_0^\infty & \hookleftarrow & H_0 \end{array} \qquad \textbf{(8.7.1)}$$

Let's study $Y_0/\mathbb{P}_0$ near the infinity, locally for the etale topology.

Lemma (8.7). *Y_0 is smooth outside the inverse image of H_0.*

The divisor of a rational function Q on $\mathbb{P}_0$ is the sum of the finite part $div(Q)_f$ and $(-d)$ times the hyperplane at infinity. We have:

$$div(Q) = div(Q)_f - d\mathbb{P}_0^\infty \quad \textbf{(8.7.1)}$$

$$div(Q)_f \cap \mathbb{P}_0^\infty = H_0$$

At a finite distance, $Y_0 = X_0$ is etale over $\mathbb{A}_0$, so smooth. At the infinity but outside the inverse image of H_0 there exist local coordinates $(z_1, \cdots, z_n)$ such that $Q = z_1^{-d}$ (here we use $(d,p) = 1$). In these coordinates, Y_0 appears as a product of a curve and a smooth space (corresponding to coordinates $z_2, \cdots, z_n$). By normality it is smooth.

Lemma (8.8). *In the etale neighborhood of a point above H_0, Y_0 is smooth on a normal singular surface, always the same.*

This time we can find local coordinates such that $Q = z_1^{-d} z_2$. Indeed, since H_0 is smooth, $div(Q)_f$ is smooth in the neighborhood of infinity and crosses $\mathbb{P}_0^\infty$ transversely. This form is independent of the chosen point and uses only two coordinates, hence the assertion.

(8.9) The following method (due to Zariski) allows one to resolve singularities on surfaces: alternately, we normalize and we blow up the (reduced) singular locus. Operators in play commute with etale localization and taking products by a smooth space. The method of Zariski, therefore, allows one to resolve singularities on a space that (like Y_0) is, locally for the etale topology, smooth on a surface. The resolution obtained from Y_0 is the Z_0 we seek.

If T is a curve on a surface S containing the singular locus and T' is the inverse image of T in the Zariski resolution S' of S, we know that if we repeatedly blow up the (reduced) singular locus of $(T')_{red}$ in S', we obtain a surface S'' such that the inverse reduced image $(T'')_{red}$ of T in S'' is a divisor with normal crossings. Again, operations in play commute with etale localization and taking products by a smooth space. Reasoning as above and observing that $(Y_0, infinity)$ is locally smooth in (S, T), we can find Z_0 such that $Z_0 - X_0$ is a divisor with normal crossings.

(8.10) We prove (8.5) (i), (ii). These assertions are geometric; this allows us to work from now on in $\bar{\mathbb{F}}_q$. Let S' be the affine space over $\bar{\mathbb{F}}_q$ that parametrizes polynomials in n variables of degree $\leqslant d$ and let S be an open in S' corresponding to the polynomials, which homogeneous part of degree d has nonzero discriminant. We denote by $Q_S \in H^0(S, \mathcal{O}[x_1, \cdots, x_n])$ the universal polynomial[50] of S and by X_S the Galois etale covering of $\mathbb{A}_S = \mathbb{A}^n \times S$ with equation $T^p - T = Q_S$ and Galois

[49] By the same reasoning as in the proof of (7.1) A) (for which I referred to Milne).

[50] $\sum_{i_1 + \cdots + i_n = d} a_{i_1, \cdots, i_n} x_1^{i_1} \cdots x_n^{i_n}$

group $\mathbb{Z}/p$. Let $\mathbb{P}_S = \mathbb{P}^n \times S$ be the projective completion of $\mathbb{A}_S$ and Y_S the normalization of $\mathbb{P}_S$ along X_s. We have, for S, a diagram similar to (8.6.1).

Expressions of Q in local coordinates (8.7) and (8.8) remain valid in this situation with parameters such that locally for the etale topology on Y_S, Y_S/S is isomorphic to the product of S (that is smooth) with one fiber. The method of canonical resolution used in (8.9) gives us a relative compactification Z_S/S of X_S/S with $Z_S - X_S$ a divisor with normal crossings relative to S

$$\begin{array}{ccc} X_S & \xhookrightarrow{u} & Z_S \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle f} \\ \mathbb{A}_S & \xhookrightarrow{a} & S \end{array}$$

(f is proper and smooth, u is an open immersion, $Z_s - X_s$ a divisor with relative normal crossings).

Let $\mathcal{F}_{j,S}$ be a E_λ-sheaf on $\mathbb{A}_S$ obtained as in (8.4) from $X_S/\mathbb{A}_S$. We have $\sigma^* E_\lambda = \oplus \mathcal{F}_{j,S}$, so

$$R^*(fu)_!(E_\lambda) = \bigoplus_j R^* a_! \mathcal{F}_{j,S}.$$

The properties of Z_S ensure that $R^i(fu)_! E_\lambda = R^i f_*(u_! E_\lambda)$ is a locally constant sheaf on S. Therefore, $R^i a_! \mathcal{F}_{j,S}$ is also locally constant. Since S is connected, it suffices to prove (8.5) (i), (ii) for a particular polynomial Q. We will take $Q = \sum_i x_i^d$. This polynomial satisfies the nonsingularity condition because $(d,p) = 1$. For this polynomial variables in the exponential sum (8.4) are separated. This corresponds to the fact that $\mathcal{F}_j$ is the tensor product of the inverse images of similar sheaves $\mathcal{F}_j^1$ on the factors of dimension one $\mathbb{A}^1$ of $\mathbb{A} = \mathbb{A}^n$. By Kunneth's formula

$$H^*(\mathbb{A}, \mathcal{F}_j) = \oplus H^*(\mathbb{A}^1, \mathcal{F}_j^1).$$

This reduces the proof of (8.5) (i), (ii) to the case when $n = 1$ and Q is x^d.

(8.11) Let's deal with this particular case. The covering X of $\mathbb{A}$ is irreducible, so for $i = 0, 2$

$$H_c^i(\mathbb{A}, E_\lambda) \xrightarrow{\sim} H_c^i(X, E_\lambda).$$

So for $i \neq 1$ and $j \neq 0$ we have[51]

$$H_c^i(X, \mathcal{F}_j) = 0.$$

Assertion (ii) follows from (2.8) or (2.12) and the fact that $u_! \mathcal{F}_j = u_* \mathcal{F}_j$. To prove (i) it remains to verify that

$$\chi_c(\mathbb{A}, \mathcal{F}_j) = 1 - d.$$

According to the Euler-Poincare formula (see expose Bourbaki 286 of February 1965, by M.Raynaud), it is equivalent to the following lemma.

Lemma (8.12). *Swan's conductor of $\mathcal{F}_j$ at infinity equals d.*

This statement is equivalent to the following.

Lemma (8.13). *Let k be a finite field of characteristic p, $y \in k[[x]]$ an element of valuation d coprime to p, L the extension of $K = k((x))$ generated by the roots of $T^p - T = y^{-1}$ and χ the following character on $Gal(L/K)$ with values in $\mathbb{Z}/p$:*

$$\chi(\sigma) = \sigma T - T.$$

Then χ has conductor $d + 1$.

By extension of the residue field we may assume that k is algebraically closed rather than finite and apply: J.P.Serre, Sur les corps locaux a corps residuel algebriquement clos, *Bull. Soc. Math. France*, 89 (1961), p. 105-154, n^o 4.4.

[51] In the original paper $\mathbb{A}$ appears instead of X.

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