Picard-Vessiot groups of Lauricella's hypergeometric systems E_C and Calabi-Yau varieties arising integral representations

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Abstract

We study the Zariski closure of the monodromy group **Mon** of Lauricella's hypergeometric function F_C. If the identity component $\mathbf{Mon}^0$ acts irreducibly, then $\overline{\mathbf{Mon}} \cap \mathbf{SL}_{2^n}(\mathbb{C})$ must be one of classical groups $\mathbf{SL}_{2^n}(\mathbb{C}), \mathbf{SO}_{2^n}(\mathbb{C})$ and $\mathbf{Sp}_{2^n}(\mathbb{C})$. We also study Calabi-Yau varieties arising from integral representations of F_C.

1 Introduction

In [3], Beukers and Heckman studied the monodromy of the generalized hypergeometric function ${}_{n+1}F_n$ from a viewpoint of differential Galois theory. They determined the differential Galois group called the Picard-Vessiot group (for Fuchsian equations, which is given by the Zariski closure of the monodromy group), and parameters for which the monodromy group is finite. In this paper, applying their method for results in [10] and [13], we study the Zariski closure of the monodromy group of Lauricella's hypergeometric function

$$F_C(a,b,c;x) = \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b)_{m_1+\cdots+m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n} m_1!\cdots m_n!} x_1^{m_1}\cdots x_n^{m_n},$$

and we also study Calabi-Yau varieties arising from integral representations of F_C.

Lauricella's hypergeometric function F_C, together with F_A, F_B and F_D, was introduced by Appell and Lauricella in the 19th century as generalizations of the Gauss hypergeometric function ${}_2F_1$. In the case of two variables, Lauricella's F_A, F_B, F_C and F_D are called Appell's F_2, F_3, F_4 and F_1 respectively. The monodromy of these functions have been studied by many authors. In [23], Sasaki showed that Picard-Vessiot groups for F_i $(i = 2, 3, 4)$ and F_D are general linear groups for general parameters. Deligne and Mostow gave a list of parameters of Lauricella's F_D that produce complex ball uniformizations, and concrete examples of non-arithmetic subgroup of unitary groups in [6]. Recently the structure of monodromy group for F_C was studied in [10] and [13]. According to them, the monodromy group **Mon** for F_C is generated by $M_0, M_1, \dots, M_n$ where M_0 is a reflection in the sense of [3] and $M_1, \dots, M_n$ form an Abelian subgroup. Applying results in [3], we can classify the Zariski closure $\overline{\mathbf{Mon}}$ as in the case of the generalized hypergeometric function ${}_{n+1}F_n$. If the identity component $\mathbf{Mon}^0$ acts irreducibly, then $\overline{\mathbf{Mon}} \cap \mathbf{SL}_{2^n}(\mathbb{C})$ must be one of classical groups $\mathbf{SL}_{2^n}(\mathbb{C}), \mathbf{SO}_{2^n}(\mathbb{C})$ and $\mathbf{Sp}_{2^n}(\mathbb{C})$ (Theorem 2.1). To study irreducibility conditions of the identity component, we introduce the reflection subgroup $\mathbf{Ref} \subset \mathbf{Mon}$, generated by gM_0g^{-1} $(g \in \mathbf{Mon})$. In Theorem 2.2, we give the necessary and sufficient condition for the irreducibility of **Ref** in terms of parameters. It is simply that at most one of $\gamma_1, \dots, \gamma_n, \alpha\beta^{-1}$ is -1 in addition to irreducibility conditions for **Mon** in Proposition 2.6. The proof is based on ideas in a work of Kato for Appell's F_4 ([17]). Moreover we prove that if the action of the identity component $\mathbf{Ref}^0$ is reducible and **Ref** is irreducible, then **Ref** and **Mon** is finite (Theorem 2.3).

In the last section, we study double coverings $V(x)$ of projective spaces associated to integral representations of $F_C(a,b,c;x)$ with $a = b = 1/2, c_k = 1$. It is known that the monodromy group for hyperelliptic curves is arithmetic, that is, finite index in $\mathbf{Sp}_{2g}(\mathbb{Z})$. Since a period integral for a hyperelliptic curve is given by Lauricella's function F_D, our varieties are regarded as the counterpart of hyperelliptic curves. By the results of the former part, we see that the Zariski closure of the monodromy group for $a = b = 1/2, c_k = 1$ is the symplectic or orthogonal group. It is interesting to study arithmeticy of these group. In the case of $n = 2$, it is well known that Appell's hypergeometric function $F_4(1/2, 1/2, 1, 1; x_1, x_2)$ is a products of Gauss's hypergeometric functions. We show that $V(x)$ is in fact a product Kummer surface, and the monodromy group contains $\Gamma(2) \times \Gamma(2)$ as a subgroup of index 2. In the case of $n = 3$, we have double octic Calabi-Yau varieties $\widetilde{V}$ of Euler number 128 by resolving singularities. For computation of Euler and Hodge numbers, we use methods in [5] and [4]. For $n \geq 4$, we do not know if there are crepant resolutions of $V(x)$.

2 Monodromy of the system E_C

2.1 Lauricella's hypergeometric function F_C

Lauricella's hypergeometric function F_C of n variables $x_1, ..., x_n$ is

$$F_C(a, b, c; x) = \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b)_{m_1+\cdots+m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n} m_1!\cdots m_n!} x_1^{m_1}\cdots x_n^{m_n},$$

where $x = (x_1, \dots, x_n)$, $c = (c_1, \dots, c_n)$, $c_1, \dots, c_n \notin \{0, -1, -2, \dots\}$, and $(c_1)_{m_1} = \Gamma(c_1 + m_1)/\Gamma(c_1)$. This series converges in the domain $\left\{(x_1, \dots, x_n) \in \mathbb{C}^n \;\middle|\; \sum_{k=1}^n \sqrt{|x_k|} < 1\right\}$. In the case of $n = 2$, the series $F_C(a, b, c; x)$ is called Appell's hypergeometric series $F_4(a, b, c_1, c_2; x_1, x_2)$. Let ∂_k $(k = 1, \dots, n)$ be the partial differential operator with respect to x_k. We set $\theta_k = x_k\partial_k$, $\theta = \sum_{k=1}^n \theta_k$. Lauricella's $F_C(a, b, c; x)$ satisfies differential equations

$$[\theta_k(\theta_k + c_k - 1) - x_k(\theta + a)(\theta + b)]\, f(x) = 0, \quad k = 1, \dots, n.$$

The system generated by them is called Lauricella's hypergeometric system $E_C(a, b, c)$ of differential equations.

Proposition 2.1 (([15])). *The system $E_C(a, b, c)$ is a holonomic system of rank 2^n with the singular locus*

$$S := \left(\prod_{k=1}^n x_k \cdot R(x) = 0\right) \subset \mathbb{C}^n, \quad R(x_1, \dots, x_n) := \prod_{\epsilon_1,\dots,\epsilon_n=\pm 1} \left(1 + \sum_{k=1}^n \epsilon_k \sqrt{x_k}\right).$$

In [1] and [18], an integral representation of $F_C(a, b, c; x)$ with generic parameters is given in terms of the twisted cycles.

Proposition 2.2 (([1], [18])). *For sufficiently small positive real numbers $x_1, \dots, x_n$, if $c_1, \dots, c_n, a - \sum c_k \notin \mathbb{Z}$, then $F_C(a, b, c; x)$ admits the following integral representation:*

$$\begin{aligned} F_C(a, b, c; x) =& \frac{\Gamma(1-a)}{\prod_k \Gamma(1-c_k) \cdot \Gamma(\sum_k c_k - a - n + 1)} \\ & \cdot \int_\Delta \prod_k t_k^{-c_k} \cdot (1 - \sum_k t_k)^{\sum_k c_k - a - n} \cdot \left(1 - \sum_k \frac{x_k}{t_k}\right)^{-b} dt_1 \wedge \cdots \wedge dt_n, \end{aligned}$$

where Δ is the twisted cycle made by an n-simplex, in Sections 3.2 and 3.3 of [1].

For our applications, we show that F_C has an Euler-type integral representation even if c_i's are positive integers.

Proposition 2.3. *We assume $c_1, \dots, c_n \in \mathbb{Z}_{\geq 1}$. Let ϵ and x_k be small positive real numbers such that*

$$0 < \epsilon < \frac{1}{n+1}, \quad 0 < x_k < \frac{\epsilon^2}{n}.$$

Then the integration on the direct product

$$C_\epsilon^n : |t_1| = |t_2| = \cdots = |t_n| = \epsilon$$

of circles expresses F_C*:*

$$F_C(a,b,c;x) = \frac{(-1)^{n+\sum_k c_k}}{(2\pi\sqrt{-1})^n}\frac{\Gamma(1-a)\prod_k \Gamma(c_k)}{\Gamma(1-a-n+\sum_k c_k)}\int_{C_\epsilon^n}\prod_k t_k^{-c_k}\cdot(1-\sum_k t_k)^{\sum_k c_k-a-n}\cdot\left(1-\sum_k\frac{x_k}{t_k}\right)^{-b} dt,$$

where $dt = dt_1\wedge\cdots\wedge dt_n$*.*

Proof. By the assumption, if $(t_1,\dots,t_n)\in C_\epsilon^n$, we have

$$\left|\sum_k \frac{x_k}{t_k}\right| \le \sum_k \frac{|x_k|}{|t_k|} < \sum_k \frac{\epsilon^2}{n}\cdot\frac{1}{\epsilon} = \epsilon < 1,$$
$$\left|\sum_k t_k\right| \le \sum_k |t_k| = n\epsilon < \frac{n}{n+1} < 1.$$

Thus the series

$$\left(1-\sum_k\frac{x_k}{t_k}\right)^{-b} = \sum_{m_1,\dots,m_n}\frac{(b)_{m_1+\cdots+m_n}}{\prod_k m_k!}\prod_k\left(\frac{x_k}{t_k}\right)^{m_k},$$
$$\left(1-\sum_k t_k\right)^{\sum_k c_k-a-n} = \sum_{p_1,\dots,p_n}\frac{(a+n-\sum c_k)_{p_1+\cdots+p_n}}{\prod_k p_k!}\prod_k t_k^{p_k}$$

uniformly converge on C_ϵ^n, and hence we have

$$\begin{aligned}&\int_{C_\epsilon^n}\prod_k t_k^{-c_k}\cdot(1-\sum_k t_k)^{\sum_k c_k-a-n}\cdot\left(1-\sum_k\frac{x_k}{t_k}\right)^{-b}dt\\ &=\sum_{m_1,\dots,m_n}\frac{(b)_{m_1+\cdots+m_n}}{\prod_k m_k!}\prod_k x_k^{m_k}\int_{C_\epsilon^n}\prod_k t_k^{-c_k-m_k}\cdot(1-\sum_k t_k)^{\sum_k c_k-a-n}dt\\ &=\sum_{m_1,\dots,m_n}\sum_{p_1,\dots,p_n}\frac{(a+n-\sum c_k)_{p_1+\cdots+p_n}(b)_{m_1+\cdots+m_n}}{\prod_k p_k!\prod_k m_k!}\prod_k x_k^{m_k}\int_{C_\epsilon^n}\prod_k t_k^{p_k-c_k-m_k}dt.\end{aligned}$$

By the residue theorem, only the terms with $p_k = c_k+m_k-1$ survive. If $p_k = c_k+m_k-1$, then

$$\frac{(a+n-\sum c_k)_{p_1+\cdots+p_n}}{\prod_k p_k!} = \frac{(a+n-\sum c_k)_{\sum_k c_k+\sum_k m_k-n}}{\prod_k(c_k+m_k-1)!} = \frac{\Gamma(a+\sum_k m_k)}{\Gamma(a+n-\sum c_k)\cdot\prod_k\Gamma(c_k+m_k)}.$$

Thus we obtain

$$\begin{aligned}&\int_{C_\epsilon^n}\prod_k t_k^{-c_k}\cdot(1-\sum_k t_k)^{\sum_k c_k-a-n}\cdot\left(1-\sum_k\frac{x_k}{t_k}\right)^{-b}dt\\ &=(2\pi\sqrt{-1})^n\sum_{m_1,\dots,m_n}\frac{(b)_{m_1+\cdots+m_n}}{\prod_k m_k!}\prod_k x_k^{m_k}\cdot\frac{\Gamma(a+\sum_k m_k)}{\Gamma(a+n-\sum c_k)\cdot\prod_k\Gamma(c_k+m_k)}\\ &=(2\pi\sqrt{-1})^n\frac{\Gamma(a)}{\Gamma(a+n-\sum c_k)\prod_k\Gamma(c_k)}\sum_{m_1,\dots,m_n}\frac{(a)_{\sum m_k}(b)_{\sum m_k}}{\prod_k(c_k)_{m_k}\prod_k m_k!}\prod_k x_k^{m_k}\\ &=(2\pi\sqrt{-1})^n\frac{\Gamma(a)}{\Gamma(a+n-\sum c_k)\prod_k\Gamma(c_k)}\cdot F_C(a,b,c;x).\end{aligned}$$

By using the reflection formula, we conclude the proposition. □

Remark 2.1. *Roughly,* C_ϵ^n *can be regarded as the limit of* $\prod_k(1-e^{-2\pi i c_k})\cdot\Delta$ *as* c_k*'s to integers.*

2.2 Monodromy representation

The monodromy representation of F_C is expressed in terms of the twisted homology groups in [10], and clear representation matrices of circuit transformations are obtained in [13]. Here we briefly review results in [10] and [13]. Let X be the complement of the singular locus S of the system $E_C(a,b,c)$. Put $\dot{x} = \left(\frac{1}{2n^2}, \dots, \frac{1}{2n^2}\right) \in X$. Let $\rho_0, \rho_1, \dots, \rho_n$ be loops in X so that

- ρ_0 turns the hypersurface $(R(x) = 0)$ around the point $\left(\frac{1}{n^2}, \dots, \frac{1}{n^2}\right)$, positively,
- ρ_k $(k = 1, \dots, n)$ turns the hyperplane $(x_k = 0)$, positively.

For explicit definitions of them, see [10].

Proposition 2.4 (([10])). *The loops $\rho_0, \rho_1, \dots, \rho_n$ generate the fundamental group $\pi_1(X, \dot{x})$. Moreover, if $n \geq 2$, then they satisfy the following relations:*

$$\rho_i\rho_j = \rho_j\rho_i \quad (i,j = 1, \dots, n), \quad (\rho_0\rho_k)^2 = (\rho_k\rho_0)^2 \quad (k = 1, \dots, n).$$

Let $\mathcal{M}_i$ be the circuit transformation corresponding to the loop ρ_i $(i = 0, \dots, n)$. To write down representation matrices of $\mathcal{M}_i$, it is convenient to regard $\mathbb{C}^{2^n}$ as $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ and take a basis

$$\mathbf{e}_{i_1,\dots,i_n} = \mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_n}, \qquad \mathbf{e}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \mathbf{e}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We align them in the pure lexicographic order of indices $I = (i_1, \dots, i_n) \in \{0,1\}^n$:

$$(0,\dots,0),\ (1,0,\dots,0),\ (0,1,\dots,0),\ (1,1,\dots,0),\ (0,0,1,\dots,0),\ \dots,(1,\dots,1).$$

We define the tensor product $A \otimes B$ of matrices A and $B = (b_{ij})_{\substack{1\leq i\leq r \\ 1\leq j\leq s}}$ as

$$A \otimes B = \begin{pmatrix} A\,b_{11} & A\,b_{12} & \cdots & A\,b_{1s} \\ A\,b_{21} & A\,b_{22} & \cdots & A\,b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A\,b_{r1} & A\,b_{r2} & \cdots & A\,b_{rs} \end{pmatrix},$$

and we put

$$\alpha = \exp(2\pi\sqrt{-1}a), \quad \beta = \exp(2\pi\sqrt{-1}b), \quad \gamma_k = \exp(2\pi\sqrt{-1}c_k) \qquad (k = 1, \dots, n).$$

Regarding α, β, γ_k just as symbols, we define an isomorphism $^\vee : \mathbb{Q}(\alpha,\beta,\gamma_k) \to \mathbb{Q}(\alpha,\beta,\gamma_k)$ of a rational function field by $\alpha \mapsto \alpha^{-1}$, $\beta \mapsto \beta^{-1}$, $\gamma_k \mapsto \gamma_k^{-1}$. If $a, b, c_k \in \mathbb{R}$, then $^\vee$ is nothing but the complex conjugation. With these notations, we have

Proposition 2.5 (([10], [13])). *For a certain basis of the solution space to $E_C(a,b,c)$, the representation matrix M_i of $\mathcal{M}_i$ $(i = 0, \dots, n)$ is written as follows. For $k = 1, \dots, n$, we have*

$$M_k = E_2 \otimes \cdots \otimes E_2 \otimes \underset{k\text{-}th}{G_k} \otimes E_2 \otimes \cdots \otimes E_2, \quad G_k = \begin{pmatrix} 1 & -\gamma_k^{-1} \\ 0 & \gamma_k^{-1} \end{pmatrix}.$$

The matrix M_0 is written as

$$M_0 = E_{2^n} - N_0, \quad N_0 = {}^t(\mathbf{0}, \dots, \mathbf{0}, v),$$

where $v \in \mathbb{C}^{2^n}$ is a column vector whose I-th entry is

$$\begin{cases} (-1)^n \frac{(\alpha-1)(\beta-1)\prod_{k=1}^n \gamma_k}{\alpha\beta} & (I = (0,\dots,0)), \\ (-1)^{n+|I|} \frac{(\alpha\beta + (-1)^{|I|}\prod_{k=1}^n \gamma_k^{i_k})\prod_{k=1}^n \gamma_k^{1-i_k}}{\alpha\beta} & (I \neq (0,\dots,0)). \end{cases}$$

Further, the intersection matrix $H = (H_{I,I'})$ defined as

$$H_{I,I'} = \begin{cases} \displaystyle\prod_{k=1}^n (-\gamma_k)^{i'_k}(1-\gamma_k)^{1-i_k-i'_k} \cdot \frac{\alpha - 1}{\alpha - \prod_{k=1}^n \gamma_k} & (I \cdot I' = (0,\dots,0)), \\ \displaystyle\frac{\alpha\beta + (-1)^{|I\cdot I'|}\prod_{k=1}^n \gamma_k^{i_k i'_k}}{(\alpha - \prod_{k=1}^n \gamma_k)(\beta - 1)} \cdot \prod_{k=1}^n (-\gamma_k)^{i'_k(1-i_k)}(1-\gamma_k)^{(1-i_k)(1-i'_k)} & (\textit{otherwise}) \end{cases}$$

satisfies ${}^tM_i \cdot H \cdot M_i^\vee = H$ and ${}^tH = (-1)^n H^\vee$. Here, $I \cdot I' = (i_1 i'_1, \dots, i_n i'_n)$, $|I| = i_1 + \cdots + i_n$ for $I = (i_1, \dots, i_n)$ and $I' = (i'_1, \dots, i'_n)$.

Remark 2.2. (1) *In particular, M_0 is lower triangular and $M_1, \dots, M_n$ are upper triangular.*
(2) *Proposition 2.5 implies that $\mathbf{e}_{1,\dots,1} = \mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_1$ is an eigenvector of M_0, that is,*

$$M_0 \mathbf{e}_{1,\dots,1} = \delta_0 \mathbf{e}_{1,\dots,1}, \quad \delta_0 = (-1)^{n+1} \frac{\gamma_1 \cdots \gamma_n}{\alpha\beta}.$$

In [13], it is also shown that the eigenspace of M_0 with eigenvalue 1 is expressed as

$$\ker N_0 = \{ w \in \mathbb{C}^{2^n} \mid {}^t w H \mathbf{e}_{1,\dots,1} = 0 \}.$$

The matrix M_0 is a "reflection" defined later, with the special eigenvalue δ_0.

For $I \in \{0,1\}^n$, we put $M^I = M_1^{i_1} M_2^{i_2} \cdots M_n^{i_n}$. By using these notations, we have

$$M^I \mathbf{e}_{j_1,\dots,j_n} = (G_1^{i_1} \mathbf{e}_{j_1}) \otimes \cdots \otimes (G_n^{i_n} \mathbf{e}_{j_n}).$$

We often use the vectors

$$\mathbf{f}_I = M^I \mathbf{e}_{1,\dots,1} = (G_1^{i_1} \mathbf{e}_1) \otimes \cdots \otimes (G_n^{i_n} \mathbf{e}_1), \quad I = (i_1, \dots, i_n) \in \{0,1\}^n.$$

Let **Mon** be the monodromy group generated by $M_0, M_1, \dots, M_n$, which is a subgroup of $\mathbf{GL}_{2^n}(\mathbb{C}) \simeq \mathbf{GL}((\mathbb{C}^2)^{\otimes n})$.

Proposition 2.6 (([15], [13]))**.** *We assume*

$$(\mathrm{irr-abc}) \qquad a - \sum_{k=1}^{n} i_k c_k, \quad b - \sum_{k=1}^{n} i_k c_k \notin \mathbb{Z}, \qquad \forall I = (i_1, \dots, i_n) \in \{0,1\}^n,$$

or equivalently,

$$(\mathrm{irr-}\alpha\beta\gamma) \qquad \alpha - \prod_{k=1}^{n} \gamma_k^{i_k}, \quad \beta - \prod_{k=1}^{n} \gamma_k^{i_k} \neq 0, \qquad \forall I = (i_1, \dots, i_n) \in \{0,1\}^n.$$

Then we have
(1) *The vectors $\mathbf{f}_I$ ($I = (i_1, \dots, i_n) \in \{0,1\}^n$) form a basis of $\mathbb{C}^{2^n} \cong (\mathbb{C}^2)^{\otimes n}$.*
(2) *The monodromy group* **Mon** *acts on $\mathbb{C}^{2^n}$ irreducibly.*

We consider the case of $a = b = \frac{1}{2}$, $c_k = 1$ (i.e., $\alpha = \beta = -1$, $\gamma_k = 1$) in detail.

Corollary 2.1. (1) *Assume that $a, b, c_k \in \mathbb{R}$. Since $^\vee$ means the complex conjugation, the intersection matrix H is a Hermitian matrix if n is even and a skew Hermitian matrix if n is odd. Further assume that $a+b \in \mathbb{Z}$, $c_k \in \frac{1}{2}\mathbb{Z}$ and $\sum_{k=1}^n c_k \in \mathbb{Z}$ (that is, $\alpha = \overline{\beta}$, $\gamma_k = \pm 1$ and $\prod_{k=1}^n \gamma_k = 1$). Then M_k and H are defined over $\mathbb{R}$. In this case, the monodromy group* **Mon** *is a subgroup of a real orthogonal/symplectic group with respect to H.*
(2) *In the case of $a = b = \frac{1}{2}$, $c_k = 1$, the representation matrices are as follows. For $k = 1, \dots, n$, we have*

$$M_k = E_2 \otimes \cdots \otimes E_2 \otimes \underset{k\text{-}th}{G_k} \otimes E_2 \otimes \cdots \otimes E_2, \quad G_k = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

The (I, I')-entry of M_0 is

$$\begin{cases} 1 & (I = I' \neq (1,\dots,1)), \\ (-1)^{n+1} & (I = I' = (1,\dots,1)), \\ (-1)^{n+1} \cdot 4 & (I = (1,\dots,1), I' = (0,\dots,0)), \\ (-1)^{n+1} \cdot 2 & (I = (1,\dots,1), I' \neq (0,\dots,0), (1,\dots,1) \text{ and } |I'| \equiv 0 \mod 2) \\ 0 & (\textit{otherwise}). \end{cases}$$

We have ${}^t M_i \cdot H \cdot M_i = H$ and ${}^t H = (-1)^n H$. Each entry of H belongs to $\mathbb{Z}[\frac{1}{2}]$.

2.3 Zariski closure and Reflection group

Let us consider the Zariski closure of the monodromy group **Mon** after Beukers and Heckman. As in [3], we call a linear map $g \in \mathbf{GL}_n(\mathbb{C})$ a *reflection* if $g - \mathrm{Id}$ has rank one (Hence a reflection may be of infinite order, and our reflections include matrices called *transvection*). We call the determinant of a reflection g the *special eigenvalue* of g. For a subgroup $G \subset \mathbf{GL}_n(\mathbb{C})$, the Zariski closure of G over complex numbers is denoted by $\overline{G}$. The connected component (in the Zariski topology) of the identity, which is a normal subgroup, is denoted by G^0. The quotient group $G/G^0 \cong \overline{G}/\overline{G}^0$ is a finite group. We apply the following Proposition with $r = M_0$ (see Remark 2.2).

Proposition 2.7 (([3])). *Suppose $G \subset \mathbf{SL}_n(\mathbb{C})$ is a connected algebraic group acting irreducibly on $\mathbb{C}^n$. Let $r \in \mathbf{GL}_n(\mathbb{C})$ be a reflection with special eigenvalue $\delta \in \mathbb{C}^\times$ which normalizes G. Then we have the following three possibilities,*
(I) *If $\delta \neq \pm 1$ then $G = \mathbf{SL}_n(\mathbb{C})$,*
(II) *If $\delta = +1$ then $G = \mathbf{SL}_n(\mathbb{C})$ or $G = \mathbf{Sp}_n(\mathbb{C})$,*
(III) *If $\delta = -1$ then $G = \mathbf{SL}_n(\mathbb{C})$ or $G = \mathbf{SO}_n(\mathbb{C})$.*

An immediate consequence is

Theorem 2.1. *Assume that $\mathbf{Mon}^0$ acts on $\mathbb{C}^{2^m}$ irreducibly. Then we have the following three possibilities,*
(I) *If $\delta_0 \neq \pm 1$ then $\mathbf{SL}_{2^n}(\mathbb{C}) \subset \overline{\mathbf{Mon}}$,*
(II) *If $\delta_0 = +1$ then $\mathbf{SL}_{2^n}(\mathbb{C}) \subset \overline{\mathbf{Mon}}$ or $\mathbf{Sp}_{2^n}(\mathbb{C}) \subset \overline{\mathbf{Mon}} \subset \mathbf{GSp}_{2^n}(\mathbb{C})$,*
(III) *If $\delta_0 = -1$ then $\mathbf{SL}_{2^n}(\mathbb{C}) \subset \overline{\mathbf{Mon}}$ or $\mathbf{SO}_{2^n}(\mathbb{C}) \subset \overline{\mathbf{Mon}} \subset \mathbf{GO}_{2^n}(\mathbb{C})$.*
Moreover we have $\overline{\mathbf{Mon}}^0 \subset \mathbf{SL}_{2^n}(\mathbb{C})$ if $a, b, c_i \in \mathbb{Q}$.

Proof. Let $\mathfrak{m}$ be the Lie algebra of $\overline{\mathbf{Mon}}$. By the assumption, $\mathfrak{m}$ acts on $\mathbb{C}^{2^n}$ irreducibly, and so is $\mathfrak{m} \cap \mathfrak{sl}_{2^n}(\mathbb{C})$. Therefore $G = (\overline{\mathbf{Mon}} \cap \mathbf{SL}_{2^n}(\mathbb{C}))^0$ ats on $\mathbb{C}^{2^n}$ irreducibly. Since G is normalized by M_0, we can apply Proposition 2.7. We have the above three cases, since $\overline{\mathbf{Mon}}$ normalizes G and the normalizer of **Sp** and **SO** in **GL** are **GSp** and **GO** respectively. If $a, b, c_k \in \mathbb{Q}$, then the image of $\det : \mathbf{Mon} \to \mathbb{C}^\times$ is finite and we have $\mathbf{Mon}^0 \subset \mathbf{SL}_{2^n}(\mathbb{C})$. □

Remark 2.3. *For $n = 2$, it was shown by Sasaki that $\overline{\mathbf{Mon}} = \mathbf{GL}_4(\mathbb{C})$ if parameters are general complex numbers in* [23]. *The same is true for $n \geq 3$ by* (I) *of the above theorem* (*see also Corollary* 2.3).

Corollary 2.2. *Assume that $\mathbf{Mon}^0$ acts on $\mathbb{C}^{2^m}$ irreducibly, and that $a, b \in \mathbb{R}$, $a + b \in \mathbb{Z}$, $c_k \in \frac{1}{2}\mathbb{Z}$ and $\sum_{k=1}^n c_k \in \mathbb{Z}$. Then we have $\overline{\mathbf{Mon}} = \mathbf{O}_{2^n}(\mathbb{C})$ if n is even, and $\overline{\mathbf{Mon}} = \mathbf{Sp}_{2^n}(\mathbb{C})$ if n is odd.*

Proof. By Corollary 2.1, we have $\mathbf{Mon} \subset \mathbf{O}_{2^n}(\mathbb{C})$ if n is even, and $\mathbf{Mon} \subset \mathbf{Sp}_{2^n}(\mathbb{C})$ if n is odd. Since $\delta_0 = (-1)^{n+1}$, the assertion follows from Theorem 2.1. □

Because of the above theorem, it is interesting to determine conditions for irreducibility of $\mathbf{Mon}^0$. We give a partial answer to this problem in the following (Corollary 2.3). Let $\mathbf{Ref} \subset \mathbf{Mon}$ be the smallest normal subgroup containing M_0, that is, a subgroup generated by reflections gM_0g^{-1} ($g \in \mathbf{Mon}$). Since $\mathbf{Ref}^0 \subset \mathbf{Mon}^0$, the irreducibility of $\mathbf{Ref}^0$ is a sufficient condition for irreducibility of $\mathbf{Mon}^0$. The reflection subgroup was introduced in [3] for the generalized hypergeometric function ${}_nF_{n-1}$, and considered in [17] for Appell's F_4 to study the finiteness condition.

Proposition 2.8. *The monodromy group* **Mon** *is finite if and only if* **Ref** *is finite.*

Proof. Let us assume that **Ref** is a finite group. Since $\{M_1^d M_0 M_1^{-d} \mid d = 1, 2, \dots\}$ is a finite set, there exist k and l ($k \neq l$) such that $M_1^k M_0 M_1^{-k} = M_1^l M_0 M_1^{-l}$, namely, $M_1^{k-l} M_0 = M_0 M_1^{k-l}$. On the other hand, we have

$$M_1^{-d} = G_1^{-d} \otimes E_2 \otimes \cdots \otimes E_2, \qquad G_1^{-d} = \begin{pmatrix} 1 & \frac{1-\gamma_1^d}{1-\gamma_1} \\ 0 & \gamma_1^d \end{pmatrix}$$

for $d = 1, 2, \dots$, and

$$M_1^{-d} M_0 \mathbf{e}_{1,\dots,1} = \delta_0 M_1^{-d} \mathbf{e}_{1,\dots,1} = \delta_0 \left(\frac{1-\gamma_1^d}{1-\gamma_1} \mathbf{e}_{0,1,\dots,1} + \gamma_1^d \mathbf{e}_{1,\dots,1}\right),$$

$$M_0 M_1^{-d} \mathbf{e}_{1,\dots,1} = M_0 \left(\frac{1-\gamma_1^d}{1-\gamma_1} \mathbf{e}_{0,1,\dots,1} + \gamma_1^d \mathbf{e}_{1,\dots,1}\right) = \frac{1-\gamma_1^d}{1-\gamma_1} M_0 \mathbf{e}_{0,1,\dots,1} + \delta_0 \gamma_1^d \mathbf{e}_{1,\dots,1}.$$

Therefore, if M_1^d commutes with M_0, we have

$$\begin{aligned}\mathbf{0} = (M_0 M_1^{-d} - M_1^{-d} M_0)\mathbf{e}_{1,\dots,1} &= \frac{1-\gamma_1^d}{1-\gamma_1}(M_0 - \delta_0 \mathrm{Id})\mathbf{e}_{0,1,\dots,1} \\ &= \frac{1-\gamma_1^d}{1-\gamma_1}\big((1-\delta_0)\mathbf{e}_{0,1,\dots,1} + (\gamma_1+\delta_0)\mathbf{e}_{1,\dots,1}\big).\end{aligned}$$

If $\gamma_1 \neq -1$, this implies $\gamma_1^d = 1$ and $M_1^d = \mathrm{Id}$. If $\gamma_1 = -1$, we have $G_1 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ and $M_1^2 = \mathrm{Id}$. By the similar argument, we see that M_k $(k = 1, \dots, n)$ are of finite order. Therefore $\mathbf{Mon}/\mathbf{Ref}$ is finite, and so is $\mathbf{Mon}$. The converse is obvious. □

Let us consider reflections

$$R_I = M^I M_0 (M^I)^{-1} = E_{2^n} - M^I N_0 (M^I)^{-1} \qquad I \in \{0,1\}^n$$

and endomorphisms $N_I = E_{2^n} - R_I = M^I N_0 (M^I)^{-1}$ of $\mathbb{C}^{2^n}$.

Lemma 2.1. (1) The image of N_I is spanned by $\mathbf{f}_I$, and we have

$$\ker N_I = M^I \cdot \ker N_0 = \{w \in \mathbb{C}^{2^n} \mid {}^t w H \mathbf{f}_I^\vee = 0\}.$$

(2) A linear map

$$\nu \ : \ \mathbb{C}^{2^n} \to \oplus_{I \in \{0,1\}^n} \mathbb{C}\mathbf{f}_I, \quad w \mapsto (\dots, N_I w, \dots)$$

is an isomorphism under the condition $(\mathrm{irr} - \alpha\beta\gamma)$.

Proof. (1) We have

$$N_I \cdot \mathbb{C}^{2^n} = M^I N_0 (M^I)^{-1} \cdot \mathbb{C}^{2^n} = M^I N_0 \cdot \mathbb{C}^{2^n} = M^I \cdot \mathbb{C}\mathbf{e}_{1,\dots,1} = \mathbb{C}\mathbf{f}_I$$

and

$$\begin{aligned} w \in \ker N_I \ \Leftrightarrow \ M^I N_0 (M^I)^{-1} w = \mathbf{0} \ &\Leftrightarrow \ N_0 (M^I)^{-1} w = \mathbf{0} \\ &\Leftrightarrow \ (M^I)^{-1} w \in \ker N_0 \\ &\Leftrightarrow \ {}^t((M^I)^{-1} w) H \mathbf{e}_{1,\dots,1} = 0 \\ &\Leftrightarrow \ {}^t w H ((M^I)^\vee \mathbf{e}_{1,\dots,1}) = 0 \ \Leftrightarrow \ {}^t w H \mathbf{f}_I^\vee = 0. \end{aligned}$$

(2) Since $\mathbf{f}_I$ are linearly independent under the condition $(\mathrm{irr} - \alpha\beta\gamma)$, we see that $\mathbf{f}_I^\vee$ are linearly independent and $\ker \nu = \mathbf{0}$. □

Theorem 2.2. *Assume that* $\mathbf{Mon}$ *is irreducible (that is,* $(\mathrm{irr} - \alpha\beta\gamma)$ *holds). The action of* $\mathbf{Ref}$ *is irreducible if and only if at most one of* $\gamma_1, \dots, \gamma_n, \alpha\beta^{-1}$ *is* -1.

We divide the proof into the following four Lemmas, where we always assume that *the action of* $\mathbf{Mon}$ *is irreducible.*

Lemma 2.2. *If none of* $\gamma_1, \dots, \gamma_n$ *is* -1*, then the action of* $\mathbf{Ref}$ *is irreducible.*

Proof. First note that if $\mathbf{Ref}$ acts on a subspace W, then W must be a direct sum of 1-dimensional subspaces $\mathbb{C}\mathbf{f}_I$ since N_I acts on W as an endomorphism. Let $W \neq \mathbf{0}$ be an irreducible $\mathbf{Ref}$-subspace. Since $\mathbf{Ref}$ is a normal subgroup, gW is also irreducible $\mathbf{Ref}$-subspace for $g \in \mathbf{Mon}$. Replacing W by $M^I W$ with $I \in \mathbb{Z}^n$ if necessary, we assume that $\mathbf{e}_{1,\dots,1} = \mathbf{f}_{0,\dots,0} \in W$. Then we have $M_1 \mathbf{e}_{1,\dots,1} \in M_1 W$, and hence

$$\begin{aligned} N_0(M_1 \mathbf{e}_{1,\dots,1}) &= -\frac{1}{\gamma_1} N_0(\mathbf{e}_{0,1,\dots,1} - \mathbf{e}_{1,\dots,1}) \\ &= -\frac{1}{\gamma_1}\Big((-\gamma_1 + (-1)^n \frac{\gamma_1 \cdots \gamma_n}{\alpha\beta})\mathbf{e}_{1,\dots,1} - (1 + (-1)^n \frac{\gamma_1 \cdots \gamma_n}{\alpha\beta})\mathbf{e}_{1,\dots,1}\Big) \\ &= \frac{1+\gamma_1}{\gamma_1}\mathbf{e}_{1,\dots,1}, \end{aligned}$$

belongs to $M_1 W$. By the assumption, this is not $\mathbf{0}$. Therefore we have $\mathbf{e}_{1,\dots,1} \in W \cap M_1 W$ and $M_1 W = W$ by the irreducibility. We see that every M_k acts on W in the same way. Hence we have $W = \mathbb{C}^{2^n}$. □

Lemma 2.3. *If exactly one of $\gamma_1, \ldots, \gamma_n$ is -1 and $\alpha\beta^{-1}$ is not equal to -1, then the action of* **Ref** *is irreducible.*

Proof. For simplicity, we may assume that

$$\gamma_1 = -1, \quad \gamma_k \neq -1 \ (k = 2, \ldots, n), \quad \beta \neq -\alpha.$$

Let $W \neq \mathbf{0}$ be an irreducible **Ref**-subspace such that $\mathbf{e}_{1,\ldots,1} \in W$ as in the proof of Lemma 2.2. By the assumption $\gamma_k \neq -1$ $(k = 2, \ldots, n)$ and the proof of Lemma 2.2, we have $M_2^{i_2} \cdots M_n^{i_n} \mathbf{e}_{1,\ldots,1} \in W$. Since

$$\begin{aligned} W \ni M_2 \mathbf{e}_{1,\ldots,1} &= \mathbf{e}_1 \otimes (G_2 \mathbf{e}_1) \otimes \cdots \otimes \mathbf{e}_1 = \mathbf{e}_1 \otimes (-\gamma_2^{-1} \mathbf{e}_0 + \gamma_2^{-1} \mathbf{e}_1) \otimes \cdots \otimes \mathbf{e}_1 \\ &= -\gamma_2^{-1} \mathbf{e}_{1,0,1,1,\ldots,1} + \gamma_2^{-1} \mathbf{e}_{1,\ldots,1}, \end{aligned}$$

and $\mathbf{e}_{1,\ldots,1} \in W$, we obtain $\mathbf{e}_{1,0,1,1,\ldots,1} \in W$. Similar arguments show that $\mathbf{e}_{1,i_2,\ldots,i_n} \in W$ for any $(i_2, \ldots, i_n) \in \{0,1\}^{n-1}$. In particular, we have $\mathbf{e}_{1,0,\ldots,0} \in W$. Note that the condition $\gamma_1 = -1$ implies $G_1 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ and

$$M_1 \mathbf{e}_{0,i_2,\ldots,i_n} = \mathbf{e}_{0,i_2,\ldots,i_n}, \quad M_1 \mathbf{e}_{1,i_2,\ldots,i_n} = \mathbf{e}_{0,i_2,\ldots,i_n} - \mathbf{e}_{1,i_2,\ldots,i_n}.$$

By Proposition 2.5, we have

$$\begin{aligned} &M_0 M_1 \mathbf{e}_{1,0,\ldots,0} = M_0(\mathbf{e}_{0,0,\ldots,0} - \mathbf{e}_{1,0,\ldots,0}) \\ &= \left(\mathbf{e}_{0,0,\ldots,0} + (-1)^n \frac{(\alpha-1)(\beta-1)\prod_{k=2}^n \gamma_k}{\alpha\beta} \mathbf{e}_{1,\ldots,1} \right) - \left(\mathbf{e}_{1,0,\ldots,0} - (-1)^{n+1} \frac{(\alpha\beta+1)\prod_{k=2}^n \gamma_k}{\alpha\beta} \mathbf{e}_{1,\ldots,1} \right) \\ &= \mathbf{e}_{0,0,\ldots,0} - \mathbf{e}_{1,0,\ldots,0} + \lambda \mathbf{e}_{1,\ldots,1}, \end{aligned}$$

where we put

$$\lambda = (-1)^n \frac{\prod_{k=2}^n \gamma_k}{\alpha\beta} \Big((\alpha-1)(\beta-1) - (\alpha\beta+1) \Big) = (-1)^n \frac{\prod_{k=2}^n \gamma_k}{\alpha\beta} (-\alpha - \beta).$$

By the condition $\beta \neq -\alpha$, we have $\lambda \neq 0$. Because of

$$\begin{aligned} W \ni M_1 M_0 M_1^{-1} \mathbf{e}_{1,0,\ldots,0} &= M_1 M_0 M_1 \mathbf{e}_{1,0,\ldots,0} \\ &= M_1(\mathbf{e}_{0,0,\ldots,0} - \mathbf{e}_{1,0,\ldots,0} + \lambda \mathbf{e}_{1,\ldots,1}) \\ &= \mathbf{e}_{1,0,\ldots,0} + \lambda \mathbf{e}_{0,1,\ldots,1} - \lambda \mathbf{e}_{1,\ldots,1}, \end{aligned}$$

$\mathbf{e}_{1,0,\ldots,0}, \mathbf{e}_{1,\ldots,1} \in W$ and $\lambda \neq 0$, we obtain $\mathbf{e}_{0,1,\ldots,1} \in W$. By using $M_2^{i_2} \cdots M_n^{i_n} \mathbf{e}_{0,1,\ldots,1} \in W$, we can show that $\mathbf{e}_{i_1,i_2,\ldots,i_n} \in W$ for all $(i_1, \ldots, i_n) \in \{0,1\}^n$. Hence we obtain $W = \mathbb{C}^{2^n}$. □

Lemma 2.4. *If at least two of $\gamma_1, \ldots, \gamma_n$ are -1, then the action of* **Ref** *is reducible.*

Lemma 2.5. *If at least one of $\gamma_1, \ldots, \gamma_n$ is -1 and $\alpha\beta^{-1}$ is -1, then the action of* **Ref** *is reducible.*

We show these lemmas by applying ideas given in the proofs of [17, Lemmas 4.2 and 4.3]. To prove the lemmas, we use relations

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \mathbf{e}_0 = \mathbf{e}_0, \quad \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \mathbf{e}_1 = \mathbf{e}_0 - \mathbf{e}_1, \quad \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} (2\mathbf{e}_1 - \mathbf{e}_0) = -(2\mathbf{e}_1 - \mathbf{e}_0). \tag{1}$$

Proof of Lemma 2.4. For simplicity, we may assume that $\gamma_1 = \gamma_2 = -1$. For each $(i_3, \ldots, i_n)$, we put

$$\begin{aligned} &g_{0,0,i_3,\ldots,i_n} = \mathbf{e}_{0,0,i_3,\ldots,i_n} = \mathbf{e}_0 \otimes \mathbf{e}_0 \otimes \mathbf{e}_{i_3} \otimes \cdots \otimes \mathbf{e}_{i_n}, \\ &g_{1,0,i_3,\ldots,i_n} = 2\mathbf{e}_{1,0,i_3,\ldots,i_n} - \mathbf{e}_{0,0,i_3,\ldots,i_n} = (2\mathbf{e}_1 - \mathbf{e}_0) \otimes \mathbf{e}_0 \otimes \mathbf{e}_{i_3} \otimes \cdots \otimes \mathbf{e}_{i_n}, \\ &g_{0,1,i_3,\ldots,i_n} = 2\mathbf{e}_{0,1,i_3,\ldots,i_n} - \mathbf{e}_{0,0,i_3,\ldots,i_n} = \mathbf{e}_0 \otimes (2\mathbf{e}_1 - \mathbf{e}_0) \otimes \mathbf{e}_{i_3} \otimes \cdots \otimes \mathbf{e}_{i_n}, \\ &g_{1,1,i_3,\ldots,i_n} = 4\mathbf{e}_{1,1,i_3,\ldots,i_n} - 2\mathbf{e}_{1,0,i_3,\ldots,i_n} - 2\mathbf{e}_{0,1,i_3,\ldots,i_n} + \mathbf{e}_{0,0,i_3,\ldots,i_n} = (2\mathbf{e}_1 - \mathbf{e}_0) \otimes (2\mathbf{e}_1 - \mathbf{e}_0) \otimes \mathbf{e}_{i_3} \otimes \cdots \otimes \mathbf{e}_{i_n}, \\ &f^{\pm}_{14;i_3,\ldots,i_n} = g_{0,0,i_3,\ldots,i_n} \pm g_{1,1,i_3,\ldots,i_n}, \qquad f^{\pm}_{23;i_3,\ldots,i_n} = g_{1,0,i_3,\ldots,i_n} \pm g_{0,1,i_3,\ldots,i_n}. \end{aligned}$$

We consider proper subspaces

$$W^{\pm} = \bigoplus_{(i_3,\ldots,i_n)} \mathbb{C} f^{\pm}_{14;i_3,\ldots,i_n} \oplus \bigoplus_{(i_3,\ldots,i_n)} \mathbb{C} f^{\pm}_{23;i_3,\ldots,i_n}$$

of $\mathbb{C}^{2^n}$ whose dimensions are $2 \cdot 2^{n-2} = 2^{n-1}$. We show that these are non-trivial **Ref**-subspaces. Note that by the definition, we have

$$\mathbf{e}_{1,\ldots,1} = \frac{1}{4}(g_{0,0,1,\ldots,1} + g_{1,0,1,\ldots,1} + g_{0,1,1,\ldots,1} + g_{1,1,1,\ldots,1}) = \frac{1}{4}(f^{+}_{14;1,\ldots,1} + f^{+}_{23;1,\ldots,1}) \in W^{+}.$$

By (1), actions of M_1 and M_2 are given as

$$M_1 : \begin{cases} g_{0,i_2,i_3,\ldots,i_n} \mapsto g_{0,i_2,i_3,\ldots,i_n} \\ g_{1,i_2,i_3,\ldots,i_n} \mapsto -g_{1,i_2,i_3,\ldots,i_n} \end{cases}, \qquad M_2 : \begin{cases} g_{i_1,0,i_3,\ldots,i_n} \mapsto g_{i_1,0,i_3,\ldots,i_n} \\ g_{i_1,1,i_3,\ldots,i_n} \mapsto -g_{i_1,1,i_3,\ldots,i_n}, \end{cases}$$

and hence

$$M_1 : \begin{cases} f^{\pm}_{14;i_3,\ldots,i_n} \mapsto f^{\mp}_{14;i_3,\ldots,i_n} \\ f^{\pm}_{23;i_3,\ldots,i_n} \mapsto -f^{\mp}_{23;i_3,\ldots,i_n} \end{cases}, \qquad M_2 : \begin{cases} f^{\pm}_{14;i_3,\ldots,i_n} \mapsto f^{\mp}_{14;i_3,\ldots,i_n} \\ f^{\pm}_{23;i_3,\ldots,i_n} \mapsto f^{\mp}_{23;i_3,\ldots,i_n}. \end{cases}$$

These imply that $M_1 W^{\pm} = W^{\mp}$, $M_2 W^{\pm} = W^{\mp}$.

We can show that $M_i W^{\pm} = W^{\pm}$ $(i = 3, \ldots, n)$. For example, if $i = 3$, we have

$$M_3 \cdot g_{i_1,i_2,0,i_4,\ldots,i_n} = g_{i_1,i_2,0,i_4,\ldots,i_n}, \quad M_3 \cdot g_{i_1,i_2,1,i_4,\ldots,i_n} = -\gamma_3^{-1} g_{i_1,i_2,0,i_4,\ldots,i_n} + \gamma_3^{-1} g_{i_1,i_2,1,i_4,\ldots,i_n}$$

and

$$\begin{aligned} M_3 \cdot f^{\pm}_{14;0,i_4,\ldots,i_n} &= f^{\pm}_{14;0,i_4,\ldots,i_n}, \quad M_3 \cdot f^{\pm}_{14;1,i_4,\ldots,i_n} = -\gamma_3^{-1} f^{\pm}_{14;0,i_4,\ldots,i_n} + \gamma_3^{-1} f^{\pm}_{14;1,i_4,\ldots,i_n}, \\ M_3 \cdot f^{\pm}_{23;0,i_4,\ldots,i_n} &= f^{\pm}_{23;0,i_4,\ldots,i_n}, \quad M_3 \cdot f^{\pm}_{23;1,i_4,\ldots,i_n} = -\gamma_3^{-1} f^{\pm}_{23;0,i_4,\ldots,i_n} + \gamma_3^{-1} f^{\pm}_{23;1,i_4,\ldots,i_n}, \end{aligned}$$

which imply $M_3 W^{\pm} = W^{\pm}$.

By Proposition 2.5 and $\gamma_1 = \gamma_2 = -1$, we have

$$\begin{aligned} M_0 \mathbf{e}_{0,0,i_3,\ldots,i_n} &= \mathbf{e}_{0,0,i_3,\ldots,i_n} - \lambda_{0;i_3,\ldots,i_n} \mathbf{e}_{1,\ldots,1}, \\ M_0 \mathbf{e}_{1,0,i_3,\ldots,i_n} &= \mathbf{e}_{1,0,i_3,\ldots,i_n} - \lambda_{1;i_3,\ldots,i_n} \mathbf{e}_{1,\ldots,1}, \\ M_0 \mathbf{e}_{0,1,i_3,\ldots,i_n} &= \mathbf{e}_{0,1,i_3,\ldots,i_n} - \lambda_{1;i_3,\ldots,i_n} \mathbf{e}_{1,\ldots,1}, \\ M_0 \mathbf{e}_{1,1,i_3,\ldots,i_n} &= \mathbf{e}_{1,1,i_3,\ldots,i_n} - \lambda_{1;i_3,\ldots,i_n} \mathbf{e}_{1,\ldots,1}, \end{aligned}$$

where

$$\begin{aligned} \lambda_{1;i_3,\ldots,i_n} &= (-1)^{n+i_3+\cdots+i_n} \frac{(\alpha\beta + (-1)^{i_3+\cdots+i_n} \prod_{k=3}^{n} \gamma_k^{i_k}) \prod_{k=3}^{n} \gamma_k^{1-i_k}}{\alpha\beta}, \\ \lambda_{0;i_3,\ldots,i_n} &= \begin{cases} (-1)^n \frac{(\alpha-1)(\beta-1)\prod_{k=3}^{n}\gamma_k}{\alpha\beta} & ((i_3,\ldots,i_n) = (0,\ldots,0)) \\ \lambda_{1;i_3,\ldots,i_n} & ((i_3,\ldots,i_n) \neq (0,\ldots,0)). \end{cases} \end{aligned}$$

Thus we obtain

$$\begin{aligned} M_0 \cdot g_{0,0,i_3,\ldots,i_n} &= g_{0,0,i_3,\ldots,i_n} - \lambda_{0;i_3,\ldots,i_n} \mathbf{e}_{1,\ldots,1}, \\ M_0 \cdot g_{1,0,i_3,\ldots,i_n} &= g_{1,0,i_3,\ldots,i_n} - (2\lambda_{1;i_3,\ldots,i_n} - \lambda_{0;i_3,\ldots,i_n}) \mathbf{e}_{1,\ldots,1}, \\ M_0 \cdot g_{0,1,i_3,\ldots,i_n} &= g_{0,1,i_3,\ldots,i_n} - (2\lambda_{1;i_3,\ldots,i_n} - \lambda_{0;i_3,\ldots,i_n}) \mathbf{e}_{1,\ldots,1}, \\ M_0 \cdot g_{1,1,i_3,\ldots,i_n} &= g_{1,1,i_3,\ldots,i_n} - \lambda_{0;i_3,\ldots,i_n} \mathbf{e}_{1,\ldots,1}, \end{aligned}$$

and

$$\begin{aligned} M_0 \cdot f^{-}_{14;i_3,\ldots,i_n} &= f^{-}_{14;i_3,\ldots,i_n} \in W^{-}, \quad M_0 \cdot f^{-}_{23;i_3,\ldots,i_n} = f^{-}_{23;i_3,\ldots,i_n} \in W^{-}, \\ M_0 \cdot f^{+}_{14;i_3,\ldots,i_n} &= f^{+}_{14;i_3,\ldots,i_n} - 2\lambda_{0;i_3,\ldots,i_n} \mathbf{e}_{1,\ldots,1} \in W^{+}, \\ M_0 \cdot f^{+}_{23;i_3,\ldots,i_n} &= f^{+}_{23;i_3,\ldots,i_n} - 2(2\lambda_{1;i_3,\ldots,i_n} - \lambda_{0;i_3,\ldots,i_n}) \mathbf{e}_{1,\ldots,1} \in W^{+}. \end{aligned}$$

These mean $M_0 W^{\pm} = W^{\pm}$.

Now we show that $W^{\pm}$ are **Ref**-subspaces. To prove this claim, it is sufficient to see that $(gM_0g^{-1})W^{\pm} = W^{\pm}$, for each generator gM_0g^{-1} ($g \in \mathbf{Mon}$) of **Ref**. Since g is represented as a product of $M_0^{\pm 1}, \ldots, M_n^{\pm 1}$, g^{-1} maps $W^{\pm}$ as follows:

$$g^{-1} : \begin{cases} W^{\pm} \to W^{\pm} & \text{(the number of } M_1^{\pm 1} \text{ and } M_2^{\pm 1} \text{ in } g \text{ is even)} \\ W^{\pm} \to W^{\mp} & \text{(the number of } M_1^{\pm 1} \text{ and } M_2^{\pm 1} \text{ in } g \text{ is odd).} \end{cases}$$

By $M_0 W^{\pm} = W^{\pm}$, we thus obtain

$$gM_0g^{-1} : \begin{cases} W^{\pm} \xrightarrow{g^{-1}} W^{\pm} \xrightarrow{M_0} W^{\pm} \xrightarrow{g} W^{\pm} & \text{(the number of } M_1^{\pm 1} \text{ and } M_2^{\pm 1} \text{ in } g \text{ is even)} \\ W^{\pm} \xrightarrow{g^{-1}} W^{\mp} \xrightarrow{M_0} W^{\mp} \xrightarrow{g} W^{\pm} & \text{(the number of } M_1^{\pm 1} \text{ and } M_2^{\pm 1} \text{ in } g \text{ is odd).} \end{cases}$$

Therefore, $W^{\pm}$ are non-trivial **Ref**-subspaces, and the proof is completed. □

Proof of Lemma 2.5. For simplicity, we assume $\gamma_1 = -1$. For each $(i_2, \ldots, i_n)$, we put

$$\begin{aligned} h_{0,i_2,i_3,\ldots,i_n} &= \mathbf{e}_{0,i_2,i_3,\ldots,i_n} = \mathbf{e}_0 \otimes \mathbf{e}_{i_2} \otimes \mathbf{e}_{i_3} \otimes \cdots \otimes \mathbf{e}_{i_n}, \\ h_{1,i_2,i_3,\ldots,i_n} &= 2\mathbf{e}_{1,i_2,i_3,\ldots,i_n} - \mathbf{e}_{0,i_2,i_3,\ldots,i_n} = (2\mathbf{e}_1 - \mathbf{e}_0) \otimes \mathbf{e}_{i_2} \otimes \mathbf{e}_{i_3} \otimes \cdots \otimes \mathbf{e}_{i_n}, \\ f^{\pm}_{12;i_2,\ldots,i_n} &= h_{0,i_2,i_3,\ldots,i_n} \pm h_{1,i_2,i_3,\ldots,i_n}. \end{aligned}$$

We consider proper subspaces

$$W^{\pm} = \bigoplus_{(i_2,\ldots,i_n)} \mathbb{C} f^{\pm}_{12;i_2,\ldots,i_n}$$

of $\mathbb{C}^{2^n}$ whose dimensions are 2^{n-1}. We show that these are non-trivial **Ref**-subspaces. Note that by the definition, we have

$$\mathbf{e}_{1,\ldots,1} = \frac{1}{2}(h_{0,1,1,\ldots,1} + h_{1,1,1,\ldots,1}) = \frac{1}{2} f^{+}_{12;1,\ldots,1} \in W^{+}.$$

By arguments similar to those in Proof of Lemma 2.4, we obtain

$$M_1 W^{\pm} = W^{\mp}, \quad M_i W^{\pm} = W^{\pm} \ (i = 2, \ldots, n).$$

We put

$$\lambda_{i_2,\ldots,i_n} = (-1)^{n+i_2+\cdots+i_n} \frac{(\alpha\beta + (-1)^{i_2+\cdots+i_n} \prod_{k=2}^{n} \gamma_k^{i_k}) \prod_{k=2}^{n} \gamma_k^{1-i_k}}{\alpha\beta}.$$

Since $\alpha + \beta = 0$ by the assumption of the lemma, $\lambda_{0,\ldots,0}$ is written as

$$\lambda_{0,\ldots,0} = (-1)^n \frac{(\alpha\beta + 1) \prod_{k=2}^{n} \gamma_k}{\alpha\beta} = (-1)^n \frac{(\alpha - 1)(\beta - 1) \prod_{k=2}^{n} \gamma_k}{\alpha\beta}.$$

By Proposition 2.5 and $\gamma_1 = -1$, we have

$$M_0 \mathbf{e}_{0,i_2,\ldots,i_n} = \mathbf{e}_{0,i_2,\ldots,i_n} + \lambda_{i_2,\ldots,i_n} \mathbf{e}_{1,\ldots,1}, \quad M_0 \mathbf{e}_{1,i_2,\ldots,i_n} = \mathbf{e}_{1,i_2,\ldots,i_n} + \lambda_{i_2,\ldots,i_n} \mathbf{e}_{1,\ldots,1},$$

and

$$M_0 \cdot h_{0,i_2,\ldots,i_n} = h_{0,i_2,\ldots,i_n} + \lambda_{i_2,\ldots,i_n} \mathbf{e}_{1,\ldots,1}, \quad M_0 \cdot h_{1,i_2,\ldots,i_n} = h_{1,i_2,\ldots,i_n} + \lambda_{i_2,\ldots,i_n} \mathbf{e}_{1,\ldots,1}.$$

These imply

$$\begin{aligned} M_0 \cdot f^{-}_{12;i_2,\ldots,i_n} &= f^{-}_{12;i_2,\ldots,i_n} \in W^{-}, \\ M_0 \cdot f^{+}_{12;i_2,\ldots,i_n} &= f^{+}_{12;i_2,\ldots,i_n} + 2\lambda_{i_2,\ldots,i_n} \mathbf{e}_{1,\ldots,1} \in W^{+}, \end{aligned}$$

and hence we obtain $M_0 W^{\pm} = W^{\pm}$.

An argument similar to that in Proof of Lemma 2.4 shows that $(gM_0g^{-1})W^{\pm} = W^{\pm}$, for each generator gM_0g^{-1} ($g \in \mathbf{Mon}$) of **Ref**. Therefore, $W^{\pm}$ are non-trivial **Ref**-subspaces. We conclude that the action of **Ref** is reducible. □

Theorem 2.3. *Assume that* **Ref** *acts on* $\mathbb{C}^{2^n}$ *irreducibly. If the action of* $\mathbf{Ref}^0$ *is reducible, then* **Ref** *is finite* (*and hence* **Mon** *is finite by* Proposition 2.8).

To prove this, we use the following simple fact.

Lemma 2.6. *Let* $G \subset \mathbf{GL}_n(\mathbb{C})$ *be a subgroup acting on a 1-dimensional subspace* $W \subset \mathbb{C}^n$*. Assume that a matrix* $g \in \mathbf{GL}_n(\mathbb{C})$ *normalizes* G *and* $gW \neq W$*. If one of the followings holds, then* G *acts on* $W \oplus gW$ *as scalar multiplications.*
(1) *g is diagonalizable, and has two eigenvalues α_1 and α_2 such that $\alpha_1 \neq \pm\alpha_2$,*
(2) *g is unipotent and $(g - E_n)^2 = 0$.*

Proof. Since G is normalized by g, we see that G acts on $g^k W$ $(k = 1, 2, \dots)$. In the cases of (1), we can write

$$W = \mathbb{C}w, \quad w = w_1 + w_2,$$

where $w_i \neq 0$ is an eigenvector of g corresponding to α_i. Then we have

$$g^2 W = \mathbb{C} \cdot (\alpha_1^2 w_1 + \alpha_2^2 w_2) \subset \mathbb{C}w_1 \oplus \mathbb{C}w_2 = W \oplus gW,$$

and $g^2 W \neq W$ since $\alpha_1^2 \neq \alpha_2^2$. Therefore a 2-dimensional space $W \oplus gW$ contains three different G-invarinat subspaces W, gW and g^2W, and this implies that G acts on $W \oplus gW$ as constants. In the case of (2), we have

$$W = \mathbb{C}w, \qquad gw = w + w', \quad gw' = w',$$

and

$$g^2 W = \mathbb{C} \cdot (w + 2w') \subset \mathbb{C}w \oplus \mathbb{C}w' = W \oplus gW.$$

By the same reason, G acts on $W \oplus gW$ as constants. □

Proof of Theorem 2.3. Let us assume that the action of $\mathbf{Ref}^0$ is not irreducible, and let W_0 be a non-trivial irreducible $\mathbf{Ref}^0$-subspace. Then there exists a reflection $r_0 = gM_0g^{-1}$ such that $r_0 W_0 \neq W_0$. Replacing W_0 by $g^{-1}W_0$, we may assume that $r_0 = M_0$. By irreducibility, we have $W_0 \cap M_0 W_0 = \mathbf{0}$ and W_0 does not contain any eigenvector of M_0. We see that $\dim W_0 = 1$ and $W_0 = \mathbb{C}w_0$ with $w_0 \notin \ker N_0$ by Proposition 2.5. Note that $M_0^d \in \mathbf{Ref}^0$ for some d since $\mathbf{Ref}/\mathbf{Ref}^0$ is finite, and hence we have $M_0^d W_0 = W_0$. Namely w_0 is an eigenvector of M_0^d, but not of M_0. If $\delta_0 = 1$, both of M_0 and M_0^d have the unique eigenspace $\ker N_0$. Therefore we have $\delta_0 \neq 1$ and $\mathbb{C}^{2^n} = \mathbb{C}\mathbf{e}_{1,\dots,1} \oplus \ker N_0$. Now we may assume that

$$w_0 = \mathbf{e}_{1,\dots,1} + \nu_0 \qquad (0 \neq \nu_0 \in \ker N_0),$$

and $M_0^d W_0 = W_0$ implies $M_0^d = \mathrm{Id}$. Therefore the special eigenvalue δ_0 is a d-th root of unity, and hence $\mathbf{Ref}^0 \subset \mathbf{SL}_{2^n}(\mathbb{C})$. Moreover, we may assume that $\gamma_2, \dots, \gamma_n \neq -1$ by Theorem 2.2.

We show that $\mathbf{Ref}^0$ acts on $\mathbb{C}\mathbf{e}_{1,\dots,1}$, dividing into three cases.

(Case 1) Assume that $\delta_0 \neq -1$. Applying Lemma 2.6 for $G = \mathbf{Ref}^0$, $W = W_0$ and $g = M_0$, we see that $\mathbf{Ref}^0$ acts on

$$W_0 \oplus M_0 W_0 = \langle \mathbf{e}_{1,\dots,1},\ \nu_0 \rangle_{\mathbb{C}}$$

as scalar multiplications. Therefore $\mathbf{Ref}^0$ acts on $\mathbb{C}\mathbf{e}_{1,\dots,1}$.

(Case 2) Next we assume that $\delta_0 = -1$, that is, $M_0 w_0 = -\mathbf{e}_{1,\dots,1} + \nu_0$. We have the following two possibilities:

(2-i) There is a $\mathbf{Ref}^0$-subspace $W_0' \not\subset \ker N_0$ different from W_0 and $M_0 W_0$.

(2-ii) Any irreducible $\mathbf{Ref}^0$-subspace different from W_0 and $M_0 W_0$ is contained in $\ker N_0$.

In the first case, W_0' is generated by $\mathbf{e}_{1,\dots,1} + \nu_0'$ with $\nu_0' \in \ker N_0$. If $\nu_0' = 0$, then $\mathbf{Ref}^0$ acts on $W_0' = \mathbb{C}\mathbf{e}_{1,\dots,1}$. If $\nu_0' \neq 0$, then the dimension of

$$W_0 + M_0 W_0 + W_0' + M_0 W_0' = \langle \mathbf{e}_{1,\dots,1},\ \nu_0,\ \nu_0' \rangle_{\mathbb{C}}$$

is at most three, and we see that $\mathbf{Ref}^0$ acts on $W_0' = \mathbb{C}\mathbf{e}_{1,\dots,1}$ by the same argument with Lemma 2.6.

Finally we consider the case (2-ii). Since $\mathbb{C}^{2^n} = \mathbb{C}\mathbf{e}_{1,\dots,1} \oplus \ker N_0$, we can write

$$M_n \nu_0 = c_n \mathbf{e}_{1,\dots,1} + \nu_n, \qquad \mathbf{e}_{1,\dots,1,0} = c_n' \mathbf{e}_{1,\dots,1} + \nu_n'$$

with $c_n, c'_n \in \mathbb{C}$ and $\nu_n,\ \nu'_n \in \ker N_0$. Then we have

$$\begin{aligned}
M_n(\pm \mathbf{e}_{1,\dots,1} + \nu_0) &= \pm \frac{1}{\gamma_n} \mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_1 \otimes (-\mathbf{e}_0 + \mathbf{e}_1) + M_n \nu_0 \\
&= \pm \frac{1}{\gamma_n} (\mathbf{e}_{1,\dots,1} - \mathbf{e}_{1,\dots,1,0}) + c_n \mathbf{e}_{1,\dots,1} + \nu_n \\
&= \pm \frac{1 \pm c_n \gamma_n}{\gamma_n} \mathbf{e}_{1,\dots,1} \mp \frac{1}{\gamma_n} (c'_n \mathbf{e}_{1,\dots,1} + \nu'_n) + \nu_n \\
&= \pm \frac{1 \pm c_n \gamma_n - c'_n}{\gamma_n} \mathbf{e}_{1,\dots,1} \mp \frac{1}{\gamma_n} \nu'_n + \nu_n.
\end{aligned}$$

Let us assume that both of $M_n W_0 = W_0$ and $M_n(M_0 W_0) = M_0 W_0$ are hold. By the above calculation, we have

$$\begin{aligned}
\begin{cases} M_n W_0 = W_0 \\ M_n(M_0 W_0) = M_0 W_0 \end{cases} &\Leftrightarrow \begin{cases} M_n(\mathbf{e}_{1,\dots,1} + \nu_0) = \text{const.} \times (\mathbf{e}_{1,\dots,1} + \nu_0) \\ M_n(-\mathbf{e}_{1,\dots,1} + \nu_0) = \text{const.} \times (-\mathbf{e}_{1,\dots,1} + \nu_0) \end{cases} \\
&\Leftrightarrow \begin{cases} \frac{1 + c_n \gamma_n - c'_n}{\gamma_n} \nu_0 = -\frac{1}{\gamma_n} \nu'_n + \nu_n \\ \frac{-1 + c_n \gamma_n + c'_n}{\gamma_n} \nu_0 = \frac{1}{\gamma_n} \nu'_n + \nu_n \end{cases} \quad \Rightarrow \quad c_n \nu_0 = \nu_n.
\end{aligned}$$

However, if $c_n \nu_0 = \nu_n$, we have

$$M_n \nu_0 = c_n \mathbf{e}_{1,\dots,1} + \nu_n = c_n(\mathbf{e}_{1,\dots,1} + \nu_0) = c_n w_0 \in W_0,$$

and hence $\nu_0 \in M_n^{-1} W_0 = W_0$. This contradicts $W_0 = \mathbb{C} w_0$. Therefore, at least one of $M_n W_0 \neq W_0$ and $M_n(M_0 W_0) \neq M_0 W_0$ must be hold. Replacing W_0 by $M_0 W_0$ if necessary, we assume that $M_n W_0 \neq W_0$. Applying Lemma 2.6 for $g = M_n$ (note that $M_n = E_2 \otimes \cdots \otimes E_2 \otimes G_n$ satisfies conditions for g in Lemma 2.6 by the assumption $\gamma_n \neq -1$), we see that $\mathbf{Ref}^0$ acts on $W_0 \oplus M_n W_0$ as constants. Therefore, there are infinitely many $\mathbf{Ref}^0$-subspaces $W \subset W_0 \oplus M_n W_0$. Let W be an irreducible $\mathbf{Ref}^0$-subspace different from 2^{n+1} subspaces

$$M^I W_0, \quad M^I(M_0 W_0), \qquad I \in \{0,1\}^n.$$

Then $(M^I)^{-1} W$ is not equal to either of W_0 and $M_0 W_0$, and hence M_0-invariant by the assumption. Therefore we have

$$R^I W = M^I M_0 (M^I)^{-1} W = M^I (M^I)^{-1} W = W, \qquad I \in \{0,1\}^n.$$

By Lemma 2.1, we have $W = \mathbb{C}\mathbf{f}_J$ for some $J \in \{0,1\}^n$ and $\mathbf{Ref}^0$ acts on $(M^J)^{-1} W = \mathbb{C}\mathbf{e}_{1,\dots,1}$.

From the above, $\mathbb{C}\mathbf{e}_{1,\dots,1}$ is an irreducible $\mathbf{Ref}^0$-subspace in any case. Now we see that each $\mathbb{C}\mathbf{f}_I$ ($I \in \{0,1\}^n$) is an irreducible $\mathbf{Ref}^0$-subspace since M^I normalizes $\mathbf{Ref}^0$. However, if $\gamma_1 = -1$, then we have

$$\begin{aligned}
M_0 \mathbf{f}_{1,0,\dots,0} &= M_0(M_1 \mathbf{e}_{1,\dots,1}) \\
&= M_0(\mathbf{e}_{0,1,\dots,1} - \mathbf{e}_{1,\dots,1}) \\
&= \mathbf{e}_{0,1,\dots,1} + (-1) \frac{(\alpha\beta + (-1)^{n-1} \prod_{k=2}^n \gamma_k)\gamma_1}{\alpha\beta} \mathbf{e}_{1,\dots,1} - \delta_0 \mathbf{e}_{1,\dots,1} \\
&= \mathbf{e}_{0,1,\dots,1} - (\gamma_1 + 2\delta_0)\mathbf{e}_{1,\dots,1} \\
&= \mathbf{f}_{1,0,\dots,0} + \mathbf{e}_{1,\dots,1} - (\gamma_1 + 2\delta_0)\mathbf{e}_{1,\dots,1} \\
&= \mathbf{f}_{1,0,\dots,0} + (1 - \gamma_1 - 2\delta_0)\mathbf{e}_{1,\dots,1} = \mathbf{f}_{1,0,\dots,0} + 2(1 - \delta_0)\mathbf{f}_{0,\dots,0}.
\end{aligned}$$

This contradicts $\delta_0 \neq 1$, and therefore we have $\gamma_1 \neq -1$. By Lemma 2.6, we see that $\mathbf{Ref}^0$ acts on $\mathbb{C}\mathbf{e}_{1,\dots,1} \oplus \mathbb{C} M_i \mathbf{e}_{1,\dots,1}$ as constants for $i = 1, \dots, n$. Applying Lemma 2.6 again, for $W = \mathbb{C} M_i \mathbf{e}_{1,\dots,1}$ and $g = M_j$, we see that $\mathbf{Ref}^0$ acts on $\mathbb{C} M_i \mathbf{e}_{1,\dots,1} \oplus \mathbb{C} M_j M_i \mathbf{e}_{1,\dots,1}$ as constants for $1 \leq i < j \leq n$. Repeating this process, we can conclude that $\mathbf{Ref}^0$ acts on $\oplus_{I \in \{0,1\}^n} \mathbb{C}\mathbf{f}_I = \mathbb{C}^{2^n}$, and $\mathbf{Ref}^0$ consists of scalar matrices. Since $\mathbf{Ref}^0 \subset \mathbf{SL}_{2^n}(\mathbb{C})$, we see that $\mathbf{Ref}^0$ is finite and so is $\mathbf{Ref}$. □

Corollary 2.3. *Assume that* $\mathbf{Ref}$ *acts on* $\mathbb{C}^{2^n}$ *irreducibly (that is, the parameters satisfy the conditions* (irr$-\alpha\beta\gamma$) *and "at most one of* $\gamma_1, \dots, \gamma_n, \alpha\beta^{-1}$ *is* -1*"). If* $\mathbf{Mon}$ *is infinite* (*for example, if* $\delta_0 = 1$), *then* $\mathbf{Mon}^0$ *is irreducible.*

3 Double coverings arising from integral representations of F_C

3.1 Double coverings

For $a = b = 1/2$ and $c_1 = \cdots = c_n = 1$, Lauricella's function $F_C(a, b, c; x)$ is a period of an algebraic variety

$$V(x) \ : \ s^2 = t_1 \cdots t_n \big(1 - \sum_{i=1}^{n} t_i\big)\big(t_1 \cdots t_n - \sum_{i=1}^{n} x_i \frac{t_1 \cdots t_n}{t_i}\big)$$

with respect to a rational n-form $\omega = \dfrac{dt_1 \wedge \cdots \wedge dt_n}{s}$. The variety $V(x)$ is a double covering of $\mathbb{P}^n$ branched along $n+2$ hyperplanes and a hypersurface of degree n. Similarly, Euler-type integrals of F_C are regarded as periods of algebraic varieties that are cyclic branched coverings of projective spaces if all parameters are rational numbers.

Note that the monodromy group is infinite and irreducible for the parameters $a = b = 1/2$ and $c_1 = \cdots = c_n = 1$. By Theorem 2.1, 2.2 and 2.3, we see that the Zariski closure of $\mathbf{Mon}$ is $\mathbf{Sp}_{2^n}(\mathbb{C})$ if n is odd, and $\mathbf{O}_{2^n}(\mathbb{C})$ if n is even. Moreover, monodromy groups are defined over rational numbers. In the following, we study varieties $V(x)$ for $n = 2$ and 3.

3.2 K3 surfaces

It is classically known (e.g. [2]) that Appell's F_4 satisfy the following formula

$$\begin{aligned} & F_4(a, c + c' - a - 1, c, c'; x(1-y), y(1-x)) \\ &= {}_2F_1(a, c + c' - a - 1, c; x)\,{}_2F_1(a, c + c' - a - 1, c'; y), \end{aligned}$$

and we see that $F_4(1/2, 1/2, 1, 1; x(1-y), y(1-x))$ is a product of elliptic integrals. However it seems that a geometric proof of the formula is not known. In any way, we can show the following.

Proposition 3.1. *For a general parameters (x_1, x_2), the minimal smooth model of*

$$V(x_1, x_2) \ : \ s^2 = t_1 t_2 (1 - t_1 - t_2)(t_1 t_2 - x_1 t_2 - x_2 t_1)$$

is a product Kummer surface with transcendental lattice $\mathrm{U}(2) \oplus \mathrm{U}(2)$ *where* $\mathrm{U}(2) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$.

Proof. Let $V(x_1, x_2)$ be a double covering of $\mathbb{P}^2$ branched along four lines and a conic:

$$L_i = \{T_i = 0\} \quad (i = 0, 1, 2), \qquad L_3 = \{T_0 - T_1 - T_2 = 0\}, \qquad Q = \{T_1 T_2 - x_1 T_0 T_2 - x_2 T_0 T_1 = 0\}.$$

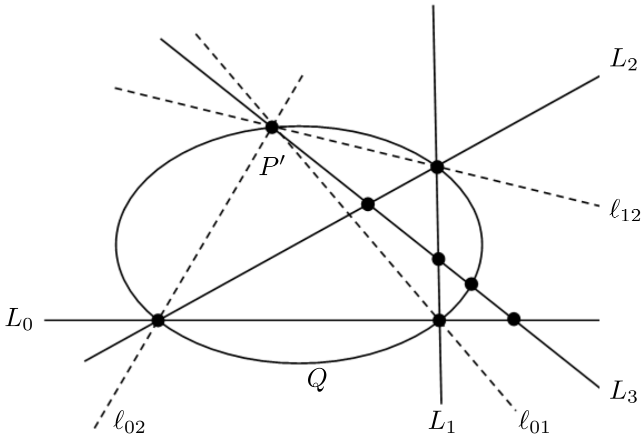
For a general parameters (x_1, x_2), it has A_1-singularities at points over

$$P_{i3} = L_i \cap L_3 \quad (i = 0, 1, 2), \qquad \{P', \ P''\} = Q \cap L_3$$

and D_4-singularities at points over

$$P_{01} = L_0 \cap L_1 \cap Q = [0:0:1], \quad P_{02} = L_0 \cap L_2 \cap Q = [0:1:0], \quad P_{12} = L_1 \cap L_2 \cap Q = [1:0:0].$$

Hence it is a double sextic with rational singularities, and the minimal resolution $S = S(x_1, x_2)$ is a K3 surface. Let us consider a pencil of lines passing through P'. For such a line ℓ, let $P_Q(\ell)$ be another intersection point with Q. The (strict) pull back $\pi^{-1}\ell$ by the projection $\pi : S \to \mathbb{P}^2$ is a double covering of ℓ branched over four points $\ell \cap L_i$ $(i = 0, 1, 2)$ and $P_Q(\ell)$. Therefore they form an elliptic fibration with 2-torsion sections $\pi^{-1}L_i$ $(i = 0, 1, 2)$ and $\pi^{-1}Q$.
Let ℓ_{ij} $(i, j = 0, 1, 2)$ be the line passing through P_{ij} and P'. These three lines and L_3 gives four I_0^*-fibers, and we obtain disjoint sixteen smooth rational curves from their components. Hence S is a Kummer surface. Let $\mathrm{NS}(S)$ be the Néron-Severi group of S, and $\rho(S) = \operatorname{rank} \mathrm{NS}(S)$ be the Picard number. Since the family has 2-dimensional moduli, we have $\rho(S) \leq 18$ for a general member S. By the Shioda-Tate formula ([25], Corollary 6.13), we see that $\rho(S) = 18$ and the Mordell-Weil rank is zero.



Moreover, the Mordell-Weil group is precisely $(\mathbb{Z}/2\mathbb{Z})^2$, since the specialization of torsion sections to a singular fiber is injective and we have only I_0^*-fibers. By the formula (22) in [25], we have

$$\text{disc NS}(S) = -(\text{disc } D_4)^4/|(\mathbb{Z}/2\mathbb{Z})^2|^2 = -4^4/4^2 = -16,$$

where D_4 is the Dynkin lattice of type D_4. Therefore the discriminant of the transcendental lattice T_S is 16. On the other hand, T_S must be of the form $\mathrm{U}(2) \oplus T'(2)$ where T' is a even lattice ([21], Corollary 4.4). Hence we have $T' = \mathrm{U}$ and S is a product Kummer surface. □

Changing our basis by the following matrix P, we have new intersection matrix $H' = {}^tPHP$ and monodromy representations $M_k' = P^{-1}M_kP$ $(k = 0, 1, 2)$:

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \qquad H' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$M_0' = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad M_1' = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_2' = \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now let us consider the Segre embedding

$$\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3, \qquad [s_0 : s_1] \times [t_0 : t_1] \mapsto [s_0t_0 : s_0t_1 : s_1t_0 : s_1t_1].$$

The image satisfies a quadratic relation

$$(s_0t_0, s_0t_1, s_1t_0, s_1t_1)H'{}^t(s_0t_0, s_0t_1, s_1t_0, s_1t_1) = 0$$

and M_k' acts on $\mathbb{P}^1 \times \mathbb{P}^1$ by

$$M_0' \cdot (s, t) = (-t^{-1}, -s^{-1}), \quad M_1' \cdot (s, t) = (s, t + 2), \quad M_2' \cdot (s, t) = (s + 2, t)$$

where $s = s_1/s_0$ and $t = t_1/t_0$. Since the congruence subgroup $\Gamma(2) \subset \mathbf{SL}_2(\mathbb{Z})$ is generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ projectively, we see that a subgroup of index 2, generated by

$$M_1', \ M_2', \ M_0'M_1M_0', \ M_0'M_2M_0',$$

is isomorphic to $\Gamma(2) \times \Gamma(2)$ as projective transformations.

Remark 3.1. (1) *The projective monodromy of* ${}_2F_1(1/2, 1/2, 1)$ *is* $\Gamma(2)/\{\pm 1\}$.
(2) *The product* $\mathbf{SL}_2(\mathbb{C}) \times \mathbf{SL}_2(\mathbb{C})$ *is a double cover of* $\mathbf{SO}_4(\mathbb{C})$.

3.3 Calabi-Yau varieties

Proposition 3.2. *For a general parameter $x = (x_1, x_2, x_3)$, we have a resolution $\widetilde{V} = \widetilde{V(x)}$ of*

$$V(x) \ : \ s^2 = t_1t_2t_3(1 - t_1 - t_2 - t_3)(t_1t_2t_3 - x_1t_2t_3 - x_2t_1t_3 - x_3t_1t_2)$$

which is a Calabi Yau 3-fold with Hodge numbers $h^{1,1}(\widetilde{V}) = 68$, $h^{2,1}(\widetilde{V}) = 4$ and the Euler characteristic $e(\widetilde{V}) = 128$.

Proof. The variety $V = V(x)$ is a double covering of $\mathbb{P}^3$ branched along the following five planes H_i and a nodal cubic surface S:

$$H_i : T_i = 0 \quad (i = 0, 1, 2, 3), \qquad H_4 : (T_0 - T_1 - T_2 - T_3) = 0,$$
$$S : T_1T_2T_3 - T_0(x_1T_2T_3 + x_2T_1T_3 + x_3T_1T_2) = 0.$$

The cubic surface S is known as the Cayley cubic, and it has four nodes

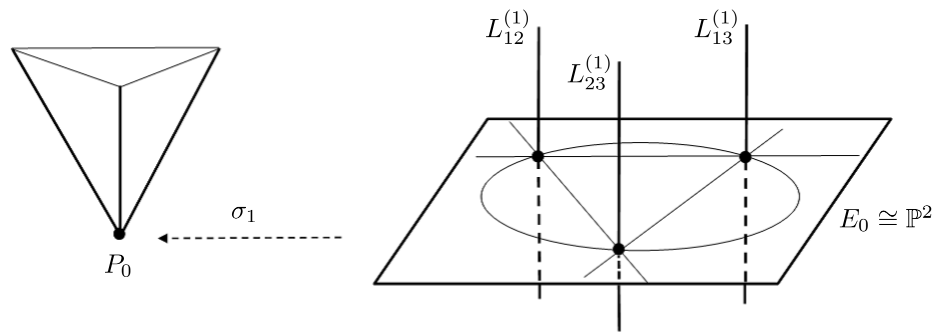
$$P_0 = [1 : 0 : 0 : 0], \quad P_1 = [0 : 1 : 0 : 0], \quad P_2 = [0 : 0 : 1 : 0], \quad P_3 = [0 : 0 : 0 : 1].$$

The branch divisor $B = H_0 + \cdots + H_4 + S$ has singularities as given in the table below.

5-fold points	$P_i = H_j \cap H_k \cap H_l \cap S, \quad (\{i, j, k, l\} = \{0, 1, 2, 3\})$
4-fold points	$H_i \cap H_j \cap H_4 \cap S, \quad (i, j = 0, 1, 2, 3)$
triple lines	$L_{ij} = H_i \cap H_j \cap S \quad (i, j = 0, 1, 2, 3)$
double curve	$S \cap H_4$ (smooth cubic curve), $\quad L_{i4} = H_i \cap H_4 \quad (i = 0, 1, 2, 3)$

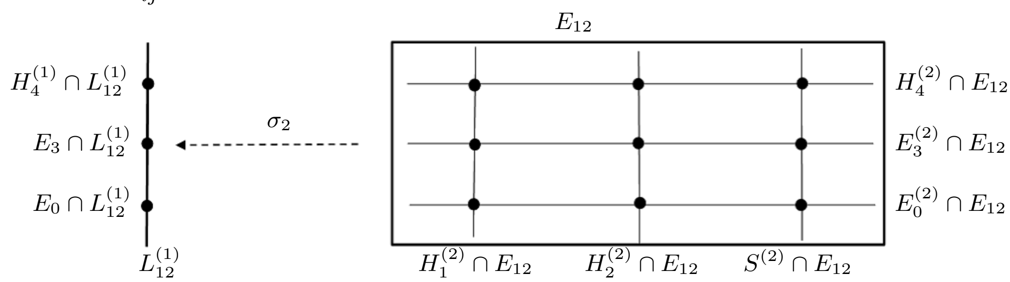
We resolve them in three steps by admissible blow-ups in [5], according to Cynk and Szemberg. (However B is not an arrangement in the sense of [5], and we can not apply formulas in [5] for the pair $(\mathbb{P}^3, B)$ directly).

(step 1) Let $\sigma_1 : U_1 \to \mathbb{P}^3$ be the blow-up at 5-fold points P_0, P_1, P_2 and P_3, and let E_i be the exceptional divisor corresponding to P_i. We denote the strict transform of a subvariety $D \subset \mathbb{P}^3$ by $D^{(1)}$, and take $B_1 = \sum H_k^{(1)} + S^{(1)} + \sum E_k$. On $E_0 \cong \mathbb{P}^2$, three lines $H_k^{(1)} \cap E_0$ $(k = 1, 2, 3)$ form a triangle with a circumscribed conic $S^{(1)} \cap E_0$. The same is true for other E_i.



Now triple lines $L_{ij}^{(1)}$ are disjoint, and there are new twelve 4-fold points as intersections of $L_{ij}^{(1)}$ and E_k.

(step 2) Let $\sigma_2 : U_2 \to U_1$ be the blow-up along six triple lines $L_{ij}^{(1)}$. Let E_{ij} be the exceptional divisor corresponding to $L_{ij}^{(1)}$, that are $\mathbb{P}^1$-bundles over $\mathbb{P}^1$.



At this point, the branch divisor

$$B_2 = \sum H_k^{(2)} + S^{(2)} + \sum E_k^{(2)} + \sum E_{ij}$$

is normal crossing, and $H_k^{(2)}$ $(k = 0,1,2,3)$ and $S^{(2)}$ are disjoint.
(step 3) Let $\sigma_3 : U_3 \to U_2$ be a blow-up along double curves of B_2. This is not unique and depends on the order of blow-ups of double curves. However it does not affect on Euler characteristics of resultant varieties. We blow up in the following order:
(i) blow up along $E_{ij} \cap H_i^{(2)}$ $(i \neq 4)$ and $E_{ij} \cap S^{(2)}$,
(ii) blow up along $E_{ij} \cap H_4^{(2)}$ and $E_{ij} \cap E_k^{(2)}$,
(iii) blow up along $H_i^{(2)} \cap H_4^{(2)}$, $H_i^{(2)} \cap E_k^{(2)}$ $(i \neq 4, k)$, $S^{(2)} \cap H_4^{(2)}$ and $S^{(2)} \cap E_k^{(2)}$,
and put

$$B_3 = \sum H_k^{(3)} + S^{(3)} + \sum E_k^{(3)} + \sum E_{ij}^{(3)}.$$

The Euler characteristics of U_i and components of B_i are changed as in the following table.

	U_i	H_i $(i \neq 4)$	H_4	S	E_i	E_{ij}
$U_0 = \mathbb{P}^3$	4	3	3	5	$-$	$-$
U_1	12	6	3	9	3	$-$
U_2	24	6	9	9	6	4
$U_3 -$ (i)	60	6	27	9	15	4
$U_3 -$ (ii)	96	6	27	9	15	4
$U_3 -$ (iii)	136	6	27	9	15	4

Let $\pi : \widetilde{V} \to U_3$ be the double covering branched along B_3. Then $\widetilde{V}$ is a Calabi-Yau variety with the Euler characteristic $e(\widetilde{V}) = 128$, and hence $h^{1,1}(\widetilde{V}) - h^{1,2}(\widetilde{V}) = 64$.

Next we compute Hodge numbers. By Proposition 2.1 in [4], we have

$$\mathrm{H}^1(\widetilde{V}, \Theta_{\widetilde{V}}) \cong \mathrm{H}^1(U_3, \Theta_{U_3}(\log B_3)) \oplus \mathrm{H}^1(U_3, \Theta_{U_3} \otimes \mathcal{L}^{-1})$$

where Θ_X is the tangent bundle, $\Theta_X(\log D)$ is the sheaf of logarithmic vector field ([4], [7]) and $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_{U_3}(B_3)$. Moreover $\mathrm{H}^1(U_3, \Theta_{U_3}(\log B_3))$ is isomorphic to the space of equisingular deformations of B in $\mathbb{P}^3$, and $h^1(\Theta_{U_3} \otimes \mathcal{L}^{-1})$ is the sum of genera of all blown-up curves (see [4]). Since blown-up curves are rational except an elliptic curve $S \cap H_4$, we have $h^1(\Theta_{U_3} \otimes \mathcal{L}^{-1}) = 1$. Let us show $h^1(\Theta_{U_3}(\log B_3)) = 3$. (Then we have $h^1(\Theta_{\widetilde{V}}) = 4$, and we can conclude that $h^{1,2}(\widetilde{V}) = 4$ by the Serre duality since $K_{\widetilde{V}} \cong \mathcal{O}_{\widetilde{V}}$.) To show this, let W be an octic surface which has similar singularities with $V(x)$. By a projective transformation, we may assume that 5-folds points of W are $P_0, \ldots, P_3$. Consequently triple lines of W must be L_{ij}. Since W has multiplicity 5 at P_i, the polynomial $F(T_0, \ldots, T_3)$ defining W is a linear combination of

$$T_i^3T_j^3T_k^2, \quad T_i^3T_j^3T_kT_l, \quad T_i^3T_j^2T_k^2T_l, \quad T_0^2T_1^2T_2^2T_3^2, \qquad \{i,j,k,l\} = \{0,1,2,3\}.$$

Moreover F belongs to ideals $(T_i^3, T_i^2T_j, T_iT_j^2, T_j^3)$ since F vanishes on L_{ij} with third order. Therefore F does not have terms $T_i^3T_j^3T_k^2$, $T_i^3T_j^3T_kT_l$, and we have $F = T_0T_1T_2T_3G$ where G is a linear combination of

$$T_i^2T_jT_k, \quad T_0T_1T_2T_3, \qquad \{i,j,k,l\} = \{0,1,2,3\}.$$

Then the singular locus of a quartic $G = 0$ must contain an elliptic curve C of degree 3 (a deformation of $S \cap H_4$). Note that C is on a certain plane H. Since 4-fold points of W are on C, double lines connecting 4-fold points (a deformation of L_{i4}) must be on H. This implies that $G = 0$ decomposes into H and a cubic surface. We see that W coming from our branch divisors $B \subset \mathbb{P}^3$, and we have $h^1(\Theta_{U_3}(\log B_3)) = 3$. □

Changing our basis by the following matrix P as in the case $n = 2$, we have $H' = {}^tPHP$ and

$M_k' = P^{-1} M_k P$ $(k = 0, 1, 2, 3)$:

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad H' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_0' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_1' = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_2' = \begin{pmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_3' = \begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

These give an integral representation of **Mon**, but we do not know arithmeticy of **Mon**, that is, the finiteness of the index $|\mathbf{Sp}(H', \mathbb{Z}) : \mathbf{Mon}|$.

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