Algebraic spectral curves over \mathbb{Q} and their tau-functions

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Abstract

Let W(z) be a $n \times n$ matrix polynomial with rational coefficients. Denote C the spectral curve det $(w \cdot \mathbf{1} - W(z)) = 0$. Under some natural assumptions about the structure of W(z) we prove that certain combinations of logarithmic derivatives of the Riemann theta-function of C of an arbitrary order starting from the third one all take rational values at the point of the Jacobi variety J(C) specified by the line bundle of eigenvectors of W(z).

1 Introduction

Let

$$W(z) = B^{0} z^{m} + B^{1} z^{m-1} + \dots + B^{m}, \quad B^{i} \in Mat_{n}(\mathbb{C}), \quad B^{0} = \operatorname{diag}(b_{1}^{0}, \dots, b_{n}^{0})$$
(1.1)

be a $n \times n$ matrix polynomial. Consider an algebraic curve C defined by the characteristic equation

$$R(z, w) := \det (w \cdot \mathbf{1} - W(z)) = 0.$$
(1.2)

It will be called *algebraic spectral curve* associated with the matrix-valued polynomial W(z). Consider the generic situation when the leading coefficient B^0 has pairwise distinct eigenvalues. In that case the Riemann surface $C \xrightarrow{z} \mathbf{P}^1$ has *n* distinct infinite points $P_1 \cup \cdots \cup P_n = z^{-1}(\infty)$ labelled by the eigenvalues. It will also be assumed that the affine part of the curve (1.2) is smooth irreducible. In sequel only such spectral curves will be considered. The genus *g* of such a spectral curve is uniquely determined by the numbers *m* and *n*, see eq. (2.4) below.

Assuming g > 0 choose a canonical basis of cycles $a_1, \ldots, a_g, b_1, \ldots, b_g \in H_1(C, \mathbb{Z})$,

$$a_i \circ a_j = b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij}, \quad i, j = 1, \dots, g.$$

$$(1.3)$$

Let $\omega_1, \ldots, \omega_g$ be the basis of holomorphic differentials on C normalized by the conditions

$$\oint_{a_j} \omega_k = 2\pi \sqrt{-1} \,\delta_{jk}, \quad j, \, k = 1, \dots, g. \tag{1.4}$$

Denote

$$B_{jk} = \oint_{b_j} \omega_k, \quad j, \, k = 1, \dots, g \tag{1.5}$$

and let

$$\theta(\mathbf{u}) = \sum_{\mathbf{n}\in\mathbb{Z}^g} \exp\left\{\frac{1}{2}\langle\mathbf{n},B\mathbf{n}\rangle + \langle\mathbf{n},\mathbf{u}\rangle\right\}$$
(1.6)

be the Riemann theta-function of the curve C associated with the chosen basis of cycles. Here $\mathbf{u} = (u_1, \ldots, u_q)$ is the vector of independent complex variables u_1, \ldots, u_q , $\mathbf{n} = (n_1, \ldots, n_q) \in \mathbb{Z}^g$,

$$\langle \mathbf{n}, B\mathbf{n} \rangle = \sum_{j,k=1}^{g} B_{jk} n_j n_k, \quad \langle \mathbf{n}, \mathbf{u} \rangle = \sum_{k=1}^{g} n_k u_k$$

Also recall that the Jacobi variety, or simply Jacobian of the curve C is defined as the quotient

$$J(C) = \mathbb{C}^g / \{2\pi\sqrt{-1}\,\mathbf{m} + B\,\mathbf{n} \,|\,\mathbf{m},\,\mathbf{n} \in \mathbb{Z}^g\}.$$

There is a natural line bundle of degree g+n-1 on the spectral curve (1.2) given by the eigenvectors of the matrix W(z). Let D_0 be the divisor of poles of a section of this line bundle and denote

$$\mathbf{u}_0 = D_0 - D_\infty - \Delta \in J(C) \tag{1.7}$$

the point of the Jacobian corresponding to the line bundle. Here D_{∞} is the divisor of poles of the function $z: C \to \mathbf{P}^1$, Δ is the Riemann divisor (see [17] for the definition). Here and below we identify divisors of degree 0 with their images in the Jacobian by means of the Abel–Jacobi map $A: C \to J(C)$,

$$A(P-Q) = \left(\int_Q^P \omega_1, \dots, \int_Q^P \omega_g\right)$$

Our interest is in the N-differentials

$$\sum_{k_1,\dots,k_N=1}^g \frac{\partial^N \log \theta(\mathbf{u}_0)}{\partial u_{k_1} \dots \partial u_{k_N}} \,\omega_{k_1}(Q_1) \dots \omega_{k_N}(Q_N), \quad Q_1,\dots,Q_N \in C$$
(1.8)

for any $N \geq 3$. For N = 2 instead of (1.8) we will be looking at the following bi-differential

$$\frac{\theta \left(P - Q - \mathbf{u}_0\right) \theta \left(P - Q + \mathbf{u}_0\right)}{\theta^2(\mathbf{u}_0) E(P, Q)^2}, \quad P, Q \in C$$
(1.9)

that differs from the second logarithmic derivative of the form (1.8) by the fundamental normalized bi-differential (see Corollary 2.12 in the J.Fay's book [17]). Here E(P,Q) is the prime-form [*ibid.*]

$$E(P,Q) = \frac{\theta[\nu](P-Q)}{\sqrt{\sum \omega_i(P)\partial_{u_i}\theta[\nu](0)}}\sqrt{\sum \omega_j(Q)\partial_{u_j}\theta[\nu](0)}$$

where ν is a non-degenerate odd half-period. The goal is to express these multi-differentials on a spectral curve in terms of the associated matrix polynomial W(z). The expression will involve the following matrix-valued function on the spectral curve C

$$\Phi(P) \equiv \Phi(z, w) = \frac{R(z, W(z)) - R(z, w)}{W(z) - w}, \quad P = (z, w) \in C.$$
(1.10)

In the right hand side it is understood that, first one has to cancel the factor W - w common for the numerator and denominator in the ratio $\frac{R(z,W)-R(z,w)}{W-w}$ and then to replace W with W(z).

Main Theorem. For the spectral curve (1.2) and for arbitrary points $Q_1 = (z_1, w_1), \ldots, Q_N = (z_N, w_N) \in C$ the following expressions hold true

$$\frac{\theta \left(Q_1 - Q_2 - \mathbf{u}_0\right)\theta \left(Q_1 - Q_2 + \mathbf{u}_0\right)}{\theta^2(\mathbf{u}_0)E(Q_1, Q_2)^2} = \operatorname{tr}\frac{\Phi(Q_1)\Phi(Q_2)}{(z_1 - z_2)^2}\frac{dz_1}{R_w(z_1, w_1)}\frac{dz_2}{R_w(z_2, w_2)}$$
(1.11)

$$\sum_{k_1,\dots,k_N=1}^{g} \frac{\partial^N \log \theta(\mathbf{u}_0)}{\partial u_{k_1} \dots \partial u_{k_N}} \omega_{k_1}(Q_1) \dots \omega_{k_N}(Q_N) =$$

$$= \frac{(-1)^{N+1}}{N} \sum_{s \in S_N} \frac{\operatorname{tr} \left[\Phi\left(Q_{s_1}\right) \dots \Phi\left(Q_{s_N}\right)\right]}{(z_{s_1} - z_{s_2}) \dots (z_{s_{N-1}} - z_{s_N}) (z_{s_N} - z_{s_1})} \frac{dz_1}{R_w(z_1, w_1)} \dots \frac{dz_N}{R_w(z_N, w_N)}$$
(1.12)

for any $N \geq 3$.

In these equations $R_w(z, w)$ is the partial derivative of the characteristic polynomial R(z, w) with respect to w, the summation in (1.12) is taken over all permutations of $\{1, 2, \ldots, N\}$.

For given m, n denote $\mathcal{W}_{m,n}$ the space of $n \times n$ matrix polynomials of the form (1.1). Functions on $\mathcal{W}_{m,n}$ will be denoted by f([W]). By $\mathbb{Z}[\mathcal{W}_{m,n}]$ we denote the ring

$$\mathbb{Z}[\mathcal{W}_{m,n}] = \mathbb{Z}\left[b_1^0, \dots, b_n^0, B_{ij}^1, \dots, B_{ij}^m, \prod_{i < j} (b_i^0 - b_j^0)^{-1}\right].$$

Let $\mathcal{C}_{m,n}$ be the space of algebraic curves of the form (1.2), (2.2). There is a natural fibration

$$\begin{array}{c}
\mathcal{W}_{m,n} \\
\downarrow \\
\mathcal{C}_{m,n}
\end{array} \tag{1.13}$$

assigning to a matrix polynomial W(z) its spectral curve C. The fiber over the point C is isomorphic to the affine part of the generalized Jacobian (see below) of the singularized curve C_{sing} obtained from C by identifying its infinite points, cf. [15], [22], [24], [29], [18]. The quotient of the fiber over the diagonal conjugations

$$W(z) \mapsto D^{-1}W(z) D, \quad D = \operatorname{diag}(d_1, \dots, d_n)$$

is naturally isomorphic to $J(C) \setminus (\theta)$.

Define a function

$$F([W]; \mathbf{t}) \in \mathbb{Z}[\mathcal{W}_{m,n}] \otimes \mathbb{Z}[[\mathbf{t}]], \quad \mathbf{t} = (t_k^a), \quad a = 1, \dots, n, \quad k \ge 0$$
(1.14)

by the infinite sum

$$F([W]; \mathbf{t}) = \sum_{N=2}^{\infty} \frac{1}{N!} \sum_{a_1, \dots, a_N=1}^{n} \sum_{k_1, \dots, k_N=0}^{\infty} F_{k_1 \dots k_N}^{a_1 \dots a_N} [W] t_{k_1}^{a_1} \dots t_{k_N}^{a_N}$$
(1.15)

where the coefficients $F_{k_1...k_N}^{a_1...a_N}[W] \in \mathbb{Q}[\mathcal{W}_{m,n}]$ are defined by the following generated series

$$\sum_{k_1,k_2=0}^{\infty} \frac{F_{k_1k_2}^{a_1a_2}[W]}{z_1^{k_1} z_2^{k_2}} = \frac{\operatorname{tr}\left[\Pi_{a_1}(z_1)\Pi_{a_2}(z_2)\right]}{(z_1 - z_2)^2} - \frac{\delta_{a_1,a_2}}{(z_1 - z_2)^2}$$
(1.16)

for any $a_1, a_2 = 1, ..., n$ and

$$\sum_{k_1,\dots,k_N=0}^{\infty} \frac{F_{k_1\dots k_N}^{a_1\dots a_N}[W]}{z_1^{k_1+2}\dots z_N^{k_N+2}} = -\frac{1}{N} \sum_{s \in S_N} \frac{\operatorname{tr}\left[\Pi_{s_1}(z_{s_1})\dots\Pi_{s_N}(z_{s_N})\right]}{(z_{s_1}-z_{s_2})\dots(z_{s_{N-1}}-z_{s_N})(z_{s_N}-z_{s_1})}$$
(1.17)

for any $N \geq 3$ and any $a_1, \ldots, a_N = 1, \ldots, n$. Here the matrix-valued series

$$\Pi_a(z) = \Pi_a([W]; z) \in Mat_n\left(\mathbb{Z}[\mathcal{W}_{m,n}]\right) \otimes \mathbb{Z}[[z^{-1}]]$$

are defined by the expansions

$$\frac{\Phi(P)}{R_w(z,w)} = \Pi_a(z), \quad P = (z,w) \to P_a, \quad a = 1,\dots,n.$$
(1.18)

Observe that the function F([W]; t) is invariant with respect to diagonal conjugations.

Main Lemma. Let $W(z) \in W_{m,n}$ be any matrix-valued polynomial such that its spectral curve (1.2) has n distinct points at infinity and it is nonsingular. Denote $\theta(\mathbf{u})$ the theta-function of the curve with respect to some basis of cycles and \mathbf{u}_0 the point (1.7) of the Jacobian specified by the line bundle of eigenvectors of W(z). Then the following equality of formal series in \mathbf{t} takes place

$$e^{F([W];\mathbf{t})} = e^{\alpha + \sum \beta_{a,i} t_i^a + \frac{1}{2} \sum \gamma_{a,i;b,j} t_i^a t_j^b} \theta \left(\sum t_k^a \mathbf{V}^{(a,k)} - \mathbf{u}_0 \right)$$
(1.19)

for some coefficients α , $\beta_{a,i}$, $\gamma_{a,i;b,j} \in \mathbb{C}$. Here the vectors $\mathbf{V}^{(a,k)} = \left(V_1^{(a,k)}, \ldots, V_g^{(a,k)}\right)$ come from the coefficients of the expansion

$$\omega_i(P) = \sum_{k=0}^{\infty} \frac{V_i^{(a,k)}}{z^{k+2}} dz, \quad z = z(P), \quad P \to P_a$$
(1.20)

of the holomorphic differentials $\omega_i(P)$ near infinity, $i = 1, \ldots, g, a = 1, \ldots, n$.

Corollary 1.1 For any matrix-valued polynomial $W(z) \in W_{m,n}$ satisfying the conditions of Main Lemma the series (1.15) has a non-zero radius of convergence when restricted onto a finite number of the indeterminates \mathbf{t} .

Definition. We say that det $(w \cdot 1 - W(z)) = 0$ is an algebraic spectral curve over \mathbb{Q} if $W(z) \in Mat_n(\mathbb{Q}) \otimes \mathbb{Q}[z]$.

Corollary 1.2 For an algebraic spectral curve over \mathbb{Q} all coefficients of expansions of the multidifferentials (1.8), (1.9) near infinity are rational numbers.

One can also derive from the Main Theorem some identities between sums of products of Riemann theta-function and its logarithmic derivatives.

Corollary 1.3 Let C be an arbitrary compact Riemann surface of genus g > 0. For any $N \ge 3$ the following identity holds true for its theta-function along with the normalised holomorphic differentials on C

$$\theta^{N}(\mathbf{u}) \sum_{i_{1},\dots,i_{N}=1}^{g} \frac{\partial^{N} \log \theta(\mathbf{u})}{\partial u_{i_{1}} \dots \partial u_{i_{N}}} \omega_{i_{1}}(Q_{1}) \dots \omega_{i_{N}}(Q_{N}) =$$

$$= -\frac{1}{N} \sum_{s \in S_{N}} \frac{\theta(Q_{s_{1}} - Q_{s_{2}} + \mathbf{u})\theta(Q_{s_{2}} - Q_{s_{3}} + \mathbf{u}) \dots \theta(Q_{s_{N-1}} - Q_{s_{N}} + \mathbf{u})\theta(Q_{s_{N}} - Q_{s_{1}} + \mathbf{u})}{E(Q_{s_{1}}, Q_{s_{2}})E(Q_{s_{2}}, Q_{s_{3}}) \dots E(Q_{s_{N-1}}, Q_{s_{N}})E(Q_{s_{N}}, Q_{s_{1}})}$$
(1.21)

for arbitrary points $Q_1, Q_2, \ldots, Q_N \in C$ and arbitrary $\mathbf{u} \in J(C) \setminus (\theta)$.

See below eqs. (2.85), (2.86) for the explicit spelling of the above identity for the cases N = 3 and N = 4. For N = 4 and $\mathbf{u} = 0$ the identity (1.21) appeared in [17] (see Proposition 2.14 there). We did not find in the literature other identities of the form (1.21).

The constructions of the present paper can be extended to algebraic spectral curves with an arbitrary ramification profile at infinity. This will be done¹ a subsequent publication.

Before we proceed to the precise constructions and to the proofs let us say few words about the main ideas behind them. First, it is the connection between spectral curves and their theta-functions with particular classes of solutions to integrable systems, see e.g. [21, 12]. Second, it is the remarkable idea that goes back to M.Sato *et al.* that suggests to consider tau-functions of integrable systems as partition functions of quantum field theories, see e.g. [6, 28]. The time variables of the integrable hierarchy play the role of coupling constants. So, the logarithmic derivatives of tau-functions can be considered as the connected correlators of the underlined quantum field theory. For the algebrogeometric solutions the tau-function essentially coincides with the theta-function of the spectral curve, up to multiplication by exponential of a quadratic form. There were quite a few interesting results in the theory of theta-functions inspired by this connection, see e.g. [20, 23]. The novelty of approach of the present work is that we are looking more on the correlators than on the tau-function. The main tools in proving the statements formulated above is in using the algorithm of [2, 3, 4] developed for efficient computation of correlators in cohomological field theories. This algorithm applied to tau-functions of algebraic spectral curves readily produces the explicit expressions for the correlators given above.

2 Main constructions and proofs

2.1 Matrix polynomials \leftrightarrow spectral curves + divisors

For a given $n \ge 2$, $m \ge 1$ for n > 2 or $m \ge 2$ for n = 2 consider the space \mathcal{W} of matrix polynomials of the form

$$\mathcal{W} = \{W(z) = z^m B + \text{lower degree terms}\}$$
(2.1)

where $B = \text{diag}(b_1, \ldots, b_n)$ is an arbitrary diagonal matrix. For any $W(z) \in \mathcal{W}$ the corresponding spectral curve C is of the form

$$\det (w \cdot \mathbf{1} - W(z)) = w^n + a_1(z)w^{n-1} + \dots + a_n(z) = 0, \quad \deg a_i(z) = m \, i, \quad i = 1, \dots, n.$$
(2.2)

¹For hyperelliptic curves with a branch point at infinity this has already been done in [14].

Assume the entries b_1, \ldots, b_n of the matrix B to be pairwise distinct. Then the Riemann surface (2.2) has n distinct points P_1, \ldots, P_n at infinity,

$$P_a = \{z \to \infty, w \to \infty, \frac{w}{z^m} \to b_a\}, \quad a = 1, \dots, n.$$

For the algebraic curve (2.2) this condition translates as follows. Let $a_i(z) = \alpha_i z^{m\,i} + \dots, i = 1, \dots, n$. Then the roots of the equation

$$b^{n} + \alpha_{1}b^{n-1} + \dots + \alpha_{n} = 0$$
(2.3)

must be pairwise distinct.

Assuming smoothness of the finite part of the curve we compute its genus

$$g = \frac{(n-1)(mn-2)}{2}.$$
(2.4)

We have a natural line bundle \mathcal{L} over C of the eigenvectors

$$W(z)\psi(P) = w\,\psi(P), \quad P = (z,w) \in C, \quad \psi(P) = (\psi^1(P),\dots,\psi^n(P))^T$$
 (2.5)

(the symbol $(.)^T$ stands for the transposition) of the matrix W(z). We will associate with this line bundle a point in the generalized Jacobian $J(C; P_1, ..., P_n)$ that can be considered as an analogue of the Jacobi variety for the singular curve obtained by gluing together all infinite points $P_1, ..., P_n$, see, e.g., [18]. It can be represented by classes of relative linear equivalence of divisors of degree zero on the curve. By definition two divisors D_1 and D_2 of the same degree belong to the same relative linear equivalence class if there exists a rational function f on the curve C with $(f) = D_1 - D_2$ satisfying $f(P_1) = f(P_2) = \cdots = f(P_n)$. There is a natural fibration

$$J(C; P_1, \dots, P_n) \to J(P) \tag{2.6}$$

associating with any divisor its class of linear equivalence.

With the line bundle \mathcal{L} we associate a divisor D_0 on the spectral curve defined by

$$D_0 = \{ P \in C \,|\, \psi^1(P) + \dots + \psi^n(P) = 0 \}$$
(2.7)

for a nonzero eigenvector. It can be considered as the divisor of poles of the section normalized by the condition

$$\psi^1 + \dots + \psi^n = 1. \tag{2.8}$$

The components of the eigenvector can be represented as

$$\psi^{j}(P) = \frac{\Delta_{ij}(z,w)}{\sum_{s=1}^{n} \Delta_{is}(z,w)}, \quad P = (z,w) \in C, \quad j = 1,\dots,n$$
(2.9)

for any *i*. Here $\Delta_{ij}(z, w)$ is the (i, j)-cofactor of the matrix $w \cdot \mathbf{1} - W(z)$. So at infinity the normalised (2.8) eigenvector behaves as

$$\psi^{i}(P) = \delta_{ia} + \mathcal{O}\left(\frac{1}{z(P)}\right), \quad P \to P_{a}.$$
 (2.10)

Lemma 2.1 The divisor D_0 of poles of the eigenvector normalised by eq. (2.8) is a nonspecial divisor on $C \setminus (P_1 \cup \cdots \cup P_n)$ of degree g + n - 1. Conversely, for any nonsingular curve C of the form

$$w^n + a_1(z)w^{n-1} + \dots + a_n(z) = 0, \quad \deg a_i(z) = m \, i, \quad i = 1, \dots, n$$
 (2.11)

with n distinct ordered points P_1, \ldots, P_n at infinity and an arbitrary nonspecial divisor $D_0 \subset C \setminus (P_1 \cup \cdots \cup P_n)$ of degree g + n - 1 there exists a unique matrix polynomial W(z) of the form (2.1) with the spectral curve coinciding with C and the divisor of poles of the normalized eigenvector coinciding with D_0 .

Proof: To prove the first part of Lemma we will use an algorithm developed in [9] for computing the poles of eigenvectors of matrix polynomials adjusting it to the present situation. Denote $e^* = (1, 1, ..., 1) \in \mathbb{C}^{n*}$ and define a polynomial

$$D(z) = e^* \wedge e^* W(z) \wedge \dots \wedge e^* W^{n-1}(z) \in \Lambda^n \mathbb{C}^{n*} \otimes \mathbb{C}[z].$$
(2.12)

We also define polynomials $q_{ij}(z)$, i, j = 1, ..., n as coefficients of the expansion of sums of cofactors $\Delta_{ij}(z, w)$ of the matrix $w \cdot \mathbf{1} - W(z)$

$$\sum_{s=1}^{n} \Delta_{is}(z,w) = w^{n-1} + q_{i2}(z)w^{n-2} + \dots + q_{i,n-1}(z)w + q_{i,n}(z), \quad i = 1,\dots,n.$$
(2.13)

 $(q_{i1}(z) = 1 \text{ for any } i = 1, \dots, n).$

Proposition 2.2 1) For any matrix polynomial of the form

 $W(z) = z^m B$ + lower degree terms, $B = \text{diag}(b_1, \dots, b_n), \quad b_i \neq b_j$ for $i \neq j$

the polynomial (2.12) has degree $\frac{mn(n-1)}{2} = g + n - 1$.

2) Assume that the spectral curve of W(z) is nonsingular and the roots z_1, \ldots, z_{g+n-1} of the polynomial D(z) are pairwise distinct. Then the rank of the rectangular matrix

$$C(z) = (q_{ij}(z))_{1 \le i \le n, \ 1 \le j \le n-1}$$

evaluated at $z = z_k$, $k = 1, \ldots, g + n - 1$, is equal to n - 1.

3) For a given $k \in \{1, 2, \dots, g + n - 1\}$ let C_k be a non-zero (n - 1)-minor of the matrix $C(z_k)$,

$$C_k = \left(q_{i_s,j}(z_k)\right)_{1 \le s, j \le n-1}$$

Denote \hat{C}_k the matrix obtained from C_k by changing the last column

$$q_{i_s,n-1}(z_k) \mapsto q_{i_s,n}(z_k), \quad s = 1, \dots, n-1$$

and put

$$w_k = -\frac{\det \hat{C}_k}{\det C_k}, \quad k = 1, \dots, g + n - 1.$$
 (2.14)

Then the poles of the eigenvector of the matrix W(z) normalized by (2.8) are at the points

$$Q_1 = (z_1, w_1), \dots, Q_{g+n-1} = (z_{g+n-1}, w_{g+n-1}) \in C \setminus \{P_1 \cup \dots P_n\}.$$
(2.15)

Let us prove that the divisor D_0 is nonspecial, i.e., that $l(D_0) = n$, where $l(D) = \dim H^0(C, \mathcal{O}(D))$. Indeed, if $l(D_0) > n$ then there exists a non-constant rational function f on the curve C with poles² at D_0 satisfying $f(P_1) = \cdots = f(P_n) = 1$. Consider the vector-function

$$\tilde{\psi}(P) = \frac{1}{f(P)}\psi(P).$$

Clearly it is again an eigenvector of the matrix W(z) satisfying the normalization (2.8). Due to uniqueness it must coincide with ψ , so f must be identically equal to 1. Such a contradiction completes the proof of the first part of Lemma.

Let us now explain the reconstruction procedure of the polynomial matrix W(z) starting from a pair (C, D_0) consisting of a curve C of the form (2.2) smooth for $|z| < \infty$ and a nonspecial positive divisor D_0 on $C \setminus (P_1 \cup \cdots \cup P_n)$ of degree g + n - 1. As by assumption $l(D_0) = n$, there exist nrational functions $\psi^1(P), \ldots, \psi^n(P)$ on C with poles at D_0 satisfying

$$\psi^{i}(P_{j}) = \delta_{ij}, \quad i, j = 1, \dots, n.$$
 (2.16)

Let R be a sufficiently large number such that no ramification neither points of the divisor D_0 occur for |z| > R. For any such z denote $(z, 1), \ldots, (z, n)$ the preimages of z on the spectral curve with respect to the natural projection $z : C \mapsto \mathbf{P}^1$ ordered in an arbitrary way. Define a $n \times n$ matrix $\Psi(z)$ whose *i*-th row is $(\psi^i((z, 1)), \ldots, \psi^i((z, n)))$. The matrix $\Psi(z)$ is invertible for any $R < |z| \le \infty$. Denote

$$w_a(z) = w((z, a)) = b_a z^m + \dots, a = 1, \dots, n$$

the branches of the algebraic function $\mu(P), P \in C$ and put

$$\hat{w}(z) = \text{diag}(w_1(z), \dots, w_n(z)), \quad |z| > R.$$

The matrix-valued function $\Psi(z)\hat{w}(z)\Psi^{-1}(z)$ is analytic for |z| > R having an *m*-th order pole at infinity. Observe that it does not depend on the ordering of the preimages of z, so it can be extended to a rational function on the complex plane. Consider its Laurent expansion at infinity

$$\Psi(z)\hat{w}(z)\Psi^{-1}(z) = Bz^m + B_1 z^{m-1} + \dots + B_m + \mathcal{O}\left(\frac{1}{z}\right)$$

where $B = \text{diag}(b_1, \ldots, b_n)$ and B_1, \ldots, B_m are some $n \times n$ matrices. Put

$$W(z) = Bz^m + B_1 z^{m-1} + \dots + B_m$$

and consider the vector-valued function $\tilde{\psi}(P)$ on the curve defined by

$$\tilde{\psi}(P) = W(z(P))\psi(P) - w(P)\psi(P).$$

It has poles only at the points of the divisor D_0 . From the definition of the matrix W(z) it readily follows that $\tilde{\psi}(P)$ vanishes at P_1, \ldots, P_n . Hence it equals zero due to the nonspeciality of the divisor D_0 . This implies that C coincides with the spectral curve of the matrix polynomial W(z). To complete the reconstruction procedure it remains to observe that the function

$$\psi^1(P) + \dots + \psi^n(P) - 1$$

having poles at D_0 vanishes at P_1, \ldots, P_n . Hence it is identically equal to 0, that is, the eigenvector $\psi(P)$ of the matrix W(z) satisfies the normalization (2.8).

²Here and below we will say that a rational function f on the curve C has poles at the points of a divisor D if $(f) + D \ge 0$.

Remark 2.3 Changing the divisor $D_0 \to D'_0 \sim D_0$ in the class of linear equivalence yields a conjugation of the matrix W(z) by a diagonal matrix

$$W(z) \to F W(z)F^{-1}, \quad F = \text{diag} (f(P_1), \dots, f(P_n))$$

where f is a rational function on C with the divisor $D_0 - D'_0$.

One can repeat the above constructions dealing with the dual line bundle \mathcal{L}^{\dagger} over C coming from the left eigenvectors of the matrix W(z)

$$\boldsymbol{\psi}^{\dagger}(P) W(z) = w \, \boldsymbol{\psi}^{\dagger}(P), \quad P = (z, w) \in C, \quad \boldsymbol{\psi}^{\dagger}(P) = (\psi_1^{\dagger}(P), \dots, \psi_n^{\dagger}(P)). \tag{2.17}$$

Let us use the same normalization

$$\psi_1^{\dagger} + \dots + \psi_n^{\dagger} = 1 \tag{2.18}$$

 \mathbf{SO}

$$\psi_i^{\dagger}(P) = \frac{\Delta_{ij}(z, w)}{\sum_{s=1}^n \Delta_{sj}(z, w)}, \quad P = (z, w) \in C, \quad i = 1, \dots n$$
(2.19)

for any choice of j (cf. eq. (2.9)). Denote D_0^{\dagger} the divisor of poles of the dual eigenvector (2.17) normalized by the condition (2.18). The following statement is an analogue of Lemma 2.1 for the dual eigenvectors.

Lemma 2.4 The divisor D_0^{\dagger} of poles of the eigenvector (2.17) normalised by eq. (2.18) is a nonspecial divisor on $C \setminus (P_1 \cup \cdots \cup P_n)$ of degree g + n - 1. Conversely, for any nonsingular curve C of the form

$$w^n + a_1(z)w^{n-1} + \dots + a_n(z) = 0, \quad \deg a_i(z) = m \, i, \quad i = 1, \dots, n$$
 (2.20)

and an arbitrary nonspecial divisor $D_0^{\dagger} \subset C \setminus (P_1 \cup \cdots \cup P_n)$ of degree g + n - 1 there exists a unique matrix polynomial W(z) of the form (2.1) with the spectral curve coinciding with C and the divisor of the normalized eigenvector (2.17) coinciding with D_0 .

The proof is similar to that of Lemma 2.1, so it will be omitted.

Remark 2.5 Of course the divisors D_0 and D_0^{\dagger} do depend on each other. The nature of this dependence will be clarified below.

We will now look at the *spectral projectors* of the matrix polynomial W(z). Consider the matrixvalued function

$$\Pi(P) = \frac{\Phi(P)}{R_w(z,w)}, \quad P = (z,w) \in C$$
(2.21)

where $R(z, w) = \det (w \cdot \mathbf{1} - W(z))$ is the characteristic polynomial of the matrix W(z) and $\Phi(P)$ is defined by (1.10). So

$$\Pi(z,w) = \frac{1}{R_w(z,w)} \frac{R(z,W) - R(z,w)}{W - w} = \frac{\sum_{i=0}^{n-1} b_i(z)w^{n-i-1}}{R_w(z,w)}$$

$$b_i(z) = \sum_{j=0}^i a_j(z)W^{i-j}(z).$$
(2.22)

Let $z \in \mathbb{C}$ be not a ramification point. Denote $w_1(z), \ldots, w_n(z)$ the points above z on the spectral curve C and put

$$\Pi_i(z) = \Pi(z, w_i(z)), \quad i = 1, \dots, n.$$
(2.23)

We will prove that these matrices are the spectral projectors

$$\Pi_i(z): \mathbb{C}^n \to \operatorname{Ker} \left(W(z) - w_i(z) \cdot \mathbf{1} \right), \quad i = 1, \dots, n$$

of W(z).

Lemma 2.6 The matrices $\Pi_1(z), \ldots, \Pi_n(z)$ are basic idempotents of the matrix W(z), i.e.

$$\Pi_i^2 = \Pi_i, \quad \Pi_i \cdot \Pi_j = 0 \quad \text{for} \quad i \neq j, \quad i, j = 1, \dots, n$$
(2.24)

$$\sum_{i=1}^{n} \Pi_{i}(z) = \mathbf{1}, \quad \sum_{i=1}^{n} w_{i}(z) \Pi_{i}(z) = W(z).$$
(2.25)

Proof: Let us first prove that

$$\sum_{i=1}^{n} w_i^r(z) \Pi_i(z) = W^r(z) \quad \text{for any} \quad 0 \le r.$$
(2.26)

We have

$$\sum_{i=1}^{n} w_i^r(z) \frac{\sum_{k=0}^{n-1} b_k(z) w_i^{n-k-1}(z)}{R_w(z, w_i(z))} = \sum_{i=1}^{n} \operatorname{res}_{w=w_i(z)} w^r \frac{\sum_{k=0}^{n-1} b_k(z) w^{n-k-1}}{R(z, w)} = -\operatorname{res}_{w=\infty} w^r \frac{\sum_{k=0}^{n-1} b_k(z) w^{n-k-1}}{R(z, w)}.$$

Using the explicit expression (2.22) along with the obvious identity

$$\left(1 - \frac{W}{w}\right)\left(1 + \frac{b_1}{w} + \dots + \frac{b_{n-1}}{w^{n-1}}\right) = 1 + \frac{a_1}{w} + \dots + \frac{a_n}{w^n}$$

we arrive at

$$\frac{\sum_{k=0}^{n-1} b_k(z) w^{n-k-1}}{R(z,w)} = \frac{1}{w-W}, \quad |w| \to \infty.$$
(2.27)

Thus

$$\mathop{\rm res}_{w=\infty} w^r \frac{\sum_{k=0}^{n-1} b_k(z) w^{n-k-1}}{R(z,w)} = -W^r$$

This proves (2.26) and, hence (2.25).

To prove (2.24) we solve the system (2.26) for r = 0, 1, ..., n-1 with respect to $\Pi_1(z), ..., \Pi_n(z)$ to obtain

$$\Pi_i(z) = \frac{\prod_{j \neq i} (W(z) - w_j(z))}{\prod_{j \neq i} (w_i(z) - w_j(z))}, \quad i = 1, \dots, n.$$

Rewriting these matrices in the basis of eigenvectors of W(z) we readily get (2.24).

It will be convenient to also consider a matrix-valued differential with the matrix entries

$$\Omega_{j}^{i}(P) = \Pi_{j}^{i}(P)dz, \quad i, j = 1, \dots, n$$
(2.28)

where the matrix $\Pi(P) = \left(\Pi_j^i(P)\right)_{1 \le i, j \le n}$ is given by (2.21), (2.22).

Proposition 2.7 For every $i \neq j$ the differential $\Omega_j^i(P)$ is holomorphic on $C \setminus (P_i \cup P_j)$ with simple poles at $P = P_i$ and $P = P_j$. The differential $\Omega_i^i(P)$ is holomorphic on $C \setminus P_i$ having a double pole at $P = P_i$ such that

$$\Omega_i^i(P) = dz + \text{regular terms}, \quad P \to P_i \tag{2.29}$$

for every $i = 1, \ldots, n$.

Proof: As it follows from the explicit expression (1.10) the entries of the matrix $\Phi(P)$ are holomorphic on the affine part $P = (z, w) \in C$, $|z| < \infty$. The differential $dz/R_w(z, w)$ is holomorphic on C. Hence the differentials $\prod_{ij}(P)$ are holomorphic on the affine part of C. Let us look at their behaviour at infinity. To this end we will use the standard realization of the spectral projectors of a $n \times n$ complex matrix with pairwise distinct eigenvalues that in our case can be formulated in the following way. Let $z \in \mathbb{C}$ be not a ramification point with respect to the projection $z : C \to \mathbb{C}$. Order the points $(z, w_1(z)), \ldots, (z, w_n(z))$ in the preimage. Like in the proof of Lemma 2.1 produce a matrix $\Psi(z)$ whose k-th column is given by the eigenvector $(\psi^1(z, w_k(z)), \ldots, \psi^n(z, w_k(z)))^T$ of W(z) normalized by the condition (2.8). Then

$$\Pi(P)|_{P=(z,w_k(z))} = \Psi(z)E_k\Psi^{-1}(z)$$
(2.30)

where the $n \times n$ matrix E_k has only one nonzero entry

$$(E_k)^i_j = \delta^i_k \delta^k_j$$

Consider now sufficiently large R; choose the order of eigenvalues over the disk |z| > R in such a way that $w_k(z) \sim b_k z^m$ for $|z| \to \infty$. Then

$$\Psi(z) = \mathbf{1} + \mathcal{O}\left(\frac{1}{z}\right), \quad \Psi^{-1}(z) = \mathbf{1} + \mathcal{O}\left(\frac{1}{z}\right), \quad z \to \infty$$
(2.31)

due to eq. (2.10). The behaviour of the differentials $\Omega_j^i(P)$ at infinity easily follows from (2.30), (2.31).

Consider now the differentials

$$\Omega_j(P) = \sum_{i=1}^n \Omega_j^i(P) \tag{2.32}$$

$$\Omega^i(P) = \sum_{j=1}^n \Omega^i_j(P).$$
(2.33)

Lemma 2.8 The differential $\Omega_j(P)$ has zeros at the points of the divisor D and at some divisor D_j^{\dagger} of degree g. It has poles at the points of the divisor $P_j + \sum_{s=1}^n P_s$. In a similar way the differential $\Omega^i(P)$ has zeros at the points of the divisor D^{\dagger} and at some divisor D_i of degree g. It has poles at the points of the divisor $P_i + \sum_{s=1}^n P_s$.

Proof: Using the representation (2.30) along with the normalization (2.8) we immediately conclude that the sum (2.32) vanishes at the points of the divisor D. The configuration of poles of this differential can be easily recovered from Proposition 2.7. The degree counting

$$\deg D + \deg D_j^{\dagger} - \deg \left(P_j + \sum_{s=1}^n P_s \right) = 2g - 2$$

yields deg $D_j^{\dagger} = g$. To derive similar statements about the differentials $\Omega^i(P)$ we use an alternative representation of the projector matrix

$$\Pi(P)|_{P=(z,w_k(z))} = \Psi^{\dagger}(z)^{-1} E_k \Psi^{\dagger}(z)$$
(2.34)

where the k-th row of the matrix $\Psi^{\dagger}(z)$ is given by the left eigenvector $\left(\psi_{1}^{\dagger}(z, w_{k}(z)), \ldots, \psi_{n}^{\dagger}(z, w_{k}(z))\right)$ normalized by the condition (2.18).

Corollary 2.9 The divisor of zeros of the differential $\Omega_j^i(P)$ on $C \setminus (P_1 \cup \ldots P_n)$ coincides with $D_i + D_j^{\dagger}$, $i, j = 1, \ldots, n$.

Proof: The matrix $\Omega_j^i(P)$ has rank one. Its columns are eigenvectors of the matrix W(z) with the same eigenvalue. Normalizing any column we obtain the same vector function $\psi(P)$

$$\psi^{i}(P) = \frac{\Omega^{i}_{j}(P)}{\sum_{k=1}^{n} \Omega^{k}_{j}(P)}, \quad i = 1, \dots, n$$
(2.35)

for any j. According to Lemma the denominator vanishes at the points of the divisor D_j^{\dagger} . Hence also the numerator must vanish at the points of this divisor. In a similar way, the rows of $\Omega_j^i(P)$ are left eigenvectors of the same matrix W(z). Normalizing them one obtains

$$\psi_j^{\dagger}(P) = \frac{\Omega_j^i(P)}{\sum_{k=1}^n \Omega_k^i(P)}, \quad j = 1, \dots, n$$
 (2.36)

for any *i*. Hence $\Omega_j^i(P)$ vanishes also at the points of the divisor D_i . Since degree of the divisor of poles of this differentials equals two and $\deg(D_i + D_j^{\dagger}) = 2g$, there are no other zeros.

Corollary 2.10 The zeros of the differential

$$\Omega(P) = \sum_{i,j=1}^{n} \Omega_j^i(P)$$
(2.37)

are at the points of the divisor $D + D^{\dagger}$. It has double poles at the infinite points P_1, \ldots, P_n and

$$\Omega(P) = \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right) dz, \quad P \to P_k, \quad k = 1, \dots, n.$$
(2.38)

The corollary suggests the following way of determining the dual divisor D^{\dagger} starting from D. For a nonspecial divisor D of degree g + n - 1 there exists a unique differential $\Omega(P)$ vanishing at the points of D and having double poles of the form (2.38) at infinity. The remaining zeros of the differential give the points of the divisor D^{\dagger} .

Corollary 2.11 The differentials $\Omega_i^i(P)$ admit the following representation

$$\Omega_j^i(P) = \psi^i(P)\psi_j^\dagger(P)\Omega(P), \quad i, j = 1, \dots, n.$$
(2.39)

Proof: Due to the previous Corollary the product (2.39) is holomorphic on $C \setminus (P_1 \cup \cdots \cup P_n)$. From Corollary 2.9 it follows that the divisor of zeroes of this product coincides with $D_i + D_j^{\dagger}$. Finally, using (2.38) along with the asymptotics of $\psi^i(P)$ and $\psi_i^{\dagger}(P)$ at infinity we complete the proof.

2.2 Generalized Jacobian and theta-functions. Proof of eq. (1.11)

The generalized Jacobian $J(C; P_1, \ldots, P_n)$ can be realized as a fiber bundle over J(C) with (n-1)dimensional fiber. For n = 2 the construction was already given in [17]; it is quite similar also for arbitrary n.

Define $J(C; P_1, \ldots, P_n)$ as the set of all pairs

$$(\mathbf{u}, \boldsymbol{\lambda}), \quad \mathbf{u} = (u_1, \dots, u_g) \in \mathbb{C}^g, \quad \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n$$

modulo the following equivalence relation $(\mathbf{u}, \boldsymbol{\lambda}) \sim (\mathbf{u}', \boldsymbol{\lambda}')$ if

$$\mathbf{u}' = \mathbf{u} + 2\pi i M + BN, \quad \lambda'_k = c \,\lambda_k \, e^{\langle N, A(P_k) \rangle}, \quad k = 1, \dots, n, \quad c \in \mathbb{C}^*, \quad M, \, N \in \mathbb{Z}^g.$$
(2.40)

The fibration (2.6) is realized by the map $(\mathbf{u}, \boldsymbol{\lambda}) \mapsto \mathbf{u}$.

To define an analogue of the Abel map

$$C \to J(C; P_1, \dots, P_n) \tag{2.41}$$

fix a pair of distinct points $P_0, Q_0 \in C \setminus (P_1 \cup \cdots \cup P_n)$ and put

$$P \mapsto (\mathbf{u}(P), \boldsymbol{\lambda}(P)), \quad u_i(P) = A_i(P) = \int_{P_0}^P \omega_i, \quad \lambda_k(P) = e^{\alpha_k(P)}, \quad \alpha_k(P) = \int_{P_0}^P \Omega_{P_kQ_0}. \quad (2.42)$$

Here and below Ω_{PQ} is the third kind differential on C having simple poles at the points P and Q with residues +1 and -1 respectively and vanishing *a*-periods. The map is extended linearly/multiplicatively on the group of divisors of a given degree. The following statement is an analogue of the Abel–Jacobi theorem.

Proposition 2.1 Two divisors D, D' of the same degree are relatively equivalent iff

$$(\mathbf{u}(D), \boldsymbol{\lambda}(D)) \sim (\mathbf{u}(D'), \boldsymbol{\lambda}(D'))$$

modulo equivalence (2.40).

An analogue of the Riemann theorem about zeros of theta-function is given by

Proposition 2.2 For a given $\mathbf{u} \in \mathbb{C}^g$ such that $\theta(\mathbf{u}) \neq 0$ and $\boldsymbol{\lambda} \in (\mathbb{C}^*)^n$ consider the function

$$F(P) = \sum_{s=1}^{n} \lambda_s \,\theta(P - P_s - \mathbf{u}) \frac{E(P, P_0) (d\zeta_0)^{1/2}}{E(P, P_s) (d\zeta_s)^{1/2}}$$
(2.43)

on the 4g-gon \tilde{C} obtained by cutting the curve C along the chosen basis of cycles $a_1, \ldots, a_g, b_1, \ldots, b_g$. Here $\zeta_0(P) = z(P) - z(P_0), \zeta_s(P) = 1/z(P)$ are local parameters³ near P_0, P_s respectively. This

³For simplicity we assume that $P_0 \in C$ is not a branch point.

function has simple poles at P_1, \ldots, P_n , a simple zero at P_0 and also zeros at some points Q_1, \ldots, Q_{g+n-1} of a divisor $D \subset C \setminus (P_1 \cup \cdots \cup P_n)$ satisfying

$$\sum_{i=1}^{g+n-1} A_j(Q_i) - \sum_{s=1}^n A_j(P_s) = u_j - \mathcal{K}_j, \quad j = 1, \dots, g$$
(2.44)

$$\exp\sum_{i=1}^{g+n-1} \alpha_s \left(Q_i\right) = \lambda_s \, e^{-\kappa_s}, \quad s = 1, \dots, n \tag{2.45}$$

where $\mathcal{K}_1, \ldots, \mathcal{K}_g$ are Riemann constants

$$\mathcal{K}_j = \frac{2\pi i + B_{jj}}{2} - \frac{1}{2\pi i} \sum_{k \neq j} \int_{a_k} A_j(P) \omega_k(P),$$

 $\kappa_1, \ldots, \kappa_n$ are given by analogous formulae,

$$\kappa_s = -\log E(P_s, P_0) - \frac{1}{2\pi i} \sum_{k=1}^g \int_{a_k} \alpha_s(P) \omega_k(P).$$
(2.46)

Recall [17] that the combination

$$\Delta := -\mathcal{K} + (g-1)P_0 \tag{2.47}$$

does not depend on the choice of the base point P_0 . This gives rise to definition of the Riemann divisor Δ that already appeared above. In particular eq. (2.44) can be rewritten in the form

$$D - \sum_{s=1}^{n} P_s - \Delta = \mathbf{u}.$$
(2.48)

Observe that, for a given $(\mathbf{u}, \boldsymbol{\lambda})$ the divisor of zeros of the function (2.43) determined by eqs. (2.44), (2.45) does not depend on the choice of the basepoint P_0 .

Proof: The function F(P) does not change its value when the point P crosses the cut along the cycle b_k while crossing the cut a_k it is multiplied by

$$\exp\left(-\frac{1}{2}B_{kk} - \int_{P_0}^P \omega_k + u_k\right).$$

So the total logarithmic residue

$$\frac{1}{2\pi\,i}\oint_{\partial\tilde{C}}d\log F(P)$$

is equal to g. Hence F(P) has g + n - 1 zeros on $C \setminus (P_1 \cup \cdots \cup P_n)$, counted with multiplicity. To prove the first equality (2.44) one has to compute the contour integral

$$\frac{1}{2\pi i} \oint_{\partial \tilde{C}} A_j(P) d\log F(P) = A_j(Q_1) + \dots + A_j(Q_{g+n-1}) - A_j(P_1) - \dots - A_j(P_n).$$

To prove eq. (2.45) we need to add more cuts: a cut from P_0 to Q_0 and also cuts from Q_0 to the infinite points P_1, \ldots, P_n . Denote \tilde{C}' the resulting polygon. Then

$$\sum_{i=1}^{g+n-1} \alpha_s(P_i) = \frac{1}{2\pi i} \oint_{\partial \tilde{C}'} \alpha_s(P) d\log F(P), \quad s = 1, \dots, n$$

up to a s-independent shift. The integral in the right hand side must be regularized at $P \to \infty$ or $P \to Q_0$. After such regularization we arrive at eqs. (2.45) up to equivalence (2.40).

We will now express the differentials (2.28) in terms of the coordinates $(\mathbf{u}_0, \boldsymbol{\lambda}_0)$.

Proposition 2.3 Let $Q_1 + \cdots + Q_{g+n-1} = D \subset C \setminus (P_1 \cup \cdots \cup P_n)$ be the divisor of poles of the eigenvector of the matrix W(z) normalised by the condition (2.8). Denote $(\mathbf{u}_0, \boldsymbol{\lambda}_0)$ the corresponding point on the generalized Jacobian (2.40)

$$\mathbf{u}_{0} = D_{0} - \sum_{a=1}^{n} P_{a} - \Delta$$

$$\lambda_{j}^{0} = \exp\left\{\sum_{s=1}^{g+n-1} \int_{P_{0}}^{Q_{s}} \Omega_{P_{j}Q_{0}} + \kappa_{j}\right\}, \quad j = 1, \dots, n$$
(2.49)

(cf. eqs. (2.44), (2.45)). Then the differentials $\Omega_j^i(P)$ of the form (2.28) are given by the following equation

$$\Omega_j^i(P) = \frac{\lambda_i^0}{\lambda_j^0} \frac{\theta(P - P_i - \mathbf{u}_0)\theta(P - P_j + \mathbf{u}_0)}{\theta^2(\mathbf{u}_0)E(P_i, P)E(P, P_j)\sqrt{d\zeta_i}\sqrt{d\zeta_j}}$$
(2.50)

Proof: Any differential $\Omega_j^i(P)$ having, for $i \neq j$ simple poles at $P = P_i$ and $P = P_j$ and, for i = j a double pole of the form (2.29) at $P = P_i$ can be written [17] as follows

$$\Omega_j^i(P) = \alpha_{ij} \frac{\theta(P - P_i - \mathbf{u}_{ij})\theta(P - P_j + \mathbf{u}_{ij})}{\theta^2(\mathbf{u}_{ij})E(P_i, P)E(P, P_j)\sqrt{d\zeta_i}\sqrt{d\zeta_j}}$$

for some $\mathbf{u}_{ij} \in J(C) \setminus (\theta)$ and some nonzero constants α_{ij} satisfying $\alpha_{ii} = 1$. According to Riemann theorem the zeros of the function $\theta(P - P_j + \mathbf{u}_{ij})$ are at the points of a divisor \mathcal{D} of degree g satisfying

$$\mathcal{D} - P_j - \Delta = -\mathbf{u}_{ij}.$$

According to the Corollary 2.9 it must coincide with the divisor D_j^{\dagger} . From Lemma 2.8 we derive the following linear equivalence

$$D_0 + D_j^{\dagger} - P_j - \sum_{a=1}^n P_a = K_C.$$

Substituting $K_C = 2\Delta$ we can rewrite it as follows

$$D_j^{\dagger} - P_j - \Delta = -D_0 + \sum_{a=1}^n P_a + \Delta = -\mathbf{u}_0.$$

Hence the condition $\mathcal{D} = D_j^{\dagger}$ implies $\mathbf{u}_{ij} = \mathbf{u}_0$ on Jac(C).

It remains to fix the constants α_{ij} . Since the rank of the matrix $\Omega_j^i(P)$ must be equal to one, we conclude that $\alpha_{ij} = \alpha_i \beta_j$ for some nonzero constants α_i , β_j . As $\alpha_{ii} = 1$ then $\beta_j = \alpha_j^{-1}$. The last condition to be used is that the sum $\sum_{i=1}^n \Omega_j^i(P)$ must vanish at the points Q_1, \ldots, Q_{g+n-1} of the divisor D. This implies that $\alpha_i = \lambda_i^0$, up to a common factor.

Corollary 2.4 The eigenvector $\psi(P)$ of the matrix W(z) normalized by the condition (2.8) is

$$\psi^{i}(P) = \frac{\lambda_{i}^{0} \frac{\theta(P - P_{i} - \mathbf{u}_{0})}{E(P, P_{i})(d\zeta_{i})^{1/2}}}{\sum_{b=1}^{n} \lambda_{b}^{0} \frac{\theta(P - P_{b} - \mathbf{u}_{0})}{E(P, P_{b})(d\zeta_{b})^{1/2}}}, \quad i = 1, \dots, n.$$
(2.51)

The dual eigenvector $\psi^{\dagger}(P)$ is given by a similar formula

$$\psi_i^{\dagger}(P) = \frac{\frac{1}{\lambda_i^0} \frac{\theta(P - P_i + \mathbf{u}_0)}{E(P_i, P)(d\zeta_i)^{1/2}}}{\sum_{b=1}^n \frac{1}{\lambda_b^0} \frac{\theta(P - P_b + \mathbf{u}_0)}{E(P_b, P)(d\zeta_b)^{1/2}}}, \quad i = 1, \dots, n.$$
(2.52)

Proof: Use (2.50) along with (2.35), (2.36).

Remark 2.5 Observe that the change

 $D_0 \mapsto D_0^{\dagger}$

corresponds to the involution

$$\left(\mathbf{u}_{0}, \lambda_{1}^{0}, \dots, \lambda_{n}^{0}\right) \mapsto \left(-\mathbf{u}_{0}, 1/\lambda_{1}^{0}, \dots, 1/\lambda_{n}^{0}\right)$$

$$(2.53)$$

on the generalized Jacobian.

We are now in a position to prove the first equation (1.11) of the Main Theorem.

Proposition 2.6 Let C be a compact Riemann surface of positive genus and $z : C \to \mathbf{P}^1$ a rational function with n simple poles at the points P_1, \ldots, P_n . Introduce the following matrix of Abelian differentials on C

$$\mathbf{\Omega}(P) = \left(\Omega_j^i(P)\right)_{1 \le i, j \le n}, \quad \Omega_j^i(P) = \frac{\lambda_i}{\lambda_j} \frac{\theta(P - P_i - \mathbf{u})\theta(P - P_j + \mathbf{u})}{\theta^2(\mathbf{u})E(P_i, P)E(P, P_j)\sqrt{d\zeta_i}\sqrt{d\zeta_j}}$$
(2.54)

where $\mathbf{u} \in J(C) \setminus (\theta)$ is an arbitrary point and $\lambda_1, \ldots, \lambda_n$ are arbitrary nonzero numbers. Then for an arbitrary pair of distinct points $P, Q \in C$ the following equation holds true

$$\operatorname{tr}\frac{\mathbf{\Omega}(P)\mathbf{\Omega}(Q)}{\left(z(P)-z(Q)\right)^2} = \frac{\theta(P-Q-\mathbf{u})\theta(P-Q+\mathbf{u})}{\theta^2(\mathbf{u})E^2(P,Q)}.$$
(2.55)

Proof: The trace of the product of the matrices $\Omega(P)$ and $\Omega(Q)$ factorizes as follows

$$\operatorname{tr} \mathbf{\Omega}(P)\mathbf{\Omega}(Q) = \sum_{i=1}^{n} \frac{\theta(P - P_i - \mathbf{u})\theta(Q - P_i + \mathbf{u})}{\theta^2(\mathbf{u})E(P_i, P)E(Q, P_i)d\zeta_i} \cdot \sum_{j=1}^{n} \frac{\theta(Q - P_j - \mathbf{u})\theta(P - P_j + \mathbf{u})}{\theta^2(\mathbf{u})E(P_j, Q)E(P, P_j)d\zeta_j}.$$
 (2.56)

For a fixed pair of distinct points $P, Q \in C$ consider the differential

$$H_{PQ}(Z) = \frac{\theta(P - Z - \mathbf{u})\theta(Q - Z + \mathbf{u})}{\theta^2(\mathbf{u})E(Z, P)E(Q, Z)\sqrt{dz(P)}\sqrt{dz(Q)}}, \quad Z \in C.$$
(2.57)

It has simple poles at Z = P and Z = Q with residues

$$\operatorname{res}_{Z=P} H_{PQ}(Z) = -\operatorname{res}_{Z=Q} H_{PQ}(Z) = \frac{\theta(P-Q-\mathbf{u})}{\theta(\mathbf{u})E(Q,P)\sqrt{dz(P)}\sqrt{dz(Q)}}.$$

We now consider the product $z(Z)H_{PQ}(Z)$. Vanishing of the sum of residues of this differential yields

$$\sum_{i=1}^{n} \frac{\theta(P - P_i - \mathbf{u})\theta(Q - P_i + \mathbf{u})}{\theta^2(\mathbf{u})E(P_i, P)E(Q, P_i)\sqrt{dz(P)}\sqrt{dz(Q)}\,d\zeta_i} = \sum_{i=1}^{n} \operatorname{res}_{Z=P_i} z(Z)\,H_{PQ}(Z) = \left[z(P) - z(Q)\right] \frac{\theta(P - Q - \mathbf{u})}{\theta(\mathbf{u})E(Q, P)\sqrt{dz(P)}\sqrt{dz(Q)}}$$
(2.58)

$$\sum_{j=1}^{n} \frac{\theta(Q - P_j - \mathbf{u})\theta(P - P_j + \mathbf{u})}{\theta^2(\mathbf{u})E(P_j, Q)E(P, P_j)\sqrt{dz(P)}\sqrt{dz(Q)}d\zeta_j} = \sum_{j=1}^{n} \operatorname{res}_{Z=P_j} z(Z) H_{QP}(Z) =$$
$$= [z(P) - z(Q)] \frac{\theta(P - Q + \mathbf{u})}{\theta(\mathbf{u})E(Q, P)\sqrt{dz(P)} \cdot \sqrt{dz(Q)}}$$

Therefore the bi-differential in the right hand side of eq. (2.56) becomes

$$\sum_{i=1}^{n} \frac{\theta(P-P_i-\mathbf{u})\theta(Q-P_i+\mathbf{u})}{\theta^2(\mathbf{u})E(P_i,P)E(Q,P_i)d\zeta_i} \cdot \sum_{j=1}^{n} \frac{\theta(Q-P_j-\mathbf{u})\theta(P-P_j+\mathbf{u})}{\theta^2(\mathbf{u})E(P_j,Q)E(P,P_j)d\zeta_j} =$$
$$= \sum_{i=1}^{n} \max_{Z=P_i} z(Z) H_{PQ}(Z) \cdot \sum_{j=1}^{n} \max_{Z=P_j} z(Z) H_{QP}(Z) dz(P)dz(Q) =$$
$$= [z(P)-z(Q)]^2 \frac{\theta(P-Q-\mathbf{u})\theta(P-Q+\mathbf{u})}{\theta^2(\mathbf{u})E^2(P,Q)}.$$

Equation (1.11) immediately follows from Proposition 2.3 and eq. (2.55).

2.3 Algebro-geometric solutions to the *n*-wave hierarchy and their tau-functions. Proof of eq. (1.12)

According to the original idea of S.P.Novikov [25] algebro-geometric (aka *finite gap*) solutions to integrable systems of PDEs are obtained by considering stationary points of a linear combination of

the commuting flows. Here we will be dealing with the n-wave system of nonlinear evolution PDEs represented in the form

$$[L_{a,k}, L_{b,l}] = 0$$

$$L_{a,k} = \frac{\partial}{\partial t_k^a} - U_{a,k}(\mathbf{t}; z), \quad a = 1, \dots, n, \quad k \ge -1$$

where

 $U_{a,k}(\mathbf{t};z) = z^{k+1}E_a + \text{lower degree terms}$

is a $n \times n$ matrix-valued polynomial in z of degree k+1 depending on the infinite number of independent variables $\mathbf{t} = (t_k^a)$. The independent variable z is often called *spectral parameter*. The above equations hold true identically in z. Here the diagonal $n \times n$ matrix E_a has only one nonzero entry

$$(E_a)_{ij} = \delta_{ia}\delta_{aj}.$$

For example, for k = -1

$$U_{a,-1} = E_a$$

and, for k = 0

 $U_{a,0} = z E_a - [E_a, Y], \quad a = 1, \dots, n$

where the diagonal entries of the $n \times n$ matrix $Y = Y(\mathbf{t})$ vanish. It turns out that the coefficients of the matrix polynomials $U_{a,k}(\mathbf{t}; z)$ can be represented as polynomials in the entries of the matrix $Y(\mathbf{t})$ and its derivatives in the variables t_0^1, \ldots, t_0^n . Thus the *n*-wave system can be considered as an infinite family of partial differential equations for the matrix $Y(\mathbf{t})$. See Appendix A below for the details about the structure of the *n*-wave hierarchy.

The following statement is crucial for computing tau-functions of solutions to the *n*-wave hierarchy.

Proposition 2.1 1) For any solution to the n-wave hierarchy there exists a unique n-tuple of matrixvalued series

$$M_b(\mathbf{t}, z) = E_b + \sum_{k \ge 1} \frac{B_{b,k}(\mathbf{t})}{z^k}, \quad b = 1, \dots, n$$

satisfying

$$[L_{a,k}, M_b] = 0 \quad \Leftrightarrow \quad \frac{\partial M_b(\mathbf{t}, z)}{\partial t_k^a} = [U_{a,k}(\mathbf{t}, z), M_b(\mathbf{t}, z)] \quad \forall a = 1, \dots, n, \quad k \ge -1$$

and also

$$M_a(\mathbf{t}, z)M_b(\mathbf{t}, z) = \delta_{ab}M_a(\mathbf{t}, z), \quad M_1(\mathbf{t}, z) + \dots + M_n(\mathbf{t}, z) = \mathbf{1}.$$

2) The (principle) tau-function $\tau(\mathbf{t})$ of this solution is determined from the following generating series in independent variables z_1 , z_2 for its second logarithmic derivatives

$$\sum_{p,q=0}^{\infty} \frac{1}{z_1^{p+2}} \frac{1}{z_2^{q+2}} \frac{\partial^2 \log \tau(\mathbf{t})}{\partial t_q^b \partial t_p^a} = \operatorname{tr} \frac{M_a(\mathbf{t}, z_1) M_b(\mathbf{t}, z_2)}{(z_1 - z_2)^2} - \frac{\delta_{ab}}{(z_1 - z_2)^2}$$
(2.59)

for any a, b = 1, ..., n.

3) The logarithmic derivatives of higher orders $N \ge 3$ of the same tau-function can be determined by the following generating series

$$\sum_{k_1,\dots,k_N \ge 0} \frac{\partial^N \log \tau(\mathbf{t})}{\partial t_{k_1}^{a_1} \dots \partial t_{k_N}^{a_N}} \frac{1}{z_1^{k_1+2} \dots z_N^{k_N+2}} = -\frac{1}{N} \sum_{s \in S_N} \frac{\operatorname{tr} \left[M_{a_{s_1}} \left(\mathbf{t}, z_{s_1} \right) \dots M_{a_{s_N}} \left(\mathbf{t}, z_{s_N} \right) \right]}{\left(z_{s_1} - z_{s_2} \right) \dots \left(z_{s_{N-1}} - z_{s_N} \right) \left(z_{s_N} - z_{s_1} \right)}$$
(2.60)

Clearly the tau-function is determined by (2.59) uniquely up to

$$\tau(\mathbf{t}) \mapsto e^{\alpha + \sum \beta_{a,k} t_k^a} \tau(\mathbf{t})$$

for some constants α and $\beta_{a,k}$.

The above proposition about construction of tau-functions of solutions to the *n*-wave integrable hierarchy is an extension to this case of the approach of [2]-[4] based on the theory of the so-called matrix resolvents. For the proofs see Appendix A below.

Remark 2.2 The construction of tau-function given in the Proposition differs from the original definition of [19], [7], [30] (we recall this definition in Appendix B below). One can proof equivalence of the two definitions following the scheme of [2]. We will not do it here as it is not needed for the proofs of the results of this paper.

We will now apply the Proposition to the finite-gap solutions of the *n*-wave hierarchy. According to the Novikov's recipe mentioned above for arbitrary choice of constants $c_{a,k}$, $a = 1, \ldots, n, -1 \le k \le N$ for any integer $N \ge 0$, we obtain a family of algebro-geometric solutions $Y(\mathbf{t})$ satisfying

$$\sum_{a=1}^{n} \sum_{k=-1}^{N} c_{a,k} \frac{\partial Y(\mathbf{t})}{\partial t_{k}^{a}} = 0.$$

For any such solution define a matrix polynomial

$$U(\mathbf{t}, z) = \sum_{a=1}^{n} \sum_{k=-1}^{N} c_{a,k} U_{a,k}(\mathbf{t}, z).$$

Recall that the coefficients of the matrix polynomial belong to the space \mathcal{Y} . From commutativity (A.1) of the flows it readily follows that the matrix $U = U(\mathbf{t}, z)$ satisfies

$$\frac{\partial U}{\partial t_l^b} = [U_{b,l}, U] \quad \text{for any} \quad b = 1, \dots, n, \quad l \ge -1.$$
(2.61)

Therefore the characteristic polynomial

$$R(z, w) = \det(w \cdot \mathbf{1} - U(\mathbf{t}, z))$$

does not depend on **t**. Its coefficients can be considered as first integrals of the differential equations (2.61). For a given N, assuming the coefficients $c_{1,N}, \ldots, c_{n,N}$ to be pairwise distinct the matrix polynomial $U(\mathbf{t}, z)$ for any **t** belongs to the family \mathcal{W} of polynomials of the form (2.1) with m = N+1. Therefore the spectral curves $C = \{R(z, w) = 0\}$ is of the form (2.2).

Our nearest goal is to construct an algebro-geometric solution to the *n*-wave system such that the matrix polynomial $U(\mathbf{t}, z)$ satisfies the initial condition

$$W(z) = U(0, z)$$
 (2.62)

so C is the spectral curve of the matrix polynomial W(z). The construction is rather standard for the theory of integrable systems. Namely, for the matrix polynomial W(z) we have constructed the spectral curve C with a nonspecial divisor D_0 . Starting from these data one can construct an algebro-geometric solution of the *n*-wave system following the I.M.Krichever's scheme [21]. We will use a vector-valued Baker-Akhiezer function $\psi(\mathbf{t}, P) = (\psi^1(\mathbf{t}, P), \dots, \psi^n(\mathbf{t}, P))^T$ meromorphic on the spectral curve $C \setminus (P_1 \cup \cdots \cup P_n)$ with poles at the points of the divisor D_0 and having essential singularities at $P = P_1, \dots, P = P_n$ of the form

$$\psi^{i}(\mathbf{t}, P) = \left(\delta_{ij} + \mathcal{O}\left(\frac{1}{z}\right)\right) e^{\phi_{j}(\mathbf{t}, z)}, \quad P \to P_{j}, \quad i, j = 1, \dots, n.$$

It is a standard fact of the theory of Baker–Akhiezer functions that $\boldsymbol{\psi}(\mathbf{t}, P)$ exists for sufficiently small $|\mathbf{t}|$ and is unique. It also exists the unique dual Baker–Akhiezer function $\boldsymbol{\psi}^{\dagger}(\mathbf{t}, P) = \left(\psi_{1}^{\dagger}(\mathbf{t}, P), \dots, \psi_{n}^{\dagger}(\mathbf{t}, P)\right)$ with poles at the divisor D_{0}^{\dagger} and essential singularities of the form

$$\psi_i^{\dagger}(\mathbf{t}, P) = \left(\delta_{ij} + \mathcal{O}\left(\frac{1}{z}\right)\right) e^{-\phi_j(\mathbf{t}, z)}, \quad P \to P_j, \quad i, j = 1, \dots, n.$$

The needed algebro-geometric solution to the *n*-wave system is uniquely specified by the condition that its wave function is expressed in terms of $\psi(\mathbf{t}, P)$. We will now obtain an explicit expression of this solution in terms of theta-functions of the spectral curve.

Proposition 2.3 Let W(z) be a matrix polynomial of the form (2.1) with a nonsingular spectral curve C. Let D_0 be the divisor of poles of the eigenvector of W(z) normalized by the condition (2.8) and denote $(\mathbf{u}_0, \boldsymbol{\lambda}^0) \in J(C, P_1, \ldots, P_n)$ the corresponding point (2.49) of the generalized Jacobian. Introduce Abelian differentials

$$\Omega_{j}^{i}(\mathbf{t}, P) = \frac{\lambda_{i}(\mathbf{t})}{\lambda_{j}(\mathbf{t})} \frac{\theta(P - P_{i} - \mathbf{u}(\mathbf{t}))\theta(P - P_{j} + \mathbf{u}(\mathbf{t}))}{\theta^{2}(\mathbf{u}(\mathbf{t}))E(P_{i}, P)E(P, P_{j})\sqrt{d\zeta_{i}}\sqrt{d\zeta_{j}}}$$

$$\mathbf{u}(\mathbf{t}) = \mathbf{u}_{0} - \sum t_{k}^{a} \mathbf{V}^{(a,k)}, \quad \lambda_{i}(\mathbf{t}) = \exp\left\{\sum t_{k}^{a} \int_{P_{i}}^{P_{0}} \Omega_{a}^{(k)}\right\} \lambda_{i}^{0}, \quad i, j = 1, \dots, n.$$

$$(2.63)$$

Here and below

$$\zeta_a = \zeta_a(Q) = \frac{1}{z(Q)}, \quad Q \in C, \quad Q \to P_a, \quad a = 1, \dots, n$$

is a natural local parameter near P_a . The principal values of the integrals are defined by the following limits

$$\int_{P_i}^{P_0} \Omega_a^{(k)} = \lim_{Q \to P_i} \left(\int_Q^{P_0} \Omega_a^{(k)} + z^{k+1}(Q) \right).$$
(2.64)

Define matrix-valued power series in z^{-1} by expanding the differentials at infinity

$$M_a(\mathbf{t}, z) = \left(\frac{\Omega_j^i(\mathbf{t}, P)}{dz}\right)_{1 \le i, j \le n}, \quad P = (z, w_a(z)) \to P_a, \quad a = 1, \dots, n$$
(2.65)

and put

$$U_{a,k}(\mathbf{t},z) = \left(z^{k+1}M_a(\mathbf{t},z)\right)_+.$$
(2.66)

This collection of matrix polynomials is an algebro-geometric solution to the n-wave system with the corresponding matrix $U(\mathbf{t}, z)$ satisfying (2.61) given by

$$U(\mathbf{t}, z) = w_1(z)M_1(\mathbf{t}, z) + \dots + w_n(z)M_n(\mathbf{t}, z).$$
(2.67)

In this formula $w_a(z)$ is the Laurent expansion of the algebraic function w(z) near $P = P_a$, a = 1, ..., n. This matrix polynomial $U(\mathbf{t}, z)$ satisfies the initial condition (2.62).

Proof: Let $\Omega(P)$ be the differential (2.37) on C constructed above. We first prove that the differentials (2.63) coincide with

$$\Omega_j^i(\mathbf{t}, P) = \psi^i(\mathbf{t}, P)\Omega(P)\psi_j^{\dagger}(\mathbf{t}, P), \quad i, j = 1, \dots, n$$
(2.68)

To this end we use the following expressions of the Baker–Akhiezer functions $\psi(\mathbf{t}, P)$ and $\psi^{\dagger}(\mathbf{t}, P)$

$$\psi^{i}(\mathbf{t},P) = \exp\left(\sum t_{k}^{a} \int_{P_{i}}^{P} \Omega_{a}^{(k)}\right) \frac{\lambda_{i}^{0} \frac{\theta(P-P_{i}+\sum t_{k}^{a} \mathbf{V}^{(a,k)}-\mathbf{u}_{0})}{\theta(\sum t_{k}^{a} \mathbf{V}^{(a,k)}-\mathbf{u}_{0})E(P,P_{i})(d\zeta_{i})^{1/2}}}{\sum_{b=1}^{n} \lambda_{b}^{0} \frac{\theta(P-P_{b}-\mathbf{u}_{0})}{\theta(\mathbf{u}_{0})E(P,P_{b})(d\zeta_{b})^{1/2}}}, \quad i = 1, \dots, n$$
(2.69)

and

$$\psi_{i}^{\dagger}(\mathbf{t},P) = \exp\left(-\sum t_{k}^{a} \int_{P_{i}}^{P} \Omega_{a}^{(k)}\right) \frac{\frac{1}{\lambda_{i}^{0}} \frac{\theta(P-P_{i}-\sum t_{k}^{a} \mathbf{V}^{(a,k)}+\mathbf{u}_{0})}{\left(\sum_{b=1}^{n} \frac{1}{\lambda_{b}^{0}} \frac{\theta(P-P_{b}+\mathbf{u}_{0})}{\theta(\mathbf{u}_{0})E(P_{b},P)(d\zeta_{b})^{1/2}}\right)}, \quad i = 1, \dots, n.$$
(2.70)

Here $\Omega_a^{(k)}$ is the normalised second kind differential on C with a unique pole at P_a of order k+2

$$\Omega_a^{(k)}(P) = dz^{k+1} + \text{regular terms}, \quad P \to P_a$$
$$\oint_{a_i} \Omega_a^{(k)} = 0, \quad V_i^{(a,k)} = \oint_{b_i} \Omega_a^{(k)}, \quad i = 1, \dots, g$$
(2.71)

for a chosen canonical basis a_i , b_j in $H_1(C, \mathbb{Z})$. Recall [17] that the *b*-periods $V_i^{(a,k)}$ of the differentials $\Omega_a^{(k)}$ coincide with the coefficients of expansions (1.20) of holomorphic differentials $\omega_i(P)$ at $P \to P_a$. Observe that for $\mathbf{t} = 0$ the functions $\psi(0, P)$ and $\psi^{\dagger}(0, P)$ coincide with the right and left eigenvectors of W(z).

The derivation of the representations (2.69), (2.70) is standard for the theory of Baker–Akhiezer functions: we check that (2.69), (2.70) are well-defined meromorphic functions on $C \setminus (P_1 \cup \cdots \cup P_n)$ with essential singularities at infinity of the needed form having poles at rhe points of the divisors D_0 and D_0^{\dagger} respectively (for the claim about the location of poles use Proposition 2.2 and Remark 2.5). With the help of these expressions it is easy to verify validity of eq. (2.68).

Define now a matrix-valued function $\Psi(\mathbf{t}, z)$ such that its *i*-th row is given by the expansions of $\psi^i(\mathbf{t}, P)$ at the infinite points P_1, \ldots, P_n . We will prove that $\Psi(\mathbf{t}, z)$ is the wave function of the solution (2.66) to the *n*-wave system. To this end we will first verify that the definition (2.65) of the matrices $M_a(z)$ can be rewritten in the form (A.28).

Introduce another matrix-valued function $\Psi^{\dagger}(\mathbf{t}, z)$ in a similar way: its *i*-th column is given by the expansions of $\psi_i^{\dagger}(\mathbf{t}, P)$ at the infinite points P_1, \ldots, P_n . Let us prove that this matrix is inverse to $\Psi(\mathbf{t}, z)$ up to multiplication on the left by a nondegenerate diagonal matrix.

Lemma 2.4 Define

$$\hat{\rho}(z) = \text{diag}\left(\rho_1(z), \dots, \rho_n(z)\right), \quad \rho_a(z) = \left(\frac{\Omega(P)}{dz}\right)_{P=(z,w_a(z))} = 1 + \mathcal{O}\left(\frac{1}{z}\right), \quad a = 1, \dots, n.$$
 (2.72)

Then

$$\Psi(\mathbf{t}, z)\hat{\rho}(z)\Psi^{\dagger}(\mathbf{t}, z) = \mathbf{1}.$$
(2.73)

Proof: The differentials (2.63) are holomorphic on $C \setminus (P_1 \cup \cdots \cup P_n)$. At infinity they behave in the same way as the differentials $\Omega_j^i(P)$ (see Proposition 2.7 above). For an arbitrary complex number z away from the ramification points of C consider the sum

$$\Omega_j^i(\mathbf{t},(z,w_1(z))+\cdots+\Omega_j^i(\mathbf{t},(z,w_n(z)))=\Psi_1^i(\mathbf{t},z)\rho_1(z)\Psi_j^{\dagger}(\mathbf{t},z)dz+\cdots+\Psi_n^i(\mathbf{t},z)\rho_n(z)\Psi_j^{\dagger}(\mathbf{t},z)dz.$$

This is a well-defined differential on \mathbf{P}^1 . It can have poles only at $z = \infty$, namely, a simple pole for $i \neq j$ and a double pole $\sim dz$ for i = j. Therefore the above sum is equal to $\delta^i_j dz$.

Corollary 2.5 The matrix series $M_a(\mathbf{t}, z)$ coincide with

$$M_a(\mathbf{t}, z) = \Psi(\mathbf{t}, z) E_a \Psi^{-1}(\mathbf{t}, z), \quad a = 1, \dots n.$$
 (2.74)

Lemma 2.6 1) The matrix $\Psi(\mathbf{t}, z)$ satisfies

$$\frac{\partial}{\partial t_k^a} \Psi(\mathbf{t}, z) = U_{a,k}(\mathbf{t}, z) \Psi(\mathbf{t}, z) \quad \forall \ a, \ b = 1, \dots, n, \quad k \ge -1$$
(2.75)

where $U_{a,k}(\mathbf{t}, z)$ are given by (2.66).

- 2) The matrix polynomials $U_{a,k}(\mathbf{t}, z)$ satisfy eqs. (A.1) of the n-wave hierarchy.
- 3) The matrix series $M_b(\mathbf{t}, z)$ satisfy

$$\frac{\partial M_b(z)}{\partial t_k^a} = \left[U_{a,k}(z), M_b(z) \right].$$
(2.76)

Proof: Let $\Psi(\mathbf{t}, z) = A(\mathbf{t}, z)e^{\phi(\mathbf{t}, z)}$ with $A(\mathbf{t}, z) = \mathbf{1} + \mathcal{O}\left(\frac{1}{z}\right)$ and denote

$$\tilde{U}_{a,k}(\mathbf{t},z) = \left(z^{k+1}M_a(\mathbf{t},z)\right)_{-}.$$

It is a power series in z^{-1} . We have

$$\frac{\partial}{\partial t_k^a}\Psi(\mathbf{t},z) - U_{a,k}(\mathbf{t},z)\Psi(\mathbf{t},z) = \left(\frac{\partial A(\mathbf{t},z)}{\partial t_k^a} \cdot A^{-1}(\mathbf{t},z) + \tilde{U}_{a,k}(\mathbf{t},z)\right)\Psi(\mathbf{t},z)$$

As the expression in the parenthesis contains only negative powers of z, the right hand side is a Baker– Akhiezer function on the curve C with the same divisor of poles and with expansion at infinity of the form

$$\left(\frac{\partial A(\mathbf{t},z)}{\partial t_k^a} \cdot A^{-1}(\mathbf{t},z) + \tilde{U}_{a,k}(\mathbf{t},z)\right) \Psi(\mathbf{t},z) = \mathcal{O}\left(\frac{1}{z}\right) e^{\phi(\mathbf{t},z)}.$$

Hence this Baker-Akhiezer function identically vanishes. This proves the first part of Lemma.

The equations (A.1) readily follow from the compatibility

$$\frac{\partial}{\partial t^a_k}\frac{\partial}{\partial t^b_l}\Psi(\mathbf{t},z) = \frac{\partial}{\partial t^b_l}\frac{\partial}{\partial t^a_k}\Psi(\mathbf{t},z).$$

Finally the eq. (2.76) follows from (2.74) and (2.75).

In a similar way one can verify validity of eq. (2.61) for the matrix $U(\mathbf{t}, z)$ defined by(2.67). It remains to prove that this matrix is polynomial in z satisfying the initial condition (2.62). To this end we consider the differentials

$$U_j^i(\mathbf{t},z)dz = w_1(z)\Omega_j^i(\mathbf{t},(z,w_1(z)) + \dots + w_n(z)\Omega_j^i(\mathbf{t},(z,w_n(z))).$$

Like in the proof of Lemma 2.4 this is a differential on \mathbf{P}^1 with poles only at infinity. Hence it must be a polynomial. Since $\Omega_i^i(0, P) = \Omega_i^i(P)$ we have U(0, z) = W(z). The Proposition is proved. \Box

We are now ready to compute the tau-function of the algebro-geometric solution. Define numbers $q_{a,k;b,l}$ as coefficients of expansions of the second kind differentials $\Omega_a^{(k)}(P)$ at $P \to P_l$

$$\Omega_a^{(k)}(P) = \delta_{ab} d\left(z^{k+1}\right) + \sum_{l \ge 0} \frac{q_{a,k;b,l}}{z^{l+2}} dz, \quad P \to P_b.$$
(2.77)

Alternatively these coefficients can be recovered from the expansions of the normalized second kind bi-differential [17]

$$\omega(P,Q) = d_P d_Q \log E(P,Q) = \left[\frac{\delta_{ab}}{(z_1 - z_2)^2} + \sum_{k,l \ge 0} \frac{q_{a,k;b,l}}{z_1^{k+2} z_2^{l+2}} \right] dz_1 dz_2$$
(2.78)
$$z_1 = z(P), \ z_2 = z(Q), \quad P \to P_a, \ Q \to P_b.$$

Proposition 2.7 Tau-function of the algebro-geometric solution constructed in Proposition 2.3 is equal to

$$\tau(\mathbf{t}) = e^{\frac{1}{2}\sum q_{a,k;b,l}t_k^a t_l^b} \theta\left(\sum t_k^a \mathbf{V}^{(a,k)} - \mathbf{u}_0\right)$$
(2.79)

up to multiplication by exponential of a linear function. Here

$$\mathbf{V}^{(a,k)} = \left(V_1^{(a,k)}, \dots, V_g^{(a,k)}\right).$$
(2.80)

Proof: We have to compute the generating function (A.33) of the second logarithmic derivatives of the tau-function

$$\sum \frac{\partial^2 \log \tau(\mathbf{t})}{\partial t_k^a \partial t_l^b} \frac{dz_1}{z_1^{k+2}} \frac{dz_2}{z_2^{l+2}} = \frac{\operatorname{tr}[M_a(\mathbf{t}, z_1)M_b(\mathbf{t}, z_2)]}{(z_1 - z_2)^2} dz_1 dz_2 - \frac{\delta_{ab}}{(z_1 - z_2)^2} dz_1 dz_2$$
(2.81)

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where $M_1(\mathbf{t}, z)$, $M_b(\mathbf{t}, z)$ are solutions to the equations (2.76) in the class of matrix-valued power series in z^{-1} uniquely specified by the conditions (2.74). Using the representation (2.65) we can rewrite the previous equation in the form

$$\sum \frac{\partial^2 \log \tau(\mathbf{t})}{\partial t_k^a \partial t_l^b} \frac{dz_1}{z_1^{k+2}} \frac{dz_2}{z_2^{l+2}} = \frac{\operatorname{tr}[\mathbf{\Omega}(\mathbf{t}, P)\mathbf{\Omega}(\mathbf{t}, Q)]}{(z(P) - z(Q))^2} - \frac{\delta_{ab}}{(z_1 - z_2)^2} dz_1 dz_2$$
$$z_1 = z(P), \ z_2 = z(Q), \quad P \to P_a, \ Q \to P_b$$

where the matrix entries of $\Omega(\mathbf{t}, P)$ are equal to (2.63). According to Proposition 2.6 the right hand side can be rewritten in the form

$$\frac{\mathrm{tr}[\mathbf{\Omega}(\mathbf{t},P)\mathbf{\Omega}(\mathbf{t},Q)]}{(z(P)-z(Q))^2} - \frac{\delta_{ab}}{(z_1-z_2)^2} dz_1 dz_2 = \frac{\theta(P-Q-\mathbf{u}(\mathbf{t}))\theta(P-Q+\mathbf{u}(\mathbf{t}))}{\theta^2(\mathbf{u}(\mathbf{t}))E^2(P,Q)} - \frac{\delta_{ab}}{(z_1-z_2)^2} dz_1 dz_2.$$
(2.82)

We will now use the following important identity [17]

$$\frac{\theta(P-Q-\mathbf{u}(\mathbf{t}))\theta(P-Q+\mathbf{u})}{\theta^2(\mathbf{u})E^2(P,Q)} = \sum_{i,j=1}^n \frac{\partial^2 \log \theta(\mathbf{u})}{\partial u_i \partial u_j} \omega_i(P)\omega_j(Q) + \omega(P,Q)$$

where $\omega(P,Q)$ is the normalized bi-differential (2.78). Using this identity we can expand the right hand side of eq. (2.82) at $P \to P_a$, $Q \to P_b$

$$\begin{split} &\frac{\theta(P-Q-\mathbf{u}(\mathbf{t}))\theta(P-Q+\mathbf{u}(\mathbf{t}))}{\theta^{2}(\mathbf{u}(\mathbf{t}))E^{2}(P,Q)} - \frac{\delta_{ab}}{(z_{1}-z_{2})^{2}}dz_{1}dz_{2} \\ &= \sum_{i,\,j=1}^{n} \frac{\partial^{2}\log\theta(\mathbf{u}(\mathbf{t}))}{\partial u_{i}\partial u_{j}} \sum_{k\geq 0} \frac{V_{i}^{(a,k)}}{z_{1}^{k+2}} \sum_{l\geq 0} \frac{V_{j}^{(b,l)}}{z_{2}^{l+2}}dz_{1}dz_{2} + \sum_{k,\,l\geq 0} \frac{q_{a,k;b,l}}{z_{1}^{k+2}z_{2}^{k+2}}dz_{1}dz_{2} \\ &= \sum_{k,\,l\geq 0} \frac{\partial^{2}\log\theta(\mathbf{u}(\mathbf{t}))}{\partial t_{k}^{a}\partial t_{l}^{b}} \frac{dz_{1}}{z_{1}^{k+2}} \frac{dz_{2}}{z_{2}^{l+2}} + \sum_{k,\,l\geq 0} \frac{q_{a,k;b,l}}{z_{1}^{k+2}z_{2}^{k+2}}dz_{1}dz_{2}. \end{split}$$

Comparing this expansion with (2.81) we arrive at the proof of the Proposition.

Remark 2.8 An expression similar to (2.79) is well known in the theory of KP equation and its reductions [28], [16], [23]. We emphasize that here our main task was to prove that eq. (2.79) is in agreement with the construction of the tau-function given in terms of Proposition 2.1.

Let us now proceed to the proof of eq. (1.12). The expression (A.40) for the N-th order logarithmic derivatives of the tau-function will be applied to the tau-function (2.79) of an algebro-geometric solution. Due to the previous Proposition the tau-function in the left hand side of eq. (A.40) for $N \geq 3$ can be replaced with the theta-function

$$\sum_{k_1,\dots,k_N\geq 0} \frac{\partial^N \log \tau(\mathbf{t})}{\partial t_{k_1}^{a_1}\dots \partial t_{k_N}^{a_N}} \frac{dz_1\dots dz_N}{z_1^{k_1+2}\dots z_N^{k_N+2}} = \sum_{k_1,\dots,k_N\geq 0} \frac{\partial^N \log \theta(\mathbf{u}(\mathbf{t}))}{\partial t_{k_1}^{a_1}\dots \partial t_{k_N}^{a_N}} \frac{dz_1\dots dz_N}{z_1^{k_1+2}\dots z_N^{k_N+2}} = \\ = (-1)^N \sum_{k_1,\dots,k_N\geq 0} \sum_{i_1,\dots,i_N=1}^g V_{i_1}^{(a_1,k_1)}\dots V_{i_N}^{(a_N,k_N)} \frac{\partial^N \log \theta(\mathbf{u}(\mathbf{t}))}{\partial u_{i_1}\dots \partial u_{i_N}} \frac{dz_1\dots dz_N}{z_1^{k_1+2}\dots z_N^{k_N+2}} = \\ = (-1)^N \sum_{i_1,\dots,i_N=1}^g \frac{\partial^N \log \theta(\mathbf{u}(\mathbf{t}))}{\partial u_{i_1}\dots \partial u_{i_N}} \omega_{i_1}(Q_1)\dots \omega_{i_N}(Q_N), \quad Q_1 \to P_{a_1},\dots,Q_N \to P_{a_N}$$

where the last multi-differential is considered as its expansion in negative powers of

$$z_1 = z(Q_1), \ldots, z_N = z(Q_N)$$

Let us now consider the right hand side of eq. (A.40) multiplying it, like above by $dz_1 \dots dz_N$

$$-\frac{1}{N}\sum_{s\in S_{N}}\frac{\operatorname{tr}\left[M_{a_{s_{1}}}\left(\mathbf{t}, z_{s_{1}}\right)\dots M_{a_{s_{N}}}\left(\mathbf{t}, z_{s_{N}}\right)\right]}{(z_{s_{1}}-z_{s_{2}})\dots (z_{s_{N-1}}-z_{s_{N}})(z_{s_{N}}-z_{s_{1}})}dz_{1}\dots dz_{N} = \\ = -\frac{1}{N}\sum_{s\in S_{N}}\frac{\operatorname{tr}\left[\mathbf{\Omega}\left(\mathbf{t}, Q_{s_{1}}\right)\dots \mathbf{\Omega}\left(\mathbf{t}, Q_{s_{N}}\right)\right]}{(z(Q_{s_{1}})-z(Q_{s_{2}}))\dots (z(Q_{s_{N-1}})-z(Q_{s_{N}}))(z(Q_{s_{N}})-z(Q_{s_{1}}))}$$

where Q_1, \ldots, Q_N are arbitrary points of C such that $Q_1 \to P_{a_1}, \ldots, Q_N \to P_{a_N}$. So eq. (A.40) implies that the two N-differentials coincide when the points Q_1, \ldots, Q_N go to infinity in all possible ways. Therefore these N-differentials coincide

$$\sum_{i_1,\dots,i_N=1}^{g} \frac{\partial^N \log \theta(\mathbf{u}(\mathbf{t}))}{\partial u_{i_1} \dots \partial u_{i_N}} \omega_{i_1}(Q_1) \dots \omega_{i_N}(Q_N) =$$

$$= \frac{(-1)^{N-1}}{N} \sum_{s \in S_N} \frac{\operatorname{tr} \left[\mathbf{\Omega} \left(\mathbf{t}, Q_{s_1} \right) \dots \mathbf{\Omega} \left(\mathbf{t}, Q_{s_N} \right) \right]}{\left(z(Q_{s_1}) - z(Q_{s_2}) \right) \dots \left(z(Q_{s_{N-1}}) - z(Q_{s_N}) \right) \left(z(Q_{s_N}) - z(Q_{s_1}) \right)}.$$

$$(2.83)$$

To complete the derivation of eq. (1.12) we set $\mathbf{t} = 0$ where

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{\Omega}(0, P) = \Phi(P) \frac{dz}{R_w(z, w)}$$

The Main Lemma and Main Theorem are proved.

2.4 Proof of Corollary 1.3

Let C be a compact Riemann surface of genus g > 0 and n, m a pair of positive integers.

Proposition 2.1 For sufficiently large n, m and an arbitrary collection of n pairwise distinct points P_1, \ldots, P_n there exist two rational functions z, w on C such that

- (i) the function z has simple poles at P_1, \ldots, P_n
- (ii) the function w has poles of order m at the same points and $w \neq P(z)$ for any polynomial P.

This is an easy consequence of Riemann–Roch theorem.

Corollary 2.2 An arbitrary compact Riemann surface of genus g > 0 can be represented as the spectral curve of a matrix W(z) of the form (1.1) for sufficiently large n and m.

According to the Corollary we can rewrite eq. (1.12) in the form

$$\sum_{i_{1},\dots,i_{N}=1}^{g} \frac{\partial^{N} \log \theta(\mathbf{u})}{\partial u_{i_{1}} \dots \partial u_{i_{N}}} \omega_{i_{1}}(Q_{1}) \dots \omega_{i_{N}}(Q_{N}) =$$

$$= \frac{(-1)^{N-1}}{N} \sum_{s \in S_{N}} \frac{\operatorname{tr} \left[\mathbf{\Omega} \left(Q_{s_{1}} \right) \dots \mathbf{\Omega} \left(Q_{s_{N}} \right) \right]}{\left(z(Q_{s_{1}}) - z(Q_{s_{2}}) \right) \dots \left(z(Q_{s_{N-1}}) - z(Q_{s_{N}}) \right) \left(z(Q_{s_{N}}) - z(Q_{s_{1}}) \right)}.$$
(2.84)

(cf. (2.83)). Here $\mathbf{u} \in J(C) \setminus (\theta)$ is the point of the Jacobian corresponding to the matrix W(z), the matrix-valued differential $\Omega(Q)$ equals

$$\mathbf{\Omega}(Q) = \Phi(Q) \frac{dz}{R_w(z, w)}.$$

Using the representation (2.54) of the matrix entries of this differential we can rewrite the numerator, for an arbitrary permutation $s \in S_N$ as follows

tr
$$\left[\mathbf{\Omega}\left(Q_{s_1}\right)\ldots\mathbf{\Omega}\left(Q_{s_N}\right)\right] =$$

$$\begin{split} &= \sum_{i_{1},\dots,i_{N}=1}^{n} \frac{\theta(Q_{s_{1}}-P_{i_{1}}-\mathbf{u})\theta(Q_{s_{1}}-P_{i_{2}}+\mathbf{u})}{\theta^{2}(\mathbf{u})E(P_{i_{1}},Q_{s_{1}})E(Q_{s_{1}},P_{i_{2}})\sqrt{d\zeta_{i_{1}}}\sqrt{d\zeta_{i_{2}}}} \frac{\theta(Q_{s_{2}}-P_{i_{2}}-\mathbf{u})\theta(Q_{s_{2}}-P_{i_{3}}+\mathbf{u})}{\theta^{2}(\mathbf{u})E(P_{i_{2}},Q_{s_{2}})E(Q_{s_{2}},P_{i_{3}})\sqrt{d\zeta_{i_{2}}}\sqrt{d\zeta_{i_{3}}}}\cdots \\ & \cdots \frac{\theta(Q_{s_{N-1}}-P_{i_{N-1}}-\mathbf{u})\theta(Q_{s_{N-1}}-P_{i_{N}}+\mathbf{u})}{\theta^{2}(\mathbf{u})E(P_{i_{N-1}},Q_{s_{N-1}})E(Q_{s_{N-1}},P_{i_{N}})\sqrt{d\zeta_{i_{N-1}}}\sqrt{d\zeta_{i_{N}}}} \frac{\theta(Q_{s_{N}}-P_{i_{N}}-\mathbf{u})\theta(Q_{s_{N}}-P_{i_{1}}+\mathbf{u})}{\theta^{2}(\mathbf{u})E(P_{i_{N-1}},Q_{s_{N-1}})E(Q_{s_{N-1}},P_{i_{N}}+\mathbf{u})} \frac{\theta(Q_{s_{N}}-P_{i_{N}}-\mathbf{u})\theta(Q_{s_{N}}-P_{i_{1}}+\mathbf{u})}{\theta^{2}(\mathbf{u})E(P_{i_{N}},Q_{s_{N}})E(Q_{s_{N}},P_{i_{N}})\sqrt{d\zeta_{i_{N}}}\sqrt{d\zeta_{i_{1}}}} \\ &= \sum_{i_{2}=1}^{n} \frac{\theta(Q_{s_{2}}-P_{i_{2}}-\mathbf{u})\theta(Q_{s_{1}}-P_{i_{2}}+\mathbf{u})}{\theta^{2}(\mathbf{u})E(P_{i_{2}},Q_{s_{2}})E(Q_{s_{1}},P_{i_{2}})d\zeta_{i_{2}}} \sum_{i_{3}=1}^{n} \frac{\theta(Q_{s_{3}}-P_{i_{3}}-\mathbf{u})\theta(Q_{s_{2}}-P_{i_{3}}+\mathbf{u})}{\theta^{2}(\mathbf{u})E(P_{i_{3}},Q_{s_{3}})E(Q_{s_{2}},P_{i_{3}})d\zeta_{i_{3}}} \cdots \\ &\cdots \sum_{i_{N}=1}^{n} \frac{\theta(Q_{s_{N}}-P_{i_{N}}-\mathbf{u})\theta(Q_{s_{N-1}}-P_{i_{N}}+\mathbf{u})}{\theta^{2}(\mathbf{u})E(Q_{s_{1}},Q_{s_{2}})E(Q_{s_{N}},P_{i_{1}})d\zeta_{i_{1}}}} \sum_{i_{1}=1}^{n} \frac{\theta(Q_{s_{1}}-P_{i_{1}}-\mathbf{u})\theta(Q_{s_{N}}-P_{i_{1}}+\mathbf{u})}{\theta^{2}(\mathbf{u})E(Q_{s_{N}},P_{i_{1}})d\zeta_{i_{1}}} \cdots \\ &\cdots \sum_{i_{N}=1}^{n} \frac{\theta(Q_{s_{N}}-P_{i_{N}}-\mathbf{u})\theta(Q_{s_{N-1}}-P_{i_{N}}+\mathbf{u})}{\theta^{2}(\mathbf{u})E(Q_{s_{1}},Q_{s_{1}})E(Q_{s_{N}},P_{i_{1}})d\zeta_{i_{1}}}} \sum_{i_{1}=1}^{n} \frac{\theta(Q_{s_{1}}-P_{i_{1}}-\mathbf{u})\theta(Q_{s_{N}}-P_{i_{1}}+\mathbf{u})}{\theta^{2}(\mathbf{u})E(Q_{s_{N}},P_{i_{1}})d\zeta_{i_{1}}}} \\ &= \sum_{i_{2}=1}^{n} \sum_{i_{2}=i_{1}}^{n} \sum_{i_{N}=1}^{n} \sum_{$$

In the above computation we have used the differential H_{PQ} defined by (2.57). The computation of residues is analogous to the one in (2.58). In the last line we use the short notation $z_s := z(Q_s)$. The Corollary 1.3 is proved.

For N = 3 the identity (1.21) takes the following explicit form

$$\frac{\theta^{3}(\mathbf{u}) \sum_{i,j,k=1}^{g} \frac{\partial^{3} \log \theta(\mathbf{u})}{\partial u_{i} \partial u_{j} \partial u_{k}} \omega_{i}(Q_{1}) \omega_{j}(Q_{2}) \omega_{k}(Q_{3}) = (2.85)}{\frac{\theta(Q_{1} - Q_{2} - \mathbf{u})\theta(Q_{2} - Q_{3} - \mathbf{u})\theta(Q_{3} - Q_{1} - \mathbf{u}) - \theta(Q_{1} - Q_{2} + \mathbf{u})\theta(Q_{2} - Q_{3} + \mathbf{u})\theta(Q_{3} - Q_{1} + \mathbf{u})}{E(Q_{1}, Q_{2})E(Q_{2}, Q_{3})E(Q_{3}, Q_{1})}$$

and for N = 4

$$\theta^{4}(\mathbf{u}) \sum_{i,j,k,l=1}^{g} \frac{\partial^{4} \log \theta(\mathbf{u})}{\partial u_{i} \partial u_{j} \partial u_{k} \partial u_{l}} \omega_{i}(Q_{1}) \omega_{j}(Q_{2}) \omega_{k}(Q_{3}) \omega_{l}(Q_{4}) =$$

$$= V_{\mathbf{u}}(Q_{1}, Q_{2}, Q_{3}, Q_{4}) + V_{\mathbf{u}}(Q_{1}, Q_{3}, Q_{2}, Q_{4}) + V_{\mathbf{u}}(Q_{1}, Q_{3}, Q_{4}, Q_{2})$$
(2.86)

where

$$V_{\mathbf{u}}(Q_1, Q_2, Q_3, Q_4) = -\frac{\theta(Q_1 - Q_2 + \mathbf{u}) \dots \theta(Q_4 - Q_1 + \mathbf{u}) + \theta(Q_1 - Q_2 - \mathbf{u}) \dots \theta(Q_4 - Q_1 - \mathbf{u})}{E(Q_1, Q_2)E(Q_2, Q_3)E(Q_3, Q_4)E(Q_4, Q_1)}$$

3 Examples

3.1 Hyperelliptic case

Consider a 2×2 matrix polynomial of the form

$$W(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & -a(z) \end{pmatrix}, \quad \deg a(z) = g + 1, \quad \deg b(z) = \deg c(z) = g, \tag{3.1}$$

the polynomial a(z) is monic. The spectral curve

$$w^2 = Q(z), \quad Q(z) = -\det W(z) = z^{2g+2} + q_1 z^{2g+1} + \dots + q_{2g+2}$$
 (3.2)

is hyperelliptic. It has two distinct points P_\pm at infinity,

$$w = \pm z^{g+1} + \dots, \quad (z, w) \to P_{\pm}.$$

We have

$$\Pi(z,w) = \frac{1}{2} \frac{w + W(z)}{w}$$
(3.3)

so the basic idempotents of the matrix W(z) take the form

$$M_{\pm}(z) = \frac{1}{2} \pm \frac{1}{2} \frac{W(z)}{w(z)}, \quad w(z) = z^{g+1} \sqrt{1 + \frac{q_1}{z} + \dots + \frac{q_{2g+2}}{z^{2g+2}}}$$

Thus eq. (1.11) for the bi-differential takes the following form

$$\frac{\theta \left(Q_1 - Q_2 - \mathbf{u}_0\right)\theta \left(Q_1 - Q_2 + \mathbf{u}_0\right)}{\theta^2(\mathbf{u}_0)E(Q_1, Q_2)^2} = \frac{b(z_1)c(z_2) + 2a(z_1)a(z_2) + b(z_2)c(z_1) + 2w_1w_2}{4(z_1 - z_2)^2w_1w_2} \, dz_1 \, dz_2,$$

 $Q_1 = (z_1, w_1), Q_2 = (z_2, w_2).$ Since

$$M_{+}(z) - M_{-}(z) = \frac{W(z)}{w(z)}$$

and the time-derivatives satisfy

$$\frac{\partial}{\partial t_k^+} + \frac{\partial}{\partial t_k^-} = 0$$

we introduce

$$\frac{\partial}{\partial t_k} = \frac{\partial}{\partial t_k^+} - \frac{\partial}{\partial t_k^-}$$

So eq. (A.40) for the tau-function of the spectral curve (3.2) reduces to

$$\sum_{k_1,\dots,k_N} \frac{\frac{\partial^N \log \tau(0)}{\partial t_{k_1}\dots \partial t_{k_N}}}{z_1^{k_1+2}\dots z_N^{k_N+2}} = -\frac{1}{N} \frac{1}{w(z_1)\dots w(z_N)} \sum_{s \in S_N} \frac{\operatorname{tr}\left[W(z_{s_1})\dots W(z_{s_N})\right]}{(z_{s_1}-z_{s_2})\dots (z_{s_{N-1}}-z_{s_N})(z_{s_N}-z_{s_1})} - \frac{2\delta_{N,2}}{(z_1-z_2)^2}$$
(3.4)

for any $N \ge 2$. First few logarithmic derivatives $F_{i_1...i_N} := \partial^N \log \tau(0) / \partial t_{k_1} \dots \partial t_{k_N}$ read

$$\begin{split} F_{00} &= -b_1c_1, \quad F_{01} = 2a_1b_1c_1 - b_2c_1 - b_1c_2, \\ F_{11} &= \frac{1}{2}(-8a_1^2b_1c_1 + 4a_2b_1c_1 + 6a_1b_2c_1 - 2b_3c_1 + b_1^2c_1^2 + 6a_1b_1c_2 - 4b_2c_2 - 2b_1c_3) \\ F_{000} &= 2(b_1c_2 - b_2c_1), \quad F_{001} = 2(a_1b_2c_1 - b_3c_1 - a_1b_1c_2 + b_1c_3) \\ F_{0000} &= 4(2a_2b_1c_1 - a_1b_2c_1 - b_3c_1 - a_1b_1c_2 + 2b_2c_2 - b_1c_3). \end{split}$$

We do not specify the genus: the above expressions are valid for any $g \ge 2$; for g = 1 one has to set $a_3 = b_3 = c_3 = 0$.

The theta-function of the spectral curve is related to the tau-function by the equation⁴

$$\tau(\mathbf{t}) = e^{\frac{1}{2}\sum_{i,j}q_{ij}t_it_j}\theta\left(\sum_k t_k \mathbf{V}^{(k)} - \mathbf{u}_0\right), \quad \mathbf{t} = (t_0, t_1, \dots)$$
(3.5)

with suitable coefficients q_{ij} (cf. eq. (2.78) above). The vectors $\mathbf{V}^{(k)} = \left(V_1^{(k)}, \ldots, V_g^{(k)}\right)$ have the form

$$V_i^{(k)} = \alpha_{i1}r_k + \alpha_{i2}r_{k-1} + \dots + \alpha_{ig}r_{k-g+1}, \quad i = 1, \dots, g, \quad k \ge 0$$
(3.6)

where the $g \times g$ matrix (α_{ij}) is the inverse, up to a factor $2\pi\sqrt{-1}$ to the matrix of *a*-periods of the following holomorphic differentials

$$(\alpha_{ij}) = 2\pi\sqrt{-1} \left(\oint_{a_j} z^{g-i} \frac{dz}{2w}\right)^{-1}$$
(3.7)

and the rational numbers r_k come from the expansion

$$\left(1 + \frac{q_1}{z} + \dots + \frac{q_{2g+2}}{z^{2g+2}}\right)^{-1/2} = \sum_{k \ge 0} \frac{r_k}{z^k};$$

The point \mathbf{u}_0 is given by

$$\mathbf{u}_0 = \sum_{j=1}^{g+1} \left(\int_{P_+}^{Q_j} \omega_1, \dots, \int_{P_+}^{Q_j} \omega_g \right) - \boldsymbol{\varpi}$$
(3.8)

$$\omega_i = (\alpha_{i1}z^{g-1} + \dots + \alpha_{ig})\frac{dz}{2w}, \quad i = 1, \dots, g$$
(3.9)

⁴Like above the eq. (3.5) holds true up to multiplication by exponential of a linear function of t_i .

and the half-period ϖ for a suitable choice of the basis of cycles (see details in [17]) has the form

$$\boldsymbol{\varpi} = \pi \sqrt{-1}(1, 0, 1, 0, \dots) + \frac{1}{2} \sum_{i=1}^{g} \left(B_{1i}, B_{2i}, \dots, B_{gi} \right).$$
(3.10)

Points of the divisor $D_0 = Q_1 + \cdots + Q_{g+1}$ of poles of the normalized eigenvector of the matrix W(z) have the form $Q_i = (z_i, w_i)$ where z_1, \ldots, z_{g+1} are roots of the equation

$$a(z) = \frac{1}{2}(b(z) + c(z))$$
(3.11)

and

$$w_i = \frac{1}{2}(c(z_i) - b(z_i)), \quad i = 1, \dots, g+1.$$
 (3.12)

The Corollary 1.3 in this particular case takes the following form

Corollary 3.1 Assume rationality of coefficients of the polynomials a(z), b(z), c(z). Then for any $N \ge 3$ and an arbitrary choice of indices $k_1, \ldots, k_N \ge 0$ one has

$$\sum_{i_1,\dots,i_N=1}^g V_{i_1}^{(k_1)} \dots V_{i_N}^{(k_N)} \frac{\partial^N \log \theta(\mathbf{u}_0)}{\partial u_{i_1} \dots \partial u_{i_N}} \in \mathbb{Q}.$$

Here $\theta = \theta(\mathbf{u}|B)$ is the theta-function (1.6) of the hyperelliptic curve (3.2) and the point \mathbf{u}_0 is given by (3.8), (3.11), (3.12).

3.2 Three-sheet Riemann surfaces

Let

$$W(z) = z^m B^0 + z^{m-1} B^1 + \dots + B^m, \quad B^k = \left(B^k_{ij}\right)_{1 \le i,j \le 3}, \quad B^0 = \operatorname{diag}\left(b^0_1, b^0_2, b^0_3\right)$$
(3.13)

be a 3×3 matrix polynomial satisfying $b_i^0 \neq b_j^0$ for $i \neq j$ and tr W(z) = 0. Let

$$w^{3} + p(z)w + q(z) = \det(w \cdot \mathbf{1} - W(z))$$

be the characteristic polynomial. The genus of the spectral curve C is equal to g = 3m - 2. The spectral projectors of W(z) are given by branches of the algebraic function

$$\Pi(z,w) = \frac{W^2 + wW + w^2 + p(z)}{3w^2 + p(z)}, \quad (z,w) \in C.$$
(3.14)

The expession for the bidifferential (1.11) takes the following form

$$\frac{\theta \left(Q_1 - Q_2 - \mathbf{u}_0\right) \theta \left(Q_1 - Q_2 + \mathbf{u}_0\right)}{\theta^2(\mathbf{u}_0) E(Q_1, Q_2)^2} = (3.15)$$

$$\frac{\operatorname{tr} \left[W_1^2 W_2^2 + (w_1 W_2 + w_2 W_1) W_1 W_2 + w_1 w_2 W_1 W_2\right] - 2(p_1 p_2 + w_1^2 p_2 + w_2^2 p_1 + 3w_1^2 w_2^2)}{(3w_1^2 + p_1)(3w_2^2 + p_2)(z_1 - z_2)^2} dz_1 dz_2$$

where $Q_i = (z_i, w_i)$ and we use short notations

$$W_i = W(z_i), \quad p_i = p(z_i), \quad i = 1, 2.$$

The first few logarithmic derivatives of the tau-function (1.19) read as follows

$$\begin{split} F_{00}^{ij}[W] &= \frac{b_{ij}^{1}b_{ji}^{1}}{(b_{i}^{0} - b_{j}^{0})^{2}}, \quad i \neq j \\ F_{10}^{ij}[W] &= \frac{b_{ij}^{2}b_{ji}^{1} + b_{ji}^{2}b_{ij}^{1}}{(b_{i}^{0} - b_{j}^{0})^{2}} - 2(b_{ii}^{1} - b_{jj}^{1})\frac{b_{ij}^{1}b_{ji}^{1}}{(b_{i}^{0} - b_{j}^{0})^{3}}, \quad i \neq j \\ F_{000}^{123}[W] &= \frac{b_{12}^{1}b_{23}^{1}b_{31}^{1} - b_{13}^{1}b_{32}^{1}b_{21}^{1}}{(b_{1}^{0} - b_{2}^{0})(b_{2}^{0} - b_{3}^{0})(b_{3}^{0} - b_{1}^{0})} \\ F_{000}^{iij}[W] &= \frac{b_{ij}^{1}b_{jk}^{1}b_{ki}^{1} - b_{ik}^{1}b_{kj}^{1}b_{ji}^{1}}{(b_{i}^{0} - b_{j}^{0})^{2}(b_{k}^{0} - b_{i}^{0})} + \frac{b_{ij}^{2}b_{ji}^{1} - b_{ji}^{2}b_{ij}^{1}}{(b_{i}^{0} - b_{j}^{0})^{2}}, \quad i \neq j, \quad k \neq i, j \end{split}$$

etc. One can also compute the derivatives of the above type for i = j using the identities

$$\sum_{a=1}^{n} \frac{\partial}{\partial t_{k}^{a}} = 0, \quad \forall \ k \ge 0.$$

A Appendix. Tau-function of the *n*-wave integrable system

The (complexified) *n*-wave system [26] is an infinite family of pairwise commuting systems of nonlinear PDEs for n(n-1) functions y_{ij} , $i \neq j$ of infinite number of independent variables t_k^a , $a = 1, \ldots, n$, $k \geq 0$ called *times*. We will often use an alternative notation for the variables $t_0^a =: x^a$, $a = 1, \ldots, n$ that will be called *spatial variables*.

The equations of the n-wave hierarchy (also called AKNS-D hierarchy, see [8]) are written as conditions of commutativity of linear differential operators

$$L_{a,k} = \frac{\partial}{\partial t_k^a} - U_{a,k}(\mathbf{y}; z), \quad a = 1, \dots, n, \quad k \ge 0$$
$$[L_{a,k}, L_{b,l}] = 0 \tag{A.1}$$

where $U_{a,k}(\mathbf{y}; z)$ is a $n \times n$ matrix-valued polynomial in z of degree k + 1 depending polynomially on the functions y_{ij} and their derivatives in x^1, \ldots, x^n . For k = 0 one has

$$U_{a,0} = z E_a - [E_a, Y]$$
(A.2)

where the diagonal $n \times n$ matrix E_a has only one nonzero entry

$$(E_a)_{ij} = \delta_{ia}\delta_{aj},$$

the $n \times n$ matrix Y has the form

$$Y = (y_{ij}), \quad y_{ii} = 0.$$

The commutativity

$$[L_{a,0}, L_{b,0}] = 0 \tag{A.3}$$

implies the system of constraints

$$\sum_{k=1}^{n} \frac{\partial y_{ij}}{\partial x^k} = 0$$

$$\frac{\partial y_{ij}}{\partial x^k} = y_{ik} y_{kj}, \quad \text{the indices} \quad i, j, k \quad \text{are pairwise distinct.}$$
(A.4)

For n = 2 the second part of the constraints is empty.

In order to construct the matrix polynomials $U_{a,k}$ for k > 0 we will use the following procedure [11] that can be considered as a generalization of the well-known AKNS construction developed for n = 2 in the seminal paper [1].

Consider an arbitrary function⁵ $Y(\mathbf{x})$ satisfying the system (A.4). We are looking for solutions to the following system of linear differential equations

$$\frac{\partial M}{\partial x^a} = [U_{a,0}, M] \quad \Leftrightarrow \quad [L_{a,0}, M] = 0, \quad a = 1, \dots, n.$$
(A.5)

for a matrix-valued function $M = M(\mathbf{x}, z)$ of the form

$$M(\mathbf{x}, z) = \sum_{k \ge 0} \frac{M_k(\mathbf{x})}{z^k}.$$
(A.6)

Compatibility of this overdetermined system of differential equations follows from (A.3). Observe that the coefficients of the characteristic polynomial det $(M(\mathbf{x}, z) - w \cdot \mathbf{1})$ of the matrix $M(\mathbf{x}, z)$ are first integrals of the system (A.5).

Proposition A.1 For an arbitrary solution $Y(\mathbf{x})$ to the system (A.3), (A.4) there exist unique matrix series of the form

$$M_a(\mathbf{x}, z) = E_a + \sum_{k \ge 1} \frac{B_{a,k}(\mathbf{x})}{z^k}, \quad a = 1, \dots, n$$
 (A.7)

satisfying (A.5) as well as the following equations

$$M_a(\mathbf{x}, z)M_b(\mathbf{x}, z) = \delta_{ab}M_a(\mathbf{x}, z), \quad M_1(\mathbf{x}, z) + \dots + M_n(\mathbf{x}, z) = \mathbf{1}.$$
 (A.8)

Proof: We begin with the recursion procedure for computing the coefficients of the expansion (A.6). Clearly M_0 must be a constant diagonal matrix. Other coefficients can be determined by the following procedure.

Lemma A.2 For an arbitrary solution $Y = Y(\mathbf{x})$ to eqs. (A.4) and an arbitrary diagonal matrix $B = \text{diag}(b_1, \ldots, b_n)$ with pairwise distinct diagonal entries there exists a unique solution

$$M = M_B(\mathbf{x}, z) = B + \sum_{k \ge 1} \frac{M_{B,k}(\mathbf{x})}{z^k}$$
(A.9)

to the system (A.5) normalized by the condition

$$\det \left(M_B(\mathbf{x}, z) - w \cdot \mathbf{1} \right) = \det \left(B - w \cdot \mathbf{1} \right). \tag{A.10}$$

⁵Here and below saying "functions" we have in mind just formal power series in the independent variables.

Proof: Split every coefficient into its diagonal and off-diagonal part

$$M_{B,k} = D_k + C_k, \quad k \ge 1.$$

Vanishing of the constant term in (A.5) implies

$$[E_a, M_{B,1}] = [B, [E_a, Y]] = [E_a, [B, Y]], \quad a = 1, \dots, n.$$

This system uniquely determines the off-diagonal part of the matrix $M_{B,1}$

$$C_1 = [B, Y].$$

To determine the diagonal part D_1 we use the coefficient of 1/z of eq. (A.10). The off-diagonal part C_1 does not contribute to this coefficient, so we obtain

$$\sum_{m=1}^{n} D_{1mm} \prod_{s \neq m} (b_s - w) = 0 \quad \Rightarrow \quad D_1 = 0.$$

We proceed by induction. Assume that the matrices $D_1, \ldots, D_{k-1}, C_1, \ldots, C_{k-1}$ are already computed so that equations (A.5), (A.10) hold true modulo $\mathcal{O}(1/z^{k-1})$ and $\mathcal{O}(1/z^k)$ respectively. From the coefficient of $1/z^{k-1}$ in (A.5) we have

$$[E_a, C_k] = \frac{\partial C_{k-1}}{\partial x^a} + [[E_a, Y], C_{k-1}]_{\text{off-diag}} - [E_a, [D_{k-1}, Y]]$$

From this equation we can compute for any $i \neq a$ the (a, i)- and (i, a)-entries of the matrix C_k . Since a is an arbitrary number between 1 and n we obtain the full off-diagonal matrix C_k . Equating to zero the coefficient of $1/z^k$ in (A.10) we obtain the diagonal matrix D_k

$$D_{kmm} = -\prod_{s \neq m} (b_s - b_m)^{-1} \times \text{coefficient of} \quad \frac{1}{z^k} \quad \text{in} \quad \det \left(B + \sum_{i \leq k-1} \frac{D_i + C_i}{z^i} - b_m \cdot \mathbf{1} \right).$$

We will now prove that there exists a matrix-valued series

$$A(\mathbf{x}, z) = \mathbf{1} + \sum_{k \ge 1} \frac{A_k(\mathbf{x})}{z^k}$$
(A.11)

such that

$$A^{-1}(\mathbf{x}, z)M_B(\mathbf{x}, z)A(\mathbf{x}, z) = B$$
(A.12)

for any diagonal matrix B.

Lemma A.3 For any $Y(\mathbf{x})$ satisfying eqs. (A.3) there exists a solution

$$\Psi(\mathbf{x}, z) = A(\mathbf{x}, z)e^{z \operatorname{diag}(x^1, \dots, x^n)}$$
(A.13)

to the following system of linear differential equations

$$\frac{\partial \Psi}{\partial x^a} = U_{a,0}(\mathbf{x}, z)\Psi(\mathbf{x}, z), \quad a = 1, \dots, n$$
 (A.14)

where the matrix series $A(\mathbf{x}, z)$ has the form (A.11).

Proof: For the coefficients $A_k = A_k(\mathbf{x})$ we obtain

$$[E_a, A_k] = \frac{\partial A_{k-1}}{\partial x^a} + [E_a, Y]A_{k-1}, \quad a = 1, \dots, n.$$

From this system we uniquely determine the off-diagonal part of the matrix A_k . Using the next equation $k \mapsto k + 1$ we arrive at

$$\frac{\partial}{\partial x^a} \left(A_k \right)_{\text{diag}} = -\left([E_a, Y] A_k \right)_{\text{diag}}$$

The off-diagonal part of A_k does not contribute to the right hand side. So the matrix A_k is determined uniquely up to adding a constant diagonal matrix.

Remark A.4 From the proof it follows that the matrix $\Psi(\mathbf{x}, z)$ is determined by eqs. (A.14) uniquely up to a multiplication on the right by a diagonal matrix series in 1/z

$$\Psi(\mathbf{x}, z) \mapsto \Psi(\mathbf{x}, z)\Delta(z), \quad \Delta(z) = \mathbf{1} + \sum_{k=0}^{\infty} \frac{\Delta^k}{z^{k+1}}, \quad \Delta^k = \operatorname{diag}\left(\Delta_1^k, \dots, \Delta_n^k\right)$$
(A.15)

Lemma A.5 For any diagonal matrix B the solution $M_B(\mathbf{x}, z)$ to the equations (A.5), (A.10) can be represented in the form

$$M_B(\mathbf{x}, z) = A(\mathbf{x}, z)BA^{-1}(\mathbf{x}, z)$$
(A.16)

where the matrix $A(\mathbf{x}, z)$ is defined in the previous Lemma.

Proof: Since

$$A(\mathbf{x}, z)B A^{-1}(\mathbf{x}, z) = \Psi(\mathbf{x}, z)B \Psi^{-1}(\mathbf{x}, z)$$

the matrix (A.16) satisfies eqs. (A.5). Obviously it also satisfies (A.10). Due to uniqueness of such a solution to (A.5), (A.10) the Lemma is proved. \Box

We are now in a position to complete the proof of Proposition A.1. Due to uniqueness the matrix $M_B(\mathbf{x}, z)$ depends linearly on $B = \text{diag}(b_1, \ldots, b_n)$. So the construction can be extended to an arbitrary diagonal matrix B (see also eq. (A.16)). Put

$$M_a(\mathbf{x}, z) = M_{E_a}(\mathbf{x}, z), \quad a = 1, \dots, n.$$
(A.17)

These matrices clearly satisfy eqs. (A.5) and (A.8). It remains to prove uniqueness.

Since the matrices $M_1(\mathbf{x}, z), \ldots, M_n(\mathbf{x}, z)$ commute pairwise due to (A.8) and $M_a \to E_a$ for $z \to \infty$, we can look for their common eigenvectors in $\mathbb{C}^n \otimes \mathbb{C}[[z^{-1}]]$. Every matrix $M_a = M_a(\mathbf{x}, z)$ has only one non-zero eigenvalue; the corresponding eigenvector

$$M_a \mathbf{f}_a = \mathbf{f}_a$$

can be normalized in such a way that $(\mathbf{f}_a)_b = \delta_{ab} + \mathcal{O}(1/z)$. It is determined uniquely up to multiplication

$$\mathbf{f}_a \mapsto c_a(z)\mathbf{f}_a, \quad c_a(z) = 1 + \mathcal{O}\left(\frac{1}{z}\right) \in \mathbb{C}\left[z^{-1}\right].$$

Denote $A(\mathbf{x}, z)$ the matrix whose columns are the eigenvectors $\mathbf{f}_1, \ldots, \mathbf{f}_n$. According to the previous arguments this matrix is uniquely defined up to a multiplication on the right by diag $(c_1(z), \ldots, c_n(z))$ and satisfies

$$M_a(\mathbf{x}, z) = A(\mathbf{x}, z) E_a A^{-1}(\mathbf{x}, z), \quad a = 1, \dots, n.$$

The Proposition is proved.

We will now slightly modify the setting of Proposition A.1 in order to apply it to the construction of the *n*-wave hierarchy. Denote \mathcal{Y} the ring of polynomials in variables y_{ij} , $\partial y_{ij}/\partial x^k$, $\partial^2 y_{ij}/\partial x^k \partial x^l$ etc. satisfying the constraints (A.4) along with their differential consequences. Elements of this ring will be denoted like $P(\mathbf{y})$ where P is a polynomial. The commuting derivations $\partial/\partial x^1$, ..., $\partial/\partial x^n$ naturally act on this ring.

So, consider equations of the form (A.5)

$$\frac{\partial M}{\partial x^a} = [zE_a - [E_a, Y], M], \quad a = 1, \dots, n$$
(A.18)

as equations for matrices

$$M = M(\mathbf{y}, z) = M_0 + \sum_{k \ge 1} \frac{M_k(\mathbf{y})}{z^k}.$$

Proposition A.6 There exists a unique collection of matrix series

$$M_{a}(\mathbf{y}, z) = E_{a} + \sum_{k \ge 1} \frac{B_{a,k}(\mathbf{y})}{z^{k}} \in Mat_{n}(\mathcal{Y}) \otimes \mathbb{C}\left[z^{-1}\right], \quad a = 1, \dots, n$$
(A.19)

satisfying (A.18) and also

$$M_a(\mathbf{y}, z)M_b(\mathbf{y}, z) = \delta_{ab}M_a(\mathbf{y}, z), \quad M_1(\mathbf{y}, z) + \dots + M_n(\mathbf{y}, z) = \mathbf{1}.$$
 (A.20)

The proof essentially repeats the above arguments so it will be omitted.

Remark A.7 In practical computations of the coefficients $B_{a,k}(\mathbf{y})$ instead of the normalization (A.10) one can alternatively use the following one:

$$B_{a,k}(0) = 0, \quad a = 1, \dots, n, \quad k \ge 1.$$

Explicitly

$$M_{a} = E_{a} + \frac{B_{a,1}}{z} + \frac{B_{a,2}}{z^{2}} + \frac{B_{a,3}}{z^{3}} + \mathcal{O}\left(\frac{1}{z^{4}}\right)$$

$$B_{a,1} = -[E_{a}, Y]$$

$$(B_{a,2})_{ij} = \begin{cases} -\frac{\partial y_{ij}}{\partial x^{a}}, & i \neq j \\ -y_{ia}y_{ai}, & j = i \neq a \\ \sum_{s} y_{as}y_{sa}, & i = j = a \end{cases}$$

$$(B_{a,3})_{ij} = \begin{cases} \frac{\partial^{2}y_{aj}}{\partial x^{a2}} - 2y_{aj}\sum_{s} y_{as}y_{sa}, & i = a, j \neq a \\ \frac{\partial^{2}y_{ia}}{\partial x^{a2}} + 2y_{ia}\sum_{s} y_{as}y_{sa}, & i = j = a \end{cases}$$

$$\sum_{s} y_{sa}\frac{\partial y_{as}}{\partial x^{a}} - \frac{\partial y_{sa}}{\partial x^{a}}y_{as}, & i = a, j \neq a \\ \frac{\partial^{2}y_{ia}}{\partial x^{a2}} - 2y_{aj}\sum_{s} y_{as}y_{sa}, & i = a, j \neq a \end{cases}$$

Define matrix-valued polynomials

$$U_{a,k}(\mathbf{y},z) = \left(z^{k+1}M_a(\mathbf{y},z)\right)_+ \in Mat_n\left(\mathcal{Y}\right) \otimes \mathbb{C}[z], \quad a = 1,\dots,n, \quad k \ge -1.$$
(A.21)

Here and below the notation ()₊ will be used for the polynomial part of a Laurent series in 1/z. The matrix-valued polynomials $U_{a,k}$ are exactly those that appear in the formulation (A.1) of equations of the *n*-wave hierarchy that can be rewritten in the following form

$$\frac{\partial Y}{\partial t_k^a} = (B_{a,k+2}(\mathbf{y}))_{\text{off-diagonal}}.$$
(A.22)

For n = 2 it coincides with the complexified nonlinear Schrödinger hierarchy, also known as the AKNS hierarchy. Observe that the t^a_{-1} -flows generate just conjugations by diagonal matrices

$$y_{ij} \mapsto \frac{\lambda_i}{\lambda_j} y_{ij}, \quad i, j = 1, \dots, n.$$
 (A.23)

Such transformations are symmetries of the n-wave hierarchy (A.22).

Remark A.8 The dependence of the functions $y_{ij}(\mathbf{x})$ is uniquely determined by their restriction onto any line

$$x^i = a_i x, \quad i = 1, \dots, n$$

for arbitrary pairwise distinct constants a_1, \ldots, a_n . Indeed, we reconstruct all partial derivatives in x^1, \ldots, x^n

$$\frac{\partial y_{ij}}{\partial x^k} = \begin{cases} y_{ik}y_{kj}, & k \neq i, j \\ \frac{y'_{ij}}{a_i - a_j} + \sum_s \frac{a_j - a_s}{a_i - a_j} y_{is}y_{sj}, & k = i \\ \frac{y'_{ij}}{a_j - a_i} + \sum_s \frac{a_i - a_s}{a_j - a_i} y_{is}y_{sj}, & k = j \end{cases}$$

starting from the derivatives $y'_{ij} = dy_{ij}/dx$ in x. So for every pair of indices (a, k) the equation (A.22) can be considered as a system of n(n-1) partial differential equations with one space variable x and one time variable t^a_k .

Let $Y(\mathbf{t})$ be a solution to the *n*-wave hierarchy. Then the matrices $M_a(\mathbf{y}, z)$ become well-defined functions $M_a(\mathbf{t}, z)$ of \mathbf{t} . So do the matrix-valued polynomials $U_{a,p} = U_{a,p}(\mathbf{t}, z)$.

Proposition A.9 The matrix-valued series $M_b = M_b(\mathbf{t}, z)$ satisfy

$$[L_{a,k}, M_b] = 0 \iff \frac{\partial M_b(\mathbf{t}, z)}{\partial t_k^a} = [U_{a,k}(\mathbf{t}, z), M_b(\mathbf{t}, z)] \quad \forall \ a, \ b = 1, \dots, n, \quad k \ge -1.$$
(A.24)

Proof: It suffices to verify validity of eq. (A.24) for the series $M_b = M_b(\mathbf{y}, z)$ and polynomials $U_{a,k}(\mathbf{y}, z)$. Let

$$\tilde{M}_b = \frac{\partial M_b}{\partial t_k^a} - [U_{a,k}, M_b].$$

It is easy to check that this matrix-valued Laurent series satisfies

$$\frac{\partial \tilde{M}_b}{\partial x^c} = \left[U_{c,0}, \tilde{M}_b \right]$$

for any c = 1, ..., n. Let us now check that the expansion of \tilde{M}_b contains only strictly negative powers of z. To this end define the matrix series

$$V_{a,k}(\mathbf{y},z) = \left(z^{k+1}M_a(\mathbf{y},z)\right)_{-}$$

so that

$$z^{k+1}M_a(\mathbf{y}, z) = U_{a,k}(\mathbf{y}, z) + V_{a,k}(\mathbf{y}, z).$$

Therefore

$$\tilde{M}_b = \frac{\partial M_b}{\partial t_k^a} + [V_{a,k}, M_b] \in Mat_n(\mathcal{Y}) \otimes z^{-1}\mathbb{C}\left[z^{-1}\right].$$

So, the series $\tilde{M}_b = \tilde{M}_b(\mathbf{y}, z)$ satisfies eqs. (A.18) and contains only strictly negative powers of z. Due to uniqueness it is equal to zero.

Lemma A.10 The matrices $M_a(\mathbf{t}, z)$ satisfy the identities (A.20)

Proof: Using eq. (A.24) we prove that

$$\frac{\partial}{\partial t_k^c} \left(M_a M_b - \delta_{ab} M_a \right) = 0.$$

Definition A.11 A wave function $\Psi = \Psi(\mathbf{t}, z)$ of the solution $Y(\mathbf{t})$ is a solution to the infinite family of systems of linear differential equations

$$\frac{\partial}{\partial t_p^a}\Psi = U_{a,p}(\mathbf{t}, z)\Psi, \quad a = 1, \dots, n, \quad p \ge -1$$
(A.25)

of the form

$$\Psi(\mathbf{t}, z) = A(\mathbf{t}, z)e^{\phi(\mathbf{t}, z)}$$

$$A(\mathbf{t}, z) = \mathbf{1} + \frac{A^0(\mathbf{t})}{z} + \frac{A^1(\mathbf{t})}{z^2} + \dots,$$

$$\phi(\mathbf{t}, z) = \sum_{k=0}^{\infty} \operatorname{diag}\left(t_k^1, \dots, t_k^n\right) z^{k+1}$$
(A.26)

where $A^0(\mathbf{t})$, $A^1(\mathbf{t})$ etc. are $n \times n$ matrix-valued functions of \mathbf{t} .

The wave-function is determined by a solution $Y(\mathbf{t})$ uniquely up to multiplication on the right by a constant diagonal matrix-valued series

$$\Psi(\mathbf{t}, z) \mapsto \Psi(\mathbf{t}, z) \Delta(z), \quad \Delta(z) = \mathbf{1} + \sum_{k=0}^{\infty} \frac{\Delta^k}{z^{k+1}}, \quad \Delta^k = \operatorname{diag}\left(\Delta_1^k, \dots, \Delta_n^k\right)$$
(A.27)

Lemma A.12 Let $(Y(\mathbf{t}), \Psi(\mathbf{t}, z))$ be a solution to the equations of the hierarchy (A.22) and its wave function. Then the matrix-valued series $M_1(\mathbf{t}, z), \ldots, M_n(\mathbf{t}, z)$ can be represented in the form

$$M_a(\mathbf{t}, z) = \Psi(\mathbf{t}, z) E_a \Psi^{-1}(\mathbf{t}, z), \quad a = 1, \dots, n.$$
(A.28)

Proof: As

$$\Psi(\mathbf{t}, z) E_a \Psi^{-1}(\mathbf{t}, z) = A(\mathbf{t}, z) E_a A^{-1}(\mathbf{t}, z) = E_a + \mathcal{O}\left(\frac{1}{z}\right),$$

the right hand side of (A.28) is a series in inverse powers of z. It satisfies the differential equations (A.24). Due to uniqueness it coincides with $M_a(\mathbf{t}, z)$.

Introduce the following generating series for the time derivatives

$$\nabla_a(z) = \sum_{k \ge -1} \frac{1}{z^{k+2}} \frac{\partial}{\partial t_k^a}, \quad a = 1, \dots, n.$$
(A.29)

Lemma A.13 The following formula holds true

$$\nabla_a(w)\Psi(\mathbf{t},z) = \frac{M_a(\mathbf{t},w)\Psi(\mathbf{t},z)}{w-z}.$$
(A.30)

The *proof* is straightforward by using (A.21) and (A.25).

We now proceed to the definition of tau-function. It is based on the following statement (cf. [2]).

Proposition A.14 For any solution Y(t) to the system (A.25) and its wave function (A.26) there exists a function $\log \tau(t)$ such that

$$\frac{\partial \log \tau(\mathbf{t})}{\partial t_p^a} = -\operatorname{res}_{z=\infty} \operatorname{tr} \left(A_z(\mathbf{t}, z) E_a A^{-1}(\mathbf{t}, z) \right) \, z^{p+1} dz \tag{A.31}$$

Proof: We need to prove symmetry of the second derivatives

$$\frac{\partial^2 \log \tau(\mathbf{t})}{\partial t_p^a \partial t_q^b} = \frac{\partial^2 \log \tau(\mathbf{t})}{\partial t_q^b \partial t_p^a}$$

or, equivalently

$$abla_a(z_1)
abla_b(z_2)\log au(\mathbf{t}) =
abla_b(z_2)
abla_a(z_1)\log au(\mathbf{t}).$$

Using eq. (A.30) one can represent the generating series for the logarithmic derivatives of $\tau(\mathbf{t})$ in the following form

$$\nabla_a(z)\log\tau(\mathbf{t}) = \operatorname{tr}\left(A_z(\mathbf{t},z)E_aA^{-1}(\mathbf{t},z)\right) = \operatorname{tr}\left(\Psi_z(\mathbf{t},z)E_a\Psi^{-1}(\mathbf{t},z)\right) - \phi_z^a(\mathbf{t},z)$$
(A.32)

where $\phi(\mathbf{t}, z) = \text{diag} \left(\phi^1(\mathbf{t}, z), \dots, \phi^n(\mathbf{t}, z) \right)$ (see eq. (A.26) above).

Lemma A.15 The second order logarithmic derivatives of the tau-function (A.31), (A.32) can be computed from the following generating series

$$\nabla_a(z_1)\nabla_b(z_2)\log\tau(\mathbf{t}) = \frac{\operatorname{tr} M_a(\mathbf{t}, z_1)M_b(\mathbf{t}, z_2) - \delta_{ab}}{(z_1 - z_2)^2}.$$
(A.33)

Before we proceed to the proof let us observe that, using $M_a(z)M_b(z) = \delta_{ab}M_a(z)$ (see eq.(A.20) above) it readily follows that the numerator in (A.33) vanishes at $z_1 = z_2$. Hence, due to its symmetry in z_1 , z_2 it is divisible by $(z_1 - z_2)^2$. Thus the right hand side is a series in inverse powers of z_1 , z_2 . *Proof:* Using the second part of eq. (A.32) we obtain

$$\sum_{p,q=0}^{\infty} \frac{1}{w^{q+2}} \frac{1}{z^{p+2}} \frac{\partial^2 \log \tau(\mathbf{t})}{\partial t^b_q \partial t^a_p} = \nabla_b(w) \operatorname{tr} \left(\Psi_z(\mathbf{t},z) E_a \Psi^{-1}(\mathbf{t},z) \right) - \nabla_b(w) \phi^a_z(\mathbf{t},z)$$

Obviously

$$\nabla_b(w)\phi^a(\mathbf{t},z) = \frac{\delta_{ab}}{w-z} \quad \Rightarrow \quad \nabla_b(w)\phi^a_z(\mathbf{t},z) = \frac{\delta_{ab}}{(w-z)^2}$$

From (A.30) obtain

$$\nabla_b(w)\Psi_z(\mathbf{t},z) = \frac{M_b(\mathbf{t},w)\Psi_z(\mathbf{t},z)}{w-z} + \frac{M_b(\mathbf{t},w)\Psi(\mathbf{t},z)}{(w-z)^2}$$

and

$$\nabla_b(w)\Psi^{-1}(\mathbf{t},z) = -\frac{\Psi^{-1}(\mathbf{t},z)M_b(\mathbf{t},w)}{w-z}$$

Therefore

$$\nabla_b(w)\operatorname{tr}\left(\Psi_z(\mathbf{t},z)E_a\Psi^{-1}(\mathbf{t},z)\right) = \operatorname{tr}\frac{M_b(\mathbf{t},w)\Psi(\mathbf{t},z)E_a\Psi^{-1}(\mathbf{t},z)}{(w-z)^2} = \operatorname{tr}\frac{M_b(\mathbf{t},w)M_a(\mathbf{t},z)}{(z-w)^2}.$$

Summarizing we arrive at

$$\sum_{p,q=0}^{\infty} \frac{1}{w^{q+2}} \frac{1}{z^{p+2}} \frac{\partial^2 \log \tau(\mathbf{t})}{\partial t_q^b \partial t_p^a} = \operatorname{tr} \frac{M_b(\mathbf{t}, w) M_a(\mathbf{t}, z)}{(z-w)^2} - \frac{\delta_{ab}}{(w-z)^2}$$

that completes the proof of Lemma and, therefore of the Proposition.

Remark A.16 The definition (A.31), (A.32) does depend on the normalization of the wave function. A change of the normalization

$$\Psi(\mathbf{t}, z) \mapsto \Psi(\mathbf{t}, z) \Delta(z), \quad \Delta(z) = \text{diag} \left(\Delta_1(z), \dots, \Delta_n(z)\right), \quad \Delta_a(z) \in \mathbb{C}[[z^{-1}]], \quad a = 1, \dots, n$$

yields

$$\operatorname{tr}\left(\Psi_{z}(\mathbf{t},z)E_{a}\Psi^{-1}(\mathbf{t},z)\right) \mapsto \operatorname{tr}\left(\Psi_{z}(\mathbf{t},z)E_{a}\Psi^{-1}(\mathbf{t},z)\right) + \frac{d}{dz}\log\Delta_{a}(z).$$

The tau-function will change as follows

$$\tau(\mathbf{t}) \mapsto e^{\sum c_{a,p} t_p^a} \tau(\mathbf{t}), \quad \frac{d}{dz} \log \Delta_a(z) = \sum \frac{c_{a,p}}{z^{p+2}}.$$
 (A.34)

We see that the logarithmic derivatives of the tau-function of order two (and, therefore, of any higher order) belong to the ring \mathcal{Y} . In particular they do not depend on the choice of a wave function of a solution $Y(\mathbf{t})$ of the hierarchy (A.22). Explicitly,

$$\frac{\partial^2 \log \tau}{\partial t_0^a \partial t_0^b} = \begin{cases} -y_{ab} y_{ba}, & b \neq a \\ \\ \sum_s y_{as} y_{sa}, & b = a \end{cases}$$
(A.35)

$$\frac{\partial^2 \log \tau}{\partial t_0^a \partial t_1^b} = \begin{cases} \frac{\partial y_{ab}}{\partial x^b} y_{ba} - y_{ab} \frac{\partial y_{ba}}{\partial x^b}, & b \neq a \\ \\ \sum_s y_{as} \frac{\partial y_{sa}}{\partial x^s} - \frac{\partial y_{as}}{\partial x^s} y_{sa}, & b = a \end{cases}$$
(A.36)

$$\frac{\partial^2 \log \tau}{\partial t_0^a \partial t_2^b} = \begin{cases} -y_{ab} \frac{\partial^2 y_{ba}}{\partial x^{b^2}} - y_{ba} \frac{\partial^2 y_{ab}}{\partial x^{b^2}} + \frac{\partial y_{ab}}{\partial x^{b}} \frac{\partial y_{ba}}{\partial x^{b}} - 3y_{ab} y_{ba} \sum_s y_{bs} y_{sb}, \quad b \neq a \\ -\sum_{s \neq a} \frac{\partial^2 \log \tau}{\partial t_0^s \partial t_2^a}, \qquad b = a \end{cases}$$
(A.37)

etc.

For computation of the derivatives of order three and higher we will need the following

Lemma A.17 The following equations hold true for all a, b = 1, ..., n

$$\nabla_a(z_1)M_b(z_2) = \frac{[M_a(z_1), M_b(z_2)]}{z_1 - z_2}.$$
(A.38)

Here and below we omit the explicit dependence on \mathbf{t} of the matrix-valued functions $M_a(\mathbf{t}, z)$. *Proof:* it easily follows from (A.24).

Proposition A.18 The following equation holds true

$$\nabla_a(z_1)\nabla_b(z_2)\nabla_c(z_3)\log\tau(\mathbf{t}) = -\operatorname{tr}\frac{[M_a(z_1), M_b(z_2)]M_c(z_3)}{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)}.$$
(A.39)

Proof: it can be easily obtained by applying the operator $\nabla_c(z_3)$ at both sides of eq. (A.33) with the help of (A.38) and then using invariance of the trace of product of matrices with respect to cyclic permutations.

Higher order logarithmic derivatives of the tau-function can be computed using the following

Proposition A.19 For the logarithmic derivatives of order $N \ge 3$ of the tau-function of any solution $Y(\mathbf{t})$ to the n-wave hierarchy (A.22) the following expression holds true

$$\sum_{k_1,\dots,k_N \ge 0} \frac{\partial^N \log \tau(\mathbf{t})}{\partial t_{k_1}^{a_1} \dots \partial t_{k_N}^{a_N}} \frac{1}{z_1^{k_1+2} \dots z_N^{k_N+2}} = -\frac{1}{N} \sum_{s \in S_N} \frac{\operatorname{tr} \left[M_{a_{s_1}} \left(z_{s_1} \right) \dots M_{a_{s_N}} \left(z_{s_N} \right) \right]}{\left(z_{s_1} - z_{s_2} \right) \dots \left(z_{s_{N-1}} - z_{s_N} \right) \left(z_{s_N} - z_{s_1} \right)}$$
(A.40)

Proof: For N = 3 eq. (A.40) coincides with (A.39). For higher N the proof is obtained by induction using (A.38). It does not differ from the proof of a similar equation given in [4], so we omit the details.

B Another [19] definition of the principal tau-function

Proposition B.1 [19] For a given pair $(Y(\mathbf{t}), \Psi(\mathbf{t}, z))$ consisting of a solution to the hierarchy (A.22) and its wave function there exists a function $\tau(\mathbf{t})$ such that

$$\nabla_a(z)\log\tau(\mathbf{t}) = \left(\frac{\partial}{\partial z} - \nabla_a(z)\right)\log\left[A(\mathbf{t}, z)_{aa}\right].$$
(B.1)

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It is easy to see that the diagonal entries of the matrix $A(\mathbf{t}, z)$ do not depend on the variables t_{-1}^a . So, according to this definition $\frac{\partial \log \tau(\mathbf{t})}{\partial t_{-1}^a} \equiv 0$.

For example,

$$\frac{\partial \log \tau}{\partial t_0^a} = -A_{aa}^0$$
$$\frac{\partial \log \tau}{\partial t_1^a} = -2A_{aa}^1 + \left(A_{aa}^1\right)^2 - \frac{\partial A_{aa}^0}{\partial t_0^a}$$

Definition B.2 The function $\tau(\mathbf{t})$ will be called the principal tau-function of the pair $(Y(\mathbf{t}), \Psi(\mathbf{t}, z))$.

Clearly the principal tau-function of a given pair (Y, Ψ) is determined uniquely up to a nonzero constant factor.

Remark B.3 There are [19] other tau-functions in the theory of the n-wave hierarchy. The principal one is selected by the following property: it is invariant with respect to diagonal conjugations

$$Y(\mathbf{t}) \mapsto \Lambda Y(\mathbf{t}) \Lambda^{-1}, \quad \Psi(\mathbf{t}, z) \mapsto \Lambda \Psi(\mathbf{t}, z) \Lambda^{-1}, \quad \Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n),$$

$$\tau(\mathbf{t}) \mapsto \tau(\mathbf{t}).$$

In other words, it does not depend on the time variables t_{-1}^a . Thus its logarithmic derivatives, starting from the second one are combinations of the functions $y_{ij}(\mathbf{t})$ and their derivatives invariant with respect to the diagonal conjugations (A.23).

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