# **KOBAYASHI HYPERBOLICITY IN DEGREE** $\ge n^{2n}$

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ABSTRACT. For a generic hypersurface  $\mathbb{X}^{n-1} \subset \mathbb{P}^n(\mathbb{C})$  of degree

$$d \ge n^{2n}$$

(1)  $\mathbb{P}^n \setminus \mathbb{X}^{n-1}$  is Kobayashi-hyperbolically imbedded in  $\mathbb{P}^n$ ;

(2)  $\mathbb{X}^{n-1}$  is Kobayashi( $\Leftrightarrow$  Brody)-hyperbolic.

(1) improves Brotbek-Deng 1804.01719:  $d \ge (n+2)^{n+3} (n+1)^{n+3} = n^{2n} n^6 (e^3 + O(\frac{1}{n})).$ 

(2) supersedes Demailly 1801.04765:  $d \ge \frac{1}{3} \left( e^1(n-1) \right)^{2n} = n^{2n} e^{2n} \left( \frac{1}{3 e^2} + O(\frac{1}{n}) \right).$ The method gives in fact  $d \ge \frac{n^{2n}}{\text{const}^n}$  for  $n \ge N(\text{const})$  with any const  $\ge 1$ .

#### 1. Introduction

The study of degeneracy properties of entire holomorphic curves contained in projective algebraic varieties is a first (deep) step towards understanding several outstanding problems of Diophantine Geometry and Number Theory. In [7], it was shown that such nonconstant entire curves in generic projective hypersurfaces  $\mathbb{X}^n \subset \mathbb{P}^{n+1}(\mathbb{C})$  of degree  $d \ge 2^{n^5}$  must land inside some fixed proper algebraic subvariety  $\mathbb{Y} \subsetneq \mathbb{X}$ , as predicted by the Green-Griffiths conjecture. There, use was made of the algebraic Morse inequalities of Trapani in order to reduce the existence of global invariant jet differentials on  $\mathbb{X}$  to the positivity of a certain intersection number on the Semple tower. To handle the complexity and the difficulties of computations not performed in [7] within the cohomology ring of this tower, Bérczi [1] introduced equivariant localization to transform fixed point formulas into iterated residue formulas which express various intersection numbers between the tautological Semple line bundles, and he improved the degree bound to  $d \ge n^{9n}$ .

In his Ph.D. (Orsay, July 2014), Darondeau adapted and strengthened these techniques to study strong algebraic degeneracy of entire curves in complements  $\mathbb{P}^n(\mathbb{C})\setminus\mathbb{X}^{n-1}$  of generic projective hypersurfaces, and in [4] he reached  $d \ge (5n)^2 n^n$ . As is known by experts, Darondeau's calculations in the complement case extend to the compact case, with the same degree bound.

Siu [13] developed a much more effective method — concerning not only degree bounds (also exponential), but mainly explicitness of jet differentials — to establish the Kobayashi hyperbolicity conjecture.

Brotbek [2] settled this conjecture for  $\mathbb{X}^n \subset \mathbb{P}^{n+1}(\mathbb{C})$  (in a somehow more general context), using a sufficiently rich subfamily of hypersurfaces inspired from Masuda-Noguchi [10, 7.4] (after explorations and exchanges with experts) on which he had the idea of using Wronskian differential operators, associated multiplier ideal sheaves and positivity of tautological line bundles on Grassmannian varieties. From Brotbek's breakthrough,

Deng [6] deduced with short arguments the degree bound  $d \ge (n+1)^{n+2} (n+2)^{2n+7} = n^{3n+9} \left(e^5 + O(\frac{1}{n})\right).$ 

By means of an essentially smaller family of hypersurfaces for which base loci of jet differentials can be easily controlled, Demailly [5] improved this to the lower degree bound  $d \ge \lfloor \frac{1}{3} (e n)^{2n+2} \rfloor$ , still for generic  $\mathbb{X}^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ .

Lowering the degree of examples of Kobayashi-hyperbolic hypersurfaces is also a competitive topic. Currently, the best existing examples in any dimension n of such hyperbolic  $\mathbb{X}^n \subset \mathbb{P}^{n+1}(\mathbb{C})$  are due to Dinh-Tuan Huynh [8]: they have any degree  $d \ge \left(\frac{n+3}{2}\right)^2$ . The reader is referred to this publication for more information.

Recently, by means of a general grassmannian technique having some proximity with arguments of Shiffmann-Zaidenberg [12], Riedl-Yang [11] discovered that a solution to the Green-Griffiths conjecture in dimension 2n implies a solution to the Kobayashi conjecture in dimension n, with a small (linear) degree bound discrepancy. This motivates to put renewed efforts into the Green-Griffiths conjecture for projective hypersurfaces.

## 2. Theorem and its proof

Therefore, the goal of this concise note is to reveal that the estimates of Bérczi [1] and of Darondeau's Ph.D. (*cf.* [4]) can be slightly modified in order to lower (improve) the current degree bounds for complex hyperbolicity.

**Theorem.** For a generic hypersurface  $\mathbb{X}^{n-1} \subset \mathbb{P}^n(\mathbb{C})$  of degree

 $d \ge n^{2n}$ 

(1)  $\mathbb{P}^n \setminus \mathbb{X}^{n-1}$  is Kobayashi-hyperbolically imbedded in  $\mathbb{P}^n$ ;

(2)  $\mathbb{X}^{n-1}$  is Kobayashi( $\Leftrightarrow$  Brody)-hyperbolic.

(1) improves [3]:  $d \ge (n+2)^{n+3} (n+1)^{n+3} = n^{2n} n^6 \left( e^3 + O(\frac{1}{n}) \right)$ . (2) supersedes [5]:  $d \ge \frac{1}{3} \left( e^1 (n-1) \right)^{2n} = n^{2n} e^{2n} \left( \frac{1}{3e^2} + O(\frac{1}{n}) \right)$ .

The method gives in fact  $d \ge \frac{n^{2n}}{\cosh^n}$  for  $n \ge N(\text{const})$  with any const  $\ge 1$ . We provide selected details only in the case const = 1.

*Proof.* The way (1) and (2) are related by induction on n is known [9, 10]. Treat only the complement case (1), since Darondeau's modus operandi pp. 1894–1921 of [4] applies mutatis mutandis to the compact case. There, take  $a_i := \left(\frac{n}{3}\right)^{n-i}$  for  $1 \le i \le n$  — think  $\left[\frac{n}{3}\right]$ ; the choice  $a_i := \left(\frac{n}{\text{const}}\right)^{n-i}$  would also work. On p. 1915:

$$\widehat{C}\left(\frac{1}{a}\right) := \prod_{1 \leq i < j \leq n} \frac{a_i/a_j - 1}{a_i/a_j - 2} \prod_{2 \leq i < j \leq n} \frac{a_i/a_j - 2}{a_i/a_j - 2 - a_i/a_{i-1}} \\ = \prod_{k=1}^{n-1} \left(\frac{(n/3)^k - 1}{(n/3)^k - 2}\right)^{n-k} \prod_{k=1}^{n-1} \left(\frac{(n/3)^k - 2}{(n/3)^k - 2 - \frac{1}{n/3}}\right)^{n-1-k} \\ = \prod_{k=1}^{n-1} \frac{\left((n/3)^k - 1\right)^{n-k}}{\left((n/3)^k - 2\right) \left((n/3)^k - 2 - \frac{1}{n/3}\right)^{n-1-k}}.$$

Introduce  $C(\frac{1}{a})$  defined similarly by replacing  $-a_i/a_{i-1}$  with  $+a_i/a_{i-1}$ :

$$C\left(\frac{1}{a}\right) = \prod_{k=1}^{n-1} \frac{\left((n/3)^k - 1\right)^{n-k}}{\left((n/3)^k - 2\right) \left((n/3)^k - 2 + \frac{1}{n/3}\right)^{n-1-k}}.$$

The quotient:

$$\frac{\widehat{C}(\frac{1}{a})}{C(\frac{1}{a})} = \prod_{k=1}^{n-1} \left( \frac{(n/3)^k - 2 + \frac{1}{n/3}}{(n/3)^k - 2 - \frac{1}{n/3}} \right)^{n-1-k} = \prod_{k=1}^{n-1} \left( \frac{1 - \frac{2}{(n/3)^k} + \frac{1}{(n/3)^{k+1}}}{1 - \frac{2}{(n/3)^k} - \frac{1}{(n/3)^{k+1}}} \right)^{n-1-k}$$

has logarithm asymptotic:

$$\log \frac{\widehat{C}(\frac{1}{a})}{C(\frac{1}{a})} = \sum_{k=1}^{n-1} \left(n-1-k\right) \left\{ \log \left[1 - \left(\frac{2}{(n/3)^k} - \frac{1}{(n/3)^{k+1}}\right)\right] - \log \left[1 - \left(\frac{2}{(n/3)^k} + \frac{1}{(n/3)^{k+1}}\right)\right] \right\}$$
$$= \sum_{k=1}^{n-1} \left(n-1-k\right) \left\{ -\frac{1}{1} \left(\frac{2}{(n/3)^k} - \frac{1}{(n/3)^{k+1}}\right)^1 - \frac{1}{2} \left(\frac{2}{(n/3)^k} - \frac{1}{(n/3)^{k+1}}\right)^2 + O\left(\frac{1}{n^3}\right) + \frac{1}{1} \left(\frac{2}{(n/3)^k} + \frac{1}{(n/3)^{k+1}}\right)^1 + \frac{1}{2} \left(\frac{2}{(n/3)^k} + \frac{1}{(n/3)^{k+1}}\right)^2 + O\left(\frac{1}{n^3}\right) \right\}$$
$$= (n-2) \left\{ \frac{1}{(n/3)^2} + \frac{1}{(n/3)^2} \right\} + O\left(\frac{1}{n^2}\right) = \frac{18}{n} + O\left(\frac{1}{n^2}\right).$$

Similar estimations yield:

$$\widehat{C}\left(\frac{1}{a}\right) = e^3 + \mathcal{O}\left(\frac{1}{n}\right).$$

Next, define  $I_0 := I_0^+ - I_0^-$  where as on pages 1896 and 1913:

$$I_0^{\pm} := \pm \sum_{\substack{k_1 + \dots + k_n = 0 \\ \pm C_{k_1,\dots,k_n} > 0}} \left[ t_1^{n-k_1} \cdots t_n^{n-k_n} \right] \left( \left( a_1 t_1 + \dots + a_n t_n \right)^{n^2} \right) \cdot C_{k_1,\dots,k_n},$$
$$C^- \left( \frac{1}{a} \right) := - \sum_{\substack{k_1 + \dots + k_n = 0 \\ C_{k_1,\dots,k_n} < 0}} C_{k_1,\dots,k_n} \left( \frac{1}{a_1} \right)^{k_1} \cdots \left( \frac{1}{a_n} \right)^{k_n} = \frac{|C| \left( \frac{1}{a} \right) - C\left( \frac{1}{a} \right)}{2}.$$

With  $\widetilde{I}_0 := \frac{n^2!}{(n!)^n} a_1^n \cdots a_n^n$ , on p. 1913:

$$I_0^- \leqslant \widetilde{I}_0 C^-\left(\frac{1}{a}\right).$$

On p. 1902:

$$I_0^+ \ge \widetilde{I}_0 \left( 1 + 3 + \frac{3^2}{2} + \mathcal{O}(\frac{1}{n}) \right).$$

From:

 $|C|\left(\frac{1}{a}\right)\left(1-\frac{18}{n}-\mathcal{O}(\frac{1}{n^2})\right) \leqslant \widehat{C}\left(\frac{1}{a}\right)\left(1-\frac{18}{n}-\mathcal{O}(\frac{1}{n^2})\right) \leqslant C\left(\frac{1}{a}\right) \leqslant |C|\left(\frac{1}{a}\right) \leqslant \widehat{C}\left(\frac{1}{a}\right) = e^3 + \mathcal{O}(\frac{1}{n}),$  it comes:

$$(0 \leqslant )$$
  $C^{-}(\frac{1}{a}) \leqslant \frac{9e^3}{n} + O(\frac{1}{n^2}) = O(\frac{1}{n}),$ 

hence:

$$\frac{1}{I_0} \leqslant \frac{1}{\widetilde{I}_0} \left( \frac{2}{17} + \mathcal{O}(\frac{1}{n}) \right).$$

For  $1 \leq p \leq n$  set as on p. 1898:

$$\widetilde{I}_p := \widetilde{I}_0 \left( 2n \left( 1 \, a_1 + \dots + n \, a_n \right) \right)^p \sum_{1 \leq i_1 < \dots < i_p \leq n} \frac{1}{a_{i_1} \cdots a_{i_p}}$$
$$= \widetilde{I}_0 \left[ \frac{n^n}{3^n} \left( 6 + \mathcal{O}(\frac{1}{n}) \right) \right]^p \left( \left( \frac{3}{n} \right)^{\frac{p(p-1)}{2}} \left( 1 + \mathcal{O}(\frac{1}{n}) \right) \right).$$

On p. 1906 using p. 1920:

$$\begin{aligned} \left| I_p \right| &\leqslant \widetilde{I}_p \cdot \left| B \right| \left( \frac{2n\mu h}{a_1}, \dots, \frac{2n\mu h}{a_n} \right) \cdot \left| C \right| \left( \frac{1}{a_1}, \dots, \frac{1}{a_n} \right) \cdot \frac{(n+1)^2 + 2}{2} \\ &\leqslant \widetilde{I}_p \cdot \left( e^{1/2} + \mathcal{O}(\frac{1}{n}) \right) \cdot \left( e^3 + \mathcal{O}(\frac{1}{n}) \right) \cdot \frac{n^2}{2} \left( 1 + \mathcal{O}(\frac{1}{n}) \right), \end{aligned}$$

hence the largest root  $\lambda(a)$  of  $I_0 d^n + I_1 d^{n-1} + \cdots + I_n$  satisfies:

$$\lambda(a) \leqslant 2 \max_{1 \leqslant p \leqslant n} \sqrt[p]{\frac{|I_p|}{I_0}} \leqslant 2 \max_{1 \leqslant p \leqslant n} \left( \frac{\tilde{I}_p}{\tilde{I}_0} \frac{n^2}{2} e^{7/2} \frac{2}{17} \left( 1 + \mathcal{O}(\frac{1}{n}) \right) \right)^{\frac{1}{p}} = 2 \frac{\tilde{I}_1}{\tilde{I}_0} n^2 \frac{e^{7/2}}{17} \left( 1 + \mathcal{O}(\frac{1}{n}) \right)$$
$$= \frac{n^n}{3^n} n^2 \frac{12 e^{7/2}}{17} \left( 1 + \mathcal{O}(\frac{1}{n}) \right)$$

By [11], (1) holds for large n in degree:

$$d \ge n^{2n} \ge n^{2n} \left(\frac{2}{3}\right)^{2n} (2n)^2 \frac{12e^{7/2}}{17} \left(1 + O(\frac{1}{n})\right)$$

and for small n, a computer concludes, while  $d \ge n^n$  is not reachable so.

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