ON THE CODIMENSION OF NOETHER-LEFSCHETZ LOCI FOR TORIC THREEFOLDS

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Abstract. In this manuscript we sharpen the lower bound on the codimension of the irreducible components of the Noether-Lefschetz locus of surfaces in projective toric threefolds given in [BG17]. We also provide a simpler proof of Theorem 4.11 in [BG17], which allows one to avoid some technical assumptions.

Let $\mathbb{P}_\Sigma$ be a projective toric threefold with orbifold singularities, β a nef class in the class group $\mathrm{Cl}(\mathbb{P}_\Sigma)$, and $\mathcal{M}_\beta$ the moduli space of surfaces in $\mathbb{P}_\Sigma$ of degree β modulo automorphisms of $\mathbb{P}_\Sigma$. The Noether-Lefschetz locus U_β with respect to β is the subscheme of $\mathcal{M}_\beta$ corresponding to quasi-smooth surfaces whose Picard number is strictly larger than the one of $\mathbb{P}_\Sigma$.

Let η be a primitive ample Cartier class and suppose that β is a Cartier divisor of the form $-K_{\mathbb{P}_\Sigma} + n\eta$, where $K_{\mathbb{P}_\Sigma}$ is the canonical divisor of $\mathbb{P}_\Sigma$ and n is a non-negative integer; in this specific case, we denote U_β by $U_\eta(n)$. Note that, to have such a β, one has to implicitly assume that $K_{\mathbb{P}_\Sigma}$ is Cartier, which forces $\mathbb{P}_\Sigma$ to be Gorenstein.

Assume the multiplication map

$$\mathrm{H}^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(\beta)) \otimes \mathrm{H}^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(n\eta)) \to \mathrm{H}^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(\beta + n\eta))$$

to be surjective; we shall refer to this as "condition ($\star$)." Hence, the very general quasi-smooth surface in the linear system defined by β has the same Picard number as $\mathbb{P}_\Sigma$, see Theorem 3.5 of [BG12]; in other words $U_\eta(n)$ is a countable union of closed subschemes of positive codimension. Bruzzo and Grassi prove the following:

Theorem 4.11 of [BG17]. *Assume that the multiplication maps*

$$\mathrm{H}^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(\beta)) \otimes \mathrm{H}^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(k\eta)) \to \mathrm{H}^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(\beta + k\eta))$$

are surjective for all positive integer k (in particular condition ($\star$) is fulfilled). Assume also the vanishings

$$\mathrm{H}^1(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(\beta - \eta)) = \mathrm{H}^2(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(\beta - \eta)) = \mathrm{H}^2(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(\beta - 2\eta)) = 0\,.$$

Then, for any irreducible component U of $U_\eta(n)$,

$$\begin{array}{lrcl} i) & \eta\ \ (-1)-\textit{regular} & \Rightarrow & \operatorname{codim} U \geq n+1; \\ ii) & \eta\ \ 0-\textit{regular} & \Rightarrow & \operatorname{codim} U \geq n. \end{array}$$

Nevertheless, they provide many examples where the chosen ample class η is 0-regular and the locus $U_\eta(n)$ has a component of codimension precisely $n+1$, see Section 5 in [BG17]. These are the natural generalizations of the examples that achieve the *minimal* bound in the classical setting; namely, the surfaces in $\mathbb{P}^3$ that contain a line [Voi89, Gre84, Gre88, Gre89]. For this reason one is led to believe that the bound in *ii)* may not be sharp: we show here that this is precisely the case.

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Theorem. *Let $\mathbb{P}_\Sigma$ be a Gorenstein projective toric threefold with orbifold singularities, η a primitive ample Cartier class which is 0-regular, and β a Cartier divisor of the form $-K_{\mathbb{P}_\Sigma} + n\eta$, where $K_{\mathbb{P}_\Sigma}$ is the canonical divisor of $\mathbb{P}_\Sigma$ and n is a non-negative integer.*
Assume condition ($\star$). If $\mathrm{H}^1(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(\beta-\eta)) = \mathrm{H}^2(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(\beta-2\eta)) = 0$, then

$$\operatorname{codim} U \geq n+1$$

for every irreducible component U of $U_\eta(n)$.

Remark 1. The bounds in Theorem 4.12, Corollary 4.13, and Theorem 4.15 of [BG17] can be accordingly upgraded. Notice that the hypoteses of each of the results just mentioned imply those of Theorem 4.11 of [BG17], and so, a fortiori, ours.

The new bound is sharp for the following threefolds and suitable choices of η, see Section 5 of [BG17]:

- toric simplicial Gorenstein threefolds with nef anticanonical bundle;
- the projective space blown-up along a line, $\hat{\mathbb{P}}^3$;
- $\mathbb{P}^1 \times \mathbb{P}^2$;
- a small resolution of the cone over a quartic surface in $\mathbb{P}^3$;
- the weighted projective space $\mathbb{P}[1,1,2,2]$.

* * *

Before proving the theorem, we first notice that, for what concerns Theorem 4.11 of [BG17], the discrepancy between the case where η is (-1)-regular and the one where it is just 0-regular has its origin in Proposition 3.5 of [BG17], which inspects the regularity (with respect to η) of the tensor powers of a 1-regular locally free sheaf $\mathcal{F}$ on $\mathbb{P}_\Sigma$.

Our idea is to use instead a natural generalization of Corollary 1.8.10 of [Laz04].

Proposition 2. *Let X be a projective variety together with an ample line bundle B which is globally generated and 0-regular. If $\mathcal{F}$ is an m-regular locally free sheaf on X, then the p-fold tensor power $\mathcal{F}^{\otimes p}$ is (pm)-regular. In particular, $\Lambda^p\mathcal{F}$ and $S^p\mathcal{F}$ are likewise (pm)-regular.*

Proof. The proof of Corollary 1.8.10 of [Laz04], which is stated for locally free sheaves on a projective space $\mathbb{P}$ which are regular with respect to $\mathcal{O}_\mathbb{P}(1)$, also works in this more general context. The 0-regularity of B ensures the existence of a linear resolution for $\mathcal{F}$ of the form

$$\cdots \to \bigoplus B^{-m-2} \to \bigoplus B^{-m-1} \to \bigoplus B^{-m} \to \mathcal{F} \to 0\,,$$

cf. Proposition 1.8.8 of [Laz04] and Corollary 3.3 of [BG17]. □

It is well-known that ample line bundles are always globally generated on projective toric varieties.

Proof of the Theorem. What follows logically corresponds to §4.1 of [BG17]. We pick a base-point-free linear system W in $\mathrm{H}^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(\beta))$ and we select a complete flag of linear subspaces

$$W = W_c \subset W_{c-1} \subset \cdots \subset W_1 \subset W_0 = \mathrm{H}^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(\beta))\,.$$

This means that the codimension of W_i is i (and in particular c is the codimension of W). We define the vector bundle M_i over $\mathbb{P}_\Sigma$ as the kernel of the natural (surjective) map $W_i \otimes \mathcal{O}_{\mathbb{P}_\Sigma} \to \mathcal{O}_{\mathbb{P}_\Sigma}(\beta)$.

Proposition 3. *M_0 is 1-regular with respect to η.*

Proof. By definition, we would like to show that $\mathrm{H}^q(\mathbb{P}_\Sigma, M_0((1-q)\eta)) = 0$ for all positive q. Consider the long exact sequence in cohomology associated with

$$0 \to M_0 \to W_0 \otimes \mathcal{O}_{\mathbb{P}_\Sigma} \to \mathcal{O}_{\mathbb{P}_\Sigma}(\beta) \to 0\,, \tag{$\dagger$}$$

that is

$$0 \to \mathrm{H}^0(\mathbb{P}_\Sigma, M_0) \to \mathrm{H}^0(\mathbb{P}_\Sigma, W_0 \otimes \mathcal{O}_{\mathbb{P}_\Sigma}) \xrightarrow{\pi} \mathrm{H}^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(\beta)) \to \mathrm{H}^1(\mathbb{P}_\Sigma, M_0) \to \cdots$$

Since π is surjective, $\mathrm{H}^1(\mathbb{P}_\Sigma, M_0) = 0$.
The vanishing of $\mathrm{H}^2(\mathbb{P}_\Sigma, M_0(-\eta))$ is obtained by tensoring ($\dagger$) by $\mathcal{O}_{\mathbb{P}_\Sigma}(-\eta)$ and considering that $\mathrm{H}^2(\mathbb{P}_\Sigma, M_0(-\eta))$ lies between two zeros in the long exact sequence

$$\cdots \to \mathrm{H}^1(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(\beta-\eta)) \to \mathrm{H}^2(\mathbb{P}_\Sigma, M_0(-\eta)) \to \mathrm{H}^2(\mathbb{P}_\Sigma, W_0 \otimes \mathcal{O}_{\mathbb{P}_\Sigma}(-\eta)) \to \cdots$$

Note that $\mathrm{H}^2(\mathbb{P}_\Sigma, W_0 \otimes \mathcal{O}_{\mathbb{P}_\Sigma}(-\eta)) = 0$ because η is 0-regular with respect to itself.
One argues similarly for $\mathrm{H}^3(\mathbb{P}_\Sigma, M_0(-2\eta))$. □

Corollary 4. *For all $i = 0, \dots, c$, $\mathrm{H}^q(\mathbb{P}_\Sigma, \wedge^p M_i(n\eta)) = 0$ if $q \geq 1$ and $n + q \geq p + i$.*

Proof. As a straightforward consequence of Mumford's Theorem [Laz04, Theorem 1.8.5] one has that a coherent sheaf $\mathcal{F}$ on $\mathbb{P}_\Sigma$ is m-regular with respect to η if and only if

$$H^q(\mathbb{P}_\Sigma, \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_\Sigma}(n\eta)) = 0\,,$$

for all integers $q > 0$, $n \geq m - q$. Hence, for $i = 0$ the statement is equivalent to $\wedge^p M_0$ being p-regular with respect to η and this is proved in Proposition 2. The proof then follows as for Lemma 2 in [Gre88]. □

Proposition 5. *If $c = \operatorname{codim} W \leq n$, then the map $W \otimes \mathrm{H}^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(n\eta)) \to \mathrm{H}^0(\mathcal{O}_{\mathbb{P}_\Sigma}(\beta + n\eta))$ is surjective.*

Proof. We consider the short exact sequence

$$0 \to M_c \to W \otimes \mathcal{O}_{\mathbb{P}_\Sigma} \to \mathcal{O}_{\mathbb{P}_\Sigma}(\beta) \to 0\,,$$

and twist it by $\mathcal{O}_{\mathbb{P}_\Sigma}(n\eta)$. We pass to the long exact sequence in cohomology:

$$\cdots \to \mathrm{H}^0(\mathbb{P}_\Sigma, W \otimes \mathcal{O}_{\mathbb{P}_\Sigma}(n\eta)) \to \mathrm{H}^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(\beta + n\eta)) \to \mathrm{H}^1(\mathbb{P}_\Sigma, M_c(n\eta)) \to \dots$$

By applying Corollary 4 with $p = q = 1$, one gets that $\mathrm{H}^1(\mathbb{P}_\Sigma, M_c(-n\eta))$ is zero. □

Finally, the proof of the Theorem follows precisely as the one of Theorem 4.11 of [BG17] with the only difference that we use the previous proposition instead of Proposition 4.10 of [BG17].

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