GALOIS POINTS FOR DOUBLE-FROBENIUS NONCLASSICAL CURVES

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ABSTRACT. We determine the distribution of Galois points for plane curves over a finite field of q elements, which are Frobenius nonclassical for different powers of q. This family is an important class of plane curves with many remarkable properties. It contains the Dickson–Guralnick–Zieve curve, which has been recently studied by Giulietti, Korchmáros, and Timpanella from several points of view. A problem posed by the second author in the theory of Galois points is modified.

1. INTRODUCTION

Let $\mathbb{F}_q$ be a finite field with $q \geq 2$, and let $\mathcal{F} \subset \mathbb{P}^2$ be the plane curve defined by $F(x,y,z) = D_1(x,y,z)/D_2(x,y,z)$, where

$$(1) \qquad D_1 = \begin{vmatrix} x & x^{q^m} & x^{q^n} \\ y & y^{q^m} & y^{q^n} \\ z & z^{q^m} & z^{q^n} \end{vmatrix} \quad \text{and} \quad D_2 = \begin{vmatrix} x & x^{q} & x^{q^2} \\ y & y^{q} & y^{q^2} \\ z & z^{q} & z^{q^2} \end{vmatrix},$$

and n and m are coprime. According to [1, p.544 and Theorem 3.4], F is a homogeneous polynomial of degree $q^n + q^m - q^2 - q$ over $\mathbb{F}_q$, which is irreducible over the algebraic closure $\overline{\mathbb{F}}_q$. In 2009, the first author characterized these curves as the unique double-Frobenius nonclassical plane curves for different powers q^n and q^m, with $\gcd(n,m) = 1$. Other significant features, such as a large number of $\mathbb{F}_{q^n}$-rational points (meeting the Stöhr–Voloch bound) and the arc property in the case $m = 1$ were also noted [1]. This important family of curves contains the newly coined Dickson–Guralnick–Zieve (DGZ) curve: case $(n,m) = (3,1)$. Additional remarkable properties of the DGZ curve, such as a large automorphism group and positive p-rank, have been recently proved by Giulietti, Korchmáros, and Timpanella [6].

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In this article, we consider Galois points for the curves $\mathcal{F}$ over $\overline{\mathbb{F}}_q$ (see [7, 9] for the definition of Galois point). The set of all Galois points for the plane curve $\mathcal{F}$ on the projective plane is denoted by $\Delta(\mathcal{F})$.

Our main result is the following.

Theorem 1. *Let $n \geq 3$ and $m \geq 1$ be integers such that $n > m$ and $\gcd(n, m) = 1$. For the (q^n, q^m)-Frobenius nonclassical curve $\mathcal{F} \subset \mathbb{P}^2$,*

$$\Delta(\mathcal{F}) = \emptyset \quad or \quad \mathbb{P}^2(\mathbb{F}_q).$$

The latter case occurs if and only if $(n, m) = (3, 1)$ or $(3, 2)$.

Since the result for the case where $(n, m) = (3, 1)$ or $(3, 2)$ gives a negative answer to the problem [3, Problem 1], posed by the second author, it is modified as follows.

Problem 1. *Let $\mathcal{C}$ be a plane curve over $\mathbb{F}_q$. Assume that $\Delta(\mathcal{C}) = \mathbb{P}^2(\mathbb{F}_q)$. Then, is it true that $\mathcal{C}$ is projectively equivalent to the Hermitian, Ballico–Hefez or the (q^n, q^m)-Frobenius nonclassical curve of type $(n, m) = (3, 1)$ or $(3, 2)$? Or, more basically, is $\mathcal{C}$ Frobenius nonclassical?*

The result in [4] is contained in this article.

2. Preliminaries

Let $\mathcal{C} \subset \mathbb{P}^2$ be an irreducible plane curve of degree d, and let $r : \hat{\mathcal{C}} \to \mathcal{C}$ be the normalization. For different points P and $Q \in \mathbb{P}^2$, the line passing through P and Q is denoted by $\overline{PQ}$. For a point $P \in \mathbb{P}^2$, $\pi_P : \mathcal{C} \dashrightarrow \mathbb{P}^1$ represents the projection from P. The composite map $\pi_P \circ r : \hat{\mathcal{C}} \to \mathbb{P}^1$ is denoted by $\hat{\pi}_P$. The ramification index at $\hat{Q} \in \hat{\mathcal{C}}$ is represented by $e_{\hat{Q}}$. When $r^{-1}(Q)$ consists of a unique point $\hat{Q} \in \hat{\mathcal{C}}$, the index $e_{\hat{Q}}$ is denoted also by e_Q.

Fact 1. *For the projection $\hat{\pi}_P$, the following holds.*

(a) *For each point $\hat{Q} \in \hat{C}$ with $Q = r(\hat{Q}) \neq P$, it follows that $e_{\hat{Q}} = \mathrm{ord}_{\hat{Q}} h_{PQ}$, where h_{PQ} is a linear polynomial defining the line $\overline{PQ}$.*

(b) *If $P \in \mathcal{C} \setminus \mathrm{Sing}(\mathcal{C})$, then $e_P = \mathrm{ord}_P h_P - 1$, where h_P is a linear polynomial defining the tangent line at P.*

(c) *If P is an ordinary singularity of $\mathcal{C}$ with multiplicity $m(P)$, that is, P has $m(P)$ tangent lines, then for each tangent line defined by $h = 0$ at P, there exists a unique point $\hat{P} \in r^{-1}(P)$ such that $e_{\hat{P}} = \mathrm{ord}_{\hat{P}} h - 1$.*

According to [1, Proposition 3.2], the following holds.

Fact 2. *Let $\mathcal{F} \subset \mathbb{P}^2$ be the (q^n, q^m)-Frobenius nonclassical curve, and let $S \subset \mathbb{P}^2$ be the set of all points R such that R is contained in some $\mathbb{F}_q$-line.*

(a) *For the case where* $m > 1$, $\mathrm{Sing}(\mathcal{F}) = \mathbb{P}^2(\mathbb{F}_{q^{n-m}})$. *Let* $Q = r(\hat{Q}) \in \mathrm{Sing}(\mathcal{F})$ *be of multiplicity* $m(Q)$.

(i) *If* $Q \notin S$, *then* $m(Q) = q^m$, $r^{-1}(Q) = \{\hat{Q}\}$, *and there exists a unique line* $T_Q \ni Q$ *such that* $\mathrm{ord}_{\hat{Q}} T_Q = q^m + 1$.

(ii) *If* $Q \in S$ *and* $Q \notin \mathbb{P}^2(\mathbb{F}_q)$, *then* $m(Q) = q^m - 1$, $r^{-1}(Q) = \{\hat{Q}\}$, *and there exists a unique line* $T_Q \ni Q$ *such that* $\mathrm{ord}_{\hat{Q}} T_Q = q^m$.

(iii) *If* $Q \in \mathbb{P}^2(\mathbb{F}_q)$, *then* $m(Q) = q^m - q$, $r^{-1}(Q)$ *consists of exactly* $m(Q)$ *points, and* Q *is an ordinary singularity. For each tangent line* T *at* Q, *the intersection multiplicity of* $\mathcal{F}$ *and* T *at* Q *is equal to* $q^n - q$.

(b) *For the case where* $m = 1$ *and* $q > 2$, $\mathrm{Sing}(\mathcal{F}) = \mathbb{P}^2(\mathbb{F}_{q^{n-1}}) \setminus \mathbb{P}^2(\mathbb{F}_q)$. *Let* $Q = r(\hat{Q}) \in \mathrm{Sing}(\mathcal{F})$ *be of multiplicity* $m(Q)$.

(i) *If* $Q \notin S$, *then* $m(Q) = q$, $r^{-1}(Q) = \{\hat{Q}\}$, *and there exists a unique line* $T_Q \ni Q$ *such that* $\mathrm{ord}_{\hat{Q}} T_Q = q + 1$.

(ii) *If* $Q \in S$ *and* $Q \notin \mathbb{P}^2(\mathbb{F}_q)$, *then* $m(Q) = q - 1$, $r^{-1}(Q) = \{\hat{Q}\}$, *and there exists a unique line* $T_Q \ni Q$ *such that* $\mathrm{ord}_{\hat{Q}} T_Q = q$.

(c) *For the case where* $(m, q) = (1, 2)$, $\mathrm{Sing}(\mathcal{F}) = \mathbb{P}^2(\mathbb{F}_{2^{n-1}}) \setminus S$. *If* $Q = r(\hat{Q}) \in \mathrm{Sing}(\mathcal{F})$ *with multiplicity* $m(Q)$, *then* $m(Q) = 2$, $r^{-1}(Q) = \{\hat{Q}\}$, *and there exists a unique line* $T_Q \ni Q$ *such that* $\mathrm{ord}_{\hat{Q}} T_Q = 3$.

If P is a Galois point, then the following holds (see [8, III.7.2]).

Fact 3. *If the projection* $\hat{\pi}_P : \hat{\mathcal{C}} \to \mathbb{P}^1$ *is a Galois covering, then we have the following.*

(a) *If* $\hat{Q}_1$ *and* $\hat{Q}_2 \in \hat{\mathcal{C}}$ *have the same image, then* $e_{\hat{Q}_1} = e_{\hat{Q}_2}$.

(b) *For each point* $\hat{Q} \in \hat{\mathcal{C}}$, *the index* $e_{\hat{Q}}$ *divides the degree* $\deg \hat{\pi}_P$.

Combining Facts 1(a), 2 and 3(a), we have the following.

Corollary 1. *Let* $\hat{\pi}_P : \hat{\mathcal{F}} \to \mathbb{P}^1$ *be the projection, and let* $Q = r(\hat{Q}) \in \mathcal{F} \setminus \{P\}$ *be a singular point.*

(i) *If* Q *is in the case (a)(i), (b)(i), or (c) of Fact 2, then* $e_{\hat{Q}} = q^m$ *or* $q^m + 1$.

(ii) *If* Q *is in the case (a)(ii) or (b)(ii) of Fact 2, then* $e_{\hat{Q}} = q^m - 1$ *or* q^m.

(iii) *Assume that* P *is a Galois point. If* Q *is in the case (a)(iii) of Fact 2, then* $e_{\hat{Q}} = 1$.

3. Frobenius nonclassicality and Galois points

Proposition 1. *Let* $\mathcal{C} \subset \mathbb{P}^2$ *be a* q*-Frobenius nonclassical curve over* $\mathbb{F}_q$. *Assume that* P *is a Galois point for* $\mathcal{C}$ *and* $Q \in \mathcal{C} \setminus \mathrm{Sing}(\mathcal{C})$ *is a ramification point for the projection from* P.

(a) *If* $P \in (\mathbb{P}^2 \setminus \mathcal{C}) \cup \mathrm{Sing}(\mathcal{C})$*, then the line* $\overline{PQ}$ *is defined over* $\mathbb{F}_q$*.*
(b) *If* $P = Q$*, then the tangent line* $T_P\mathcal{C}$ *at* P *is defined over* $\mathbb{F}_q$*.*
(c) *If* $P \in \mathcal{C}\setminus\mathrm{Sing}(\mathcal{C})$ *and there exists a point* $Q' \in (\mathcal{C}\setminus\mathrm{Sing}(\mathcal{C}))\cap(\overline{PQ}\setminus\{P, Q\})$*, then the line* $\overline{PQ}$ *is defined over* $\mathbb{F}_q$*.*

Proof. We prove assertions (a) and (c). Let Q^q and L^q be the q-Frobenius images of the point Q and the line $L := \overline{PQ}$, respectively. Considering Fact 3(a), we need only prove the claim under the assumption that $Q^q \neq P$. Since Q is a ramification point, it follows from Fact 1(a) that the line L is tangent to $\mathcal{C}$ at Q. It follows that L^q is tangent to $\mathcal{C}$ at Q^q. Now the q-Frobenius nonclassicality of $\mathcal{C}$ implies that Q^q lies on L. The assumption $Q^q \neq P$ and Fact 3(a) imply that L is tangent to $\mathcal{C}$ at Q^q. Hence $L = L^q$, that is, L is an $\mathbb{F}_q$-line.

We consider the case where P is a ramification point of $\hat{\pi}_P$. By the Frobenius nonclassicality, $P^q \in T_P\mathcal{C}$. If $P^q \neq P$, then Fact 3(a) implies that P^q is a ramification point. Therefore, the tangent line $T_{P^q}\mathcal{C}$ is the same as $T_P\mathcal{C}$. Similar to the above proof, $T_P\mathcal{C}$ is $\mathbb{F}_q$-rational. □

Corollary 2. *Let* $\mathcal{F} \subset \mathbb{P}^2$ *be the* (q^n, q^m)*-Frobenius nonclassical curve over* $\mathbb{F}_q$*. Assume that* P *is a Galois point for* $\mathcal{F}$ *and* $Q \in \mathcal{F}\setminus\mathrm{Sing}(\mathcal{F})$ *is a ramification point for the projection from* P*.*

(a) *If* $P \in (\mathbb{P}^2 \setminus \mathcal{F}) \cup \mathrm{Sing}(\mathcal{F})$*, then the line* $\overline{PQ}$ *is defined over* $\mathbb{F}_q$*.*
(b) *If* $P = Q$*, then the tangent line* $T_P\mathcal{F}$ *at* P *is defined over* $\mathbb{F}_q$*.*
(c) *If* $P \in \mathcal{F} \setminus \mathrm{Sing}(\mathcal{F})$ *and there exists a point* $Q' \in (\mathcal{F} \setminus \mathrm{Sing}(\mathcal{F})) \cap (\overline{PQ} \setminus \{P, Q\})$*, then the line* $\overline{PQ}$ *is defined over* $\mathbb{F}_q$*.*

Proof. By Proposition 1, the line $\overline{PQ}$ is $\mathbb{F}_{q^n}$-rational and $\mathbb{F}_{q^m}$-rational. Since n and m are coprime, the line is defined over $\mathbb{F}_q$. □

4. Inner smooth Galois points

Let $P \in \mathcal{F} \setminus \mathrm{Sing}(\mathcal{F})$ be an inner Galois point. It follows from Fact 1(b) that $e_P = I_P(\mathcal{F}, T_P\mathcal{F}) - 1$ for the projection $\hat{\pi}_P$, where $I_P(\mathcal{F}, T_P\mathcal{F})$ is the intersection multiplicity of the curve $\mathcal{F}$ and the tangent line $T_P\mathcal{F}$ at P. Assume that $(m, q) \neq (1, 2)$. Then $I_P(\mathcal{F}, T_P\mathcal{F}) \geq q^m \geq 3$ ([1, Theorem 2.6]), and hence P is a ramification point. By Corollary 2(b), the tangent line $T_P\mathcal{F}$ is $\mathbb{F}_q$-rational. If $m > 1$, then there exists an ordinary singularity on $T_P\mathcal{F}$ (Fact 2(a)), by Corollary 1(iii), and this is a contradiction to Fact 3(a). If $m = 1$ and $q > 2$, then $T_P\mathcal{F}$ contains a singular point Q with index $e_Q = q$ or $q - 1$ (Corollary 1(ii)). It follows from Fact 3(b) that e_Q divides $\deg \hat{\pi}_P = q^n - q^2 - 1$. This is impossible.

Assume that $(m, q) = (1, 2)$ and $n > 3$. Since $n - m > 2$, there exists a singular point Q such that there does not exist an $\mathbb{F}_2$-line containing Q (Fact 2(c)). Therefore, $\overline{PQ}$ is not an $\mathbb{F}_2$-line. Since $\deg \hat{\pi}_P = q^n - q^2 - 1$, it follows from Corollary 1(i) and Fact 3(b) that $e_Q = 3$. Since the tangent line at the point Q is $\mathbb{F}_{2^{n-1}}$-rational, the line $\overline{PQ}$ is $\mathbb{F}_{2^{n-1}}$-rational. There exist at least two $\mathbb{F}_2$-lines intersecting $\overline{PQ}$ at points of $\mathcal{F}$ different from P. It follows from Fact 2(c) that such points are smooth points. According to Corollary 2(c), the line $\overline{PQ}$ is $\mathbb{F}_q$-rational. This is a contradiction.

5. The case where $n - m > 2$

Assume that $n - m > 2$ and $P \in (\mathbb{P}^2 \setminus \mathcal{F}) \cup \mathrm{Sing}(\mathcal{F})$ is a Galois point. Since $n - m > 2$, there exists a singular point $Q \neq P$ not contained in any $\mathbb{F}_q$-line (Fact 2). Then Q is a ramification point for $\hat{\pi}_P$ with index $e_Q = q^m$ or $q^m + 1$ and the line $\overline{PQ}$ is not $\mathbb{F}_q$-rational. By Corollary 2(a), $\overline{PQ}$ does not contain a smooth point. It follows from Fact 3(a) and Corollary 1 that $\overline{PQ}$ is an $\mathbb{F}_{q^{n-m}}$-line and $\overline{PQ} \cap \mathcal{F}$ consists of only singular points. By considering the intersection points given by $\overline{PQ}$ and $\mathbb{F}_q$-lines, there exists a point $Q' \in \overline{PQ}$ such that Q' is not $\mathbb{F}_q$-rational and is contained in some $\mathbb{F}_q$-line. It follows from Proposition 1 that $e_{Q'} = q^m - 1$ or q^m. By Fact 3(a), $e_{Q'} = e_Q$ and hence $e_{Q'} = e_Q = q^m$. Note that the number of points in $\overline{PQ} \cap \mathcal{F}$ is equal to $q^{n-m} + 1$ or q^{n-m}.

Assume that $P \in \mathbb{P}^2 \setminus \mathcal{F}$. Then $q^n + q^m - q^2 - q = e_Q(q^{n-m} + 1)$, or $e_Q q^{n-m}$. This is impossible.

Assume that $P \in \mathrm{Sing}(\mathcal{F}) \setminus \mathbb{P}^2(\mathbb{F}_q)$. Since $e_Q = q^m$ divides $\deg \hat{\pi}_P$, by Fact 2, the multiplicity $m(P)$ is equal to q^m. Then $q^n - q^2 - q = e_Q(q^{n-m} + 1)$, $e_Q q^{n-m}$, or $e_Q(q^{n-m} - 1)$. This is also impossible.

Assume that $P \in \mathrm{Sing}(\mathcal{F}) \cap \mathbb{P}^2(\mathbb{F}_q)$. By Fact 2, $m > 1$. It follows from Fact 2(a) that $\deg \hat{\pi}_P = q^n - q^2$ and $(\overline{PQ} \setminus \{P\}) \cap \mathcal{F}$ consists of exactly q^{n-m} singular points. According to [1, Remark 3.3], $\overline{PQ}$ is a tangent line at P. Since P is an ordinary singularity, by Fact 1(c), the fiber of the point corresponding to $\overline{PQ}$ for $\hat{\pi}_P$ contains exactly $q^{n-m} + 1$ points. It follows that $e_Q(q^{n-m} + 1) = q^n - q^2$. This is impossible.

6. The case where $n - m = 2$

Assume that $n - m = 2$, $(m, q) \neq (1, 2)$ and $P \in (\mathbb{P}^2 \setminus \mathcal{F}) \cup \mathrm{Sing}(\mathcal{F})$ is a Galois point. Let Q be a singular point different from P that is contained in $\mathbb{P}^2(\mathbb{F}_{q^2}) \setminus \mathbb{P}^2(\mathbb{F}_q)$ (Fact 2). Since any $\mathbb{F}_{q^2}$-line contains an $\mathbb{F}_q$-point, Q is a ramification point with index $q^m - 1$ or q^m.

Assume that $m > 1$. It follows from Fact 3(a) and Corollary 1(iii) that $\overline{PQ} \setminus \{P\}$ does not contain an ordinary singularity. Therefore, $\overline{PQ} \setminus \{P\}$ does not contain an $\mathbb{F}_q$-rational point. In particular, $\overline{PQ}$ is not $\mathbb{F}_q$-rational. By Corollary 2(a), $\overline{PQ}$ does not contain a smooth point. It follows from Fact 3(a) and Corollary 1 that $\overline{PQ}$ is an $\mathbb{F}_{q^2}$-line. Since any $\mathbb{F}_{q^2}$-line contains an $\mathbb{F}_q$-point, the point P must be an $\mathbb{F}_q$-point, and hence a point with multiplicity $q^m - q$. Then $\overline{PQ}$ is a tangent line at P. Furthermore, the set $(\mathcal{F} \cap \overline{PQ}) \setminus \{P\}$ consists of exactly q^2 $\mathbb{F}_{q^2}$-rational but not $\mathbb{F}_q$-rational points. Note that there exists a unique point $\hat{P} \in r^{-1}(P)$ such that the image $\hat{\pi}_P(\hat{P})$ corresponds to the line $\overline{PQ}$. Therefore, the fiber $\hat{\pi}_P^{-1}(\overline{PQ})$ contains exactly $q^2 + 1$ points. It follows that $e_Q(q^2 + 1) = q^n - q^2$. This is impossible.

Assume that $m = 1$. Then $n = 3$. If $\overline{PQ}$ contains a smooth point, then by Corollary 2(a), $\overline{PQ}$ is $\mathbb{F}_q$-rational. If $\overline{PQ}$ does not contain a smooth point, then it follows from Fact 3(a) and Corollary 1 that $\overline{PQ}$ is $\mathbb{F}_{q^2}$-rational. Therefore, P is $\mathbb{F}_{q^2}$-rational. Assume that $P \in \mathbb{P}^2(\mathbb{F}_{q^2}) \setminus \mathbb{P}^2(\mathbb{F}_q) = \mathrm{Sing}(\mathcal{F})$. Note that the tangent line T at P is defined over $\mathbb{F}_q$, and the tangent line at each singular point R in $T \cap (\mathbb{P}^2(\mathbb{F}_{q^2}) \setminus \mathbb{P}^2(\mathbb{F}_q))$ is the same as T. Then $e_R = q$. It follows from Fact 3(b) that e_R divides $\deg \hat{\pi}_P = q^3 - q^2 - (q - 1)$. This is a contradiction.

In conclusion, it follows that if $\Delta(\mathcal{F}) \neq \emptyset$, then $m = 1$ and $\Delta(\mathcal{F}) \subset \mathbb{P}^2(\mathbb{F}_q) \subset \mathbb{P}^2 \setminus \mathcal{F}$.

Remark 1. When $(n, m, q) = (3, 1, 2)$, the curve $\mathcal{F}$ is given by

$$F(x, y, z) = (x^2 + xz)^2 + (x^2 + xz)(y^2 + yz) + (y^2 + yz)^2 + z^4$$

(see [1, p.542] or [6, Remark 1]). In this case, it is known that the claim follows ([2, Theorem 4]).

7. The case where $n - m = 1$

Assume that $n - m = 1$ and $P \in \mathbb{P}^2 \setminus \mathcal{F}$ is a Galois point. Since all singular points are ordinary singularities (Fact 2), by Corollary 1(iii), the projection from P is not ramified at such points. Therefore, there exists a smooth point $Q \in \mathcal{F}$ that is a ramification point. By Corollary 2(a), the line $\overline{PQ}$ is $\mathbb{F}_q$-rational. However, there exists an ordinary singularity in $\overline{PQ}$. This is a contradiction to Fact 3(a).

Assume that $n - m = 1$, $m > 2$ and $P \in \mathrm{Sing}(\mathcal{F})$ is a Galois point. Let T be a tangent line at P. By Facts 1(c) and 2(a)(iii), there exists a ramification point $\hat{P} \in r^{-1}(P)$ contained in the fiber of the point corresponding to T for the projection $\hat{\pi}_P$. Since $m > 2$, there exists a point $Q \in (\mathcal{F} \cap T) \setminus \{P\}$ (Fact 2(a)(iii)). By [1, Remark 3.3], since T is not an $\mathbb{F}_q$-line, Q is a smooth point. By Fact 3(a), Q is a ramification point. It follows from Corollary 2(a) that T is $\mathbb{F}_q$-rational. This is a contradiction.

Accordingly, it follows that if $\Delta(\mathcal{F}) \neq \emptyset$, then $m = 2$ and $\Delta(\mathcal{F}) \subset \mathrm{Sing}(\mathcal{F}) = \mathbb{P}^2(\mathbb{F}_q)$.

8. The case where $(n, m) = (3, 1)$ or $(3, 2)$

It is easily verified that the projective linear group $PGL(3, \mathbb{F}_q)$ acts on $\mathcal{F}$ by the definitions of D_1 and D_2 as in (1) (see also [6, Lemma 4.1]). Therefore, the matrices

$$\sigma_{\gamma,\beta} = \begin{pmatrix} 1 & 0 & 0 \\ \gamma & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \tau_\mu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix} \in PGL(3, \mathbb{F}_q)$$

act on $\mathcal{F}$, where $\gamma, \beta \in \mathbb{F}_q$ and $\mu^{q-1} = 1$. Note that a rational function x/z is fixed by the actions of $\sigma_{\gamma,\beta}$ and τ_μ. This implies that $\pi_P \circ \sigma_{\gamma,\beta} = \pi_P$ and $\pi_P \circ \tau_\mu = \pi_P$, where $P = (0 : 1 : 0)$. Note also that $\deg \pi_P = q^3 - q^2$ if and only if $(n, m) = (3, 1)$ or $(3, 2)$ (Fact 2). Considering the action of $PGL(3, \mathbb{F}_q)$, it is inferred that $\mathbb{P}^2(\mathbb{F}_q) \subset \Delta(\mathcal{F})$ holds, if $(n, m) = (3, 1)$ or $(3, 2)$.

Remark 2. (1) If $(n, m) = (3, 1)$ or $(3, 2)$, then the associated Galois group G_P of a Galois point $P \in \Delta(\mathcal{F})$ is isomorphic to the semidirect product $\mathbb{F}_q^{\oplus 2} \rtimes \mathbb{F}_q^*$, where the action of $\mathbb{F}_q^*$ on $\mathbb{F}_q^{\oplus 2}$ is given by $(\mu, (\gamma, \beta)) \mapsto (\mu\gamma, \mu\beta)$.
(2) All points in $\mathbb{P}^2(\mathbb{F}_q)$ are quasi-Galois points which are not Galois points, if $(n, m) \neq (3, 1)$ or $(3, 2)$ (see [5] for the definition of quasi-Galois point).

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