

COMBINATORIAL PROOFS OF THE NEWTON-GIRARD AND CHAPMAN-COSTAS-SANTOS IDENTITIES

SAJAL KUMAR MUKHERJEE AND SUDIP BERA

ABSTRACT. In this paper we give combinatorial proofs of some well known identities and obtain some generalizations. We give a visual proof of a result of Chapman and Costas-Santos regarding the determinant of sum of matrices. Also we find a new identity expressing permanent of sum of matrices. Besides, we give a graphical interpretation of Newton-Girard identity.

1. INTRODUCTION

In this paper we give combinatorial proofs of some well known identities and obtain some generalizations. Combinatorial proofs give more insight into “why” the result is true rather than “how” [2, 6, 7]. In Section 2, we give a graphical interpretation of Newton-Girard identity. A combinatorial proof of Newton-Girard identity was first given by Doron Zeilberger in [8]. In this paper we have obtained the Newton-Girard identity as a trivial corollary of our more general formulation. Our general formulation gives a relation between weighted sum of closed walk and weighted sum of linear subdigraph of wighted digraph Γ . So from our formulation the Newton-Girard identity is exactly same with the relation between weighted sum of closed walk and weighted sum of linear subdigraph of the weighted digraph consisting isolated loops only. However, to the best of our knowledge there is no such general relation between weighted sum of closed walk and weighted sum of linear subdigraph of weighted digraph. In this work we get a new identity expressing the relation between walk and linear subdigraph.

For a weighted digraph Γ , a *linear subdigraph* γ is a collection of pairwise vertex-disjoint cycles. The number of cycles contained in γ is denoted by $c(\gamma)$. The weight of a linear subdigraph γ is denoted by $w(\gamma)$. Denote the set of all linear subdigraph of length r by L_r . Let us denote the sum of the weights of all closed walks of length r by c_r . And define $\ell_r \triangleq \sum_{\ell \in L_r} (-1)^{c(\ell)} w(\ell)$. Then our general formulation says that for a weighted digraph Γ with n vertices,

- (1) $c_r + c_{r-1}\ell_1 + c_{r-2}\ell_2 + \cdots + c_{r-n}\ell_n = 0, r > n$
- (2) $c_r + c_{r-1}\ell_1 + c_{r-2}\ell_2 + \cdots + r\ell_r = 0, 1 \leq r \leq n.$

Like Newton-Girard identity, our general formulation can also be thought as a graph theoretic analogue of the relation between power sum symmetric functions and elementary symmetric functions. So our work motivates the investigation of possible graph theoretic interpretations

2010 *Mathematics Subject Classification.* 05A19; 05A05; 05C30; 05C38.

Key words and phrases. determinants; digraphs; Newton-Girard identity; combinatorial proof.

for other standard bases of symmetric functions such as complete homogeneous and Schur functions.

Denote by $M_{m,n}$, the set of $m \times n$ matrices over an arbitrary field F and by M_n the set $M_{n,n}$. Consider the N -tuple of $n \times n$ matrices $S := (A_1, A_2, \dots, A_N)$ and $Q \in M_n$. If $N \geq n + 1$, then the identity of Costas-Santos [5] says that

$$\sum_{k=0}^N (-1)^k \sum_{\substack{\sigma \subset [N] \\ |\sigma|=k}} \det \left(Q + \sum_{i \in \sigma} A_i \right) = 0.$$

Before that in [4], Chapman proved that

$$\sum_{k=0}^N (-1)^k \sum_{\substack{\sigma \subset [N] \\ |\sigma|=k}} \det \left(\sum_{i \in \sigma} A_i \right) = 0.$$

However, there is no formula to the best of our knowledge for the permanent of sum of matrices. In this work we get a new formula for the permanent of sum of matrices. In section 3, we give a visual and combinatorial proof of above identities. Moreover our method recovers a similar identity for the permanent, which we think to be new.

2. GRAPHICAL INTERPRETATION OF NEWTON-GIRARD IDENTITY

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be roots of the polynomial $f(x) = x^n + e_1x^{n-1} + e_2x^{n-2} + \dots + e_{n-1}x + e_n = 0$. Suppose $p_r = \alpha_1^r + \alpha_2^r + \dots + \alpha_n^r$ ($r = 0, 1, \dots, n$) so $p_0 = n$. Then Newton-Girard identity says that

- (1) $p_r + e_1p_{r-1} + e_2p_{r-2} + \dots + p_1e_{r-1} + re_r = 0$ ($r \leq n$)
- (2) $p_r + e_1p_{r-1} + e_2p_{r-2} + \dots + p_1e_{r-1} + e_n p_{r-n} = 0$ ($r > n$).

In this section we give a graphical interpretation of Newton-Girard identity. In fact the Newton-Girard identity can be deduced as a very special case of our more general set up. Before that, let us briefly describe some graph theoretic concepts. See [3] for details. Let Γ be a weighted digraph. A *linear subdigraph* γ , of Γ is a collection of pairwise vertex-disjoint cycles. A loop is a cycle of length 1. So loop around a single vertex is also considered to be a cycle. The weight of a linear subdigraph γ , written as $w(\gamma)$ is the product of the weights of all its edges. The number of cycles contained in γ is denoted by $c(\gamma)$. The length of a linear subdigraph γ , denoted by $L(\gamma)$ is the number of edges present in γ . Denote the set of all linear subdigraph of length r by L_r . A *walk* in a digraph Γ from a vertex u to a vertex v is a sequence of vertices $u = x_0, x_1, \dots, x_{k-1}, x_k = v$ such that (x_i, x_{i+1}) is an edge for $i = 0, 1, 2, \dots, k-1$. The walk is called closed if $u = v$. The length $L(\tilde{w})$ of a walk \tilde{w} is the number of edges present in that walk. The weight $w(\tilde{w})$ of a walk \tilde{w} is the product of all weights of edges present in that walk. Note that when we talk about a closed walk, its initial and final point is automatically specified. Let us denote the sum of the weights of all closed walks of length r by c_r . And define $\ell_r \triangleq \sum_{\ell \in L_r} (-1)^{c(\ell)} w(\ell)$. Now we state the theorem.

Theorem 2.1. *For a weighted digraph Γ with n vertices, the following identities hold.*

- (1) $c_r + c_{r-1}\ell_1 + c_{r-2}\ell_2 + \dots + c_{r-n}\ell_n = 0, r > n$

$$(2) \quad c_r + c_{r-1}\ell_1 + c_{r-2}\ell_2 + \cdots + r\ell_r = 0, 1 \leq r \leq n.$$

Proof. First we prove the case $r > n$. To prove this consider all ordered pair (c, γ) , where c is a closed walk and γ is a linear subdigraph (possibly empty), such that $L(c) + L(\gamma) = r$. Define the weight W of (c, γ) to be $W((c, \gamma)) = (-1)^{c(\gamma)}w(c)w(\gamma)$. Note that the left hand side of (1) is precisely equal to $\sum_{(c, \gamma)} W((c, \gamma))$, where the summation runs over all ordered pairs (c, γ) as described above.

Now the crucial observation is that, since $r > n$, either c and γ share a common vertex or c is not a “simple” closed walk (here simple means the graph structure of the closed walk is a directed cycle). Now take a particular pair (c, γ) satisfying the above conditions. Suppose that x is the initial and terminal vertex of c . Start moving from x along c . There are two possibilities: either, first we meet a vertex y which is a vertex of γ or, we complete a closed directed cycle \acute{c} which is a subwalk of c and during this journey from x up to the completion of \acute{c} we have not met any vertex of γ . Now if the first case holds, we form a new ordered pair $(\tilde{c}, \tilde{\gamma})$, where $\tilde{c} = \widehat{xy}|_c \odot \gamma_y \odot \widehat{yx}|_c$ and $\tilde{\gamma} = \gamma \setminus \{\gamma_y\}$, where $\widehat{xy}|_c$ is the walk from x to y along c and γ_y is the directed cycle of γ containing the vertex y . Note that $W((\tilde{c}, \tilde{\gamma})) = -W((c, \gamma))$. Now if the second case holds, then form a new ordered pair $(\tilde{\tilde{c}}, \tilde{\tilde{\gamma}})$, where $\tilde{\tilde{c}}$ is formed by removing the directed cycle \acute{c} from c and $\tilde{\tilde{\gamma}} = \gamma \cup \acute{c}$. Note also that $W((\tilde{\tilde{c}}, \tilde{\tilde{\gamma}})) = -W((c, \gamma))$. It is easy to see that, this is in fact an involution and it is sign reversing by the above observation. This completes the proof.

Now we prove the case $r \leq n$. Let $A = \{(c, \gamma) : c \text{ is a closed walk of length } \geq 1 \text{ and } \gamma \text{ is linear subdigraph (possibly empty) and } L(c) + L(\gamma) = r\}$. Consider the following sum $S = \sum_{(c, \gamma) \in A} W((c, \gamma)) + r\ell_r$. Note that the left hand side of (2) is precisely equal to S .

Consider the subset of A consisting of ordered pair (c, γ) satisfying the conditions: either $c \cap \gamma \neq \phi$ or c is not a simple closed walk. Call this subcollection *BAD*. So the *GOOD* members of A are the ordered pairs (c, γ) satisfying $c \cap \gamma = \phi$ and c is a simple closed walk. Now observe that, the weights of the *BAD* members cancel among themselves just like the previous case (case, $r > n$). Now let us see, how a *GOOD* member looks like. As a directed graph it is just a disjoint collection of distinct cycles with vertex set $\{v_1, v_2, \dots, v_r\}$ i. e. it is a linear subdigraph $\dot{\gamma}$ with vertex set $\{v_1, v_2, \dots, v_r\}$. Now for this fixed linear subdigraph $\dot{\gamma}$ with vertex set $\{v_1, v_2, \dots, v_r\}$, we claim that there are precisely r *GOOD* members (c, γ) . For the proof, take any vertex say v_i from $\dot{\gamma}$. Consider the cycle c in $\dot{\gamma}$ containing the vertex v_i . Let $\gamma_1 = \dot{\gamma} \setminus \{c\}$. Now the cycle c can be thought of as a closed walk c_{v_i} starting and ending at the vertex v_i . So we get a *GOOD* member (c_{v_i}, γ_1) . Since v_i is arbitrary the claim follows.

The main observation is that the sum of the weights of all the good members, found in this way from $\dot{\gamma}$ is $r(-1)^{c(\dot{\gamma})-1}w(\dot{\gamma})$. This cancels with the term $r(-1)^{c(\dot{\gamma})}w(\dot{\gamma})$ in the equation $S = \sum_{(c, \gamma) \in A} W((c, \gamma)) + r\ell_r$. \square

Corollary 1 (Newton-Girard identity). *Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be roots of the polynomial $f(x) = x^n + e_1x^{n-1} + e_2x^{n-2} + \cdots + e_{n-1}x + e_n = 0$. Suppose $p_r = \alpha_1^r + \alpha_2^r + \cdots + \alpha_n^r$ ($r = 0, 1, \dots$) so $p_0 = n$. Then Newton-Girard identity says that*

$$(1) \quad p_r + e_1p_{r-1} + e_2p_{r-2} + \cdots + p_1e_{r-1} + re_r = 0 (r \leq n)$$

$$(2) \quad p_r + e_1 p_{r-1} + e_2 p_{r-2} + \cdots + p_1 e_{r-1} + e_n p_{r-n} = 0 (r > n)$$

Proof. Just apply the above theorem on the weighted digraph consisting of n disjoint loops with weight $\alpha_1, \alpha_2, \dots, \alpha_n$ respectively, we get the Newton-Girard identity. \square

3. COMBINATORIAL PROOF OF CHAPMAN-COSTAS-SANTOS IDENTITY

In this section, we prove interesting identities about the determinant and permanent of sum of matrices. Before that we have to develop some necessary background. For details we refer [1]. Let G be a weighted, acyclic digraph. We call $A = \{A_1, A_2, \dots, A_n\}$ to be the initial set of vertices and $B = \{B_1, B_2, \dots, B_n\}$ to be the terminal set of vertices (not necessarily disjoint) of the graph G . To A and B , associate the *path matrix* $M = (m_{ij})$, where $m_{ij} = \sum_{P: A_i \rightarrow B_j} w(P)$, [$w(P)$ is the product of weights of all edges involved in the path P]. A *path system* \mathcal{P} from A to B consists of a permutation σ and n paths $P_i : A_i \rightarrow B_{\sigma(i)}$, with $\text{sgn}(\mathcal{P}) = \text{sgn}(\sigma)$. The *weight* of \mathcal{P} is $w(\mathcal{P}) = \prod_{i=1}^n w(P_i)$. Now it easy to see that $\det(M) = \sum_{\mathcal{P}} \text{sgn}(\mathcal{P}) w(\mathcal{P})$ and $\text{per}(M) = \sum_{\mathcal{P}} w(\mathcal{P})$.

For the N -tuple of $n \times n$ matrices $S := (A_1, A_2, \dots, A_N)$ and an $n \times n$ matrix $Q = (q_{i,j})$ with $N \geq n + 1$, Costas-Santos [5] proved

$$\sum_{k=0}^N (-1)^k \sum_{\substack{\sigma \subset [N] \\ |\sigma|=k}} \det \left(Q + \sum_{i \in \sigma} A_i \right) = 0$$

and before that Chapman [4] proved

$$\sum_{k=0}^N (-1)^k \sum_{\substack{\sigma \subset [N] \\ |\sigma|=k}} \det \left(\sum_{i \in \sigma} A_i \right) = 0$$

by using algebraic manipulations. In this section we give a purely combinatorial proof of his identity and the same identity for permanent.

Theorem 3.1. *Consider the N -tuple of n -by- n matrices $S := (A_1, A_2, \dots, A_N)$. If $N \geq n + 1$, then the following relations hold.*

$$(1) \quad \sum_{k=0}^N (-1)^k \sum_{\substack{\sigma \subset [N] \\ |\sigma|=k}} \det \left(\sum_{i \in \sigma} A_i \right) = 0.$$

$$(2) \quad \sum_{k=0}^N (-1)^k \sum_{\substack{\sigma \subset [N] \\ |\sigma|=k}} \text{per} \left(\sum_{i \in \sigma} A_i \right) = 0.$$

Proof. For the sake of simplicity we take $N = 3$, $n = 2$ and we prove these identities for three matrices $A = (a_{i,j})_{2 \times 2}$, $B = (b_{i,j})_{2 \times 2}$, $C = (c_{i,j})_{2 \times 2}$. The proofs of these two identities follow from the digraph in Figure 1 and the *Principle of Inclusion and Exclusion (PIE)*.

Look at the boxes labeled by **A**, **B**, **C** in Figure 1. From this figure, clearly the path matrix (paths from vertex set $X = \{x_1, x_2\}$ to the vertex set $Y = \{y_1, y_2\}$) of the graph is the matrix $A + B + C$. Let $\mathcal{P}_{ABC} = \{\mathcal{P}, \text{ a path system from } X \text{ to } Y \text{ such that each box}$

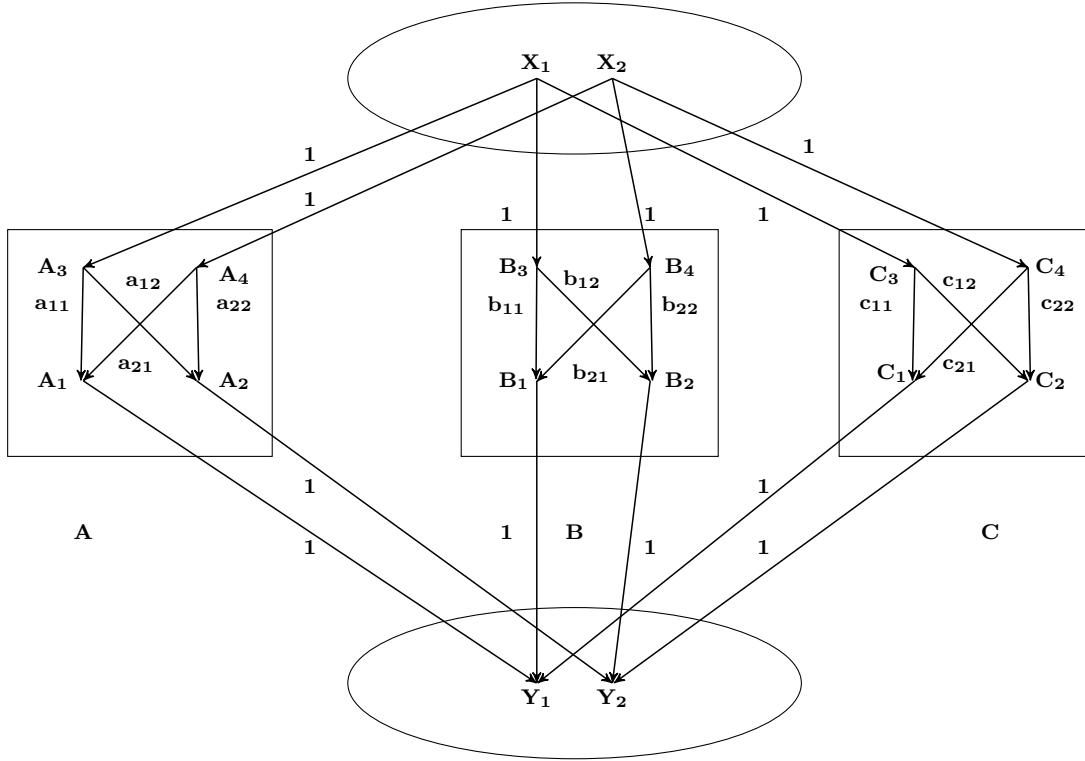


FIGURE 1. First box, second box and third box are denoted by **A**, **B**, **C** respectively.

contains at least 2 intermediate vertices of some paths in \mathcal{P} . Then $\sum_{\mathcal{P} \in \mathcal{P}_{ABC}} \text{sgn}(\mathcal{P})w(\mathcal{P}) = 0$, as $\mathcal{P}_{ABC} = \emptyset$ (because each path system contains exactly two paths which can not pass through all three boxes simultaneously). Now we compute this empty sum by *PIE*. The signed sum of weights of all possible path systems is $\det(A + B + C)$. The signed sum of weights of all possible path systems, whose underlying paths do not pass through box **C** is $\det(A + B)$. Similarly, the signed sum of weights of all possible path systems, whose underlying paths do not pass through box **B** is $\det(A + C)$ and the signed sum of weights of all possible path systems, whose underlying paths do not pass through box **A** is $\det(B + C)$. Again the signed sum of weights of all possible path systems, whose underlying paths pass through neither **B** nor **C** is $\det(A)$. Proceeding this way and using the *PIE* we get,

$$\begin{aligned} & \det(A + B + C) - \det(A + B) - \det(A + C) - \det(B + C) + \det(A) + \det(B) + \det(C) \\ &= \sum_{\mathcal{P} \in \mathcal{P}_{ABC}} \text{sgn}(\mathcal{P})w(\mathcal{P}) = 0. \end{aligned}$$

Similarly,

$$\text{per}(A + B + C) - \text{per}(A + B) - \text{per}(A + C) - \text{per}(B + C) + \text{per}(A) + \text{per}(B) + \text{per}(C) = 0.$$

Now if we add one more directed edge from each vertex $X_i (i = 1, 2)$ to each vertex $Y_j (j = 1, 2)$ in the directed graph in Figure 1 with weight $q_{i,j}$ and applying above argument

we can prove Costas-Santos's identity

$$\sum_{k=0}^N (-1)^k \sum_{\substack{\sigma \subset [N] \\ |\sigma|=k}} \det \left(Q + \sum_{i \in \sigma} A_i \right) = 0.$$

Similarly we can prove the new identity

$$\sum_{k=0}^N (-1)^k \sum_{\substack{\sigma \subset [N] \\ |\sigma|=k}} \text{per} \left(Q + \sum_{i \in \sigma} A_i \right) = 0.$$

□

Acknowledgement. We would like to thank Prof. Arvind Ayyer and Prof. Darij Grinberg for valuable suggestions in the preparation of this paper. The second author was supported by Department of Science and Technology grant EMR/2016/006624 and partly supported by UGC Centre for Advanced Studies.

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(Sajal Kumar Mukherjee) DEPARTMENT OF MATHEMATICS, VISVA-BHARATI, SANTINIKETAN-731235, INDIA.

E-mail address: shyamal.sajalmukherjee@gmail.com

(Sudip Bera) DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF SCIENCE, BANGALORE 560 012

E-mail address: sudipbera@iisc.ac.in