## BACKWARD ORBITS IN THE UNIT BALL

LEANDRO AROSIO AND LORENZO GUERINI

ABSTRACT. We show that, if  $f: \mathbb{B}^q \to \mathbb{B}^q$  is a holomorphic self-map of the unit ball in  $\mathbb{C}^q$  and  $\zeta \in \partial \mathbb{B}^q$  is a boundary repelling fixed point with dilation  $\lambda > 1$ , then there exists a backward orbit converging to  $\zeta$  with step log  $\lambda$ . Morever, any two backward orbits converging to the same boundary repelling fixed point stay at finite distance. As a consequence there exists a unique canonical pre-model  $(\mathbb{B}^k, \ell, \tau)$  associated with  $\zeta$  where  $1 \leq k \leq q, \tau$  is a hyperbolic automorphism of  $\mathbb{B}^k$ , and whose image  $\ell(\mathbb{B}^k)$  is precisely the set of starting points of backward orbits with bounded step converging to  $\zeta$ . This answers questions in [8] and [3, 4].

### 1. INTRODUCTION

For a holomorphic self-map  $f: \mathbb{B}^q \to \mathbb{B}^q$  of the unit ball of  $\mathbb{C}^q$ , there is a strong interplay among the notions of boundary repelling fixed points, backward orbits with bounded step and pre-models.

We start by recalling some definitions and elementary properties. Denote by  $k_{\mathbb{B}^q}$ the Kobayashi distance of the ball. A fundamental property of  $k_{\mathbb{B}^q}$  is the following generalization to several complex variables of the classical Schwarz-Pick Lemma: for every holomorphic self map  $f: \mathbb{B}^q \to \mathbb{B}^q$  we have

$$k_{\mathbb{B}^q}(f(z), f(w)) \le k_{\mathbb{B}^q}(z, w), \qquad \forall z, w \in \mathbb{B}^q$$

In particular every  $\gamma \in Aut(\mathbb{B}^q)$  is an isometry with respect to  $k_{\mathbb{B}^q}$ . Recall that  $Aut(\mathbb{B}^q)$  acts transitively on  $\mathbb{B}^q$ .

The Koranyi region with vertex  $\zeta \in \partial \mathbb{B}^q$  and amplitude M > 1 is defined as

$$K(\zeta, M) := \left\{ z \in \mathbb{B}^q \colon k_{\mathbb{B}^q}(0, z) + \lim_{w \to \zeta} (k_{\mathbb{B}^q}(z, w) - k_{\mathbb{B}^q}(0, w)) < 2 \log M \right\}.$$

Koranyi regions are a several variables generalization of the classical Stolz regions in the unit disc, but if q > 1 they are non-tangential to  $\partial \mathbb{B}^q$  only in the complex normal direction, while they are tangent to  $\partial \mathbb{B}^q$  in the complex tangential directions. If  $\eta$ denotes the ray connecting the origin to  $\zeta$ , then Koranyi regions are comparable to the sets of the form  $A(\eta, L) := \{z \in \mathbb{B}^q : k_{\mathbb{B}}(z, \zeta) < L\}$ . In particular a sequence converging to  $\zeta$  is contained in a Koranyi region  $K(\zeta, M)$  for some M > 1 if and only if it is contained in a region  $A(\eta, L)$  for some L > 0.

A point  $\zeta \in \partial \mathbb{B}^q$  is a boundary regular fixed point if

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- (1)  $K \lim_{z \to \zeta} f(z) = \zeta$ , which means by definition that if  $(z_n)$  is a sequence converging to  $\zeta$  inside a Koranyi region  $K(\zeta, M)$  then  $f(z_n) \to \zeta$ ,
- (2) the dilation  $\lambda$  of f at  $\zeta$ , defined as

$$\log \lambda := \liminf_{z \to \zeta} \left( k_{\mathbb{B}^q}(0, z) - k_{\mathbb{B}^q}(0, f(z)) \right)$$

satisfies  $\lambda < +\infty$ . If  $\lambda > 1$  (resp. = 1, resp. < 1) the point  $\zeta$  is repelling, (resp. indifferent, resp. attracting).

A sequence  $(z_n)$  in  $\mathbb{B}^q$  is a backward orbit if  $f(z_{n+1}) = z_n$  for all  $n \ge 0$ . The step of  $(z_n)$ is  $\sigma(z_n) := \lim_{n \to +\infty} k_{\mathbb{B}^q}(z_n, z_{n+1}) \in (0, +\infty]$ . A pre-model for f is a triple  $(\Lambda, h, \varphi)$ , where  $\Lambda$  is a complex manifold called the base space,  $h: \Lambda \to \mathbb{B}^q$  is a holomorphic mapping called the *intertwining mapping* and  $\varphi: \Lambda \to \Lambda$  is an automorphism, such that the following diagram commutes:



We say that a pre-model  $(\Lambda, h, \varphi)$  is associated with the boundary repelling point  $\zeta$  if for some (and hence for any)  $x \in \Lambda$  we have  $\lim_{n\to\infty} h(\varphi^{-n}(x)) = \zeta$ .

Poggi-Corradini [9] (see also Bracci [6]) showed, in the case of the unit disc  $\mathbb{D} \subset \mathbb{C}$ , that given a boundary repelling fixed point  $\zeta \in \partial \mathbb{D}$  one can find a backward orbit with step  $\log \lambda$  converging to  $\zeta$  and use such orbit to obtain an essentially unique pre-model  $(\mathbb{D}, h, \tau)$  associated with  $\zeta$ , where  $\tau$  is a hyperbolic automorphism of the disc with dilation  $\lambda$  at its repelling point.

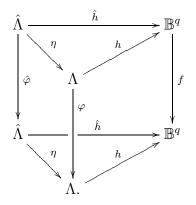
This result was partially generalized by Ostapuyk [8] in the unit ball  $\mathbb{B}^q$ . She proved that given an *isolated* boundary repelling fixed point  $\zeta$  one obtains with a similar method a pre-model  $(\mathbb{D}, h, \tau)$  associated with  $\zeta$ , where  $\tau$  is a hyperbolic automorphism of the disc with dilation  $\lambda$  at its repelling point. Since such pre-model has no uniqueness property and is one-dimensional, it is asked in [8, Question 8] what is the structure of the *stable subset*  $S(\zeta)$ , that is the subset of starting points of backward orbits with bounded step converging to  $\zeta$ , and whether one can find a "best possible" pre-model associated with  $\zeta$ .

In recent works [3, 4] a partial answer to such questions was given using the theory of canonical pre-models (see also [5]). To state such results we need to introduce some definitions. An automorphism  $\tau$  of the ball  $\mathbb{B}^q$  without inner fixed points (that is, nonelliptic) has either one or two fixed points at the boundary. If  $\tau$  has one fixed point it is called *parabolic*, and the fixed point is indifferent. If  $\tau$  has two fixed points, then it is called *hyperbolic*, and in this case one of the fixed points is repelling with dilation  $\mu > 1$  and the other one is attracting with dilation  $\frac{1}{\mu}$ . A hyperbolic automorphism  $\tau$  is conjugated to the following automorphism of the Siegel model for the unit ball  $\mathbb{H}^q = \{(z, w) \in \mathbb{C} \times \mathbb{C}^{q-1}, \operatorname{Im}(z) > ||w||^2\}$ ,

$$\tau(z,w) = \left(\frac{1}{\mu}z, \frac{e^{it_1}}{\sqrt{\mu}}w_1, \dots, \frac{e^{it_{q-1}}}{\sqrt{\mu}}w_{q-1}\right),\$$

where  $t_j \in \mathbb{R}$  for  $1 \leq j \leq q - 1$ .

If  $(y_n)$  is a backward orbit for f, we denote  $[y_n]$  the family of all backward orbits  $(z_n)$  for f such that the sequence  $(k_{\mathbb{B}^q}(z_n, y_n))$  is bounded. If  $(\hat{\Lambda}, \hat{h}, \hat{\varphi})$  and  $(\Lambda, h, \varphi)$  are pre-models for f, a morphism  $\hat{\eta} \colon (\hat{\Lambda}, \hat{h}, \hat{\varphi}) \to (\Lambda, h, \varphi)$  is given by a holomorphic map  $\eta \colon \hat{\Lambda} \to \Lambda$  such that the following diagram commutes:



In other words, there exists a morphism  $\hat{\eta}: (\hat{\Lambda}, \hat{h}, \hat{\varphi}) \to (\Lambda, h, \varphi)$  if and only if the pre-model  $(\hat{\Lambda}, \hat{h}, \hat{\varphi})$  "factors through" the pre-model  $(\Lambda, h, \varphi)$ . If  $\eta: \hat{\Lambda} \to \Lambda$  is a biholomorphism, we say that  $\hat{\eta}$  is an isomorphism.

It was shown in [3, 4] that every class  $[y_n]$  of backward orbits with bounded step converging to  $\zeta$  gives rise in a natural way to a *canonical* pre-model  $(\mathbb{B}^k, \ell, \tau)$ , where k is an integer satisfying  $1 \leq k \leq q$  and possibly depending on the class  $[y_n]$ , and  $\tau$  is a hyperbolic automorphism of the ball  $\mathbb{B}^k$  with dilation  $\mu \geq \lambda$  at its repelling point. Such model satisfies  $\ell(\tau^{-n}(x)) \in [y_n]$  for all  $x \in \mathbb{B}^k$ , and is thus associated with  $\zeta$ . Moreover the canonical pre-model satisfies the following universal property: if  $(\Lambda, h, \varphi)$ is a pre-model such that  $h(\varphi^{-n}(x)) \in [y_n]$  for all  $x \in \Lambda$ , then there exists a unique morphism  $\hat{\eta}: (\Lambda, h, \varphi) \to (\mathbb{B}^k, \ell, \tau)$ . It is easy to see that any pre-model  $(\hat{\Lambda}, \hat{h}, \hat{\varphi})$  such that  $\hat{h}(\hat{\varphi}^{-n}(x)) \in [y_n]$  for all  $x \in \hat{\Lambda}$  and which satisfies the same universal property has to be isomorphic to  $(\mathbb{B}^k, \ell, \tau)$ , hence the name "canonical".

The following questions were left open in [3, 4].

- (1) Does a non-isolated boundary repelling fixed point admit a backward orbit with bounded step converging to it (and hence an associated canonical pre-model)?
- (2) Is it possible that a boundary repelling fixed point  $\zeta$  is associated with two distinct canonical pre-models, corresponding to two different classes of backward orbits with bounded step converging to  $\zeta$ ?
- (3) Is the dilation of a canonical pre-model associated with  $\zeta$  always equal to  $\lambda$ ?

In the main result of this paper we give an answer to these three questions, showing that every boundary repelling fixed point  $\zeta$  is associated with exactly one (up to isomorphisms) canonical pre-model  $(\mathbb{B}^k, \ell, \tau)$ , which has dilation  $\lambda$ . The dimension k of the canonical pre-model is thus a dynamical invariant naturally associated with the fixed point  $\zeta$ .

**Theorem 1.** Let  $f: \mathbb{B}^q \to \mathbb{B}^q$  be a holomorphic self-map, and let  $\zeta \in \partial \mathbb{B}^q$  be a boundary repelling fixed point with dilation  $\lambda > 1$ . Then there exist an integer  $1 \le k \le q$  and a pre-model  $(\mathbb{B}^k, \ell, \tau)$  associated with  $\zeta$  such that

- (1)  $\tau$  is a hyperbolic automorphism of  $\mathbb{B}^q$  with dilation  $\lambda$  at its repelling point R,
- (2)  $\ell(\mathbb{B}^k)$  coincides with the stable subset  $\mathcal{S}(\zeta)$ ,
- (3) Universal property: if  $(\Lambda, h, \varphi)$  is a pre-model associated with  $\zeta$ , then there exists a morphism  $\hat{\eta}: (\Lambda, h, \varphi) \to (\mathbb{B}^k, \ell, \tau)$ ,
- (4)  $K \lim_{z \to R} \ell(z) = \zeta.$

Theorem 1 follows from [4, Theorem 1.3] once we prove the following two results.

**Theorem 2.** Let  $f : \mathbb{B}^q \to \mathbb{B}^q$  be a holomorphic self-map, and let  $\zeta \in \partial \mathbb{B}^q$  be a boundary repelling fixed point with dilation  $\lambda > 1$ . Then there exists a backward orbit  $(z_n)$  with step  $\log \lambda$  converging to  $\zeta$ .

**Proposition 1.** Let  $(x_n)$  and  $(y_n)$  be two backward orbits with bounded step, both converging to the boundary repelling fixed point  $\zeta \in \partial \mathbb{B}^q$ . Then

$$\lim_{n \to \infty} k_{\mathbb{B}^q}(x_n, y_n) < \infty.$$

Indeed, these two results imply that the family of backward orbits with bounded step converging to  $\zeta$  is not empty and is a unique equivalence class  $[y_n]$ . The pre-model  $(\mathbb{B}^k, \ell, \tau)$  is then the canonical pre-model given by  $[y_n]$ .

The method of proof of Theorem 2 is inspired by the proofs in [9, 8]. To get rid of the problems posed by boundary repelling points close to  $\zeta$  we use horospheres to define stopping times of the iterative process instead of euclidean balls centered at  $\zeta$ . This approach thus works also when the boundary fixed point  $\zeta$  is not isolated. On the other hand additional work has to be done to show that the iterative process still converges to a backward orbit.

## 2. Proof of Theorem 2

Without loss of generality we may assume that  $\zeta = e_1 = (1, 0, ..., 0)$ . Recall that the horosphere of center  $e_1$ , pole 0, and radius R > 0 is defined as

$$E_0(e_1, R) := \left\{ z \in \mathbb{B}^q \colon \frac{|1 - (z, e_1)|^2}{1 - ||z||^2} < R \right\}$$
$$= \left\{ z \in \mathbb{B}^q \colon \lim_{w \to e_1} \left( k_{\mathbb{B}^q}(z, w) - k_{\mathbb{B}^q}(0, w) \right) < \log R \right\}.$$

For all  $k \in \mathbb{Z}$ , let  $r_k = \left(\frac{\lambda^k - 1}{\lambda^k + 1}, 0, \dots, 0\right)$  and denote  $E_k := E_0\left(e_1, \frac{1}{\lambda^k}\right)$ . Notice that  $r_k \in \partial E_k$ . We have that (see e.g. [8])

$$\lim_{k \to \infty} k_{\mathbb{B}^q}(r_k, f(r_k)) = \log \lambda.$$
(1)

Furthermore, by the Julia's lemma (see e.g. [1]) we have

$$f(E_k) \underset{4}{\subset} E_{k-1}.$$
 (2)

If f has no interior fixed points, then it admits a Denjoy-Wolff point p such that the sequence  $(f^n)$  converges to p uniformly on compact subsets. Since the dilation of f at p is less than or equal to 1, it immediately follows that p is different from  $e_1$ . If f admits interior fixed points, it admits a limit manifold M which is a holomorphic retract of  $\mathbb{B}^q$  (see e.g. [1, Theorem 2.1.29]). By [2, Proposition 3.4] it follows that  $e_1 \notin \overline{M}$ . Hence, by conjugating the map f with an automorphism of the ball fixing  $e_1$ , we may further assume that

$$M \cap \overline{E}_0 = \emptyset$$

Hence, in both cases, for every  $k \ge 0$  there exists a first  $n(k) \ge 0$  so that  $f^{n(k)}(r_k) \notin \overline{E}_0$ . By (2) we have that n(k) > k. We write  $z_k := f^{n(k)}(r_k)$ .

**Lemma 1.** The sequence  $(z_k)$  is bounded away from  $e_1$ .

Proof of Lemma 1. Suppose instead that there exists a subsequence  $z_{k_i} \to e_1$ . Let  $b_i = z_{k_i}$  and  $a_i = f^{n(k_i)-1}(r_{k_i}) \in \overline{E}_0$ . Since  $k_{\mathbb{B}^q}(a_i, b_i) \leq k_{\mathbb{B}^q}(r_{k_i}, f(r_{k_i}))$  it follows by (1) that

$$\limsup_{i \to \infty} k_{\mathbb{B}^q}(a_i, b_i) \le \log \lambda.$$

It follows that the sequence  $(a_i)$  also converges to  $e_1$ .

By the definition of  $\lambda$ , we also have

$$\liminf_{i \to \infty} k_{\mathbb{B}^q}(a_i, b_i) \ge \liminf_{i \to \infty} \left( k_{\mathbb{B}^q}(0, a_i) - k_{\mathbb{B}^q}(0, b_i) \right)$$
$$\ge \liminf_{z \to e_1} \left( k_{\mathbb{B}^q}(0, z) - k_{\mathbb{B}^q}(0, f(z)) \right)$$
$$= \log \lambda.$$

We conclude that

$$\lim_{i \to \infty} k_{\mathbb{B}^q}(a_i, b_i) = \log \lambda.$$
(3)

Let  $\gamma_i \in Aut(\mathbb{B}^q)$  be such that  $\gamma_i(a_i) = 0$  and  $\gamma_i(e_1) = e_1$ . Such an automorphism can be obtained composing a parabolic automorphism fixing  $e_1$  and a hyperbolic automorphism fixing  $e_1$  and  $-e_1$ , with dilation  $\mu$  at  $e_1$  which satisfies  $1 \leq \mu < \lambda$  since  $a_i \in \overline{E}_0 \setminus \overline{E}_1$ . Hence the dilation of  $\gamma_i$  at  $e_1$  is  $\mu$ , which implies that

$$\overline{E}_0 \subset \gamma_i(\overline{E}_0) \subset \overline{E}_{-1}.$$

Since  $k_{\mathbb{B}^q}(0, a_i) \to \infty$ , it follows that  $k_{\mathbb{B}^q}(\gamma_i(0), 0) \to \infty$ . Since the sequence  $(\gamma_i(0))$  is contained in  $\overline{E}_{-1}$ , we obtain  $\gamma_i(0) \to e_1$ .

Let  $0 < \alpha < 1$  and set

$$c_i := -\alpha \frac{\gamma_i(0)}{\|\gamma_i(0)\|}.$$

If we write  $\beta := \log \frac{1+\alpha}{1-\alpha}$ , then for every *i* we have  $k_{\mathbb{B}}(0, c_i) = \beta$ . Since  $\gamma_i(0) \to e_1$  it follows that  $c_i \to c_{\infty} = (-\alpha, 0, \dots, 0)$ .

By (3) the sequence  $\gamma_i(b_i)$  is relatively compact in  $\mathbb{B}^q$ . By taking a subsequence of  $k_i$  if necessary, we may assume that  $\gamma_i(b_i) \to b_\infty \in \mathbb{B}^q$ . Notice that since  $b_i \notin \overline{E}_0$ , we must also have  $\gamma_i(b_i) \notin \overline{E}_0$ . It follows that  $b_\infty \notin E_0$ .

We claim that  $k_{\mathbb{B}^q}(c_{\infty}, b_{\infty}) < \log \lambda + \beta$ . Indeed, since  $k_{\mathbb{B}^q}(0, b_{\infty}) = \log \lambda$  and  $k_{\mathbb{B}}(0, c_{\infty}) = \beta$ , we get by triangular inequality that  $k_{\mathbb{B}^q}(c_{\infty}, b_{\infty}) \leq \log \lambda + \beta$ . Equality

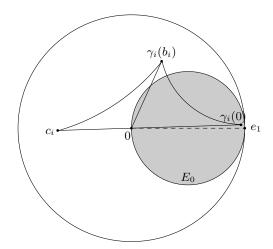


FIGURE 1. The position of the points at the *i*-th step, when q = 1.

holds if and only if  $b_{\infty}$  is contained in the real geodesic connecting the origin to  $e_1$ . But this is not possible since such geodesic is contained in the horosphere  $E_0$ . Let  $\delta > 0$  be such that  $k_{\mathbb{B}^q}(c_{\infty}, b_{\infty}) < \log \lambda + \beta - 2\delta$ .

By the last inequality and by the definition of  $\lambda$  we may choose *i* big enough such that  $k_{\mathbb{B}^q}(c_i, \gamma_i(b_i)) < \log \lambda + \beta - \delta$  and

$$k_{\mathbb{B}^{q}}(\gamma_{i}(0), 0) - k_{\mathbb{B}^{q}}(\gamma_{i}(0), \gamma_{i}(b_{i})) = k_{\mathbb{B}^{q}}(0, a_{i}) - k_{\mathbb{B}^{q}}(0, b_{i}) \ge \log \lambda - \delta.$$

We conclude that

$$k_{\mathbb{B}^{q}}(\gamma_{i}(0), c_{i}) - k_{\mathbb{B}^{q}}(\gamma_{i}(0), \gamma_{i}(b_{i})) = k_{\mathbb{B}^{q}}(\gamma_{i}(0), 0) + k_{\mathbb{B}^{q}}(0, c_{i}) - k_{\mathbb{B}^{q}}(\gamma_{i}(0), \gamma_{i}(b_{i}))$$
$$\geq \beta + \log \lambda - \delta$$
$$> k_{\mathbb{B}^{q}}(c_{i}, \gamma_{i}(b_{i})),$$

contradicting the triangular inequality.

We are now ready to conclude the proof of Theorem 2. Since the sequence  $(z_k) = (f^{n(k)}(r_k))$  is bounded away from  $e_1$  and contained in  $\overline{E}_{-1}$ , we can extract a subsequence  $k_0(h)$  such that  $(f^{n(k_0(h))}(r_{k_0(h)}))$  converges to a point  $w_0 \in \mathbb{B}^q$ . Now consider the sequence  $(f^{n(k_0(h))-1}(r_{k_0(h)}))$ . Since

$$k_{\mathbb{B}^q}(f^{n(k_0(h))}(r_{k_0(h)}), f^{n(k_0(h))-1}(r_{k_0(h)})) \le k_{\mathbb{B}^q}(r_{k_0(h)}, f(r_{k_0(h)})) \to \log \lambda,$$

we can extract a subsequence  $k_1(h)$  of  $k_0(h)$  such that  $(f^{n(k_1(h))-1}(r_{k_1(h)}))$  converges to a point  $w_1 \in \mathbb{B}^q \cap \overline{E}_0$  and  $f(w_1) = w_0$ . Iterating this procedure we obtain for all  $j \geq 1$  a subsequence  $(k_j(h))$  of  $(k_{j-1}(h))$  such that  $(f^{n(k_j(h))-j}(r_{k_j(h)}))$  converges to a point  $w_j \in \mathbb{B}^q \cap \overline{E}_0$  and  $f(w_j) = w_{j-1}$ . We notice that, since n(k) > k, the expression  $(f^{n(k_j(h))-j}(r_{k_j(h)}))$  is well defined for h large enough.

The backward orbit  $(w_j)$  has bounded step since, for all  $j \ge 0$ ,

$$k_{\mathbb{B}^{q}}(w_{j-1}, w_{j}) = \lim_{h \to \infty} k_{\mathbb{B}^{q}}(f^{n(k_{j}(h))-j-1}(r_{k_{j}(h)}), f^{n(k_{j}(h))-j}(r_{k_{j}(h)}))$$
  
$$\leq \lim_{h \to \infty} k_{\mathbb{B}^{q}}(r_{k_{j}(h)}, f(r_{k_{j}(h)}))$$
  
$$= \log \lambda.$$

We are left with showing that  $w_j \to e_1$ . Since the sequence  $(w_j)_{j\geq 1}$  is contained in  $\overline{E}_0$ it is enough to show that there is no subsequence  $(w_{m(j)})$  converging to a point  $x \in \mathbb{B}^q$ . Assume by contradiction that such a subsequence exists. Then there exists a compact set  $K \subset \overline{E}_0 \cap \mathbb{B}^q$  containing the sequence  $(w_{m(j)})$ . Recall that if f has no interior fixed points its Denjoy-Wolff point is different from  $e_1$ , and that if f has fixed points then the limit manifold M does not intersect  $\overline{E}_0$ . Hence there exists an integer  $N \ge 0$  such that  $f^n(K) \cap K = \emptyset$  for all  $n \ge N$ . But this is a contradiction since  $(w_j)$  is a backward orbit.

# 3. Proof of Proposition 1

Given a backward orbit  $(x_n)$  one can always assume that it is indexed by integers  $n \in \mathbb{Z}$ , defining for all  $n \ge 0$ ,  $x_{-n} := f^n(x_0)$ .

**Lemma 2.** Let  $(x_n)$  and  $(y_n)$  be two backward orbits with bounded step, both converging to  $e_1$ . Then  $\lim_{n\to+\infty} k_{\mathbb{B}^q}(x_n, y_n) < \infty$  if and only if

$$\lim_{n \to +\infty} \inf_{m \in \mathbb{Z}} k_{\mathbb{B}^q}(x_n, y_m) < \infty.$$

*Proof.* Notice first that  $\inf_{m \in \mathbb{Z}} k_{\mathbb{B}^q}(x_n, y_m)$  is non-decreasing in n, therefore the limit for  $n \to +\infty$  exists (possibly infinite). Suppose now that there exists C > 0 such that

$$\inf_{m \in \mathbb{Z}} k_{\mathbb{B}^q}(x_n, y_m) < C, \qquad \forall n \in \mathbb{Z}.$$

The forward orbit of the point  $y_0$  is bounded away from the bondary repelling fixed point  $e_1$ . Since  $\lim_{m\to+\infty} x_m = e_1$ , we may assume that there exists N > 0 such that  $k_{\mathbb{B}^q}(x_N, y_m) \ge C$  for all m < 0. Given such N, since  $\lim_{m\to+\infty} y_m = e_1$ , we may also find M > 0 such that  $k_{\mathbb{B}^q}(x_N, y_m) \ge C$  for all m > M. For every  $n \ge N$  we may find an integer  $\alpha(n)$  such that  $k_{\mathbb{B}^q}(x_n, y_{\alpha(n)}) < C$ . By the properties of the Kobayashi distance we deduce that

$$k_{\mathbb{B}^q}(x_N, y_{\alpha(n)-n+N}) < C,$$

which implies that  $-N \leq \alpha(n) - n \leq M - N$ . It follows that we may find a divergent sequence  $n_k \geq N$  so that  $\alpha(n_k) - n_k = \alpha \in \mathbb{Z}$ . Notice that for every k we have

$$k_{\mathbb{B}^q}(x_{n_k}, y_{n_k+\alpha}) = k_{\mathbb{B}^q}(x_{n_k}, y_{\alpha(n_k)}) < C,$$

which implies that for every  $n \in \mathbb{Z}$ , we have  $k_{\mathbb{B}^q}(x_n, y_{n+\alpha}) < C$ . Finally since  $(y_n)$  has bounded step we deduce that

$$k_{\mathbb{B}^q}(x_n, y_n) \le k_{\mathbb{B}^q}(x_n, y_{n+\alpha}) + k_{\mathbb{B}^q}(y_{n+\alpha}, y_n) \le C + |\alpha|\sigma(y_n).$$

The other implication is trivial.

The following lemma is essentially contained in [8], even if not explicitly stated.

**Lemma 3.** Let  $(z_n)$  be a backward orbit with bounded step converging to  $e_1$ . Then there exists M > 0 so that

$$z_n \in K_0(\tau, M), \quad \forall n \ge 0.$$

*Proof.* By the definition of dilation  $\lambda$  we have

$$\liminf_{n \to \infty} \left( k_{\mathbb{B}^q}(0, z_{n+1}) - k_{\mathbb{B}^q}(0, z_n) \right) \ge \log \lambda,$$

and therefore that

$$\liminf_{n \to \infty} \frac{1 - \|z_n\|}{1 - \|z_{n+1}\|} \ge \lambda.$$

If we write  $t_n := 1 - ||z_n||$ , and take  $\lambda^{-1} < c < 1$  we conclude that there exists  $n_0 \in \mathbb{N}$  such that

$$t_{n+1} \leq ct_n, \qquad \forall n \geq n_0.$$

By shifting the sequence  $(z_n)$  if necessary, we may assume that  $n_0 = 0$ . We conclude that

$$t_{n+k} \le c^k t_n, \qquad \forall n, k \ge 0.$$
(4)

The result of the lemma follows from (4) exactly as in the elliptic case of [8, Theorem 1.8].

Let  $(x_n)_{n\in\mathbb{Z}}$  and  $(y_n)_{n\in\mathbb{Z}}$  be two backward orbits with bounded step, both converging to the boundary repelling fixed point  $e_1 \in \partial \mathbb{B}^q$ . Recall that we denote by  $\eta$  the ray connecting the origin to  $e_1$ . By Lemma 3 the sequence  $(x_n)_{n\in\mathbb{N}}$  is contained in a Korányi region, and thus it is contained in a region  $A(\eta, L) := \{z \in \mathbb{B}^q : k_{\mathbb{B}^q}(z, \eta) < L\}$  for some L > 0 (see e.g. [7, Lemma 2.5]). We claim that there exists R > 0 such that

$$A(\eta, L) \subset \left\{ z \in \mathbb{B}^q \colon \inf_{m \in \mathbb{Z}} k_{\mathbb{B}^q}(z, y_m) < R \right\}.$$

Once the claim is proved, the result follows by Lemma 2.

It is enough to show that there exists a constant S > 0 such that for all  $w \in \eta$ ,  $\inf_{m \in \mathbb{Z}} k_{\mathbb{B}^q}(w, y_m) < S$ . Since  $(y_n)_{n \in \mathbb{N}}$  is contained in a Korányi region, there exists a constant C > 0 such that  $k_{\mathbb{B}^q}(y_n, \eta) < C$  for all  $n \in \mathbb{N}$ . Let  $a_n$  be a point in  $\eta$  such that  $k_{\mathbb{B}^q}(a_n, y_n) < C$ . Clearly  $a_n \to e_1$ . Let w be a point in the portion of  $\eta$  which connects  $a_0$  to  $e_1$ . There exists n(w) such that w belongs to the portion of  $\eta$  which connects  $a_n$ to  $a_{n+1}$ . Hence

$$\inf_{m} k_{\mathbb{B}^{q}}(w, y_{m}) \leq C + k_{\mathbb{B}^{q}}(a_{n}, a_{n+1}) \leq C + 2C + k_{\mathbb{B}^{q}}(y_{n}, y_{n+1}) \leq 3C + \sigma(y_{n}),$$

concluding the proof of Proposition 1.

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DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ROMA "TOR VERGATA", VIA DELLA RICERCA SCIENTIFICA 1, 00133 ROMA, ITALY

*E-mail address*: arosio@mat.uniroma2.it

KORTEWEG DE VRIES INSTITUTE FOR MATHEMATICS, UNIVERSITY OF AMSTERDAM, SCIENCE PARK 107, 1090GE AMSTERDAM, THE NETHERLANDS

*E-mail address*: lorenzo.guerini92@gmail.com