Interior gradient and Hessian estimates for the Dirichlet problem of semi-linear degenerate elliptic systems: a probabilistic approach^{*}

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Abstract

In this paper, we give interior gradient and Hessian estimates for systems of semi-linear degenerate elliptic partial differential equations on bounded domains, using both tools of backward stochastic differential equations and quasi-derivatives.

Key Words. semi-linear degenerate elliptic partial differential equations, quasi-derivatives, backward stochastic differential equations, Dirichlet problems

1 Introduction

Let d, d_1 , and k be given positive integers. Let D be a bounded domain in a d-dimensional Euclidean space \mathbb{R}^d . Consider the Dirichlet problem for a system of semi-linear degenerate elliptic partial differential equations (PDEs) of second order

$$\begin{cases} \mathcal{L}u(x) + f(x, u(x), \nabla u(x)\sigma(x)) = 0, & x \in D; \\ u(x) = g(x), & x \in \partial D, \end{cases}$$
(1.1)

where $u := (u_1, \cdots, u_k)^*, \mathcal{L}u := (\mathcal{L}u_1, \cdots, \mathcal{L}u_k)^*,$

$$\mathcal{L}u_m(x) := \sum_{i,j=1}^d a_{i,j}(x)\partial_{i,j}^2 u_m(x) + \sum_{i=1}^d b_i(x)\partial_i u_m(x), \quad m = 1, \cdots, k,$$

with $a = \frac{1}{2}\sigma\sigma^*$. Here and in the following, the asterisk * in the superscript means the transpose.

Let W be a d_1 -dimensional Wiener process in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\{\mathcal{F}_t, t \geq 0\}$ being the augmented natural filtration. The probabilistic solution of (1.1) is given by Peng [18] as

$$u(x) = Y_0(x), \qquad x \in \overline{D}, \tag{1.2}$$

where $\{(Y_t(x), Z_t(x)), 0 \le t \le \tau\}$ is the unique adapted solution to the backward stochastic differential equation (BSDE)

$$\begin{cases} dY_t = -f(X_t(x), Y_t, Z_t) \, dt + Z_t \, dW_t, & t \in [0, \tau), \\ Y_\tau = g(X_\tau(x)), \end{cases}$$
(1.3)

with $\{X_t(x), t \ge 0\}$ being the solution to the stochastic differential equation (SDE)

$$X_t = x + \int_0^t \sigma(X_s) dW_s + \int_0^t b(X_s) ds, \qquad (1.4)$$

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and $\tau := \tau(x) := \inf\{t > 0, X_t(x) \notin D\}$ is the first exit time of $X_t(x)$ from D, under the assumption that the coefficients b, σ, f , and g and the domain D are all sufficiently smooth.

When the coefficients $(b, g, \sigma) : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{d \times d_1}$ and $f : \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d_1} \to \mathbb{R}^k$ are uniformly Lipschitz continuous in all of their arguments, one has to consider weak solutions for the associated PDEs. One weak solution is the notion of viscosity solutions. The function u defined by (1.2) is shown by Darling and Pardoux [4] to be the unique continuous viscosity solution of (1.1), when k = 1. Another weak solution of PDEs is the notion of Sobolev solutions. Ouknine and Turpin [16] gave a representation for the Sobolev solutions of degenerate parabolic PDEs through FBSDEs. They were inspired by the work of Bally and Matoussi [1], in which the Sobolev solutions of semi-linear SPDEs are described via BDSDEs. Later, Feng, Wang and Zhao [6] studied the existence, uniqueness and the probabilistic representation of the Sobolev solutions of quasi-linear parabolic and elliptic PDEs in \mathbb{R}^d . We are interested in those conditions which yield further regularity of u.

Using a deterministic approach, Caffarelli [3] obtained a priori $W^{2,p}$ estimates for the viscosity solutions of second order, uniformly elliptic, fully non-linear equations: $F(D^2u, x) = f(x)$ in a unit ball. Freidlin [7] obtained an early probabilistic result that the solution is smooth for degenerate linear elliptic equations (1.1) with $f(x) = 0, x \in \overline{D}$ if the boundary data g is sufficiently smooth. Peng [18] showed that the classical solution of the nondegenerate quasi-linear elliptic PDE has a probabilistic interpretation $u(x) = Y_0(x), x \in \overline{D}$. Darling and Pardoux [4] proved that when f is monotone in $y, u(\cdot) = Y_0(\cdot)$ is a bounded and continuous viscosity solution to the Dirichlet problem for a class of semi-linear elliptic PDEs. Later, Briand and Hu [2] gives a stability result for BSDEs with random terminal time which is associated to a system of semi-linear elliptic PDEs by partially relaxing the monotonicity assumption on the coefficient.

Moreover, to obtain in a probabilistic way the gradient estimates of the solution of second order PDEs, the now well-known theory of stochastic flows plays a crucial role. For example, Pardoux and Peng [17] used the tool of BSDEs to investigate the regularity properties of the solution of parabolic PDEs, and they proved that the solution of BSDEs $\{Y_s^{t,x}, (s,t,x) \in [0,T]^2 \times \mathbb{R}^d\}$ has a version whose trajectories belong to $C^{0,0,2}([0,T]^2 \times \mathbb{R}^d)$. Tang [19] extended their context to incorporate random coefficients, and proved that, when f(t, x, y, z) is linear in z, the regularity of the solution of BSDEs can be derived from those of the coefficients of FBSDEs. These works take advantage of the Cauchy problem where the space variable takes values over the whole space. The methodology is difficult to be adapted to the Dirichlet problem of elliptic PDEs in a bounded domain: estimating the gradient of the function u through directly differentiating the expression (1.2) involves the differentiation of the exit time $\tau(x)$ with respect to x, while the function $\tau(x)$ is not necessarily differentiable with respect to x. To get around such a difficulty, Delarue [5] established a priori Hölder estimate of Krylov and Safonov type for the viscosity solution of a degenerate quasi-linear elliptic PDE, where the Hölder bound does not depend on the regularity of σ and f. He extended that of Krylov and Safonov [15], by building a special type of SDEs with σ depending on both u and its gradient ∇u .

Alternative powerful tool is that of quasi-derivative. It was first introduced by Krylov [8], to find a different condition on coefficients such that c is sufficiently large compared to first derivatives of σ and b with respect to x under which u is twice continuously differentiable in \mathbb{R}^d . This condition weakens the known conditions mentioned in [12, page 257] where no quasi-derivative is used. Since then this technique has been applied to investigate the smoothness of solution of various elliptic and parabolic PDEs. Krylov [9, 11, 14] applied different quasi-derivative methods to study the interior regularity of harmonic functions of degenerate elliptic operators. Later Krylov [10] obtained $C^{1,1}$ -regularity of the solution up to the boundary for the Dirichlet problem of degenerate Bellman equations under the boundary value assumption that $g \in C^4(\overline{D})$ (see [10, Assumption 1.3, page 67] where g should be required to lie in $C^4(\overline{D})$, though there it was only supposed to lie in $C^3(\overline{D})$; and see Krylov's own exposition [14, page 2] for this point), which holds true for the degenerate linear elliptic equations (see [10, Theorem 2.1, page 74]). The ideas are based on adding a 4-dimensional process y_t to the original d-dimensional processes x_t (see [10, page 83]) such that the augmented process $z_t = (x_t, y_t)$ never leaves a surface in \mathbb{R}^{d+4} . In this way, he can get rid of the dependence of the first exit time on the initial point and use the techniques of [8] to obtain moment estimates of quasi-derivatives in the whole space. Recently, Zhou [21] introduced the notion of the second quasi-derivative to estimate the derivatives up to the second order of u inside the domain under the weaker boundary value assumption that $g \in C^{1,1}(\overline{D})$. Under the weaker boundary regularity assumption of $g \in C^{0,1}(\overline{D})$, it has been illustrated (see, for instance, [14, page 58-63]) that, the first-order derivatives of u fail to be bounded up to the boundary in both PDE methods and quasi-derivative methods, even for the Laplacian equation (i.e., $\mathcal{L} = \Delta$). He commented that for $g \in C^{1,1}(\overline{D})$, one can only expect to

prove interior $C^{1,1}$ -regularity (see [21, page 3065]). His proof relied on a probabilistic interpretation of the linear degenerate elliptic PDEs. He introduced two local martingales with the help of quasiderivatives and their auxiliary processes to formulate first and second derivatives of u, respectively (see [21, Theorem 2.2]). Besides, instead of adding four more dimensions as [10, page 83], he constructed two families of local super-martingales to bound the moments of quasi-derivatives near the boundary and in the interior of the domain, respectively (see [21, Lemmas 3.3 and 3.4]). All these existing works which employ the method of quasi-derivatives discussed either the linear second-order PDEs or the so-called Bellman equation (which is a fully nonlinear PDE) arising from optimal stochastic control problems.

In our context for a k-dimensional vector-valued nonlinear function f, we use both tools of BSDEs and quasi-derivatives to establish the gradient and Hessian estimates for the solution u to the Dirichlet problem for a system of semi-linear degenerate second-order partial differential equations (1.1).

Our objective is to establish the counterpart of Zhou's estimates [21, Theorem 3.1] for a system of semi-linear elliptic PDEs, which is precisely stated in Theorem 2.9 at the end of Section 2 below.

In contrast to Zhou [21], we have new difficulties. In fact, for the gradient estimate, we need to calculate the difference $|Y_0^{\delta}(x+\delta\xi_0)-Y_0(x)|$ between the solutions of the perturbed BSDE (in (2.12)) and the unperturbed one and appeal to the BSDE estimates. As a consequence, new barrier functions are introduced near the boundary and in the interior of the domain (see Lemmas 3.2 and 3.4), so as to bound higher moment estimates of quasi-derivatives, which leads to that the process $(\psi_{(\xi_t)}\psi^{-1})^p$ is considered in a better space. For the Hessian estimate, we estimate the second-order difference $|Y_0^{\delta} - 2Y_0 + Y_0^{-\delta}|$. To deal with the nonlinearity of f in ∇u , we use the technical skills developed in the estimates of BSDEs by Pardoux and Peng [17, Theorem 2.9]. Finally, we emphasize that we consider a system of semi-linear degenerate elliptic PDEs rather than a single equation, where u, f and g take values in \mathbb{R}^k , although it must be said that our interior estimates are not sharper than those of [21, Theorem 3.1] in some sense.

The paper is organized as follows: In Section 2, we set notations and list the standing assumptions. Then we introduce some standard estimates for the solution of random terminal BSDEs, and recall the concept of the quasi-derivative and some known basic results. We end up with the statement of our main results. In Section 3, we build four barrier functions to get some moment estimates of the quasi-derivatives and derive generalized assertions at last. In Section 4, we use the BSDE estimates to establish the interior gradient and Hessian estimates of u in (1.1) under the aforementioned assumptions, and then show the existence and uniqueness of u in (1.1).

2 Preliminaries and Statement of the Main Results

Let $\mathcal{A} := \{ \alpha = (\alpha_1, \cdots, \alpha_d) : \alpha_i, i = 1, \cdots, d \text{ are nonnegative integers} \}$ be the set of multi-indices. For any $\alpha \in \mathcal{A}$ and $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, denote

$$|\alpha| := \sum_{i=1}^{n} \alpha_i, \quad \partial^{\alpha} := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d}.$$

In a Euclidean space \mathbb{E} , denote by $\langle \cdot, \cdot \rangle$ the inner product, and the norm by $|\cdot|_{\mathbb{E}}$ or simply by $|\cdot|$ when no confusion is made. Let \mathfrak{B} be the set of all skew-symmetric $d_1 \times d_1$ matrices. Denote by ||A|| the norm of a matrix A, which is defined to be the square root of the sum of all the squared components, i.e. $||A||^2$ is the trace of AA^* .

Denote by $C(\overline{D})$ the Banach spaces of continuous functions g in \overline{D} equipped with the norm

$$|g|_0 = \sup_{x \in \overline{D}} |g(x)|,$$

and by $C^m(\overline{D})$ with m = 1, 2 the Banach spaces of once (for m = 1) or twice (for m = 2) continuously differentiable functions g in \overline{D} equipped with the respective norm:

$$|g|_1 := |g|_0 + |g_x|_0, \quad |g|_2 := |g|_1 + |g_{xx}|_0,$$

where g_x is the gradient vector of g, and g_{xx} is the Hessian matrix of g. For $\beta \in (0, 1]$, the Hölder space $C^{m,\beta}(\overline{D})$ is the Banach subspace of $C^m(\overline{D})$ consisting of all functions g with the norm

$$|g|_{m,\beta} := |g|_m + [g]_{m,\beta}, \quad \text{where } [g]_{m,\beta} := \sum_{|\alpha|=m} \sup_{x,y\in\overline{D}} \frac{|\partial^{\alpha}g(x) - \partial^{\alpha}g(y)|}{|x-y|^{\beta}}.$$

For a function $f \in C^{0,1}(\overline{D} \times \mathbb{R}^k \times \mathbb{R}^{k \times d_1})$, define

$$\|f(\cdot)\|_{0,1} := |f(\cdot,0,0)|_0 + [f]_{0,1,x}, \quad \text{where } [f]_{0,1,x} := \sup_{\substack{x,x'\in\overline{D}\\(y,z)\in\mathbb{R}^k\times\mathbb{R}^{k\times d_1}}} \frac{|f(x,y,z) - f(x',y,z)|}{|x-x'|}.$$

For a function $f \in C^{1,1}(\overline{D} \times \mathbb{R}^k \times \mathbb{R}^{k \times d_1})$, define

$$[f]_{1,1} := \sup_{\Xi, \Xi' \in \overline{D} \times \mathbb{R}^k \times \mathbb{R}^{k \times d_1}} \frac{|\partial f(\Xi) - \partial f(\Xi')|}{|\Xi - \Xi'|}.$$

Denote by $H^2_{k,\rho}(\overline{D})$ the weighted Sobolev space, equipped with the norm:

$$|\varphi|_{k,\rho} := \left\{ \sum_{0 \le |\alpha| \le k} \int_D |\partial^{\alpha} \varphi(x)|^2 \rho^{-1}(x) dx \right\}^{\frac{1}{2}},$$

where $\rho(x) := (1 + |x|^2)^q$, $q \ge 2$ is a weight function.

For $\{\mathcal{F}_t\}$ -stopping time τ and some real number β , $\mathcal{M}_{\beta}(0,\tau;V)$ denotes the Hilbert space of all progressively measurable processes X taking values in the Euclidean space V, such that

$$||X||_{\mathcal{M}_{\beta}} := E\left[\int_0^\tau e^{2\beta s} |X_s|^2 ds\right]^{\frac{1}{2}} < \infty.$$

Denote by $\mathbb{M}^m(D, \sigma, b)$ the set of real-valued *m*-times continuously differentiable functions given on *D* such that for any $v \in \mathbb{M}^m(D, \sigma, b)$ the process $\{v(X_t), [0, \tau]\}$ is a local martingale relative to \mathcal{F}_t for any $x \in D$. We write \mathbb{M}^m for $\mathbb{M}^m(D, \sigma, b)$ whenever no confusion is made.

For $y, z \in \mathbb{R}^d$ and the d_2 -dimensional column vector function ϕ , set $\phi := (\phi_1, \cdots, \phi_{d_2})^*$, and

$$\phi_{(y)} := \left(\sum_{i=1}^d \partial_i \phi_m \cdot y_i\right)_{1 \le m \le d_2}^*, \quad \phi_{(y)(z)} := \left(\sum_{i,j=1}^d \partial_{ij}^2 \phi_m \cdot y_i z_j\right)_{1 \le m \le d_2}^*.$$

Write E_x for the expectation of a functional of the underlying process which takes value x at the initial time 0, and $N(K_1, K_2, \cdots)$ for a constant N to indicate its dependence on K_1, K_2, \cdots whenever necessary.

We introduce the following assumptions with constants p = 1, 2 and q = 0, 1. The assumptions (H1) - (H3) are necessary for the well-posedness of solutions to SDEs.

(H1) σ and b are twice continuously differentiable in \mathbb{R}^d .

(H2) The domain $D \in C^4$ is bounded in \mathbb{R}^d . There is a function $\psi \in C^4$ such that (i) $\psi(x) > 0$ for $x \in D$, (ii) $\psi(x) = 0$ and $|\psi_x(x)| \ge 1$ for $x \in \partial D$, and (iii) the following inequality holds true:

$$\mathcal{L}\psi(x) := \sum_{i,j=1}^{d} a_{ij}(x)\partial_{ij}^2\psi(x) + \sum_{i=1}^{d} b_i(x)\partial_i\psi(x) \le -1 \quad \text{for } x \in D.$$

$$(2.1)$$

In what follows, we write $D_{\lambda} := \{x \in D : \psi(x) > \lambda\}$ for $\lambda \in (0, 1)$.

(H3) There exists a constant $K_0 > 0$, such that

$$\sum_{i=1}^{d} \sum_{j=1}^{d_1} |\sigma_{ij}|_2 + \sum_{i=1}^{d} |b_i|_2 + |\psi|_4 \le K_0.$$

The assumptions (H4) - (H8) are necessary for the smoothness.

(H4) $f \in C^{0,1}(\overline{D} \times \mathbb{R}^k \times \mathbb{R}^{k \times d_1})$, and there exist constants $L, L_0 > 0$, such that

$$|f(x,y,z) - f(x,\bar{y},\bar{z})| \le L|y - \bar{y}| + L_0|z - \bar{z}|, \quad x \in \overline{D}; y, \bar{y} \in \mathbb{R}^k; z, \bar{z} \in \mathbb{R}^{k \times d_1}.$$

(H5) There exists a constant $\mu \in \mathbb{R}$ such that

$$\langle f(x,y,z) - f(x,\bar{y},z), (y-\bar{y}) \rangle \le -\mu |y-\bar{y}|^2, \quad x \in \overline{D}; y,\bar{y} \in \mathbb{R}^k; z \in \mathbb{R}^{k \times d_1}.$$

 $(H6)_q \ g \in C^{q,1}(\overline{D}).$

(H7) There exists constants β and ϑ such that

 $0 < \mu < L, \ -\mu + 2L_0^2 < 2\beta < 0, \ \text{and} \ 2\vartheta = -2\mu + L_0^2,$

where L and L_0 are the Lipschitz constants of f with respect to y and z respectively in (H4), and μ is the monotonicity constant of f in (H5).

 $(H8) \ f \in C^{1,1}(\overline{D} \times \mathbb{R}^k \times \mathbb{R}^{k \times d_1}), \text{ and for any } (x, y, z) \in \overline{D} \times \mathbb{R}^k \times \mathbb{R}^{k \times d_1}, \ f_y(x, y, z) \leq -\mu.$

The assumptions (H9) and $(H10)_p$ are necessary for controlling the moments of quasi-derivatives.

(H9) The inequality $\langle an, n \rangle > 0$ holds for any unitary normal vector n at ∂D .

 $(H10)_p$ There exist functions $(\rho, M): D \to \mathbb{R}^d \times \mathbb{R}$ and $Q(\cdot, \cdot): D \times \mathbb{R}^d \to \mathfrak{B}$, such that (i) (ρ, M) is bounded in D_{λ} for any $\lambda \in (0, 1)$, and (ii) $Q(\cdot, y)$ is bounded in D_{λ} for any $(\lambda, y) \in (0, 1) \times \mathbb{R}^{d}$ and $Q(x,\cdot)$ is a linear function for any $x \in D$. Furthermore, we have for β satisfying (H7)

$$2p(4p-1) \left\| \sigma_{(y)}(x) + \langle \rho(x), y \rangle \sigma(x) + \sigma(x)Q(x,y) \right\|^{2} + 4p \left\langle y, b_{(y)}(x) + 2\langle \rho(x), y \rangle b(x) \right\rangle$$

$$\leq (-4p\beta - 1) + 2pM(x) \langle a(x)y, y \rangle, \quad \forall (x,y) \in D \times \mathbb{R}^{d} \text{ with } |y| = 1.$$
(2.2)

Remark 2.1. (i) It is easy to see that (H4) implies (H5) for $L \leq -\mu$. Thus, (H5) gives some additional restriction only for $L > -\mu$ (see [20, page 363]). (ii) For $f \in C^1(\overline{D} \times \mathbb{R}^k \times \mathbb{R}^{k \times d_1})$, conditions (H4) and (H5) yield $||f_z|| \leq L_0$.

(iii) (H4) implies that f is continuous in x and thus $|f(\cdot, 0, 0)|_0 < \infty$, and moreover that f is Lipschitz continuous in (x, y, z). Consequently, (H4) implies that $||f(\cdot)||_{0,1} < \infty$. In addition to the assumption that all the first-order partial derivatives of f are globally Lipschitz continuous in (x, y, z), (H8) implies that $[f]_{1,1} < \infty$.

(iv) (H9) and $(H10)_p$ are conditions for guaranteeing that the moments of quasi-derivatives near the boundary or in the interior of the domain do not grow too fast. (H9) implies a is non-degenerate along the normal to the boundary. However, if σ is a constant and D is a bounded domain, (H9) implies that a is uniformly non-degenerate. $(H10)_p$ is weaker than the non-degenerate condition. Indeed, assume that M = 1, $\rho = Q = 0$, $p = \frac{1}{2}$ and $d = d_1 = 1$ for the sake of simplicity, then we will have $2\beta + 1 + 2b'(x) + |\sigma'(x)|^2 \le a(x)$, where the sum of the terms on the left hand side of the inequality may be negative. As for the necessity, we have to admit that $(H10)_{\frac{1}{2}}$ is stronger than [21, Assumption 3.2] for the linear case. But our $(H10)_p$ can be used for higher moment estimates of quasi-derivatives. In fact, if we take f = cu, $\sigma = x$ and $b = b_1 x$, where c and b_1 are constants, then as [21, Remark 3.2], the condition $1 + 2b_1 \leq c$ is necessary for u having Lipschitz continuous derivatives. However, when $p = \frac{1}{2}$, the conditions (H4), (H5), (H7) and (H10) $\frac{1}{5}$ imply $2\beta > -c$ and $1 + 2b_1 < c$, which are stronger than the necessary condition in [21, Assumption 3.2].

Note that throughout the paper constants K and N may differ in different inequalities.

BSDEs in Random Durations Revisited 2.1

BSDEs with random terminal times have been studied by Peng [18], Darling and Pardoux [4], and Briand and Hu [2]. See also Yong and Zhou [20, page 360] for a relevant exposition and related references therein. In this subsection, we give some priori estimates for the solutions of BSDEs and represent the solutions of a system of second order semi-linear elliptic PDEs through BSDEs.

We consider the Itô stochastic equation

$$X_t = x + \int_0^t \sigma(X_s) \, dW_s + \int_0^t b(X_s) \, ds, \quad t \ge 0.$$
(2.3)

In view of (H1), it has a unique solution $\{X_t, t \geq 0\}$ for any $x \in D$. We have the following four inequalities: $E\tau(x) < \infty$ (from (H2) and Lemma 3.1), $|f(\cdot, 0, 0)| < \infty$ (from (H4)), $|g|_0 < \infty$ (from $(H6)_0$ and $\vartheta < 0$ (from (H7)), all of which yield the following key assumption of [4, Theorem 3.4]: for some $\rho \in (\vartheta, 0)$,

$$E_x \left[e^{2\varrho\tau} |g(X_\tau)|^2 + \int_0^\tau e^{2\varrho s} |f(X_s, 0, 0)|^2 ds \right] < \infty, \quad \forall x \in D.$$
(2.4)

Therefore, according to [4, Theorem 3.4], the following BSDE

$$\begin{cases} dY_t = -f(X_t, Y_t, Z_t)dt + Z_t dW_t, & t \in [0, \tau); \\ Y_\tau = g(X_\tau) \end{cases}$$
(2.5)

has a unique pair $(Y, Z) \in \mathcal{M}_{\vartheta}(0, \tau; \mathbb{R}^k \times \mathbb{R}^{k \times d_1})$ if $(H2), (H4), (H5), (H6)_0$ and (H7) are all satisfied.

Lemma 2.2. [4, Proposition 4.3 and Corollary 4.4.1] Assume that f is Lipschitz with respect to (y, z)(also made in (H4)). Let the assumption (H5) and the inequality (2.4) be satisfied for some $\varrho > \vartheta$. Then BSDE (2.5) admits a unique adapted solution $(Y, Z) \in \mathcal{M}_{\vartheta}(0, \tau; \mathbb{R}^k \times \mathbb{R}^{k \times d_1})$. Moreover, there exists a constant K > 0 such that

$$\|(Y_{\cdot}, Z_{\cdot})\|_{\mathcal{M}_{\vartheta}[0,\tau]}^{2} \leq KE\left[|g(X_{\tau})|^{2} e^{2\gamma\tau}\right] + KE\int_{0}^{\tau} e^{2\gamma s} |f(X_{s}, 0, 0)|^{2} ds < \infty,$$

and for any $p \geq 2$,

$$E\left[\sup_{t\in[0,\tau]}|Y_t|^p\right] + E\int_0^\tau |Y_s|^p ds + E\left[\left(\int_0^\tau \|Z_s\|^2 ds\right)^{\frac{p}{2}}\right] \le K(|g|_0^p + |f(\cdot,0,0)|_0^p).$$
(2.6)

Remark 2.3. As was shown in [2] for the one-dimensional case of BSDE (2.5), there is a unique solution $(Y, Z) \in \mathcal{M}_{-\mu}(0, \tau; \mathbb{R}^k \times \mathbb{R}^{k \times d_1})$ without the 'structural' conditions on the coefficient f in (H7), which links the constant μ of monotonicity to the Lipschitz constant L_0 of f in z.

Lemma 2.4. [4, Lemma 6.2 and Proposition 6.3] Under the assumptions of Lemma 2.2, if we set $u(x) := Y_0(x)$ for $x \in \overline{D}$, then we have $Y_t = u(X_t)$ for $0 \le t \le \tau$ and u is bounded and continuous in \overline{D} .

2.2 Introduction of Quasi-derivative

Conventionally, to obtain the gradient and Hessian estimates of u in a probabilistic approach, we differentiate formula (1.2) with respect to x. For the Dirichlet problem of elliptic equations in a domain, a crucial trouble is that the first exit time $\tau = \tau(x)$ is not necessarily continuous (let alone the differentiability). To overcome this difficulty, we introduce the so-called first and second quasi-derivatives of X_t with respect to x along the vector ξ_0 and η_0 respectively.

Definition 2.1. Let $x \in D$, $\xi_0 \in \mathbb{R}^d$, and (ξ_t, ξ_t^0) be adapted continuous processes defined on $[0, \tau]$ taking values in $\mathbb{R}^d \times \mathbb{R}$ such that $\xi_t|_{t=0} = \xi_0$. We call ξ_t a first quasi-derivative of X_t along the direction ξ_0 at point x if the following process

$$v_{(\mathcal{E}_t)}(X_t) + \xi_t^0 v(X_t), \quad 0 \le t \le \tau$$

is a local martingale for any $v \in \mathbb{M}$, and the associated process ξ_t^0 is called a first adjoint process of ξ_t .

Additionally, let $\eta_0 \in \mathbb{R}^d$, and (η_t, η_t^0) be adapted continuous processes defined on $[0, \tau]$ taking values in $\mathbb{R}^d \times \mathbb{R}$ such that $\eta_t|_{t=0} = \eta_0$. We call η_t a second quasi-derivative of X_t associate with ξ_t and ξ_t^0 along the direction of η_0 at point x if the following process

$$v_{(\xi_t)(\xi_t)}(X_t) + v_{(\eta_t)}(X_t) + 2\xi_t^0 v_{(\xi_t)}(X_t) + \eta_t^0 v(X_t), \quad 0 \le t \le \tau$$

is a local martingale for any $v \in \mathbb{M}^2$, and the associated process η_t^0 is called a second adjoint process of η_t .

The notion of the first quasi-derivative can be found in [11, Definition 2.2] and [14, Definition 3.1.1], while the notion of the second quasi-derivative can be found in [21, Definition 2.3]. The following examples of the first quasi-derivative ξ_t and the second quasi-derivative η_t can be found in [21, Theorem 2.1].

Lemma 2.5. Let the scalar processes r_t and \tilde{r}_t , the \mathbb{R}^{d_1} -valued processes π_t and $\tilde{\pi}_t$, and the \mathfrak{B} -valued processes P_t and \tilde{P}_t be all progressively measurable such that for any finite positive time T,

$$\int_{0}^{T} \left(|(r_t, \pi_t, P_t)|^4 + |(\tilde{r}_t, \tilde{\pi}_t, \tilde{P}_t)|^2 \right) dt < \infty.$$
(2.7)

For $x \in D$ and $(\xi_0, \eta_0) \in \mathbb{R}^d \times \mathbb{R}^d$, denote by the processes ξ_t and η_t solutions of the following linear SDEs: for $t \in [0, \infty)$,

$$\xi_t = \xi_0 + \int_0^t [\sigma_{(\xi_s)} + r_s \sigma + \sigma P_s] \, dW_s + \int_0^t [b_{(\xi_s)} + 2r_s b - \sigma \pi_s] \, ds, \tag{2.8}$$

$$\eta_{t} = \eta_{0} + \int_{0}^{t} [\sigma_{(\eta_{s})} + \tilde{r}_{s}\sigma + \sigma\tilde{P}_{s} + \sigma_{(\xi_{s})(\xi_{s})} + 2r_{s}\sigma_{(\xi_{s})} - r_{s}^{2}\sigma + 2\sigma_{(\xi_{s})}P_{s} + 2r_{s}\sigma P_{s} + \sigma P_{s}^{2}] dW_{s} + \int_{0}^{t} [b_{(\eta_{s})} + 2\tilde{r}_{s}b - \sigma\tilde{\pi}_{s} + b_{(\xi_{s})(\xi_{s})} + 4r_{s}b_{(\xi_{s})} - 2\sigma_{(\xi_{s})}\pi_{s} - 2r_{s}\sigma\pi_{s} - 2\sigma P_{s}\pi_{s}] ds,$$

$$(2.9)$$

where in σ , b and their derivatives we have dropped the argument X_s . The processes ξ_t^0 and η_t^0 can be taken to be

$$\xi_t^0 = \int_0^t \pi_s dW_s, (2.10)$$

$$\eta_t^0 = (\xi_t^0)^2 - \langle \xi^0 \rangle_t + \int_0^t \tilde{\pi}_s dW_s.$$
(2.11)

Then ξ_t is a first quasi-derivative of X_t along the direction of ξ_0 at x and ξ_t^0 is a first adjoint process for ξ_t , and η_t is a second quasi-derivative of X_t associated with ξ_t along the direction of η_0 at x and η_t^0 is a second adjoint process for η_t .

The proof of Lemma 2.5 can be found in [11, Lemma 3.1] and [21, Theorem 2.1]. Indeed the auxiliary process (r_t, \tilde{r}_t) relates with a time-change. The process $(\pi_t, \tilde{\pi}_t)$ relates with a measure-transformation via Girsanov's theorem, and the processes (P_t, \tilde{P}_t) relates with a rotation of the driving Wiener process. Since all these transformations preserve the property of the local martingale and quasi-derivatives have additivity, we can easily arrive at the above results.

We also find that the quasi-derivatives ξ_t and η_t enjoy some freedom due to the presence of these auxiliary processes. Hence, sometimes we can turn the quasi-derivatives in such a way that they become tangent to the boundary when and where X_t hit it (see examples in [14, page 54-58]). In this case, it remains for us to estimate the moments of the quasi-derivatives, since the directional derivatives of u along the quasi-derivatives on the boundary coincide with that of the boundary data g.

Let δ be a small positive constant. Consider the following forward-backward stochastic differential equation (FBSDE)

$$\begin{cases} dX_{t}^{\delta} = \left[\left(1 + 2\delta r_{t} + \delta^{2}\tilde{r}_{t} \right) b(X_{t}^{\delta}) - \left(1 + 2\delta r_{t} + \delta^{2}\tilde{r}_{t} \right)^{\frac{1}{2}}\sigma(X_{t}^{\delta}) \left(\delta\pi_{t} + \frac{1}{2}\delta^{2}\tilde{\pi}_{t} \right) e^{\delta P_{t}} e^{\frac{1}{2}\delta^{2}\tilde{P}_{t}} \right] dt \\ + \left(1 + 2\delta r_{t} + \delta^{2}\tilde{r}_{t} \right)^{\frac{1}{2}}\sigma(X_{t}^{\delta}) e^{\delta P_{t}} e^{\frac{1}{2}\delta^{2}\tilde{P}_{t}} dW_{t}, \quad t \in [0, \tau^{\delta}]; \\ dY_{t}^{\delta} = \left[-f(X_{t}^{\delta}, Y_{t}^{\delta}, Z_{t}^{\delta}) \left(1 + 2\delta r_{t} + \delta^{2}\tilde{r}_{t} \right) - \tilde{Z}_{t}^{\delta} \left(\delta\pi_{t} + \frac{1}{2}\delta^{2}\tilde{\pi}_{t} \right) \right] dt + \tilde{Z}_{t}^{\delta} dW_{t}, \quad t \in [0, \tau^{\delta}], \\ \tilde{Z}_{t}^{\delta} := Z_{t}^{\delta} \left(1 + 2\delta r_{t} + \delta^{2}\tilde{r}_{t} \right)^{\frac{1}{2}} e^{\delta P_{t}} e^{\frac{1}{2}\delta^{2}\tilde{P}_{t}}; \qquad X_{0}^{\delta} = x + \delta\xi_{0} + \frac{1}{2}\delta^{2}\eta_{0}, \quad Y_{\tau^{\delta}}^{\delta} = g(X_{\tau^{\delta}}^{\delta}), \end{cases}$$

$$(2.12)$$

where τ^{δ} is the first exit time of X_t^{δ} from D.

Remark 2.6. (i) As $e^{\delta P_t}$ is an orthogonal matrix, $d\tilde{W}_t = \int_0^t e^{\delta P_s} dW_s$ is a Wiener process for any δ . (ii) Note that $(X^0, Y^0, Z^0) = (X, Y, Z)$ is the solution of (2.3) and (2.5). Therefore, $(X^{\delta}, Y^{\delta}, Z^{\delta})$ is a perturbation to (X, Y, Z).

Lemma 2.7. Let (H1)-(H5), $(H6)_0$ and (H7) be satisfied. Without loss of generality, we may assume the coefficients (r_t, π_t, P_t) and $(\tilde{r}_t, \tilde{\pi}_t, \tilde{P}_t)$ be bounded if condition (2.7) is satisfied. Then, there is a sufficiently small δ such that (see [8, page 520])

$$0 \le 1 + 2\delta r_t + \delta^2 \tilde{r}_t \le 2, \quad |\delta \pi_t| + \frac{1}{2}\delta^2 |\tilde{\pi}_t| \le 1, \quad \exp\left(\delta P_t + \frac{1}{2}\delta^2 \tilde{P}_t\right) \le 2,$$

and FBSDE (2.12) has a unique solution $(X^{\delta}, Y^{\delta}, Z^{\delta})$.

Moreover, when $\{X_t(x), t \ge 0\}$ is a unique solution of (2.3), and $\{(\xi_t(\xi_0), \eta_t(\eta_0)), t \ge 0\}$ is a unique solution of (2.8) and (2.9), for $p \ge 2$, $T \ge 1$, and $(x, \xi_0, \eta_0) \in D \times \mathbb{R}^d \times \mathbb{R}^d$, we have (see [22, Theorems 3.1 and 3.2]),

$$\lim_{\delta \to 0} E \left[\sup_{0 \le t \le \tau^{\delta} \land \tau \land T} \left| \frac{X_t^{\delta}(x + \delta\xi_0) - X_t(x)}{\delta} - \xi_t \right|^p \right] = 0,$$
(2.13)

and

$$\lim_{\delta \to 0} E \left[\sup_{0 \le t \le \tau^{\delta} \land \tau^{-\delta} \land \tau \land T} \left| \frac{X_t^{\delta}(x + \delta\xi_0 + \frac{1}{2}\delta^2\eta_0) - 2X_t(x) + X_t^{-\delta}(x - \delta\xi_0 + \frac{1}{2}\delta^2\eta_0)}{\delta^2} - \eta_t \right|^p \right] = 0.$$

$$(2.14)$$

Lemma 2.8. Under the assumptions of Lemma 2.7, we have

$$u(X_t^{\delta}) = Y_t^{\delta}, \quad t \in [0, \tau^{\delta}]$$

Proof. The proof is similar to that of Lemma 2.4. Thanks to Itô's formula, we get $u^{\epsilon}(X_t^{\epsilon,\delta}) = Y_t^{\epsilon,\delta}$ by regularizing the equations with smooth coefficients. Then, the desired result is a consequence of the stability of BSDEs provided in [2, Theorem 2.4] and the stability of weak solutions of degenerate elliptic PDEs in [12, Theorem 4.6.1].

Our main result is stated in the following theorem.

Theorem 2.9. Let assumptions (H1)-(H5), (H7) and (H9) be satisfied. Let (X, Y) be the unique solution of (1.4) and (1.3), and u be defined by (1.2). Then we have the following assertions.

(i) Under the assumptions $(H6)_0$ and $(H10)_1$, we have $u \in C^{0,1}_{loc}(D) \cap C(\overline{D})$, and for any $\xi_0 \in \mathbb{R}^d$ and a.e. $x \in D$,

$$|u_{(\xi_0)}(x)| \le N\left(|\xi_0| + \frac{|\psi_{(\xi_0)}(x)|}{\psi^{\frac{3}{4}}(x)}\right) \left(|g|_{0,1} + ||f(\cdot)||_{0,1}\right)$$
(2.15)

where $N = N(K_0, d, d_1, k, D, L, L_0, \mu)$.

(ii) Assume $(H6)_1$, (H8) and $(H10)_2$ hold. Then $u \in C^{1,1}_{loc}(D) \cap C^{0,1}(\overline{D})$, and for any $\xi_0 \in \mathbb{R}^d$ and a.e. $x \in D$,

$$|u_{(\xi_0)(\xi_0)}(x)| \le N\left(|\xi_0|^2 + \frac{\psi_{(\xi_0)}^2(x)}{\psi^{\frac{7}{4}}(x)}\right) [|g|_{1,1} + ||f(\cdot)||_{0,1} + [f]_{1,1}(1 + |g|_1^2 + ||f(\cdot)||_{0,1}^2)]$$

where $N = N(K_0, d, d_1, k, D, L, L_0, \mu)$. Furthermore, u is the unique solution in $C^{1,1}_{loc}(D) \cap C^{0,1}(\overline{D})$ of the semi-linear system of PDEs

$$\begin{cases} \mathcal{L}u(x) + f(x, u(x), \nabla u(x)\sigma(x)) = 0, & a.e. \ x \in D; \\ u(x) = g(x), & x \in \partial D. \end{cases}$$
(2.16)

3 Moment Estimates of Quasi-derivatives

In this section, we construct barrier functions in the spirit of [21, Lemmas 3.3 and 3.4] to estimate quasi-derivatives, which are used in the gradient and Hessian estimates.

The following estimates on the first exit time can be found in [21, Lemma 3.1] and [9, Lemma 2.1].

Lemma 3.1. Let (H2) be satisfied and $\tau(x)$ be the first exit time of X_t from D. Then we have for $x \in D$,

$$E[\tau(x)] \le \psi(x) \le |\psi|_0, \quad E[\tau^2(x)] \le 2|\psi|_0\psi(x) \le 2|\psi|_0^2, \\ E[\tau^p(x)] \le N(|\psi|_0^p), \quad \forall p > 2.$$

Given $\lambda, \, \delta_1 \in (0, 1)$ with $\delta_1 < \lambda^2$, define

$$D_{\delta_1}^{\lambda} := \{ x \in D : \delta_1 < \psi(x) < \lambda \}, \quad D_{\lambda^2} := \{ x \in D : \psi(x) > \lambda^2 \}.$$

For $x \in D_{\delta_1}^{\lambda}$, define $\tau_1 := \tau_{D_{\delta_1}^{\lambda}}(x)$ be the first exit time of X_t from $D_{\delta_1}^{\lambda}$. For $x \in D_{\lambda^2}$, define $\tau_2 := \tau_{D_{\lambda^2}}(x)$ be the first exit time of X_t from D_{λ^2} .

Krylov [10, Section 3] introduced the method of dividing the whole domain D into two parts to estimate the moments of quasi-derivatives separately. Since ψ vanishes at the boundary, it is not convenient to construct coefficients of the quasi-derivatives, such as r, π and P, uniformly in the whole domain. Zhou [21] constructed two families of local super-martingales to estimate moments of quasiderivatives near the boundary and in the interior of the domain, separately. We still use his notions of quasi-derivatives ξ_t and η_t . In our more general BSDE context (see next section for more details), as higher moment estimates of quasi-derivatives are necessary, we could not use his original barrier functions B_1 and B_2 in [21, Lemmas 3.3 and 3.4], and instead we consider four new barrier functions in this section. See [21, Remark 3.5] for the motivation of building barrier functions. Actually our main difficulty lies in the term $E_{x,\xi_0}[u_{(\xi_{\tau})}(X_{\tau})]$ in our gradient estimate of u. So we should try to construct ξ_t such that either ξ_τ is tangent to ∂D at X_τ almost surely, or $|u_{(\xi_\tau)}(X_\tau)|$ is bounded by a nonnegative local super-martingale $\{B(X_t,\xi_t), t \in [0,\tau]\}$. In our Hessian estimate of u, the same difficulty exists around both terms $E_{x,\xi_0}[u_{(\xi_{\tau})}(\chi_{\tau})]$ and $E_{x,\eta_0}[u_{(\eta_{\tau})}(X_{\tau})]$. Similarly, we need to construct ξ_t such that either ξ_τ is tangent to the boundary, or $|u_{(\xi_\tau)(\xi_\tau)}(X_\tau)|$ is bounded by another nonnegative local super-martingale $\{B(X_t, \xi_t), t \in [0, \tau]\}$. Here as mentioned in [21, page 5], η_{τ} is not necessarily tangent to ∂D at X_{τ} , for η_{τ} can be represented as the sum of the tangential and the normal components.

Define the three functions

$$\begin{aligned} \varphi(x) &:= \lambda^2 + \psi(x) - \frac{1}{4\lambda} \psi^2(x), \quad x \in D^{\lambda}_{\delta_1}; \\ B_1(x,y) &:= \left(\lambda + \sqrt{\psi(x)} + \psi(x)\right) |y|^4 + K_1 \varphi^{\frac{7}{2}}(x) \frac{\psi^4_{(y)}(x)}{\psi^3(x)}, \quad (x,y) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d, \end{aligned}$$

where $K_1 \in [1, \infty)$ is a constant depending only on K_0 ; and

$$B_2(y) := \lambda^{\frac{3}{4}} |y|^4, \quad y \in \mathbb{R}^d.$$

In this section, for simplicity of exposition, we shall omit the argument X_t in the coefficients σ , b, ψ and their derivatives whenever no confusion is made.

Lemma 3.2. Let (H3) and (H9) be satisfied. Define X_t by (2.3) and the first quasi-derivative ξ_t by (2.8), where for $(\tilde{x}, y) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d$

$$\begin{split} A(\tilde{x}) &:= \sum_{i=1}^{d_1} \psi_{(\sigma_i)}^2(\tilde{x}), \quad \bar{\rho}(\tilde{x}, y) := -\frac{1}{A(\tilde{x})} \sum_{i=1}^{d_1} \psi_{(\sigma_i)}(\tilde{x})(\psi_{(\sigma_i)})_{(y)}(\tilde{x}), \\ r(\tilde{x}, y) &:= \bar{\rho}(\tilde{x}, y) + \frac{\psi_{(y)}(\tilde{x})}{\psi(\tilde{x})}, \quad \tilde{r}(\tilde{x}, y) := \frac{\psi_{(y)}^2(\tilde{x})}{\psi^2(\tilde{x})}, \\ \pi_i(\tilde{x}, y) &:= \frac{4\psi_{(\sigma_i)}(\tilde{x})\psi_{(y)}(\tilde{x})}{\varphi(\tilde{x})\psi(\tilde{x})}, \quad i = 1, \cdots, d_1, \\ P_{ij}(\tilde{x}, y) &:= \frac{1}{A(\tilde{x})} [\psi_{(\sigma_j)}(\tilde{x})(\psi_{(\sigma_i)})_{(y)}(\tilde{x}) - \psi_{(\sigma_i)}(\tilde{x})(\psi_{(\sigma_j)})_{(y)}(\tilde{x})], \quad i, j = 1, \cdots, d_1; \\ r_t &:= r(X_t, \xi_t), \ \pi_t := (\pi_i(X_t, \xi_t))_{i=1, \cdots, d_1}, \ P_t := (P_{ij}(X_t, \xi_t))_{i,j=1, \cdots, d_1}, \\ \tilde{r}_t &:= \tilde{r}(X_t, \xi_t), \ \tilde{\pi}_t = 0, \ \tilde{P}_t = 0, \quad \forall (x, \xi_0, t) \in D_{\delta_1}^{\lambda} \times \mathbb{R}^d \times [0, \tau_1]. \end{split}$$

Then for sufficiently small λ , we have for $(x, \xi_0) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d$,

(i) the process
$$(B_1(X_t,\xi_t), B_1^{\frac{1}{2}}(X_t,\xi_t)), t \in [0,\tau_1]$$
 is a local super-martingale;
(ii) we have $E_{x,\xi_0} \int_0^{\tau_1} \left(|\xi_t|^4 + \frac{\psi_{(\xi_t)}^4(X_t)}{\psi^4(X_t)} \right) dt \le N(K_0,\lambda) B_1(x,\xi_0)$ and
 $E_{x,\xi_0} \int_0^{\tau_1} (|r_t|^4 + |\pi_t|^4 + ||P_t||^4 + |\tilde{r}_t|^2) dt \le N(K_0,\lambda) B_1(x,\xi_0);$

(iii) we have
$$E_{x,\xi_0} \left[\sup_{0 \le t \le \tau_1} \left(|\xi_t|^4 + \frac{\psi_{(\xi_t)}^4(X_t)}{\psi^4(X_t)} \right) \right] \le N(K_0,\lambda) B_1(x,\xi_0)$$
 and
 $E_{x,\xi_0} \left[\sup_{0 \le t \le \tau_1} (|r_t|^4 + |\pi_t|^4 + \|P_t\|^4 + |\tilde{r}_t|^2) \right] \le N(K_0,\lambda) B_1(x,\xi_0).$

Proof. First, in view of (H9), there exists a constant $\delta' > 0$, such that for $x \in \partial D$

$$A := \sum_{i=1}^{d_1} \psi_{(\sigma^i)}^2 = 2\langle a\psi_x, \psi_x \rangle = 2|\psi_x|^2 \langle an, n \rangle \ge 2\delta'.$$

Here we use the fact that ψ_x has the same direction of n and $|\psi_x| \ge 1$ near the boundary by continuity. Assume that $A \ge 1$ without loss of generality by replacing ψ by $\psi/2\delta'$ if necessary.

On the one hand, let

$$\bar{\sigma}_t := \sigma_{(\xi_t)} + r_t \sigma + \sigma P_t, \quad \bar{b}_t := b_{(\xi_t)} + 2r_t b - \sigma \pi_t, \quad t \ge 0.$$

Then we have

$$d\xi_t = \bar{b}_t dt + \bar{\sigma}_t dW_t, \quad t \ge 0.$$

Using (H3) and $\varphi \geq \lambda^2$, we reduce that

$$|\sigma \pi_t| \le K \left| \frac{\psi_{(\xi_t)}}{\psi} \right|, \quad t \in [0, \tau_1],$$

where K is a constant depending on K_0 and λ . So, we have

$$\|\bar{\sigma}_t\| \le K\left(|\xi_t| + \frac{|\psi_{(\xi_t)}|}{\psi}\right), \quad |\bar{b}_t| \le K\left(|\xi_t| + \frac{|\psi_{(\xi_t)}|}{\psi}\right), \quad t \in [0, \tau_1].$$
(3.2)

Using Itô's formula, we have

$$d\left[\left(\lambda+\sqrt{\psi}+\psi\right)|\xi_{t}|^{4}\right]$$

$$=\left(\lambda+\sqrt{\psi}+\psi\right)\left(4|\xi_{t}|^{3}\bar{b}_{t}dt+6|\xi_{t}|^{2}\|\bar{\sigma}_{t}\|^{2}dt+4|\xi_{t}|^{3}\bar{\sigma}_{t}dW_{t}\right)$$

$$+|\xi_{t}|^{4}\left[\left(1+2\sqrt{\psi}\right)\frac{\mathcal{L}\psi}{2\sqrt{\psi}}-\frac{A}{8\psi^{\frac{3}{2}}}\right]dt+|\xi_{t}|^{4}\left(1+2\sqrt{\psi}\right)\frac{\psi(\sigma)}{2\sqrt{\psi}}dW_{t}$$

$$+4|\xi_{t}|^{3}\bar{\sigma}_{t}\left(1+2\sqrt{\psi}\right)\frac{\psi(\sigma)}{2\sqrt{\psi}}dt$$

$$=\Gamma_{1}(X_{t},\xi_{t})dt+\Lambda_{1}(X_{t},\xi_{t})dW_{t}, \quad t\in[0,\tau_{1}].$$

Set $\Gamma_1(X_t, \xi_t) := I_1 + I_2 + I_3 + I_4$. Since $\lambda^2 \leq \varphi \leq 2\lambda$ and $\psi \leq 2\varphi$, applying (3.2) and Young's inequality, we have

$$I_{1} = \lambda \left(4|\xi_{t}|^{3}\bar{b}_{t} + 6|\xi_{t}|^{2}\|\bar{\sigma}_{t}\|^{2} \right)$$

$$\leq \lambda K \left[|\xi_{t}|^{3} \left(|\xi_{t}| + \frac{|\psi_{(\xi_{t})}|}{\psi} \right) + |\xi_{t}|^{2} \left(|\xi_{t}| + \frac{|\psi_{(\xi_{t})}|}{\psi} \right)^{2} \right]$$

$$\leq \left(3\lambda K\psi^{\frac{3}{2}} + \lambda K\frac{3}{4} + \frac{1}{32} \right) \frac{|\xi_{t}|^{4}}{\psi^{\frac{3}{2}}} + \left(\lambda K2^{\frac{9}{2}}\varphi^{2} + 32 \cdot 2^{\frac{3}{2}}K^{2} \right) \varphi^{\frac{5}{2}} \frac{|\psi_{(\xi_{t})}|^{4}}{\psi^{4}},$$

$$I_{2} = \left(\sqrt{\psi} + \psi\right) \left(4|\xi_{t}|^{3}\bar{b}_{t} + 6|\xi_{t}|^{2}\|\bar{\sigma}_{t}\|^{2}\right)$$

$$\leq 2K\sqrt{\psi} \left(3|\xi_{t}|^{4} + |\xi_{t}|^{3}\frac{|\psi_{(\xi_{t})}|}{\psi} + 2|\xi_{t}|^{2}\frac{|\psi_{(\xi_{t})}|^{2}}{\psi^{2}}\right)$$

$$\leq \left(6K\psi^{\frac{5}{2}} + \frac{3}{2}K\psi^{\frac{1}{2}} + \frac{1}{32}\right)\frac{|\xi_{t}|^{4}}{\psi^{\frac{3}{2}}} + \left(2^{\frac{11}{2}}K\varphi^{\frac{5}{2}} + 2^{\frac{17}{2}}K^{2}\right)\varphi^{\frac{5}{2}}\frac{|\psi_{(\xi_{t})}|^{4}}{\psi^{4}}.$$

Since $\mathcal{L}\psi \leq -1$ and $A \geq 1$, we have

$$I_3 = |\xi_t|^4 \left[\left(1 + 2\sqrt{\psi} \right) \frac{\mathcal{L}\psi}{2\sqrt{\psi}} - \frac{A}{8\psi^{\frac{3}{2}}} \right] \le -\frac{|\xi_t|^4}{8\psi^{\frac{3}{2}}}.$$

Applying (3.2) and Young's inequality, we have

$$I_{4} = 4|\xi_{t}|^{3}\bar{\sigma}_{t}\left(1+\sqrt{\psi}\right)\frac{\psi_{(\sigma)}}{2\sqrt{\psi}} \leq 2K\frac{|\xi_{t}|^{3}}{\sqrt{\psi}}\left(|\xi_{t}|+\frac{|\psi_{(\xi_{t})}|}{\psi}\right)$$
$$\leq 2K\frac{|\xi_{t}|^{4}}{\sqrt{\psi}} + 2K\frac{|\xi_{t}|^{3}|\psi_{(\xi_{t})}|}{\psi^{\frac{3}{2}}} \leq 2K\psi\frac{|\xi_{t}|^{4}}{\psi^{\frac{3}{2}}} + \frac{3}{128}\frac{|\xi_{t}|^{4}}{\psi^{\frac{3}{2}}} + 2^{17}K^{4}\frac{|\psi_{(\xi_{t})}|^{4}}{\psi^{\frac{3}{2}}}.$$

On the other hand, by definition of r and P, we get

$$\sum_{i} \langle \psi_{xx} \sigma_i, P \sigma_i \rangle = \operatorname{tr}(\sigma \sigma^* \psi_{xx} P) = 0$$

and

$$(\psi_{(\sigma_j)})_{(\xi_t)} + r_t \psi_{(\sigma_j)} + \sum_i \psi_{(\sigma_i)} P_t^{ij} = \frac{\psi_{(\xi_t)}}{\psi} \psi_{(\sigma_j)}, \quad t \in [0, \tau_1].$$

By Itô's formula, we have

$$d\psi_{(\xi_t)} = \sum_i \frac{\psi_{(\xi_t)}}{\psi} \psi_{(\sigma_i)} dW_t^i + \left[(\mathcal{L}\psi)_{(\xi_t)} + 2r_t \mathcal{L}\psi - \sum_i \psi_{(\sigma_i)} \pi_t^i \right] dt, \quad t \in [0, \tau_1].$$
(3.3)

Since $\varphi \leq 2\lambda$, $\psi \leq 2\varphi$ and $\mathcal{L}\psi \leq -1$, after removing the negative terms, we have

$$d\left(\varphi^{\frac{7}{2}}\frac{\psi_{(\xi_{t})}^{4}}{\psi^{3}}\right) = \frac{7}{2}\varphi^{\frac{5}{2}}\left(1-\frac{\psi}{2\lambda}\right)\mathcal{L}\psi\frac{\psi_{(\xi_{t})}^{4}}{\psi^{3}}dt - \frac{7}{2}\varphi^{\frac{5}{2}}\frac{\psi_{(\sigma)}^{2}}{4\lambda}\frac{\psi_{(\xi_{t})}^{4}}{\psi^{3}}dt + \frac{35}{8}\varphi^{\frac{3}{2}}\left(1-\frac{\psi}{2\lambda}\right)^{2}\psi_{(\sigma)}^{2}\frac{\psi_{(\xi_{t})}^{4}}{\psi^{3}}dt + 4\varphi^{\frac{7}{2}}\frac{\psi_{(\xi_{t})}^{3}}{\psi^{3}}\left[(\mathcal{L}\psi)_{(\xi_{t})} + 2\bar{\rho}\mathcal{L}\psi\right]dt + 5\varphi^{\frac{7}{2}}\mathcal{L}\psi\frac{\psi_{(\xi_{t})}^{4}}{\psi^{4}}dt - 16\varphi^{\frac{5}{2}}\frac{\psi_{(\sigma)}^{2}\psi_{(\xi_{t})}^{4}}{\psi^{4}}dt + \frac{7}{2}\varphi^{\frac{5}{2}}\left(1-\frac{\psi}{2\lambda}\right)\psi_{(\sigma)}\frac{\psi_{(\xi_{t})}^{4}}{\psi^{3}}dW_{t} + \varphi^{\frac{7}{2}}\frac{\psi_{(\xi_{t})}^{4}\psi_{(\sigma)}}{\psi^{4}}dW_{t} + \frac{7}{2}\left(1-\frac{\psi}{2\lambda}\right)\varphi^{\frac{5}{2}}\frac{\psi_{(\sigma)}^{2}\psi_{(\xi_{t})}^{4}}{\psi^{4}}dt \\ \leq -\frac{15}{4}\varphi^{\frac{5}{2}}\frac{A\psi_{(\xi_{t})}^{4}}{\psi^{4}}dt + 4K\varphi^{\frac{7}{2}}\frac{\psi_{(\xi_{t})}^{3}}{\psi^{3}}|\xi_{t}|dt + \Lambda_{2}(X_{t},\xi_{t})dW_{t}, \qquad (3.4)$$

where

$$\Lambda_2(X_t,\xi_t) := \frac{7}{2}\varphi^{\frac{5}{2}} \left(1 - \frac{\psi}{2\lambda}\right)\psi_{(\sigma)}\psi_{(\xi_t)}^4\psi^{-3} + \varphi^{\frac{7}{2}}\psi_{(\xi_t)}^4\psi_{(\sigma)}\psi^{-4}.$$

Collecting all the above estimates and choosing K_1 such that $K_1 \ge K$, and letting λ be sufficiently small, we get

$$dB_1(X_t,\xi_t) \le \left(-\frac{|\xi_t|^4}{64\psi^{\frac{3}{2}}} - \frac{1}{2}\varphi^{\frac{5}{2}}\frac{\psi_{(\xi_t)}^4}{\psi^4}\right)dt + dm_t \le dm_t, \quad t \in [0,\tau_1],$$
(3.5)

where $m_t := \Lambda_1(X_t, \xi_t) dW_t + K_1 \Lambda_2(X_t, \xi_t) dW_t, t \in [0, \tau_1]$ is a local martingale. It follows that the process $\{B_1(X_t, \xi_t), t \in [0, \tau_1]\}$ is a local super-martingale.

Also, since $f(x) = \sqrt{x}$ is concave, the process $\{B_1^{\frac{1}{2}}(X_t, \xi_t), t \in [0, \tau_1]\}$ is a local super-martingale. Thus Assertion (i) is proved.

By definition, we know that for $(x, \xi_0) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d$

$$E_{x,\xi_0} \int_0^{\tau_1} \left(|(r_t, \pi_t, P_t)|^4 + |\tilde{r}_t|^2 \right) dt \le N E_{x,\xi_0} \int_0^{\tau_1} \left(|\xi_t|^4 + \frac{\psi_{(\xi_t)}^4}{\psi^4} \right) dt.$$

From (3.5), there exists a sufficiently small positive λ_0 , such that

$$\lambda_0 E_{x,\xi_0} \int_0^{\tau_1} \left(|\xi_t|^4 + \frac{\psi_{(\xi_t)}^4}{\psi^4} \right) dt \le B_1(x,\xi_0) - E_{x,\xi_0} B_1(X_{\tau_1},\xi_{\tau_1}) \le B_1(x,\xi_0),$$

which yields Assertion (ii).

Using Itô's formula, from (3.2), we have

$$\begin{aligned} d|\xi_{t}|^{4} &= 2|\xi_{t}|^{2} \left(2\langle\xi_{t}, \bar{b}_{t}\rangle + \|\bar{\sigma}_{t}\|^{2} \right) dt + |\langle\xi_{t}, \bar{\sigma}_{t}\rangle|^{2} dt + 4|\xi_{t}|^{2} \langle\xi_{t}, \bar{\sigma}_{t} dW_{t}\rangle \\ &\leq N \left(|\xi_{t}|^{4} + \frac{\psi_{(\xi_{t})}^{4}}{\psi^{4}} \right) dt + 4|\xi_{t}|^{2} \langle\xi_{t}, \bar{\sigma}_{t} dW_{t}\rangle, \quad t \in [0, \tau_{1}]. \end{aligned}$$

Using Assertion (ii) and the BDG inequality, for $\tau_n = \tau_1 \wedge \inf\{t \ge 0 : |\xi_t| \ge n\}$, we have

$$E_{\xi_{0}}\left[\sup_{0\leq t\leq \tau_{n}}|\xi_{t}|^{4}\right] \leq |\xi_{0}|^{4} + NE_{x,\xi_{0}}\int_{0}^{\tau_{n}}\left(|\xi_{t}|^{4} + \frac{\psi_{(\xi_{t})}^{4}}{\psi^{4}}\right)dt \\ + 4E_{x,\xi_{0}}\left[\sup_{0\leq t\leq \tau_{n}}\left|\int_{0}^{\tau_{n}}|\xi_{t}|^{2}\langle\xi_{t},\bar{\sigma}_{t}dW_{t}\rangle\right|\right] \\ \leq NB_{1}(x,\xi_{0}) + NE_{x,\xi_{0}}\left[\left(\int_{0}^{\tau_{n}}|\xi_{t}|^{6}\left(|\xi_{t}|^{2} + \frac{\psi_{(\xi_{t})}^{2}}{\psi^{2}}\right)dt\right)^{\frac{1}{2}}\right].$$

Since the last expectation is rewritten and estimated (using twice Cauchy inequality, and then Assertion (ii)) as follows

$$\leq E_{x,\xi_0} \left[\left(\int_0^{\tau_n} N^2 \left(|\xi_t|^4 + |\xi_t|^2 \frac{\psi_{(\xi_t)}^2}{\psi^2} \right) dt \right)^{\frac{1}{2}} \sup_{0 \le t \le \tau_n} |\xi_t|^2 \right]$$

$$\leq \frac{1}{2} E_{\xi_0} \left[\sup_{0 \le t \le \tau_n} |\xi_t|^4 \right] + \frac{1}{2} N^2 E_{x,\xi_0} \int_0^{\tau_n} \left(|\xi_t|^4 + |\xi_t|^2 \frac{\psi_{(\xi_t)}^2}{\psi^2} \right) dt$$

$$\leq \frac{1}{2} E_{\xi_0} \left[\sup_{0 \le t \le \tau_n} |\xi_t|^4 \right] + \frac{1}{2} N^2 E_{x,\xi_0} \int_0^{\tau_n} \left(\frac{5}{4} |\xi_t|^4 + \frac{\psi_{(\xi_t)}^4}{\psi^4} \right) dt$$

$$\leq \frac{1}{2} E_{\xi_0} \left[\sup_{0 \le t \le \tau_n} |\xi_t|^4 \right] + N^2 N B_1(x,\xi_0),$$

we conclude the following

$$E_{\xi_0}\left[\sup_{0\le t\le \tau_n} |\xi_t|^4\right] \le NB_1(x,\xi_0), \quad \forall (x,\xi_0) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d.$$
(3.6)

In view of (3.3) and using Itô's formula to the term $\psi_{(\xi_t)}\psi^{-1}$, we find that the relevant local martingale is vanishing. Then using Itô's formula to the term $\psi_{(\xi_t)}^4\psi^{-4}$, we only need to consider the drift term. Hence, by Young's inequality, we have for $A \ge 1$

$$d\left(\frac{\psi_{(\xi_t)}}{\psi}\right)^4 = 4\frac{\psi_{(\xi_t)}^3}{\psi^3} \left[\frac{(\mathcal{L}\psi)_{(\xi_t)} + 2\bar{\rho}\mathcal{L}\psi}{\psi} + \frac{\psi_{(\xi_t)}\mathcal{L}\psi}{\psi^2} - \frac{4A\psi_{(\xi_t)}}{\varphi\psi^2}\right] dt$$
$$\leq 4K\frac{|\psi_{(\xi_t)}|^3|\xi_t|}{\psi^4} dt + 4\frac{\psi_{(\xi_t)}^4}{\psi^5}\mathcal{L}\psi dt$$
$$\leq 3\frac{\psi_{(\xi_t)}^4}{\psi^5} dt + NK^4\frac{|\xi_t|^4}{\psi} dt + 4\frac{\psi_{(\xi_t)}^4}{\psi^5}\mathcal{L}\psi dt, \quad t \in [0, \tau_1].$$

In fact, by (3.5), we have $E \int_0^{\tau_1} |\xi_t|^4 \psi^{-1} dt \leq NB_1(x,\xi_0)$. Hence, by $\mathcal{L}\psi \leq -1$, we obtain

$$E_{x,\xi_0}\left[\sup_{0\le t\le \tau_n} \left(\frac{\psi_{(\xi_t)}}{\psi}\right)^4\right] \le NB_1(x,\xi_0), \quad \forall (x,\xi_0)\in D^{\lambda}_{\delta_1}\times\mathbb{R}^d.$$
(3.7)

By (3.6) and (3.7), letting $n \to \infty$, we conclude

$$E_{x,\xi_0}\left[\sup_{0\le t\le \tau_1}\left(|\xi_t|^4 + \frac{\psi_{(\xi_t)}^4}{\psi^4}\right)\right] \le NB_1(x,\xi_0), \quad \forall (x,\xi_0)\in D^{\lambda}_{\delta_1}\times \mathbb{R}^d.$$

Also, by definition, we have for $(x, \xi_0) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d$

$$E_{x,\xi_0} \left[\sup_{0 \le t \le \tau_1} \left(|(r_t, \pi_t, P_t)|^4 + |\tilde{r}_t|^2 \right) \right] \le N E_{x,\xi_0} \left[\sup_{0 \le t \le \tau_1} \left(|\xi_t|^4 + \frac{\psi_{(\xi_t)}^4}{\psi^4} \right) \right].$$
(3.8)

Thus Assertion (iii) is proved.

Remark 3.3. (i) In our Lemma 3.2, to make the term $\varphi^{\frac{5}{2}}A\psi^4_{(\xi_t)}\psi^{-4}$ in (3.4) negative, $\pi_i(\tilde{x}, y) = 4\psi_{(\sigma_i)}(\tilde{x})\psi_{(y)}(\tilde{x})(\varphi(\tilde{x})\psi(\tilde{x}))^{-1}$ is the double of that of [21, Lemma 3.3].

(ii) The above estimate (3.8) is new to quasi-derivatives, and will be used to estimate the gradient and Hessian matrix of u.

Lemma 3.4. Let (H3) and (H10)₁ be satisfied. Define X_t by (2.3) and the first quasi-derivative ξ_t by (2.8), where for $(\tilde{x}, y) \in D_{\lambda^2} \times \mathbb{R}^d$

$$r(\tilde{x}, y) := \langle \rho(\tilde{x}), y \rangle, \ \pi(\tilde{x}, y) := \frac{M(\tilde{x})}{2} \sigma^*(\tilde{x}) y, \ P(\tilde{x}, y) := Q(\tilde{x}, y);$$

$$r_t := r(X_t, \xi_t), \ \pi_t := \pi(X_t, \xi_t), \ P_t := P(X_t, \xi_t), \ \forall (x, \xi_0, t) \in D_{\lambda^2} \times \mathbb{R}^d \times [0, \tau_2].$$

(3.9)

Then for sufficiently small λ , we have for $(x, \xi_0) \in D_{\lambda^2} \times \mathbb{R}^d$,

(i) the process
$$(e^{4\beta t}B_2(\xi_t), e^{2\beta t}B_2^{\frac{1}{2}}(\xi_t)), t \in [0, \tau_2]$$
 is a local super-martingale;
(ii) $E_{\xi_0} \int_0^{\tau_2} e^{4\beta t} |\xi_t|^4 dt \le N(K_0, \lambda) B_2(\xi_0);$
(iii) $E_{x,\xi_0} \left[\sup_{0 \le t \le \tau_2} e^{4\beta t} (|\xi_t|^4 + |r_t|^4 + |\pi_t|^4 + ||P_t||^4) \right] \le N(K_0, \lambda) B_2(\xi_0).$

Proof. First of all, replacing K_0 by

$$\max\left\{K_0, \sup_{\tilde{x}\in D_{\lambda^2}} |\rho(\tilde{x})|, \sup_{\substack{\tilde{x}\in D_{\lambda^2}, \\ y\in \mathbb{R}^d}} \frac{\|Q(\tilde{x}, y)\|}{|y|}, \sup_{\tilde{x}\in D_{\lambda^2}} M(\tilde{x})\right\}.$$

By Itô's formula, we have

$$d|\xi_t|^4 = \Gamma_2(X_t, \xi_t)dt + \sum_{m=1}^{d_1} \Lambda_2^m(X_t, \xi_t)dW_t^m, \quad t \in [0, \tau_2],$$

where

$$\Lambda_2(X_t,\xi_t) = 4|\xi_t|^2 \langle \xi_t, \sigma_{(\xi_t)} + r_t \sigma + \sigma P_t \rangle.$$

In view of $(H10)_1$, we have

$$\Gamma_2(X_t,\xi_t) = |\xi_t|^2 \left[4\langle \xi_t, b_{(\xi_t)} + 2r_t b - \sigma \pi_t \rangle + 6 \|\sigma_{(\xi_t)} + r_t \sigma + \sigma P_t\|^2 \right] \le (-4\beta - 2)|\xi_t|^4.$$

 So

$$d\left(e^{4\beta t}|\xi_t|^4\right) \le -2e^{4\beta t}|\xi_t|^4 dt + e^{4\beta t} \sum_{m=1}^{d_1} \Lambda_2^m(X_t,\xi_t) dW_t^m, \quad t \in [0,\tau_2].$$
(3.10)

Therefore, the process $\{e^{4\beta t}B_2(\xi_t), t \in [0, \tau_2]\}$ is a local super-martingale. In view of the concavity of the squared root function, the process $\{e^{2\beta t}B_2^{\frac{1}{2}}(\xi_t), t \in [0, \tau_2]\}$ is a local super-martingale. Assertion (i) is proved.

In view of (3.10), there exists a constant N > 0 such that

$$E_{\xi_0} \int_0^{\tau_2} e^{4\beta t} |\xi_t|^4 dt \le N \left(B_2(\xi_0) - E_{\xi_0} e^{4\beta \tau_2} B_2(\xi_{\tau_2}) \right) \le N B_2(\xi_0), \quad \forall \xi_0 \in \mathbb{R}^d,$$

which implies Assertion (ii).

Using Assertion (ii) and the BDG inequality, for $\tau_n := \tau_2 \wedge \inf\{t \ge 0 : |\xi_t| \ge n\}$, we have

$$E_{\xi_{0}}\left[\sup_{0\leq t\leq \tau_{n}}e^{4\beta t}|\xi_{t}|^{4}\right] \leq NB_{2}(\xi_{0}) + E_{\xi_{0}}\left[\sup_{0\leq t\leq \tau_{n}}\left|\int_{0}^{t}e^{4\beta t}\Lambda_{2}(X_{t},\xi_{t})dW_{t}\right|\right] \\ \leq NB_{2}(\xi_{0}) + NE_{\xi_{0}}\left[\left(\int_{0}^{\tau_{n}}e^{8\beta t}|\xi_{t}|^{8}dt\right)^{\frac{1}{2}}\right].$$

Since the last expectation is written and estimated (using Cauchy inequality, and then Assertion (ii)) as follows

$$\leq E_{\xi_0} \left[\left(\sup_{0 \le t \le \tau_n} e^{2\beta t} |\xi_t|^2 \right) \left(\int_0^{\tau_n} N^2 e^{4\beta t} |\xi_t|^4 dt \right)^{\frac{1}{2}} \right] \\ \leq \frac{1}{2} E_{\xi_0} \left[\sup_{0 \le t \le \tau_n} e^{4\beta t} |\xi_t|^4 \right] + \frac{N^2}{2} E_{\xi_0} \int_0^{\tau_n} e^{4\beta t} |\xi_t|^4 dt \\ \leq \frac{1}{2} E_{\xi_0} \left[\sup_{0 \le t \le \tau_n} e^{4\beta t} |\xi_t|^4 \right] + N^2 N B_2(\xi_0),$$

we conclude the following

$$E_{\xi_0}\left[\sup_{0\leq t\leq \tau_n} e^{4\beta t} |\xi_t|^4\right] \leq NB_2(\xi_0), \quad \forall \xi_0 \in \mathbb{R}^d.$$

Thus, letting $n \to \infty$, we have

$$E_{\xi_0}\left[\sup_{0\le t\le \tau_2} e^{4\beta t} |\xi_t|^4\right] \le NB_2(\xi_0), \quad \forall \xi_0 \in \mathbb{R}^d.$$

By definition, we have for $t \in [0, \tau_2]$,

$$|(r_t, \pi_t, P_t)|^4 \le K_0 |\xi_t|^4.$$

Hence, we have for $(x, \xi_0) \in D_{\lambda^2} \times \mathbb{R}^d$

$$E_{x,\xi_0} \left[\sup_{0 \le t \le \tau_2} e^{4\beta t} |(r_t, \pi_t, P_t)|^4 \right] \le K_0 E_{\xi_0} \left[\sup_{0 \le t \le \tau_2} e^{4\beta t} |\xi_t|^4 \right] \le N B_2(\xi_0),$$

which proves Assertion (iii).

As shown in both Lemmas 3.2 and 3.4, both barrier functions B_1 and B_2 play a crucial role in the fourth-order moment estimates of first quasi-derivatives ξ_t , which are used to estimate the gradient of u.

To estimate the Hessian of u, the second quasi-derivative is introduced, and the eighth-order moment estimates of first quasi-derivatives have to be considered due to standard BSDE estimates for second-order difference. Define both functions

$$B_{3}(x,y) := \left(\lambda + \sqrt{\psi(x)} + \psi(x)\right) |y|^{8} + K_{1}\varphi^{\frac{15}{2}}(x)\frac{\psi_{(y)}^{8}(x)}{\psi^{7}(x)}, \quad (x,y) \in D_{\delta_{1}}^{\lambda} \times \mathbb{R}^{d},$$

where $K_1 \in [1, \infty)$ is a constant depending only on K_0 ; and

$$B_4(y) := \lambda^{\frac{3}{4}} |y|^8, \quad y \in \mathbb{R}^d.$$

With the help of both barrier functions B_3 and B_4 , we can extend both Lemmas 3.2 and 3.4 to estimate the eighth-order moment of ξ_t and the fourth-order moment of η_t .

Lemma 3.5. Let the assumptions of Lemma 3.2 be satisfied. Define X_t by (2.3), the first quasiderivative ξ_t by (2.8), and the second quasi-derivatives η_t by (2.9). For $(x,\xi_0,t) \in D_{\delta_1}^{\lambda} \times \mathbb{R}^d \times [0,\tau_1]$, let the coefficients r_t , \tilde{r}_t , $\tilde{\pi}_t$, P_t , \tilde{P}_t be defined as (3.1) in Lemma 3.2 and define $\pi_t := 8[\psi_{(\sigma)}\psi_{(\xi_t)}(\varphi\psi)^{-1}](X_t)$. Then for sufficiently small λ , we have for $(x,\xi_0) \in D_{\delta_1}^{\lambda} \times \mathbb{R}^d$ and $\eta_0 = 0$

(i) the process
$$(B_3(X_t, \xi_t), B_3^{\frac{1}{2}}(X_t, \xi_t)), t \in [0, \tau_1]$$
 is a local super-martingale;
(ii) $E_{x,\xi_0} \left[\sup_{0 \le t \le \tau_1} \left(|\xi_t|^8 + \frac{\psi_{(\xi_t)}^8(X_t)}{\psi^8(X_t)} \right) + \int_0^{\tau_1} |\xi_t|^8 + \frac{\psi_{(\xi_t)}^8(X_t)}{\psi^8(X_t)} dt \right] \le N(K_0, \lambda) B_3(x, \xi_0);$
(iii) $E_0 \left[\sup_{0 \le t \le \tau_1} |\eta_t|^4 \right] \le N(K_0, \lambda) B_3(x, \xi_0).$

Proof. Repeating the arguments between (3.2) and (3.4), we have the analogue of (3.5)

$$dB_3(X_t,\xi_t) \le \left(-\frac{1}{64} \frac{|\xi_t|^8}{\psi^{\frac{3}{2}}} - \frac{1}{2} \varphi^{\frac{13}{2}} \frac{\psi^8_{(\xi_t)}}{\psi^8} \right) dt + dm_t \le dm_t, \quad t \in [0,\tau_1],$$

where $\{m_t, t \in [0, \tau_1]\}$ is a local martingale. Then, following the arguments next to formula (3.5), we can prove Assertion (i).

Analogous to the proof of Assertions (ii) and (iii) of Lemma 3.2, we have Assertion (ii). Now we estimate the moments of the second quasi-derivative η_t . By (2.9), we have

$$d\eta_t = \left[\sigma_{(\eta_t)} + G(X_t, \xi_t)\right] dW_t + \left[b_{(\eta_t)} + H(X_t, \xi_t)\right] dt, \quad t \in [0, \infty),$$

with

$$\begin{aligned} G(X_t,\xi_t) &= \sigma_{(\xi_t)(\xi_t)} + 2r_t \sigma_{(\xi_t)} + \left(2\sigma_{(\xi_t)} + 2r_t \sigma + \sigma P_t\right) P_t + (\tilde{r}_t - r_t^2), \\ H(X_t,\xi_t) &= b_{(\xi_t)(\xi_t)} + 4r_t b_{(\xi_t)} + 2\tilde{r}_t b - 2\sigma_{(\xi_t)} \pi_t - 2r_t \sigma \pi_t - 2\sigma P_t \pi_t, \quad t \ge 0. \end{aligned}$$

Then, we have the estimates

$$||G|| \le N|\xi_t| \left(|\xi_t| + \frac{|\psi_{(\xi_t)}|}{\psi} \right), \quad |H| \le N \left(|\xi_t|^2 + \frac{\psi_{(\xi_t)}^2}{\psi^2} \right), \quad t \in [0, \tau_1].$$

Hence, Itô's formula implies

$$d\left(e^{2\varphi}|\eta_{t}|^{4}\right) \leq e^{2\varphi}\left[2\left(1-\frac{\psi}{2\lambda}\right)\mathcal{L}\psi+\left(1-\frac{\psi}{2\lambda}\right)^{2}A+N-\frac{A}{2\lambda}\right]|\eta_{t}|^{4}dt + Ne^{2\varphi}\left(|\xi_{t}|^{8}+\frac{\psi_{(\xi_{t})}^{8}}{\psi^{8}}\right)dt+2e^{2\varphi}|\eta_{t}|^{4}(1-\frac{\psi}{2\lambda})\psi_{(\sigma)}dW_{t} + 4e^{2\varphi}|\eta_{t}|^{2}\langle\eta_{t},[\sigma_{(\eta_{t})}+G(X_{t},\xi_{t})]dW_{t}\rangle, \quad t\in[0,\tau_{1}].$$

For sufficiently small λ , there exists a positive constant λ_0 , such that for $(x, \xi_0) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d$ and $\eta_0 = 0$

$$E_{x,0}\left[e^{2\varphi}|\eta_t|^4\right] + \lambda_0 E_{x,0} \int_0^t e^{2\varphi}|\eta_s|^4 ds \le N E_{x,\xi_0} \int_0^t e^{2\varphi} \left(|\xi_t|^8 + \frac{\psi_{(\xi_t)}^8}{\psi^8}\right) ds.$$
(3.11)

Next, using Assertion (ii), formula (3.11), BDG inequality and Cauchy inequality, for $\tau_n := \tau_1 \land \{t \ge 0 : e^{\varphi} |\eta_t|^2 \ge n\}$ and $\eta_0 = 0$, we have

$$E_{x,0} \left[\sup_{0 \le t \le \tau_n} \left| \int_0^t 2e^{2\varphi} \left(1 - \frac{\psi}{2\lambda} \right) \psi_{(\sigma)} |\eta_s|^4 dW_s \right| \right] \\ \le \frac{1}{3} E_{x,0} \left[\sup_{0 \le t \le \tau_n} e^{2\varphi} |\eta_t|^4 \right] + 3N^2 E_{x,0} \int_0^{\tau_n} e^{2\varphi} |\eta_t|^4 dt,$$
(3.12)

and

$$E_{x,\xi_{0},0}\left[\sup_{0\leq t\leq \tau_{n}}\left|\int_{0}^{t}4e^{2\varphi}|\eta_{t}|^{2}\langle\eta_{t},[\sigma_{(\eta_{t})}+G(X_{t},\xi_{t})]dW_{t}\rangle\right|\right]$$

$$\leq E_{x,\xi_{0},0}\left\{\sup_{0\leq t\leq \tau_{n}}(e^{\varphi}|\eta_{t}|^{2})\left[\int_{0}^{\tau_{n}}N^{2}e^{2\varphi}|\eta_{t}|^{4}+N^{3}e^{2\varphi}|\eta_{t}|^{2}|\xi_{t}|^{2}\left(|\xi_{t}|^{2}+\frac{|\psi_{(\xi_{t})}|^{2}}{\psi^{2}}\right)dt\right]^{\frac{1}{2}}\right\}$$

$$\leq \frac{1}{3}E_{x,0}\left[\sup_{0\leq t\leq \tau_{n}}e^{2\varphi}|\eta_{t}|^{4}\right]+N^{2}E_{x,0}\int_{0}^{\tau_{n}}e^{2\varphi}|\eta_{t}|^{4}dt$$

$$+2N^{4}E_{x,\xi_{0}}\int_{0}^{\tau_{n}}e^{2\varphi}\left(|\xi_{t}|^{8}+\frac{|\psi_{(\xi_{t})}|^{8}}{\psi^{8}}\right)dt.$$

$$\leq \frac{1}{3}E_{x,0}\left[\sup_{0\leq t\leq \tau_{n}}e^{2\varphi}|\eta_{t}|^{4}\right]+(N^{2}N+2N^{4})E_{x,\xi_{0}}\int_{0}^{\tau_{n}}e^{2\varphi}\left(|\xi_{t}|^{8}+\frac{|\psi_{(\xi_{t})}|^{8}}{\psi^{8}}\right)dt$$

$$\leq \frac{1}{3}E_{x,0}\left[\sup_{0\leq t\leq \tau_{n}}e^{2\varphi}|\eta_{t}|^{4}\right]+(N^{2}N+2N^{4})NB_{3}(x,\xi_{0}).$$
(3.13)

Then by Assertion (ii), formulas (3.12) and (3.13), we have

$$E_{x,0} \left[\sup_{0 \le t \le \tau_n} e^{2\varphi} |\eta_t|^4 \right] \le NB_3(x,\xi_0) + E_{x,0} \left[\sup_{0 \le t \le \tau_n} \left| \int_0^t 2e^{2\varphi} \left(1 - \frac{\psi}{2\lambda} \right) \psi_{(\sigma)} |\eta_s|^4 dW_s \right| \right] \\ + E_{x,\xi_0,0} \left[\sup_{0 \le t \le \tau_n} \left| \int_0^t 4e^{2\varphi} |\eta_t|^2 \langle \eta_t, [\sigma_{(\eta_t)} + G(X_t,\xi_t)] dW_t \rangle \right| \right] \\ \le NB_3(x,\xi_0), \quad \forall (x,\xi_0) \in D_{\delta_1}^{\lambda} \times \mathbb{R}^d.$$

Thus, letting $n \to \infty$, Assertion (iii) is proved.

Lemma 3.6. Let (H3) and (H10)₂ be satisfied. Define X_t by (2.3), the first quasi-derivative ξ_t by (2.8), and the second quasi-derivatives η_t by (2.9). For $(\tilde{x}, y, t) \in D_{\lambda^2} \times \mathbb{R}^d \times [0, \infty)$, define functions $r(\tilde{x}, y)$, $\pi(\tilde{x}, y)$, $P(\tilde{x}, y)$ by (3.9). For $(x, \xi_0, t) \in D_{\lambda^2} \times \mathbb{R}^d \times [0, \tau_2]$ and $\eta_0 = 0$, let the coefficients r_t , π_t and P_t be defined by (3.9) in Lemma 3.4, and define

$$\tilde{r}_t := r(X_t, \eta_t), \quad \tilde{\pi}_t := \pi(X_t, \eta_t), \quad \tilde{P}_t := P(X_t, \eta_t).$$
(3.14)

Then for sufficiently small λ , we have for $(x, \xi_0) \in D_{\lambda^2} \times \mathbb{R}^d$ and $\eta_0 = 0$,

(i) the process
$$(e^{8\beta t}B_4(\xi_t), e^{4\beta t}B_4^{\frac{1}{2}}(\xi_t)), t \in [0, \tau_2]$$
 is a local super-martingale;
(ii) $E_{\xi_0} \left[\sup_{0 \le t \le \tau_2} e^{8\beta t} |\xi_t|^8 \right] + E_{\xi_0} \int_0^{\tau_2} e^{8\beta t} |\xi_t|^8 dt \le N(K_0, \lambda) B_4(\xi_0);$
(iii) $E_{x,0} \left[\sup_{0 \le t \le \tau_2} e^{8\beta t} (|\eta_t|^4 + |\tilde{r}_t|^4) \right] \le N(K_0, \lambda) B_4(\xi_0).$

Proof. By Itô's formula, we have

$$d|\xi_t|^8 = \Gamma_3(X_t, \xi_t)dt + \Lambda_3(X_t, \xi_t)dW_t, \quad t \in [0, \tau_2],$$

where by $(H10)_2$, we have

$$\Gamma_3(X_t,\xi_t) = 8|\xi_t|^6 \left\langle \xi_t, \left(b_{(\xi_t)} + 2r_t b - \sigma \pi_t \right) \right\rangle + 28|\xi_t|^6 \|\sigma_{(\xi_t)} + r_t \sigma + \sigma P_t \|^2 \le (-8\beta - 4)|\xi_t|^2.$$

Then repeating the arguments next to formula (3.10), we can prove Assertion (i).

Similarly, Assertion (ii) can be proved in the same way as Lemma 3.4.

Now, we estimate the moments of the second quasi-derivative η_t . By (2.9), we have

$$d\eta_t = \left[\sigma_{(\eta_t)} + \tilde{r}_t \sigma + \sigma \tilde{P}_t + \sigma_{(\xi_t)(\xi_t)} + 2r_t \sigma_{(\xi_t)} - r_t^2 \sigma + 2\sigma_{(\xi_t)} P_t + 2r_t \sigma P_t + \sigma P_t^2 \right] dW_t \\ + \left[b_{(\eta_t)} + 2\tilde{r}_t b - \sigma \tilde{\pi}_t + b_{(\xi_t)(\xi_t)} + 4r_t b_{(\xi_t)} - 2\sigma_{(\xi_t)} \pi_t - 2r_t \sigma \pi_t - 2\sigma P_t \pi_t \right] dt \\ = \left[\tilde{\sigma}_t + G_t \right] dW_t + \left[\tilde{b}_t + H_t \right] dt, \quad t \ge 0,$$

where

$$\begin{split} \tilde{\sigma}_{t} &= \sigma_{(\eta_{t})} + \tilde{r}_{t}\sigma + \sigma\tilde{P}_{t}, \quad \tilde{b}_{t} = b_{(\eta_{t})} + 2\tilde{r}_{t}b - \sigma\tilde{\pi}_{t}, t \geq 0, \\ G_{t} &= \sigma_{(\xi_{t})(\xi_{t})} + 2r_{t}\sigma_{(\xi_{t})} - r_{t}^{2}\sigma + (2\sigma_{(\xi_{t})} + 2r_{t}\sigma + \sigma P_{t})P_{t}, \\ H_{t} &= b_{(\xi_{t})(\xi_{t})} + 4r_{t}b_{(\xi_{t})} - 2(\sigma_{(\xi_{t})} + r_{t}\sigma - \sigma P_{t})\pi_{t}, \quad t \geq 0. \end{split}$$

Assuming that

$$\sup_{\tilde{x}\in D_{\lambda^2}} |\rho(\tilde{x})| \le K_0, \quad \sup_{\tilde{x}\in D_{\lambda^2}} \|Q(\tilde{x},y)\| \le K_0 |y|, \quad \sup_{\tilde{x}\in D_{\lambda^2}} M(\tilde{x}) \le K_0,$$

by (H3), it is not hard to see that

$$||G_t|| \le N|\xi_t|^2, \quad |H_t| \le N|\xi_t|^2, \quad t \in [0, \tau_2].$$
 (3.15)

So, using $(H10)_2$ and formula (3.15), we have

$$d(e^{8\beta t}|\eta_{t}|^{4}) = |\eta_{t}|^{2} \left[8\beta|\eta_{t}|^{2} + 4\langle\eta_{t},\tilde{b}_{t} + H_{t}\rangle + 6\|\tilde{\sigma}_{t} + G_{t}\|^{2}\right] e^{8\beta t} dt +4|\eta_{t}|^{2} e^{8\beta t} \langle\eta_{t}, (\tilde{\sigma}_{t} + G_{t})dW_{t}\rangle \leq \left[(4\beta - 1)|\eta_{t}|^{4} + N|\xi_{t}|^{8}\right] e^{8\beta t} dt + 4|\eta_{t}|^{2} e^{8\beta t} \langle\eta_{t}, (\tilde{\sigma}_{t} + G_{t})dW_{t}\rangle, \quad t \in [0, \tau_{2}],$$

where $4\beta - 1 \leq -1$. Then following the arguments next to formula (3.11), we prove Assertion (iii).

We have more moment estimates for quasi-derivatives.

Corollary 3.7. In addition to (H3) and (H9), let $(H10)_p$ be satisfied for some positive p. Define both functions

$$B_{2p-1}(x,y) := \left(\lambda + \sqrt{\psi(x)} + \psi(x)\right) |y|^{4p} + K_1 \varphi^{\frac{8p-1}{2}}(x) \frac{\psi_{(y)}^{4p}(x)}{\psi^{4p-1}(x)}, \quad (x,y) \in D_{\delta_1}^{\lambda} \times \mathbb{R}^d,$$

where $K_1 \in [1, \infty)$ is a constant depending only on K_0 ; and

$$B_{2p}(y) := \lambda^{\frac{3}{4}} |y|^{4p}, \quad y \in \mathbb{R}^d.$$

Define X_t by (2.3), the first quasi-derivative ξ_t by (2.8), and the second quasi-derivative η_t by (2.9). For $(x,\xi_0,t) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d \times [0,\tau_1]$, define

$$\pi_t := 4p[\psi_{(\sigma)}\psi_{(\xi_t)}(\varphi\psi)^{-1}](X_t)$$

and choose other coefficients r_t , \tilde{r}_t , $\tilde{\pi}_t$, P_t , \tilde{P}_t defined as (3.1) in Lemma 3.2. For $(x, \xi_0, t) \in D_{\lambda^2} \times \mathbb{R}^d \times [0, \tau_2]$, choose the coefficients r_t , \tilde{r}_t , π_t , $\tilde{\pi}_t$, P_t , \tilde{P}_t defined as (3.9) and (3.14) in Lemmas 3.4 and 3.6. Then for sufficiently small λ , we have

(i) for $(x,\xi_0) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d$ and $\eta_0 = 0$, the process $(B_{2p-1}(X_t,\xi_t), B^{\frac{1}{2}}_{2p-1}(X_t,\xi_t)), t \in [0,\tau_1]$ is a local super-martingale;

$$E_{x,\xi_{0},0}\left[\sup_{0\leq t\leq \tau_{1}}\left(|\xi_{t}|^{4p}+\frac{\psi_{(\xi_{t})}^{4p}(X_{t})}{\psi^{4p}(X_{t})}+|\eta_{t}|^{2p}\right)\right]+E_{x,\xi_{0}}\int_{0}^{\tau_{1}}\left(|\xi_{t}|^{4p}+\frac{\psi_{(\xi_{t})}^{4p}(X_{t})}{\psi^{4p}(X_{t})}\right)dt$$

$$\leq N(K_{0},\lambda)B_{2p-1}(x,\xi_{0});$$
(3.16)

(ii) for $(x,\xi_0) \in D_{\lambda^2} \times \mathbb{R}^d$ and $\eta_0 = 0$, the process $(e^{4p\beta t}B_{2p}(\xi_t), e^{2p\beta t}B_{2p}^{\frac{1}{2}}(\xi_t)), t \in [0,\tau_2]$ is a local super-martingale;

$$E_{x,\xi_0,0} \left[\sup_{0 \le t \le \tau_2} e^{4p\beta t} \left(|\xi_t|^{4p} + |\eta_t|^{2p} \right) \right] + E_{x,\xi_0} \int_0^{\tau_2} e^{4p\beta t} |\xi_t|^{4p} dt \le N(K_0,\lambda) B_{2p}(\xi_0).$$
(3.17)

4 Interior Gradient and Hessian Estimates

In this section, we prove Theorem 2.9. We begin with a standard BSDE estimate and estimate the derivative in two regions near the boundary and in the interior of the domain.

Remark 4.1. Without loss of generality, we may assume that $u \in C^1(\overline{D})$ when investigating the gradient estimate of u, and $u \in C^2(\overline{D})$ when investigating the Hessian estimate of u. In fact, to find the gradient estimate of the solution such as $\lim_{\delta \to 0} |Y_0^{\delta} - Y_0|\delta^{-1} \leq N$, we can choose smooth coefficients $(\sigma^{\epsilon}, f^{\epsilon}, g^{\epsilon})$ such that they converge to (σ, f, g) . Here we need to notice that (b, σ) is smooth enough under assumptions. Assume $(\sigma^{\epsilon})^2 \geq (\sigma)^2 + \epsilon I$, then the nondegenerate elliptic equation with $(b, \sigma^{\epsilon}, f^{\epsilon}, g^{\epsilon})$ has a classical solution $u^{\epsilon}(x) = Y_0^{\epsilon}$. Moreover, due to the solution of a BSDE has continuous dependence on parameter (see [2, Theorem 2.4]), we have

$$\lim_{\delta \to 0} \lim_{\epsilon \to 0} |Y_0^{\delta} - Y_0^{\epsilon,\delta}| \delta^{-1} = 0 \quad \text{and} \quad \lim_{\delta \to 0} \lim_{\epsilon \to 0} |Y_0 - Y_0^{\epsilon}| \delta^{-1} = 0.$$

Therefore, our problem is reduced to the estimate of $\lim_{\delta \to 0} |Y_0^{\epsilon,\delta} - Y_0^{\epsilon}|\delta^{-1}$, and thus we can assume $u \in C^1$ to derive the gradient estimate. By the way, we can also assume that $g \in C^1$ and $f \in C^1$ with bounded partial derivatives in (x, y, z) when estimating the gradient of u. Noting that $\mathcal{L}^{\epsilon}\psi \leq \mathcal{L}\psi + \epsilon\psi_{xx}/2 \leq -1/2$, we have $E\tau^{\epsilon}(x) \leq N$.

Similarly, we may assume $u, g \in C^2$ and $f \in C^2$ with bounded first and second order partial derivatives in (x, y, z) when investigating Hessian estimates.

4.1 Interior Gradient Estimate

Define the first quasi-derivative ξ_t by (2.8), X_t by (2.3) and X_t^{δ} by (2.12). Let τ and τ^{δ} be the first exit time from D of X_t and X_t^{δ} , respectively; and τ_1 and τ_2 be the first exit time of X_t from $D_{\delta_1}^{\lambda}$ and D_{λ^2} , respectively. Set

$$\gamma_T^{\delta,n} := \tau \wedge \tau^\delta \wedge k_n \wedge T$$

where $k_n := \inf\{t \ge 0; |\xi_t| \ge n\}$ and $T \in [1, \infty)$. Set

$$\gamma_0 = \tau^{\delta} \wedge T, \quad \gamma_{1,T}^{\delta,n} := \tau_1 \wedge \gamma_T^{\delta,n}, \text{ and } \quad \gamma_{2,T}^{\delta,n} = \tau_2 \wedge \gamma_T^{\delta,n}.$$

For simplicity of notation, write

$$\gamma := \gamma_T^{\delta,n}, \quad \gamma_1 := \gamma_{1,T}^{\delta,n}, \quad \text{and} \quad \gamma_2 := \gamma_{2,T}^{\delta,n}$$

whenever no confusion is made. Also, write for $(x, \xi_0, t) \in D \times \mathbb{R}^d \times [0, \gamma]$,

$$E_{x,\xi_0}[X_t^{\delta} - X_t] := E[X_t^{\delta}(x + \delta\xi_0) - X_t(x)].$$

The main result of this section is stated as follows.

Theorem 4.2. Define u by (1.2). Let X be the unique solution of SDE (1.4), (Y, Z) be the unique solution of BSDE (1.3) and $(X^{\delta}, Y^{\delta}, Z^{\delta})$ be the unique solution of FBSDE (2.12) with $(\eta_0, \tilde{r}_t, \tilde{\pi}_t, \tilde{P}_t)$ vanishing. Let assumptions (H1)-(H5), (H6)₀, (H7), (H9) and (H10)₁ be satisfied. Then $u \in C^{0,1}_{loc}(D) \cap C(\overline{D})$,

$$\frac{\lim_{T \to \infty} \lim_{\delta \to 0} E_{x,\xi_0} \left[\sup_{t \le \gamma_{1,T}^{\delta,n} \land \gamma_{2,T}^{\delta,n}} \left| \frac{Y_t^{\delta} - Y_t}{\delta} \right|^2 \right] \le N \left(|\xi_0|^2 + \frac{|\psi_{(\xi_0)}(x)|^2}{\psi^{\frac{3}{2}}(x)} \right) (|g|_{0,1}^2 + ||f(\cdot)||_{0,1}^2), \ \forall (x,\xi_0) \in D \times \mathbb{R}^d.$$
(4.1)

In particular, for any $\xi_0 \in \mathbb{R}^d$ and a.e. $x \in D$,

$$|u_{(\xi_0)}(x)| \le N\left(|\xi_0| + \frac{|\psi_{(\xi_0)}(x)|}{\psi^{\frac{3}{4}}(x)}\right) (|g|_{0,1} + ||f(\cdot)||_{0,1})$$
(4.2)

where $N = N(K_0, d, d_1, k, D, \mu, L, L_0)$.

The proof of Theorem 4.2 consists of the following sequel of propositions. The following proposition is an analogue to Lemma 2.2.

Proposition 4.3. Under the assumptions of Theorem 4.2 and the condition (2.7), there exists a constant N such that for sufficiently small δ

(i)
$$E \left[\sup_{0 \le t \le \gamma_0} |Y_t^{\delta}|^2 \right] + E \int_0^{\gamma_0} \left(|Y_t^{\delta}|^2 + \|\tilde{Z}_t^{\delta}\|^2 \right) ds \le N(|g|_0^2 + |f(\cdot, 0, 0)|_0^2);$$

(ii) $E \left[\sup_{0 \le t \le \gamma_0} |Y_t^{\delta}|^{2p} \right] + E \left[\left(\int_0^{\gamma_0} \|\tilde{Z}_t^{\delta}\|^2 ds \right)^p \right] \le N(|g|_0^{2p} + |f(\cdot, 0, 0)|_0^{2p}), \quad p \ge 2.$

Proof. Using Itô's formula, we have

$$\begin{aligned} d|Y_t^{\delta}|^2 &\geq \left\{ (1+2\delta r_t) \left[2\mu - 2\left(1+\epsilon\right) L_0^2 - \epsilon' \right] - (1+\epsilon'') \left| \delta \pi_t \right|^2 \right\} |Y_t^{\delta}|^2 dt - \frac{1}{2(1+\epsilon)} \|Z_t^{\delta}\|^2 dt \\ &+ \left[1 - \frac{1}{1+\epsilon''} \right] \|\tilde{Z}_t^{\delta}\|^2 dt - (1+2\delta r_t) \frac{1}{\epsilon'} |f(X_t^{\delta}, 0, 0)|^2 dt + 2Y_t^{\delta} \tilde{Z}_t^{\delta} dW_t. \end{aligned}$$

For $x \in D$, if we choose δ is small enough, we have

$$|\delta r_t| < \frac{1}{2}, \quad |\delta \pi_t| < \epsilon', \quad \text{and} \quad ||Z_t^{\delta}|| \le N ||\tilde{Z}_t^{\delta}||,$$

and from Lemma 2.8 and the BDG inequality, we have

$$E\left[\sup_{0\leq t\leq\gamma_{0}}|Y_{t}^{\delta}|^{2}\right]+E\int_{0}^{\gamma_{0}}|Y_{t}^{\delta}|^{2}ds+E\int_{0}^{\gamma_{0}}\|\tilde{Z}_{t}^{\delta}\|^{2}ds$$
$$\leq N\left(E\left[|u(X_{\gamma_{0}}^{\delta})|^{2}\right]+E\int_{0}^{\gamma_{0}}|f(X_{s}^{\delta},0,0)|^{2}ds\right),$$

which yields Assertion (i). Assertion (ii) can be proved in the same way.

The following lemma is about estimating the directional derivatives of the solutions along first quasi-derivatives on the boundary. When u taking values in \mathbb{R}^k , we can still use the methods provided in [22] where concerning the case of dimension one. We refer the reader to arguments (4.3)-(4.14) in [22] for details, but we still prove it for the sake of completeness.

Lemma 4.4. Let the assumptions of Theorem 4.2 be satisfied. Assume $u \in C^1(\overline{D})$. Then, there exists a constant N such that

(i) for $(x,\xi_0) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d$,

$$\frac{\lim_{T \to \infty} \lim_{\delta \to 0} E_{x,\xi_0} \left[|u_{(\xi_{\gamma_{1,T}^{\delta,n}})}(X_{\gamma_{1,T}^{\delta,n}})|^2 e^{2\beta \gamma_{1,T}^{\delta,n}} \right] \le N B_1^{\frac{1}{2}}(x,\xi_0) + \sup_{\substack{y \in \partial D_{\delta_1}^{\lambda} \\ 0 \ne \zeta \in \mathbb{R}^d}} \frac{|u_{(\zeta)}(y)|^2}{B_1^{\frac{1}{2}}(y,\zeta)} B_1^{\frac{1}{2}}(x,\xi_0); \quad (4.3)$$

(ii) for $(x, \xi_0) \in D_{\lambda^2} \times \mathbb{R}^d$,

$$\frac{\lim_{T \to \infty} \lim_{\delta \to 0} E_{x,\xi_0} \left[|u_{(\xi_{\gamma_{2,T}^{\delta,n}})}(X_{\gamma_{2,T}^{\delta,n}})|^2 e^{2\beta \gamma_{2,T}^{\delta,n}} \right] \le N B_2^{\frac{1}{2}}(\xi_0) + \sup_{\substack{y \in \partial D_{\lambda^2} \\ 0 \ne \zeta \in \mathbb{R}^d}} \frac{|u_{(\zeta)}(y)|^2}{B_2^{\frac{1}{2}}(\zeta)} B_2^{\frac{1}{2}}(\xi_0).$$
(4.4)

Proof. First, for $(x, \xi_0) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d$,

$$E_{x,\xi_{0}}\left[\left|u_{(\xi_{\gamma_{1}})}(X_{\gamma_{1}})\right|^{2}e^{2\beta\gamma_{1}}\right]$$

$$\leq E_{x,\xi_{0}}\left[\frac{|u_{(\xi_{\gamma_{1}})}(X_{\gamma_{1}})|^{2}}{B_{1}^{\frac{1}{2}}(X_{\gamma_{1}},\xi_{\gamma_{1}})}B_{1}^{\frac{1}{2}}(X_{\gamma_{1}},\xi_{\gamma_{1}})\right]$$

$$\leq E_{x,\xi_{0}}\left[\left(\frac{|u_{(\xi_{\gamma_{1}})}(X_{\gamma_{1}})|^{2}}{B_{1}^{\frac{1}{2}}(X_{\gamma_{1}},\xi_{\gamma_{1}})}-\frac{|u_{(\xi_{\gamma_{1}})}(X_{\tau_{1}})|^{2}}{B_{1}^{\frac{1}{2}}(X_{\tau_{1}},\xi_{\gamma_{1}})}\right)B_{1}^{\frac{1}{2}}(X_{\gamma_{1}},\xi_{\gamma_{1}})\right]$$

$$+E_{x,\xi_{0}}\left[\frac{|u_{(\xi_{\gamma_{1}})}(X_{\tau_{1}})|^{2}}{B_{1}^{\frac{1}{2}}(X_{\tau_{1}},\xi_{\gamma_{1}})}B_{1}^{\frac{1}{2}}(X_{\gamma_{1}},\xi_{\gamma_{1}})\right]$$

$$= J_{1}(\delta,n,T)+J_{2}(\delta,n,T).$$

Note that the function $(\tilde{x}, y) \to |u_{(y)}(\tilde{x})|^2 B_1^{-\frac{1}{2}}(\tilde{x}, y)$ is continuous from $D_{\delta_1}^{\lambda} \times S_1$ to \mathbb{R} with S_1 being the unitary ball of \mathbb{R}^d . Weierstrass approximation theorem asserts that there is a polynomial $W(\tilde{x}, y), (\tilde{x}, y) \in D_{\delta_1}^{\lambda} \times S_1$ such that

$$\max_{\tilde{x} \in D_{\delta_1}^{\lambda}, y \in S_1} \left| \frac{|u_{(y)}(\tilde{x})|^2}{B_1^{\frac{1}{2}}(\tilde{x}, y)} - W(\tilde{x}, y) \right| \le 1.$$

From Assertion (i) of Lemma 3.2, it follows that

$$\begin{aligned} J_{1}(\delta,n,T) &\leq E_{x,\xi_{0}} \left[|W(X_{\gamma_{1}},\xi_{\gamma_{1}}) - W(X_{\tau_{1}},\xi_{\gamma_{1}})| B_{1}^{\frac{1}{2}}(X_{\gamma_{1}},\xi_{\gamma_{1}}) \right] + 2E_{x,\xi_{0}} B_{1}^{\frac{1}{2}}(X_{\gamma_{1}},\xi_{\gamma_{1}}) \\ &\leq NE_{x,\xi_{0}} \left[|X_{\gamma_{1}} - X_{\tau_{1}}| B_{1}^{\frac{1}{2}}(X_{\gamma_{1}},\xi_{\gamma_{1}}) \right] + 2B_{1}^{\frac{1}{2}}(x,\xi_{0}) \\ &\leq \frac{N^{2}}{4} E_{x,\xi_{0}} \left[|X_{\gamma_{1}} - X_{\tau_{1}}|^{2} \right] B_{1}^{\frac{1}{2}}(x,\xi_{0}) + \frac{E_{x,\xi_{0}} B_{1}(X_{\gamma_{1}},\xi_{\gamma_{1}})}{B_{1}^{\frac{1}{2}}(x,\xi_{0})} + 2B_{1}^{\frac{1}{2}}(x,\xi_{0}) \\ &\leq \frac{N^{2}}{4} E_{x,\xi_{0}} \left[|X_{\gamma_{1}} - X_{\tau_{1}}|^{2} \right] B_{1}^{\frac{1}{2}}(x,\xi_{0}) + 3B_{1}^{\frac{1}{2}}(x,\xi_{0}) \\ &\leq \frac{N^{2}}{4} B_{1}^{\frac{1}{2}}(x,\xi_{0}) \left(E_{x,\xi_{0}} |\tau_{1} - \tau_{1} \wedge \tau^{\delta}| + E_{x,\xi_{0}} |\tau_{1} - \tau_{1} \wedge T| + E_{x,\xi_{0}} |\tau_{1} - \tau_{1} \wedge k_{n}| \right) \\ &\quad + 3B_{1}^{\frac{1}{2}}(x,\xi_{0}). \end{aligned}$$

Due to [22, Theorem 3.3], we have

$$\overline{\lim_{\delta \to 0}} E_{x,\xi_0}(\tau_1 - \tau_1 \wedge \tau^\delta) = 0.$$

Thus,

$$\overline{\lim_{\substack{T \to \infty \\ n \to \infty}} \overline{\lim_{\delta \to 0}} J_1(\delta, n, T) \le 3B_1^{\frac{1}{2}}(x, \xi_0)$$

From Assertion (i) in Lemma 3.2, we have

$$J_2(\delta, n, T) \le \sup_{\substack{y \in \partial D_{\delta_1}^{\lambda} \\ 0 \ne \zeta \in \mathbb{R}^d}} \frac{|u_{(\zeta)}(y)|^2}{B_1^{\frac{1}{2}}(y, \zeta)} B_1^{\frac{1}{2}}(x, \xi_0) \ .$$

So, we have Assertion (i).

Second, Assertion (ii) can be proved for $(x, \xi_0) \in D_{\lambda^2} \times \mathbb{R}^d$ in the same way.

Combined with Lemmas 2.8 and 4.4, we have the following immediate consequence.

Proposition 4.5. Let the assumptions of Theorem 4.2 be satisfied. Assume $u \in C^1(\overline{D})$. Then there exists a constant N such that

(i) for $(x, \xi_0) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d$,

$$\frac{\lim_{T \to \infty} \lim_{\delta \to 0} \frac{1}{\delta^2} E_{x,\xi_0} \left[|Y_{\gamma_{1,T}^{\delta,n}}^{\delta} - Y_{\gamma_{1,T}^{\delta,n}}|^2 e^{2\beta\gamma_{1,T}^{\delta,n}} \right] \le N B_1^{\frac{1}{2}}(x,\xi_0) + \sup_{\substack{y \in \partial D_{\lambda_1}^{\lambda} \\ 0 \ne \zeta \in \mathbb{R}^d}} \frac{|u_{(\zeta)}(y)|^2}{B_1^{\frac{1}{2}}(y,\zeta)} B_1^{\frac{1}{2}}(x,\xi_0) ; \quad (4.5)$$

(ii) for $(x, \xi_0) \in D_{\lambda^2} \times \mathbb{R}^d$,

$$\frac{\lim_{T \to \infty} \lim_{\delta \to 0} \frac{1}{\delta^2} E_{x,\xi_0} \left[|Y_{\gamma_{2,T}^{\delta,n}}^{\delta} - Y_{\gamma_{2,T}^{\delta,n}}|^2 e^{2\beta\gamma_{2,T}^{\delta,n}} \right] \le N B_2^{\frac{1}{2}}(\xi_0) + \sup_{\substack{y \in \partial D_{\lambda^2} \\ 0 \ne \zeta \in \mathbb{R}^d}} \frac{|u_{(\zeta)}(y)|^2}{B_2^{\frac{1}{2}}(\zeta)} B_2^{\frac{1}{2}}(\xi_0). \quad (4.6)$$

Proof. First, for $(x,\xi_0) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d$, we have

$$E_{x,\xi_{0}}\left[|Y_{\gamma_{1}}^{\delta}-Y_{\gamma_{1}}|^{2}e^{2\beta\gamma_{1}}\right] = E_{x,\xi_{0}}\left[|u(X_{\gamma_{1}}^{\delta})-u(X_{\gamma_{1}})|^{2}e^{2\beta\gamma_{1}}\right] \\ \leq E_{x,\xi_{0}}\left[\delta^{2}\left|\frac{u(X_{\gamma_{1}}^{\delta})-u(X_{\gamma_{1}})}{\delta}-u_{(\xi_{\gamma_{1}})}(X_{\gamma_{1}})\right|^{2}e^{2\beta\gamma_{1}}\right] + E_{x,\xi_{0}}\left[\delta^{2}\left|u_{(\xi_{\gamma_{1}})}(X_{\gamma_{1}})\right|^{2}e^{2\beta\gamma_{1}}\right].$$

And, due to Mean Value Theorem, we get

$$E_{x,\xi_{0}}\left[\left|\frac{u(X_{\gamma_{1}}^{\delta})-u(X_{\gamma_{1}})}{\delta}-u_{(\xi_{\gamma_{1}})}(X_{\gamma_{1}})\right|^{2}e^{2\beta\gamma_{1}}\right] \le E_{x,\xi_{0}}\left[\left|\widehat{u}_{x}\left(\frac{X_{\gamma_{1}}^{\delta}-X_{\gamma_{1}}}{\delta}-\xi_{\gamma_{1}}\right)\right|^{2}e^{2\beta\gamma_{1}}\right]+E_{x,\xi_{0}}\left[\left|\widehat{u}_{x}-u_{x}(X_{\gamma_{1}})\right|^{2}|\xi_{\gamma_{1}}|^{2}e^{2\beta\gamma_{1}}\right],$$

where

$$\widehat{u}_x := \int_0^1 u_x (X_{\gamma_1} + \lambda (X_{\gamma_1}^{\delta} - X_{\gamma_1})) d\lambda.$$

As u_x is continuous and $\lim_{\delta \to 0} E_{x,\xi_0}[|X_{\gamma_1}^{\delta} - X_{\gamma_1}|^2] = 0$, by the dominated convergence theorem and (2.13) with p = 2, we have

$$\lim_{\delta \to 0} E_{x,\xi_0} \left[\left| \frac{u(X_{\gamma_1}^{\delta}) - u(X_{\gamma_1})}{\delta} - u_{(\xi_{\gamma_1})}(X_{\gamma_1}) \right|^2 e^{2\beta\gamma_1} \right] = 0.$$

Then, using formula (4.3), we prove Assertion (i).

Second, repeating the arguments for $(x, \xi_0) \in D_{\lambda^2} \times \mathbb{R}^d$, Assertion (ii) is proved.

Note that our definition of B_1 and B_2 preceding Lemmas 3.2 is different from those of [21]. However, we can easily check out the same relations as [21] between both barrier functions B_1 and B_2 . **Lemma 4.6.** For sufficiently small λ , we have

$$\begin{cases} B_1(x,y) \ge 4B_2(y), & \forall (x,y) \in \{x : \psi(x) = \lambda\} \times \mathbb{R}^d; \\ 4B_1(x,y) \le B_2(y), & \forall (x,y) \in \{x : \psi(x) = \lambda^2\} \times \mathbb{R}^d. \end{cases}$$

$$(4.7)$$

The following lemma helps to estimate the unknown terms on the right hand sides of formulas (4.5) and (4.6).

Lemma 4.7. Assume that $g \in C^1(\overline{D})$, $f \in C^1(\overline{D} \times \mathbb{R}^k \times \mathbb{R}^{k \times d_1})$ has bounded partial derivatives in x, and $u \in C^1(\overline{D})$. Moreover, assume that

$$\frac{|u_{(\xi_0)}(x)|^2}{B_1^{\frac{1}{2}}(x,\xi_0)} \le \sup_{\substack{y \in \partial D_{\delta_1}^{\lambda} \\ 0 \neq \zeta \in \mathbb{R}^d}} \frac{|u_{(\zeta)}(y)|^2}{B_1^{\frac{1}{2}}(y,\zeta)} + N(|g|_0^2 + \|f(\cdot)\|_{0,1}^2), \quad \forall (x,\xi_0) \in D_{\delta_1}^{\lambda} \times \mathbb{R}^d \setminus \{0\},$$
(4.8)

and

$$\frac{u_{(\xi_0)}(x)|^2}{B_2^{\frac{1}{2}}(\xi_0)} \le \sup_{\substack{y \in \partial D_{\lambda^2} \\ 0 \neq \zeta \in \mathbb{R}^d}} \frac{|u_{(\zeta)}(y)|^2}{B_2^{\frac{1}{2}}(\zeta)} + N(|g|_0^2 + \|f(\cdot)\|_{0,1}^2), \quad \forall (x,\xi_0) \in D_{\lambda^2} \times \mathbb{R}^d \setminus \{0\},$$
(4.9)

where $N = N(K_0, d, d_1, k, D, \mu, L, L_0)$, and B_1 and B_2 are defined preceding Lemmas 3.2. Then we have

$$|u_{(\xi_0)}(x)|^2 \le N\left(|\xi_0|^2 + \frac{|\psi_{(\xi_0)}(x)|^2}{\psi^{\frac{3}{2}}(x)}\right) (|g|_1^2 + ||f(\cdot)||_{0,1}^2), \quad \forall (x,\xi_0) \in D \times \mathbb{R}^d.$$
(4.10)

In particular, N does not depend on $|u_x|_0$.

Proof. The proof is same to the arguments next to [21, relation (3.24)].

Set $N_1 := N(|g|_0^2 + ||f(\cdot)||_{0,1}^2)$. In view of formulas (4.7), (4.8) and (4.9), we have for $(x, \xi_0) \in \{x \in D : \psi(x) = \lambda\} \times \mathbb{R}^d \setminus \{0\}$

$$\frac{|u_{(\xi_0)}(x)|^2}{B_1^{\frac{1}{2}}(x,\xi_0)} \leq \frac{|u_{(\xi_0)}(x)|^2}{2B_2^{\frac{1}{2}}(\xi_0)} \leq \sup_{\substack{\psi(x)=\lambda^2\\0\neq\xi_0\in\mathbb{R}^d}} \frac{|u_{(\xi_0)}(x)|^2}{2B_2^{\frac{1}{2}}(\xi_0)} + \frac{1}{2}N_1$$

$$\leq \sup_{\substack{\psi(x)=\lambda^2\\0\neq\xi_0\in\mathbb{R}^d}} \frac{|u_{(\xi_0)}(x)|^2}{4B_1^{\frac{1}{2}}(x,\xi_0)} + \frac{N_1}{2}$$

$$\leq \sup_{\substack{\psi(x)=\lambda\\0\neq\xi_0\in\mathbb{R}^d}} \frac{|u_{(\xi_0)}(x)|^2}{4B_1^{\frac{1}{2}}(x,\xi_0)} + \sup_{\substack{\psi(x)=\delta_1\\0\neq\xi_0\in\mathbb{R}^d}} \frac{|u_{(\xi_0)}(x)|^2}{4B_1^{\frac{1}{2}}(x,\xi_0)} + \frac{3N_1}{4},$$

which implies that

$$\sup_{\substack{\psi(x)=\lambda\\0\neq\xi_0\in\mathbb{R}^d}}\frac{|u_{(\xi_0)}(x)|^2}{B_1^{\frac{1}{2}}(x,\xi_0)} \le \sup_{\substack{\psi(x)=\delta_1\\0\neq\xi_0\in\mathbb{R}^d}}\frac{|u_{(\xi_0)}(x)|^2}{3B_1^{\frac{1}{2}}(x,\xi_0)} + N_1.$$
(4.11)

Meanwhile, in view of formulas (4.7), (4.8) and (4.11), we have for $(x, \xi_0) \in \{x : \psi(x) = \lambda^2\} \times \mathbb{R}^d \setminus \{0\}$,

$$\frac{|u_{(\xi_0)}(x)|^2}{B_2^{\frac{1}{2}}(\xi_0)} \leq \frac{|u_{(\xi_0)}(x)|^2}{2B_1^{\frac{1}{2}}(x,\xi_0)} \leq \sup_{\substack{\psi(x)=\lambda\\0\neq\xi_0\in\mathbb{R}^d}} \frac{|u_{(\xi_0)}(x)|^2}{2B_1^{\frac{1}{2}}(x,\xi_0)} + \sup_{\substack{\psi(x)=\delta_1\\0\neq\xi_0\in\mathbb{R}^d}} \frac{|u_{(\xi_0)}(x)|^2}{2B_1^{\frac{1}{2}}(x,\xi_0)} + \frac{1}{2}N_1$$

$$\leq \sup_{\substack{\psi(x)=\delta_1\\0\neq\xi_0\in\mathbb{R}^d}} \frac{2|u_{(\xi_0)}(x)|^2}{3B_1^{\frac{1}{2}}(x,\xi_0)} + N_1.$$

Therefore, taking the supremum, we have

$$\sup_{\substack{\psi(x)=\lambda^2\\0\neq\xi_0\in\mathbb{R}^d}}\frac{|u_{(\xi_0)}(x)|^2}{B_2^{\frac{1}{2}}(\xi_0)} \le \sup_{\substack{\psi(x)=\delta_1\\0\neq\xi_0\in\mathbb{R}^d}}\frac{2|u_{(\xi_0)}(x)|^2}{3B_1^{\frac{1}{2}}(x,\xi_0)} + N_1.$$
(4.12)

Combining (4.8) and (4.11), we get, for $(x, \xi_0) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d \setminus \{0\}$,

$$\frac{|u_{(\xi_0)}(x)|^2}{B_1^{\frac{1}{2}}(x,\xi_0)} \le \sup_{\substack{\psi(y)=\delta_1\\0\neq\xi_0\in\mathbb{R}^d}} \frac{4|u_{(\xi_0)}(y)|^2}{3B_1^{\frac{1}{2}}(y,\xi_0)} + 2N_1.$$
(4.13)

Combining (4.9) and (4.12), we get, for $(x, \xi_0) \in D^2_{\lambda} \times \mathbb{R}^d \setminus \{0\}$,

$$\frac{|u_{(\xi_0)}(x)|^2}{B_2^{\frac{1}{2}}(\xi_0)} \le \sup_{\substack{\psi(y)=\delta_1\\0\neq\xi_0\in\mathbb{R}^d}} \frac{2|u_{(\xi_0)}(y)|^2}{3B_1^{\frac{1}{2}}(y,\xi_0)} + 2N_1.$$
(4.14)

Thus, it remains to estimate

$$\overline{\lim_{\delta_1 \to 0}} \sup_{\substack{\psi(y) = \delta_1 \\ 0 \neq \xi_0 \in \mathbb{R}^d}} \frac{|u_{(\xi_0)}(y)|^2}{B_1^{\frac{1}{2}}(y,\xi_0)}.$$

Notice that for each δ_1 , there exist $y(\delta_1) \in \{x : \psi(x) = \delta_1\}$ and $\xi_0(\delta_1) \in \{\xi_0 : |\xi_0| = 1\}$, such that

$$\sup_{\substack{\psi(y)=\delta_1\\0\neq\xi_0\in\mathbb{R}^d}}\frac{|u_{(\xi_0)}(y)|^2}{B_1^{\frac{1}{2}}(y,\xi_0)}=\frac{|u_{(\xi_0(\delta_1))}(y(\delta_1))|^2}{B_1^{\frac{1}{2}}(y(\delta_1),\xi_0(\delta_1))}.$$

A subsequence of $(y(\delta_1), \xi_0(\delta_1))$ converges to some (z, ζ) , as $\delta_1 \to 0$, such that $z \in \partial D$ and $|\zeta| = 1$. If $\psi_{(\zeta)}(z) \neq 0$, we have

$$\lim_{\delta_1 \to 0} \sup_{\substack{\psi(y) = \delta_1 \\ 0 \neq \xi_0 \in \mathbb{R}^d}} \frac{|u_{(\xi_0)}(y)|^2}{B_1^{\frac{1}{2}}(y,\xi_0)} = \lim_{\delta_1 \to 0} \frac{|u_{(\xi_0(\delta_1))}(y(\delta_1))|^2}{B_1^{\frac{1}{2}}(y(\delta_1),\xi_0(\delta_1))} = 0.$$
(4.15)

If $\psi_{(\zeta)}(z) = 0$, we have

$$\overline{\lim_{\delta_1 \to 0}} \sup_{\substack{\psi(y) = \delta_1 \\ 0 \neq \xi_0 \in \mathbb{R}^d}} \frac{|u_{(\xi_0)}(y)|^2}{B_1^{\frac{1}{2}}(y,\xi_0)} = \overline{\lim_{\delta_1 \to 0}} \frac{|u_{(\xi_0(\delta_1))}(y(\delta_1))|^2}{B_1^{\frac{1}{2}}(y(\delta_1),\xi_0(\delta_1))} = \frac{|g_{(\zeta)}(z)|^2}{\sqrt{\lambda}} \le N|g|_1^2.$$
(4.16)

From (4.13), (4.14), (4.15) and (4.16), we have

$$\begin{cases} \frac{|u_{(\xi_0)}(x)|^2}{B_1^{\frac{1}{2}}(x,\xi_0)} \le N(|g|_1^2 + \|f(\cdot)\|_{0,1}^2), & \forall (x,\xi_0) \in D^\lambda \times \mathbb{R}^d \setminus \{0\};\\ \frac{|u_{(\xi_0)}(x)|^2}{B_2^{\frac{1}{2}}(\xi_0)} \le N(|g|_1^2 + \|f(\cdot)\|_{0,1}^2), & \forall (x,\xi_0) \in D_{\lambda^2} \times \mathbb{R}^d \setminus \{0\}. \end{cases}$$

Notice that $D^{\lambda} \cup D_{\lambda^2} = D$, and

$$\begin{cases} B_1^{\frac{1}{2}}(x,\xi_0) \le N\left(|\xi_0|^2 + \frac{|\psi_{(\xi_0)}(x)|^2}{\psi^{\frac{3}{2}}(x)}\right), & \forall (x,\xi_0) \in D^\lambda \times \mathbb{R}^d, \\ B_2^{\frac{1}{2}}(\xi_0) \le N|\xi_0|^2, & \forall \xi_0 \in \mathbb{R}^d. \end{cases}$$

We then have the desired last assertion.

Now, we are ready to complete the proof of Theorem 4.2.

Proof of Theorem 4.2. Step 1. By Itô's formula, we have

$$d\left(|Y_{t}^{\delta} - Y_{t}|^{2}e^{2\beta t}\right) = e^{2\beta t}2\beta|Y_{t}^{\delta} - Y_{t}|^{2}dt + e^{2\beta t}2\left\langle(Y_{t}^{\delta} - Y_{t}), \left[-f(X_{t}^{\delta}, Y_{t}^{\delta}, Z_{t}^{\delta})(1 + 2\delta r_{t}) - \tilde{Z}_{t}^{\delta}\delta\pi_{t} + f(X_{t}, Y_{t}, Z_{t})\right]\right\rangle dt + e^{2\beta t}\left\|\tilde{Z}_{t}^{\delta} - Z_{t}\right\|^{2}dt + e^{2\beta t}2\left\langle\left(Y_{t}^{\delta} - Y_{t}\right), \left(\tilde{Z}_{t}^{\delta} - Z_{t}\right)dW_{t}\right\rangle, \quad t \in [0, \gamma].$$
(4.17)

Then we calculate by parts:

First, in view of assumptions (H4) and (H5), and Cauchy inequality, we have for $\epsilon \in (0, 1)$

$$\begin{split} & e^{2\beta t} 2\left\langle \left(Y_{t}^{\delta}-Y_{t}\right), \left[-f(X_{t}^{\delta},Y_{t}^{\delta},Z_{t}^{\delta})\left(1+2\delta r_{t}\right)\right]\right\rangle \\ \geq & e^{2\beta t}\left(1+2\delta r_{t}\right)\left[2\mu|Y_{t}^{\delta}-Y_{t}|^{2}-2L_{0}|Y_{t}^{\delta}-Y_{t}|\|Z_{t}^{\delta}-Z_{t}\|-2\left\langle (Y_{t}^{\delta}-Y_{t}),f(X_{t}^{\delta},Y_{t},Z_{t})\right\rangle\right] \\ \geq & e^{2\beta t}(1+2\delta r_{t})|Y_{t}^{\delta}-Y_{t}|^{2}\left(2\mu-2(1+\epsilon)L_{0}^{2}\right)-\frac{1}{2(1+\epsilon)}e^{2\beta t}(1+2\delta r_{t})\|Z_{t}^{\delta}-Z_{t}\|^{2} \\ & -e^{2\beta t}2(1+2\delta r_{t})\left\langle (Y_{t}^{\delta}-Y_{t}),f(X_{t}^{\delta},Y_{t},Z_{t})\right\rangle. \end{split}$$

By Cauchy inequality, we have for $\epsilon_1 \in (0, 1)$

$$-e^{2\beta t}2\left\langle (Y_t^{\delta}-Y_t), \tilde{Z}_t^{\delta}\delta\pi_t \right\rangle \ge -e^{2\beta t}\epsilon_1|Y_t^{\delta}-Y_t|^2 - \frac{1}{\epsilon_1}e^{2\beta t}|\delta\pi_t|^2 \|\tilde{Z}_t^{\delta}\|^2.$$

Let f be continuously differentiable and have bounded partial derivatives with respect to (x, y, z) (see Remark 4.1). By Cauchy inequality, we have for $\epsilon_2, \epsilon_3 \in (0, 1)$

$$\begin{split} &-e^{2\beta t}2(1+2\delta r_t)\left\langle (Y_t^{\delta}-Y_t),f(X_t^{\delta},Y_t,Z_t)\right\rangle + e^{2\beta t}2\left\langle (Y_t^{\delta}-Y_t),f(X_t,Y_t,Z_t)\right\rangle \\ &= -e^{2\beta t}4\delta r_t\left\langle (Y_t^{\delta}-Y_t),f(X_t^{\delta},Y_t,Z_t)\right\rangle \\ &-e^{2\beta t}2\left\langle (Y_t^{\delta}-Y_t),\left[f(X_t^{\delta},Y_t,Z_t)-f(X_t,Y_t,Z_t)\right]\right\rangle \\ &\geq -e^{2\beta t}4|\delta r_t|\cdot|Y_t^{\delta}-Y_t|\left[|f(X_t^{\delta},0,0)|+L|Y_t|+L_0||Z_t||\right] \\ &-e^{2\beta t}2||f(\cdot)||_{0,1}\cdot|Y_t^{\delta}-Y_t|\cdot|X_t^{\delta}-X_t| \\ &\geq -e^{2\beta t}(\epsilon_2+\epsilon_3)|Y_t^{\delta}-Y_t|^2 - \frac{16}{\epsilon_2}e^{2\beta t}\delta^2 r_t^2\left(L^2|Y_t|^2+L_0^2||Z_t||^2\right) - \frac{8}{\epsilon_2}e^{2\beta t}\delta^2 r_t^2|f(\cdot,0,0)|_0^2 \\ &-\frac{||f(\cdot)||_{0,1}^2}{\epsilon_3}e^{2\beta t}|X_t^{\delta}-X_t|^2. \end{split}$$

Second, by Cauchy inequality, we have

$$\begin{aligned} &-\frac{1}{2(1+\epsilon)}e^{2\beta t}(1+2\delta r_t)\left\|Z_t^{\delta}-Z_t\right\|^2\\ &= -\frac{1}{2(1+\epsilon)}e^{2\beta t}(1+2\delta r_t)\left\|\tilde{Z}_t^{\delta}-Z_t-\tilde{Z}_t^{\delta}+Z_t^{\delta}\right\|^2\\ &\geq -\frac{(1+2\delta r_t)}{1+\epsilon}e^{2\beta t}\left\|\tilde{Z}_t^{\delta}-Z_t\right\|^2 - \frac{(1+2\delta r_t)}{1+\epsilon}e^{2\beta t}\left\|Z_t^{\delta}\right\|^2\left|\sqrt{1+2\delta r_t}e^{\delta P_t}-1\right|^2,\end{aligned}$$

where

$$(1+2\delta r_t) \|Z_t^{\delta}\|^2 \left| \sqrt{1+2\delta r_t} e^{\delta P_t} - 1 \right|^2 \le 2 \|\tilde{Z}_t^{\delta}\|^2 \left[(\sqrt{1+2\delta r_t} - 1)^2 + (e^{\delta P_t} - 1)^2 \right]$$

$$\le N \|\tilde{Z}_t^{\delta}\|^2 \left(\delta^2 r_t^2 + \delta^2 \|P_t\|^2 \right) + o(\delta^3).$$

Third, condition (2.7) is satisfied for $(x,t) \in D_{\delta_1}^{\lambda} \times [0,\gamma_1]$. For fixed $(x,\xi_0) \in D_{\delta_1}^{\lambda} \times \mathbb{R}^d$, we can always choose a small δ such that $x + \delta\xi_0 \in D_{\delta_1}^{\lambda}$. In view of Lemma 2.2, Assertion (iii) in Lemma 3.2, Proposition 4.3 and Hölder inequality, we have

$$\begin{split} & E_{x,\xi_0} \int_0^{\gamma_1} |(r_s,\pi_s,P_s)|^2 |(Y_s,Z_s,Y_s^{\delta},\tilde{Z}_s^{\delta})|^2 e^{2\beta s} ds \\ & \leq \quad E_{x,\xi_0} \left[\sup_{0 \le t \le \gamma_1} |(r_t,\pi_t,P_t)|^2 \int_0^{\gamma_1} |(Y_s,Z_s,Y_s^{\delta},\tilde{Z}_s^{\delta})|^2 ds \right] \\ & \leq \quad E_{x,\xi_0} \left[\sup_{0 \le t \le \gamma_1} |(r_t,\pi_t,P_t)|^4 \right]^{\frac{1}{2}} \cdot E_{x,\xi_0} \left[\left(\int_0^{\gamma_1} |(Y_s,Z_s,Y_s^{\delta},\tilde{Z}_s^{\delta})|^2 ds \right)^2 \right]^{\frac{1}{2}} \\ & \leq \quad NB_1^{\frac{1}{2}}(x,\xi_0) (|g|_0^2 + |f(\cdot,0,0)|_0^2), \quad (x,\xi_0) \in D_{\delta_1}^{\lambda} \times \mathbb{R}^d. \end{split}$$

Step 2. Choosing δ small enough, by (*H*7), we have for $t \in [0, \gamma_1]$

$$2\beta - (\epsilon_1 + \epsilon_2 + \epsilon_3) + (1 + 2\delta r_t) \left[2\mu - 2(1 + \epsilon)L_0^2 \right] > 0, \quad and \quad \left| \frac{1 + 2\delta r_t}{1 + \epsilon} \right| < 1.$$

Combining all the above estimates, we have for $(x, \xi_0, t) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d \times [0, \gamma_1]$,

$$E_{x,\xi_{0}}\left[|Y_{t}^{\delta}-Y_{t}|^{2}e^{2\beta t}\right] + E_{x,\xi_{0}}\int_{t}^{\gamma_{1}}e^{2\beta s}|Y_{s}^{\delta}-Y_{s}|^{2}ds + E_{x,\xi_{0}}\int_{t}^{\gamma_{1}}e^{2\beta s}\|\tilde{Z}_{s}^{\delta}-Z_{s}\|^{2}ds$$

$$\leq E_{x,\xi_{0}}\left[|Y_{\gamma_{1}}^{\delta}-Y_{\gamma_{1}}|^{2}e^{2\beta\gamma_{1}}\right] + E_{x,\xi_{0}}\int_{t}^{\gamma_{1}}\left[\frac{1}{\epsilon_{1}}e^{2\beta s}\delta^{2}|\pi_{s}|^{2}\|\tilde{Z}_{s}^{\delta}\|^{2} + \frac{16}{\epsilon_{2}}e^{2\beta s}\delta^{2}r_{s}^{2}(L^{2}|Y_{s}|^{2} + L_{0}^{2}\|Z_{s}\|^{2}) + \frac{N}{1+\epsilon}e^{2\beta s}\delta^{2}\|\tilde{Z}_{s}^{\delta}\|^{2}(r_{s}^{2} + \|P_{s}\|^{2}) + \frac{\|f(\cdot)\|_{0,1}^{2}}{\epsilon_{3}}e^{2\beta s}|X_{s}^{\delta}-X_{s}|^{2} + \frac{8}{\epsilon_{2}}e^{2\beta s}\delta^{2}r_{s}^{2}|f(\cdot,0,0)|_{0}^{2}]ds + o(\delta^{3})$$

$$\leq E_{x,\xi_{0}}\left[|Y_{\gamma_{1}}^{\delta}-Y_{\gamma_{1}}|^{2}e^{2\beta\gamma_{1}}\right] + N\delta^{2}E_{x,\xi_{0}}\int_{t}^{\gamma_{1}}|(r_{s},\pi_{s},P_{s})|^{2}|(Y_{s},Z_{s},\tilde{Z}_{s}^{\delta})|^{2}ds + E_{x,\xi_{0}}\int_{t}^{\gamma_{1}}\frac{\|f(\cdot)\|_{0,1}^{2}}{\epsilon_{3}}e^{2\beta s}|X_{s}^{\delta}-X_{s}|^{2}ds + N|f(\cdot,0,0)|_{0}^{2}E\int_{t}^{\gamma_{1}}\delta^{2}\left(|\xi_{t}|^{2}+\frac{\psi_{(\xi_{t})}^{2}}{\psi^{2}}\right)ds + o(\delta^{3})$$

$$\leq E_{x,\xi_{0}}\left[|Y_{\gamma_{1}}^{\delta}-Y_{\gamma_{1}}|^{2}e^{2\beta\gamma_{1}}\right] + E_{x,\xi_{0}}\int_{t}^{\gamma_{1}}\frac{\|f(\cdot)\|_{0,1}^{2}}{\epsilon_{3}}e^{2\beta s}|X_{s}^{\delta}-X_{s}|^{2}ds + N|f(\cdot,0,0)|_{0}^{2}E\int_{t}^{\gamma_{1}}\delta^{2}\left(|\xi_{t}|^{2}+\frac{\psi_{(\xi_{t})}^{2}}{\psi^{2}}\right)ds + o(\delta^{3})$$

$$\leq E_{x,\xi_{0}}\left[|Y_{\gamma_{1}}^{\delta}-Y_{\gamma_{1}}|^{2}e^{2\beta\gamma_{1}}\right] + E_{x,\xi_{0}}\int_{t}^{\gamma_{1}}\frac{\|f(\cdot)\|_{0,1}^{2}}{\epsilon_{3}}e^{2\beta s}|X_{s}^{\delta}-X_{s}|^{2}ds + N\delta^{2}B_{1}^{\frac{1}{2}}(x,\xi_{0})(|g|_{0}^{2}+|f(\cdot,0,0)|_{0}^{2}) + o(\delta^{3}).$$
(4.18)

By Cauchy inequality, we have

$$E_{x,\xi_0} \int_t^{\gamma_1} e^{2\beta s} |X_s^{\delta} - X_s|^2 ds \le E_{x,\xi_0} \int_t^{\gamma_1} 2\left(\delta^2 \left|\frac{X_s^{\delta} - X_s}{\delta} - \xi_s\right|^2 + \delta^2 |\xi_s|^2\right) ds.$$

Dividing δ^2 in both sides and let $\delta \to 0$, due to Dominated convergence theorem, formula (2.13) and Assertion (iii) in Lemma 3.2, we know that

$$\lim_{\delta \to 0} \frac{1}{\delta^2} E_{x,\xi_0} \int_t^{\gamma_1} \|f(\cdot)\|_{0,1}^2 \epsilon_3^{-1} e^{2\beta s} |X_s^{\delta} - X_s|^2 ds \le N B_1^{\frac{1}{2}}(x,\xi_0) \|f(\cdot)\|_{0,1}^2.$$
(4.19)

For t = 0, we have

$$\lim_{\delta \to 0} \left| \frac{Y_0^{\delta} - Y_0}{\delta} \right| = |u_{(\xi_0)}(x)|.$$
(4.20)

In view of formulas (4.5), (4.18), (4.19) and (4.20), we have

$$\frac{|u_{(\xi_0)}(x)|^2}{B_1^{\frac{1}{2}}(x,\xi_0)} \le \sup_{(y,\zeta)\in\partial D_{\delta_1}^{\lambda}\times\mathbb{R}^d\setminus\{0\}} \frac{|u_{(\zeta)}(y)|^2}{\sqrt{B_1(y,\zeta)}} + N(|g|_0^2 + \|f(\cdot)\|_{0,1}^2), \quad \forall (x,\xi_0)\in D_{\delta_1}^{\lambda}\times\mathbb{R}^d\setminus\{0\}.$$
(4.21)

Step 3. In view of Assertion (iii) in Lemma 3.4, we have for $(x, \xi_0, t) \in D_{\lambda^2} \times \mathbb{R}^d \times [0, \gamma_2]$

$$\begin{split} & E_{x,\xi_0} \left[|Y_t^{\delta} - Y_t|^2 e^{2\beta t} \right] + E_{x,\xi_0} \int_t^{\gamma_2} e^{2\beta s} |Y_s^{\delta} - Y_s|^2 ds + E_{x,\xi_0} \int_t^{\gamma_2} e^{2\beta s} \|\tilde{Z}_s^{\delta} - Z_s\|^2 ds \\ & \leq E_{x,\xi_0} \left[|Y_{\gamma_2}^{\delta} - Y_{\gamma_2}|^2 e^{2\beta \gamma_2} \right] + E_{x,\xi_0} \int_t^{\gamma_2} [\frac{1}{\epsilon_1} e^{2\beta s} \delta^2 |\pi_s|^2 \|\tilde{Z}_s^{\delta}\|^2 + \frac{16}{\epsilon_2} e^{2\beta s} \delta^2 r_s^2 (L^2 |Y_s|^2 + L_0^2 \|Z_s\|^2) \\ & \quad + \frac{N}{1+\epsilon} e^{2\beta s} \delta^2 \|\tilde{Z}_s^{\delta}\|^2 (r_s^2 + \|P_s\|^2) + \frac{\|f(\cdot)\|_{0,1}^2}{\epsilon_3} e^{2\beta s} |X_s^{\delta} - X_s|^2 \\ & \quad + \frac{8}{\epsilon_2} e^{2\beta s} \delta^2 r_s^2 |f(X_s^{\delta}, 0, 0)|^2] ds + o(\delta^3) \\ & \leq E_{x,\xi_0} \left[|Y_{\gamma_2}^{\delta} - Y_{\gamma_2}|^2 e^{2\beta \gamma_2} \right] + E_{x,\xi_0} \int_t^{\gamma_2} \frac{\|f(\cdot)\|_{0,1}^2}{\epsilon_3} e^{2\beta s} |X_s^{\delta} - X_s|^2 ds \\ & \quad + N \delta^2 E_{x,\xi_0} \int_t^{\gamma_2} e^{2\beta s} |(r_s, \pi_s, P_s)|^2| (Y_s, Z_s, \tilde{Z}_s^{\delta})|^2 ds \\ & \quad + |f(\cdot, 0, 0)|_0^2 E_{x,\xi_0} \int_t^{\gamma_2} \delta^2 e^{2\beta s} |\xi_t|^2 ds + o(\delta^3) \\ & \leq E_{x,\xi_0} \left[|Y_{\gamma_2}^{\delta} - Y_{\gamma_2}|^2 e^{2\beta \gamma_2} \right] + E_{x,\xi_0} \int_t^{\gamma_2} \frac{\|f(\cdot)\|_{0,1}^2}{\epsilon_3} e^{2\beta s} |X_s^{\delta} - X_s|^2 ds \\ & \quad + N \delta^2 E_{x,\xi_0} \int_t^{\gamma_2} \delta^2 e^{2\beta s} |\xi_t|^2 ds + o(\delta^3) \\ & \leq E_{x,\xi_0} \left[|Y_{\gamma_2}^{\delta} - Y_{\gamma_2}|^2 e^{2\beta \gamma_2} \right] + E_{x,\xi_0} \int_t^{\gamma_2} \frac{\|f(\cdot)\|_{0,1}^2}{\epsilon_3} e^{2\beta s} |X_s^{\delta} - X_s|^2 ds \\ & \quad + N \delta^2 B_2^{\frac{1}{2}}(\xi_0) (|g|_0^2 + |f(\cdot, 0, 0)|_0^2) + o(\delta^3). \end{split}$$

Then repeating the arguments in Step 2, and using formula (4.6), we conclude that

$$\frac{|u_{(\xi_0)}(x)|^2}{B_2^{\frac{1}{2}}(\xi_0)} \le \sup_{(y,\zeta)\in\partial D_{\lambda^2}\times\mathbb{R}^d\setminus\{0\}} \frac{|u_{(\zeta)}(y)|^2}{B_2^{\frac{1}{2}}(\zeta)} + N(|g|_0^2 + \|f(\cdot)\|_{0,1}^2), \quad \forall (x,\xi_0)\in D_{\lambda^2}\times\mathbb{R}^d\setminus\{0\}.$$
(4.22)

Step 4. Finally, due to Lemma 4.7 and Remark 4.1, we have for $\xi_0 \in \mathbb{R}^d$,

$$|u_{(\xi_0)}(x)| \le N\left(|\xi_0| + \frac{|\psi_{(\xi_0)}(x)|}{\psi^{\frac{3}{4}}(x)}\right) (|g|_{0,1} + ||f(\cdot)||_{0,1}), \quad a.e. \ in \ D.$$

$$(4.23)$$

We can prove (4.1) using the same arguments in **Steps 2** and **3** and the BDG inequality. The proof is complete. \Box

Before ending this section, we provide some estimates as follows which play an important role in the next subsection. We emphasize that when \tilde{r} is not vanishing, Theorem 4.2 still holds.

Corollary 4.8. Let (Y, Z) be the unique solution of BSDE (1.3) and (Y^{δ}, Z^{δ}) be the unique solution of FBSDE (2.12), with $(\tilde{\pi}_t, \tilde{P}_t)$ vanishing. Let assumptions (H1)-(H5) and (H7) be satisfied. If $f \in C^1(\overline{D} \times \mathbb{R}^k \times \mathbb{R}^{k \times d_1})$ with bounded partial derivatives and $g, u \in C^1(\overline{D})$, we have for sufficiently small $\delta > 0$, there exists a constant $N = N(K_0, d, d_1, k, D, \mu, L, L_0)$ such that

$$E_{x,\xi_{0}}\left[\left(\int_{0}^{\gamma_{1}}e^{2\beta t}|Y_{t}^{\delta}-Y_{t}|^{2}dt\right)^{2}+\left(\int_{0}^{\gamma_{1}}e^{2\beta t}\|\tilde{Z}_{t}^{\delta}-Z_{t}\|^{2}dt\right)^{2}+\sup_{0\leq t\leq\gamma_{1}}(e^{4\beta t}|Y_{t}^{\delta}-Y_{t}|^{4})\right]$$

$$\leq N\delta^{4}B_{3}^{\frac{1}{2}}(x,\xi_{0})(|g|_{1}^{4}+\|f(\cdot)\|_{0,1}^{4})+\delta^{4}o(\delta,n,T), \quad \forall (x,\xi_{0})\in D_{\delta_{1}}^{\lambda}\times\mathbb{R}^{d},$$
(4.24)

and

$$E_{x,\xi_{0}}\left[\left(\int_{0}^{\gamma_{2}}e^{2\beta t}|Y_{t}^{\delta}-Y_{t}|^{2}dt\right)^{2}+\left(\int_{0}^{\gamma_{2}}e^{2\beta t}\|\tilde{Z}_{t}^{\delta}-Z_{t}\|^{2}dt\right)^{2}+\sup_{0\leq t\leq\gamma_{2}}(e^{4\beta t}|Y_{t}^{\delta}-Y_{t}|^{4})\right]$$

$$\leq N\delta^{4}B_{4}^{\frac{1}{2}}(\xi_{0})(|g|_{1}^{4}+\|f(\cdot)\|_{0,1}^{4})+\delta^{4}o(\delta,n,T), \quad \forall (x,\xi_{0})\in D_{\lambda^{2}}\times\mathbb{R}^{d},$$
(4.25)

where $o(\delta, n, T)$ is an infinitesimal as first $\delta \to 0$ and then $T, n \to \infty$.

4.2 Interior Hessian Estimate

In this subsection, let (X, Y, Z) be the unique solution of (2.3) and (2.5), (ξ, η) be the unique solution of (2.8) and (2.9), and $(X^{\delta}, Y^{\delta}, Z^{\delta})$ be the unique solution of FBSDE (2.12). Define u by (1.2). Choose $(\eta_0, \tilde{\pi}_t, \tilde{P}_t)$ to be vanishing for simplicity.

Theorem 4.9. Let assumptions (H1)-(H5), $(H6)_1$, (H7)-(H9) and $(H10)_2$ be satisfied. Then $u \in C^{1,1}_{loc}(D) \cap C^{0,1}(\overline{D})$, such that for $\xi_0 \in \mathbb{R}^d$ and a.e. $x \in D$,

$$|u_{(\xi_0)(\xi_0)}(x)| \le N\left(|\xi_0|^2 + \frac{\psi_{(\xi_0)}^2(x)}{\psi^{\frac{7}{4}}(x)}\right) [|g|_{1,1} + ||f(\cdot)||_{0,1} + [f]_{1,1}(1 + |g|_1^2 + ||f(\cdot)||_{0,1}^2)]$$

where $N = N(K_0, d, d_1, k, D, L_0, L, \mu)$.

The following lemma is analogous to [21, Lemma 3.2].

Lemma 4.10. Let $f \in C^1(\overline{D} \times \mathbb{R}^k \times \mathbb{R}^{k \times d_1})$ with bounded partial derivatives, $g \in C^2(\overline{D})$, $u \in C^1(\overline{D})$, (H4), (H5), (H7) and $E\tau(x) \leq \psi(x)$, $x \in D$ be satisfied. Then we have

$$|u_{(n)}(y)| \le N(|g|_2 + |f(\cdot, 0, 0)|_0), \quad \forall y \in \partial D,$$

where n := n(y) is the unitary inward normal on ∂D and the positive constant N depends on the quadruple (K_0, L_0, L, μ) . Note that N does not depend on $|u_x|_0$.

Proof. Fix a $y \in \partial D$, and choose $\epsilon_0 > 0$ so that $x := y + \epsilon n \in D$ for $0 < \epsilon \leq \epsilon_0$. Set $\tilde{Y}_t := Y_t - g(X_t)$ and $\tilde{Z}_t := Z_t - \nabla g(X_t)\sigma(X_t)$, for $t \in [0, \tau]$, where X is the unique solution to (2.3) and (Y, Z) is the unique solution to (2.5). As $g \in C^2(\overline{D})$, (Y, \tilde{Z}) is the unique solution to the BSDE

$$\begin{cases} d\tilde{Y}_{t} = [-f(X_{t}, \tilde{Y}_{t} + g(X_{t}), \tilde{Z}_{t} + \nabla g(X_{t})\sigma(X_{t})) - \mathcal{L}g(X_{t})]dt + \tilde{Z}_{t}dW_{t}, \quad t \in [0, \tau), \\ \tilde{Y}_{\tau} = 0. \end{cases}$$
(4.26)

With the analogue of (2.6), we have

$$E\left[\sup_{0\le t\le \tau} |\tilde{Y}_t|^2\right] + E\int_0^\tau \|\tilde{Z}_t\|^2 dt \le N(|g|_2^2 + |f(\cdot, 0, 0)|_0^2)\psi(x).$$
(4.27)

Then, by formulas (1.2), (4.26) and (4.27), we have

$$u(x) = g(x) + E_x \int_0^\tau \mathcal{L}g(X_t) + f(X_t, \tilde{Y}_t + g(X_t), \tilde{Z}_t + \nabla g(X_t)\sigma(X_t))dt$$

$$\leq g(x) + N(|g|_2 + |f(\cdot, 0, 0)|_0)\psi(x).$$

Since u(y) = g(y) and $\psi(y) = 0$, we have

$$\frac{u(y+\epsilon n)-u(y)}{\epsilon} \leq \frac{g(y+\epsilon n)-g(y)}{\epsilon} + N(|g|_2 + |f(\cdot,0,0)|_0) \left[\frac{\psi(y+\epsilon n)-\psi(y)}{\epsilon}\right].$$

Letting $\epsilon \to 0$, we get

$$u_{(n)}(y) \le N(|g|_2 + |f(\cdot, 0, 0)|_0)$$

Replacing u with -u yields an estimate of $u_{(n)}$ from below. The proof is complete.

Now we are ready to prove Theorem 4.9.

Proof of Theorem 4.9. Set $\nabla X_t^{\delta} := X_t^{\delta} - X_t$ and $\nabla X_t^{-\delta} := X_t - X_t^{-\delta}$. Define $\nabla Y_t^{\delta}, \nabla Y_t^{-\delta}, \nabla Z_t^{\delta}$ and $\nabla Z_t^{-\delta}$ in the same way. Set $\bar{\gamma} := \tau \wedge \tau^{\delta} \wedge \tau^{-\delta} \wedge k_n \wedge k'_n \wedge T$, where $k_n := \inf\{t \ge 0; |\xi_t| \ge n\}$, $k'_n := \inf\{t \ge 0; |\eta_t| \ge n\}$ and $T \in [1, \infty)$. **Step 1.** Set $\bar{\gamma}_1 := \bar{\gamma} \wedge \tau_1$. Write

$$\begin{split} \mathrm{II}_{1} &:= 2 \langle (\nabla Y_{t}^{\delta} - \nabla Y_{t}^{-\delta}), (2\delta r_{t}) [f(X_{t}^{\delta}, Y_{t}^{\delta}, Z_{t}^{\delta}) - f(X_{t}^{-\delta}, Y_{t}^{-\delta}, Z_{t}^{-\delta})], \\ \mathrm{II}_{2} &:= 2 \langle (\nabla Y_{t}^{\delta} - \nabla Y_{t}^{-\delta}), (\delta^{2} \tilde{r}_{t}) [f(X_{t}^{\delta}, Y_{t}^{\delta}, Z_{t}^{\delta}) + f(X_{t}^{-\delta}, Y_{t}^{-\delta}, Z_{t}^{-\delta})] \rangle, \\ \mathrm{II}_{3} &:= 2 \langle (\nabla Y_{t}^{\delta} - \nabla Y_{t}^{-\delta}), (\tilde{Z}_{t}^{\delta} - \tilde{Z}_{t}^{-\delta}) \delta \pi_{t} \rangle, \\ \mathrm{II}_{4} &:= 2 \langle (\nabla Y_{t}^{\delta} - \nabla Y_{t}^{-\delta}), [f(X_{t}^{\delta}, Y_{t}^{\delta}, Z_{t}^{\delta}) - 2f(X_{t}, Y_{t}, Z_{t}) + f(X_{t}^{-\delta}, Y_{t}^{-\delta}, Z_{t}^{-\delta})] \rangle. \end{split}$$

By Itô's formula, we have for $(x, \xi_0, t) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d \times [0, \overline{\gamma}_1]$

$$\begin{aligned} d\left(\left|\nabla Y_{t}^{\delta}-\nabla Y_{t}^{-\delta}\right|^{2}e^{4\beta t}\right) \\ &= -(\mathrm{II}_{1}+\mathrm{II}_{2}+\mathrm{II}_{3}+\mathrm{II}_{4})e^{4\beta t}dt+4\beta\left|\nabla Y_{t}^{\delta}-\nabla Y_{t}^{-\delta}\right|^{2}e^{4\beta t}dt \\ &+\left\|\tilde{Z}_{t}^{\delta}-2Z_{t}+\tilde{Z}_{t}^{-\delta}\right\|^{2}e^{4\beta t}dt+2\left\langle\left(\nabla Y_{t}^{\delta}-\nabla Y_{t}^{-\delta}\right),\left(\tilde{Z}_{t}^{\delta}-2Z_{t}+\tilde{Z}_{t}^{-\delta}\right)e^{4\beta t}dW_{t}\right\rangle.\end{aligned}$$

First, we estimate the term II₁. Let $f \in C^1(\overline{D} \times \mathbb{R}^k \times \mathbb{R}^{k \times d_1})$ with bounded partial derivatives and (H4) be satisfied. By Cauchy inequality, we have for $\epsilon \in (0, 1)$

$$\begin{aligned} \mathrm{II}_{1} &\leq \epsilon |\nabla Y_{t}^{\delta} - \nabla Y_{t}^{-\delta}|^{2} + 4\delta^{2}r_{t}^{2}\epsilon^{-1}|f(X_{t}^{\delta}, Y_{t}^{\delta}, Z_{t}^{\delta}) - f(X_{t}^{-\delta}, Y_{t}^{-\delta}, Z_{t}^{-\delta})|^{2} \\ &\leq \epsilon |\nabla Y_{t}^{\delta} - \nabla Y_{t}^{-\delta}|^{2} + N\delta^{2}r_{t}^{2}||f(\cdot)||_{0,1}^{2}\epsilon^{-1}|X_{t}^{\delta} - X_{t}^{-\delta}|^{2} \\ &+ N\delta^{2}r_{t}^{2}\epsilon^{-1}(|Y_{t}^{\delta} - Y_{t}^{-\delta}|^{2} + ||Z_{t}^{\delta} - Z_{t}^{-\delta}||^{2}). \end{aligned}$$

In view of the estimates in Lemmas 3.1 and 3.5 and formula (2.13), we have for $(x, \xi_0) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d$,

$$E_{x,\xi_{0}}\left[\left(\int_{0}^{\bar{\gamma}_{1}}|X_{t}^{\delta}-X_{t}^{-\delta}|^{2}dt\right)^{2}\right]$$

$$\leq NE_{x,\xi_{0}}\left[\left(\int_{0}^{\bar{\gamma}_{1}}(|X_{t}^{\delta}-X_{t}|^{2}+|X_{t}-X_{t}^{-\delta}|^{2})dt\right)^{2}\right]$$

$$\leq NE_{x,\xi_{0}}\left[\sup_{0\leq t\leq \bar{\gamma}_{1}}|X_{t}^{\delta}-X_{t}|^{4}\cdot\bar{\gamma}_{1}^{2}\right]$$

$$\leq N\delta^{4}E_{x,\xi_{0}}\left[\sup_{0\leq t\leq \bar{\gamma}_{1}}\left|\frac{X_{t}^{\delta}-X_{t}}{\delta}-\xi_{t}\right|^{8}\right]^{\frac{1}{2}}+N\delta^{4}E_{\xi_{0}}\left[\sup_{0\leq t\leq \bar{\gamma}_{1}}|\xi_{t}|^{8}\right]^{\frac{1}{2}}$$

$$\leq N\delta^{4}B_{3}^{\frac{1}{2}}(x,\xi_{0})+o(\delta^{5}). \qquad (4.28)$$

In view of formula (4.24), we have for $(x, \xi_0) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d$,

$$E_{x,\xi_{0}}\left[\left(\int_{0}^{\tilde{\gamma}_{1}}|Y_{t}^{\delta}-Y_{t}^{-\delta}|^{2}e^{2\beta t}dt\right)^{2}\right] \leq NE_{x,\xi_{0}}\left[\left(\int_{0}^{\tilde{\gamma}_{1}}|Y_{t}^{\delta}-Y_{t}|^{2}e^{2\beta t}dt\right)^{2}\right] \\ \leq N\delta^{4}B_{3}^{\frac{1}{2}}(x,\xi_{0})(|g|_{1}^{4}+||f(\cdot)||_{0,1}^{4})+\delta^{4}o(\delta,n,T), \quad (4.29)$$

and

$$E_{x,\xi_{0}}\left[\left(\int_{0}^{\bar{\gamma}_{1}} \|Z_{t}^{\delta} - Z_{t}^{-\delta}\|^{2} e^{2\beta t} dt\right)^{2}\right]$$

$$\leq NE_{x,\xi_{0}}\left[\left(\int_{0}^{\bar{\gamma}_{1}} (\|Z_{t}^{\delta} - \tilde{Z}_{t}^{\delta}\|^{2} + \|\tilde{Z}_{t}^{\delta} - Z_{t}\|^{2} e^{2\beta t}) dt\right)^{2}\right]$$

$$\leq NE_{x,\xi_{0}}\left[\left(\int_{0}^{\bar{\gamma}_{1}} \|Z_{t}^{\delta} - \tilde{Z}_{t}^{\delta}\|^{2} dt\right)^{2}\right] + N\delta^{4}B_{3}^{\frac{1}{2}}(x,\xi_{0})(|g|_{1}^{4} + \|f(\cdot)\|_{0,1}^{4}) + \delta^{4}o(\delta, n, T). \quad (4.30)$$

Using Taylor expansion to deal with the first term in the preceding inequality, we have

$$1 - (1 + 2\delta r_t + \delta^2 \tilde{r}_t)^{\frac{1}{2}} e^{\delta P_t}$$

$$= \left[1 - (1 + 2\delta r_t + \delta^2 \tilde{r}_t)^{\frac{1}{2}}\right] + (1 + 2\delta r_t + \delta^2 \tilde{r}_t)^{\frac{1}{2}} \left[1 - e^{\delta P_t}\right]$$

$$= \left[-\frac{1}{2}(2\delta r_t + \delta^2 \tilde{r}_t) + \frac{1}{8}(2\delta r_t + \delta^2 \tilde{r}_t)^2 + o(\delta^3)\right]$$

$$+ (1 + 2\delta r_t + \delta^2 \tilde{r}_t)^{\frac{1}{2}} \left[-\delta P_t - \frac{1}{2}\delta^2 P_t^2 + o(\delta^3)\right].$$
(4.31)

In view of Assertion (ii) in Lemma 3.5 and Proposition 4.3, for sufficiently small δ and $t \in [0, \bar{\gamma}_1]$, we have

$$E_{x,\xi_{0}}\left[\left(\int_{0}^{\tilde{\gamma}_{1}} \|Z_{t}^{\delta} - \tilde{Z}_{t}^{\delta}\|^{2} dt\right)^{2}\right]$$

$$= E_{x,\xi_{0}}\left[\left(\int_{0}^{\tilde{\gamma}_{1}} \|Z_{t}^{\delta}\|^{2} (1 - (1 + 2\delta r_{t} + \delta^{2}\tilde{r}_{t})^{\frac{1}{2}} e^{\delta P_{t}})^{2} dt\right)^{2}\right]$$

$$= E_{x,\xi_{0}}\left[\left(\int_{0}^{\tilde{\gamma}_{1}} \|Z_{t}^{\delta}\|^{2} (\delta^{2}|(r_{t}, P_{t})|^{2} + o(\delta^{3})) dt\right)^{2}\right]$$

$$\leq N\delta^{4} E_{x,\xi_{0}}\left[\sup_{0 \le t \le \tilde{\gamma}_{1}} |(r_{t}, P_{t})|^{4} \cdot \left(\int_{0}^{\tilde{\gamma}_{1}} \|\tilde{Z}_{t}^{\delta}\|^{2} dt\right)^{2}\right] + o(\delta^{6})$$

$$\leq N\delta^{4} B_{3}^{\frac{1}{2}}(x,\xi_{0}) (|g|_{0}^{4} + |f(\cdot, 0, 0)|_{0}^{4}) + o(\delta^{6}). \tag{4.32}$$

Collecting the above estimates (4.28), (4.29), (4.30) and (4.32), we have

$$E_{x,\xi_{0}}\left[\left(\int_{0}^{\bar{\gamma}_{1}}e^{2\beta t}|(X_{t}^{\delta},Y_{t}^{\delta},Z_{t}^{\delta})-(X_{t}^{-\delta},Y_{t}^{-\delta},Z_{t}^{-\delta})|^{2}dt\right)^{2}\right]$$

$$\leq N\delta^{4}B_{3}^{\frac{1}{2}}(x,\xi_{0})(1+|g|_{1}^{4}+\|f(\cdot)\|_{0,1}^{4})+\delta^{4}o(\delta,n,T).$$
(4.33)

Thus, in view of Assertion (iii) in Lemma 3.2 and formula (4.33), we have for $(x, \xi_0) \in D_{\delta_1}^{\lambda} \times \mathbb{R}^d$

$$E_{x,\xi_{0}} \int_{t}^{\gamma_{1}} \Pi_{1} e^{4\beta s} ds$$

$$\leq E_{x,\xi_{0}} \int_{t}^{\bar{\gamma}_{1}} \epsilon |\nabla Y_{s}^{\delta} - \nabla Y_{s}^{-\delta}|^{2} e^{4\beta s} ds$$

$$+ E_{x,\xi_{0}} \int_{0}^{\bar{\gamma}_{1}} N \delta^{2} r_{t}^{2} \epsilon^{-1} [\|f(\cdot)\|_{0,1}^{2} |X_{t}^{\delta} - X_{t}^{-\delta}|^{2} + |(Y_{t}^{\delta}, Z_{t}^{\delta}) - (Y_{t}^{-\delta}, Z_{t}^{-\delta})|^{2}] e^{4\beta t} dt$$

$$\leq E_{x,\xi_{0}} \int_{t}^{\bar{\gamma}_{1}} \epsilon |\nabla Y_{s}^{\delta} - \nabla Y_{s}^{-\delta}|^{2} e^{4\beta s} ds + N \delta^{4} B_{3}^{\frac{1}{2}}(x,\xi_{0}) (|g|_{1}^{2} + \|f(\cdot)\|_{0,1}^{2}) + \delta^{4} o(\delta, n, T).$$

$$(4.34)$$

Second, we estimate the term II₂. By (H4) and Cauchy inequality, we have for $\epsilon \in (0, 1)$

$$\begin{split} \mathrm{II}_2 &\leq \quad \epsilon \left| \nabla Y_t^{\delta} - \nabla Y_t^{-\delta} \right|^2 + \delta^4 \tilde{r}_t^2 \epsilon^{-1} \left| f(X_t^{\delta}, Y_t^{\delta}, Z_t^{\delta}) + f(X_t^{-\delta}, Y_t^{-\delta}, Z_t^{-\delta}) \right|^2 \\ &\leq \quad \epsilon \left| \nabla Y_t^{\delta} - \nabla Y_t^{-\delta} \right|^2 + \delta^4 \tilde{r}_t^2 \epsilon^{-1} \left[|f(\cdot, 0, 0)|_0^2 + L_0^2 |Y_t^{\delta}|^2 + L_0^2 ||Z_t^{\delta}||^2 \right]. \end{split}$$

In view of Assertion (ii) in Lemma 3.5 and Proposition 4.3, we have for $(x, \xi_0) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d$

$$E_{x,\xi_{0}} \int_{t}^{\gamma_{1}} \Pi_{2} e^{4\beta s} ds$$

$$\leq E_{x,\xi_{0}} \int_{t}^{\bar{\gamma}_{1}} \epsilon |\nabla Y_{s}^{\delta} - \nabla Y_{s}^{-\delta}|^{2} e^{4\beta s} ds$$

$$+ E_{x,\xi_{0}} \int_{0}^{\bar{\gamma}_{1}} \delta^{4} \tilde{r}_{t}^{2} \epsilon^{-1} \left[|f(\cdot,0,0)|_{0}^{2} + L_{0}^{2} |Y_{t}^{\delta}|^{2} + L_{0}^{2} ||Z_{t}^{\delta}||^{2} \right] e^{4\beta t} dt$$

$$\leq E_{x,\xi_{0}} \int_{t}^{\bar{\gamma}_{1}} \epsilon |\nabla Y_{s}^{\delta} - \nabla Y_{s}^{-\delta}|^{2} e^{4\beta s} ds + N \delta^{4} B_{3}^{\frac{1}{2}}(x,\xi_{0}) (|g|_{0}^{2} + |f(\cdot,0,0)|_{0}^{2}). \quad (4.35)$$

Third, we estimate the term II₃. Using Cauchy inequality, we have for $\epsilon \in (0, 1)$

$$II_3 \le \epsilon \left| \nabla Y_t^{\delta} - \nabla Y_t^{-\delta} \right|^2 + 4\delta^2 |\pi_t|^2 \epsilon^{-1} \left\| \tilde{Z}_t^{\delta} - Z_t \right\|^2.$$

In view of Assertion (ii) in Lemma 3.5 and formula (4.24), we have

$$E_{x,\xi_{0}} \int_{0}^{\bar{\gamma}_{1}} |\pi_{t}|^{2} \left\| \tilde{Z}_{t}^{\delta} - Z_{t} \right\|^{2} e^{4\beta t} dt$$

$$\leq E_{x,\xi_{0}} \left[\sup_{0 \le t \le \bar{\gamma}_{1}} |\pi_{t}|^{4} e^{4\beta t} \right]^{\frac{1}{2}} \cdot E_{x,\xi_{0}} \left[\left(\int_{0}^{\bar{\gamma}_{1}} \left\| \tilde{Z}_{t}^{\delta} - Z_{t} \right\|^{2} e^{2\beta t} dt \right)^{2} \right]^{\frac{1}{2}}$$

$$\leq NB_{3}^{\frac{1}{4}}(x,\xi_{0}) \cdot \left[N\delta^{4}(|g|_{1}^{4} + \|f(\cdot)\|_{0,1}^{4}) B_{3}^{\frac{1}{2}}(x,\xi_{0}) + \delta^{4}o(\delta,n,T) \right]^{\frac{1}{2}}$$

$$\leq N\delta^{2}B_{3}^{\frac{1}{2}}(x,\xi_{0})(|g|_{1}^{2} + \|f(\cdot)\|_{0,1}^{2}) + \delta^{2}o^{\frac{1}{2}}(\delta,n,T), \quad \forall (x,\xi_{0}) \in D_{\delta_{1}}^{\lambda} \times \mathbb{R}^{d}.$$

Hence, we have for $(x,\xi_0) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d$

$$E_{x,\xi_{0}} \int_{t}^{\tilde{\gamma}_{1}} \Pi_{3} e^{4\beta s} ds$$

$$\leq E_{x,\xi_{0}} \int_{t}^{\tilde{\gamma}_{1}} \epsilon |\nabla Y_{s}^{\delta} - \nabla Y_{s}^{-\delta}|^{2} e^{4\beta s} ds + N E_{x,\xi_{0}} \int_{0}^{\tilde{\gamma}_{1}} \delta^{2} |\pi_{t}|^{2} \left\| \tilde{Z}_{t}^{\delta} - Z_{t} \right\|^{2} e^{4\beta t} dt$$

$$\leq E_{x,\xi_{0}} \int_{t}^{\tilde{\gamma}_{1}} \epsilon |\nabla Y_{s}^{\delta} - \nabla Y_{s}^{-\delta}|^{2} e^{4\beta s} ds + N \delta^{4} B_{3}^{\frac{1}{2}}(x,\xi_{0}) (|g|_{1}^{2} + \|f(\cdot)\|_{0,1}^{2}) + \delta^{4} o^{\frac{1}{2}}(\delta, n, T).$$

$$(4.36)$$

Fourth, we estimate the term II_4 . Set

$$\Xi_{\lambda,t}^{\delta} := (X_t + \lambda \nabla X_t^{\delta}, Y_t + \lambda \nabla Y_t^{\delta}, Z_t + \lambda \nabla Z_t^{\delta}).$$

We have

$$\begin{split} f(X_t^{\delta}, Y_t^{\delta}, Z_t^{\delta}) &- 2f(X_t, Y_t, Z_t) + f(X_t^{-\delta}, Y_t^{-\delta}, Z_t^{-\delta}) \\ &= \int_0^1 \left[f_x(\Xi_{\lambda,t}^{\delta}) - f_x(\Xi_{-\lambda,t}^{-\delta}) \right] d\lambda \nabla X_t^{\delta} + \int_0^1 f_x(\Xi_{-\lambda,t}^{-\delta}) d\lambda (\nabla X_t^{\delta} - \nabla X_t^{-\delta}) \\ &+ \int_0^1 \left[f_y(\Xi_{\lambda,t}^{\delta}) - f_y(\Xi_{-\lambda,t}^{-\delta}) \right] d\lambda \nabla Y_t^{\delta} + \int_0^1 f_y(\Xi_{-\lambda,t}^{-\delta}) d\lambda (\nabla Y_t^{\delta} - \nabla Y_t^{-\delta}) \\ &+ \int_0^1 \left[f_z(\Xi_{\lambda,t}^{\delta}) - f_z(\Xi_{-\lambda,t}^{-\delta}) \right] d\lambda \nabla Z_t^{\delta} + \int_0^1 f_z(\Xi_{-\lambda,t}^{-\delta}) d\lambda (\nabla Z_t^{\delta} - \nabla Z_t^{-\delta}). \end{split}$$

Set

$$\begin{split} \Pi_{4.1} &:= |\Xi_{\lambda,t}^{\delta} - \Xi_{-\lambda,t}^{-\delta}|^2 (|\nabla X_t^{\delta}|^2 + |\nabla Y_t^{\delta}|^2) \\ \Pi_{4.2} &:= |\nabla X_t^{\delta} - \nabla X_t^{-\delta}|^2, \\ \Pi_{4.3} &:= 2 \langle (\nabla Y_t^{\delta} - \nabla Y_t^{-\delta}), \int_0^1 [f_z(\Xi_{\lambda,t}^{\delta}) - f_z(\Xi_{-\lambda,t}^{-\delta})] d\lambda \nabla Z_t^{\delta} + \int_0^1 f_z(\Xi_{-\lambda,t}^{-\delta}) d\lambda (\nabla Z_t^{\delta} - \nabla Z_t^{-\delta}) \rangle. \end{split}$$

When $f \in C^2$ with bounded first and second order partial derivatives (see Remark 4.1), we have for $\epsilon \in (0, 1)$

$$II_{4} \leq (3\epsilon - 2\mu) \left| \nabla Y_{t}^{\delta} - \nabla Y_{t}^{-\delta} \right|^{2} + N[f]_{1,1}^{2}II_{4,1} + \|f(\cdot)\|_{0,1}^{2}\epsilon^{-1}II_{4,2} + II_{4,3}.$$

By Hölder's inequality, Assertion (iii) in Lemma 3.2, formulas (2.13) and (4.33), we have

$$E_{x,\xi_{0}} \int_{0}^{\tilde{\gamma}_{1}} \Pi_{4,1} e^{4\beta t} dt$$

$$\leq E_{x,\xi_{0}} \left[\sup_{0 \le t \le \tilde{\gamma}_{1}} \left(|\nabla X_{t}^{\delta}|^{4} + |\nabla Y_{t}^{\delta}|^{4} e^{4\beta t} \right) \right]^{\frac{1}{2}} \cdot E_{x,\xi_{0}} \left[\left(\int_{0}^{\tilde{\gamma}_{1}} \left| \Xi_{\lambda,t}^{\delta} - \Xi_{-\lambda,t}^{-\delta} \right|^{2} e^{2\beta t} dt \right)^{2} \right]^{\frac{1}{2}}$$

$$\leq \left\{ \delta^{2} E_{x,\xi_{0}} \left[\sup_{0 \le t \le \tilde{\gamma}_{1}} \left| \frac{X_{t}^{\delta} - X_{t}}{\delta} - \xi_{t} \right|^{4} + |\xi_{t}|^{4} \right]^{\frac{1}{2}} + N \delta^{2} B_{3}^{\frac{1}{4}}(x,\xi_{0}) (|g|_{1}^{2} + ||f(\cdot)||_{0,1}^{2}) + \delta^{2} o^{\frac{1}{2}}(\delta,n,T) \right\}$$

$$\cdot \left\{ N \delta^{4} B_{3}^{\frac{1}{2}}(x,\xi_{0}) [(1 + |g|_{1}^{4} + ||f(\cdot)||_{0,1}^{4}) + \delta^{4} o(\delta,n,T)] \right\}^{\frac{1}{2}}$$

$$\leq N \delta^{4} B_{3}^{\frac{1}{2}}(x,\xi_{0}) (1 + |g|_{1}^{4} + ||f(\cdot)||_{0,1}^{4}) + \delta^{4} o(\delta,n,T), \quad \forall (x,\xi_{0}) \in D_{\delta_{1}}^{\lambda} \times \mathbb{R}^{d}.$$

$$(4.37)$$

Hence, we have for $(x, \xi_0) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d$

$$N[f]_{1,1}^2 E_{x,\xi_0} \int_0^{\bar{\gamma}_1} II_{4,1} e^{4\beta t} dt \le N \delta^4 B_3^{\frac{1}{2}}(x,\xi_0) (1+|g|_1^4+\|f(\cdot)\|_{0,1}^4) [f]_{1,1}^2 + \delta^4 o(\delta,n,T).$$
(4.38)

In view of Assertion (iii) in Lemma 3.5 and formula (2.14), we have for $\eta_0 = 0$

$$E_{x,\xi_{0}} \int_{0}^{\bar{\gamma}_{1}} \Pi_{4.2} dt \leq E_{x,\xi_{0}} \left[\sup_{0 \leq t \leq \bar{\gamma}_{1}} \left| \nabla X_{t}^{\delta} - \nabla X_{t}^{-\delta} \right|^{4} \right]^{\frac{1}{2}} \cdot E_{x,\xi_{0}} [\bar{\gamma}_{1}^{2}]^{\frac{1}{2}} \\ \leq N \delta^{4} E_{x,\xi_{0},0} \left[\sup_{0 \leq t \leq \bar{\gamma}_{1}} \left| \frac{\nabla X_{t}^{\delta} - \nabla X_{t}^{-\delta}}{\delta^{2}} - \eta_{t} \right|^{4} + \left| \eta_{t} \right|^{4} \right]^{\frac{1}{2}} \\ \leq N \delta^{4} B_{3}^{\frac{1}{2}}(x,\xi_{0}) + o(\delta^{5}), \quad \forall (x,\xi_{0}) \in D_{\delta_{1}}^{\lambda} \times \mathbb{R}^{d}.$$

$$(4.39)$$

Fifth, we need to treat the hardest term II_{4.3} consisting of ∇Z_t^{δ} . From the estimates (4.34)-(4.39) and by (*H*7), we choose positive constants ϵ_1 , ϵ_2 and ϵ such that

 $0 < \epsilon_1 < 3[8(1+2c_p^2)]^{-1}, \quad \epsilon_2 = c_p^2/4, \quad \text{and} \quad -4\beta + 6\epsilon - 2\mu + 4L_0^2 < 0,$

where c_p is the constant in the BDG inequality. We have for $(x, \xi_0) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d$,

$$\begin{split} & E_{x,\xi_{0}}\left[\left|\nabla Y_{t}^{\delta}-\nabla Y_{t}^{-\delta}\right|^{2}e^{4\beta t}\right]+E_{x,\xi_{0}}\int_{t}^{\tilde{\gamma}_{1}}\left\|\tilde{Z}_{s}^{\delta}-2Z_{s}+\tilde{Z}_{s}^{-\delta}\right\|^{2}e^{4\beta s}ds\\ &\leq E_{x,\xi_{0}}\left[\left|\nabla Y_{\tilde{\gamma}_{1}}^{\delta}-\nabla Y_{\tilde{\gamma}_{1}}^{-\delta}\right|^{2}e^{4\beta \tilde{\gamma}_{1}}\right]+N_{1}\delta^{4}B_{3}^{\frac{1}{2}}(x,\xi_{0})+\delta^{4}o(\delta,n,T)\\ &+E_{x,\xi_{0}}\int_{t}^{\tilde{\gamma}_{1}}\left(-4\beta+6\epsilon-2\mu\right)\left|\nabla Y_{s}^{\delta}-\nabla Y_{s}^{-\delta}\right|^{2}e^{4\beta s}ds+E_{x,\xi_{0}}\int_{t}^{\tilde{\gamma}_{1}}e^{4\beta s}\Pi_{4.3}ds\\ &\leq E_{x,\xi_{0}}\left[\left|\nabla Y_{\tilde{\gamma}_{1}}^{\delta}-\nabla Y_{\tilde{\gamma}_{1}}^{-\delta}\right|^{2}e^{4\beta \tilde{\gamma}_{1}}\right]+N_{1}\delta^{4}B_{3}^{\frac{1}{2}}(x,\xi_{0})+\delta^{4}o(\delta,n,T)\\ &+\epsilon_{1}E_{x,\xi_{0}}\left[\sup_{t\in[0,\tilde{\gamma}_{1}]}\left|\nabla Y_{t}^{\delta}-\nabla Y_{t}^{-\delta}\right|^{2}e^{4\beta t}\right]+E\int_{t}^{\tilde{\gamma}_{1}}\frac{\epsilon_{2}}{c_{p}^{2}}\left\|\nabla Z_{s}^{\delta}-\nabla Z_{s}^{-\delta}\right\|^{2}e^{4\beta s}ds\\ &+N[f]_{1,1}^{2}E_{x,\xi_{0}}\left[\left(\int_{t}^{\tilde{\gamma}_{1}}\left|\Xi_{\lambda,s}^{\delta}-\Xi_{-\lambda,s}\right|^{2}e^{2\beta s}ds\right)^{2}\right]^{\frac{1}{2}}\cdot E_{x,\xi_{0}}\left[\left(\int_{t}^{\tilde{\gamma}_{1}}\left\|\nabla Z_{s}^{\delta}\right\|^{2}e^{2\beta s}ds\right)^{2}\right]^{\frac{1}{2}}\\ &\leq E_{x,\xi_{0}}\left[\left|\nabla Y_{\tilde{\gamma}_{1}}^{\delta}-\nabla Y_{\tilde{\gamma}_{1}}^{-\delta}\right|^{2}e^{4\beta \tilde{\gamma}_{1}}\right]+N_{1}\delta^{4}B_{3}^{\frac{1}{2}}(x,\xi_{0})+\delta^{4}o(\delta,n,T)\\ &+\epsilon_{1}E_{x,\xi_{0}}\left[\sup_{t\in[0,\tilde{\gamma}_{1}]}\left|\nabla Y_{t}^{\delta}-\nabla Y_{t}^{-\delta}\right|^{2}e^{4\beta t}\right]+E_{x,\xi_{0}}\int_{t}^{\tilde{\gamma}_{1}}\frac{\epsilon_{2}}{c_{p}^{2}}\left\|\nabla Z_{s}^{\delta}-\nabla Z_{s}^{-\delta}\right\|^{2}e^{4\beta s}ds, \end{split}$$

where

$$N_{1} := N[|g|_{1}^{2} + ||f(\cdot)||_{0,1}^{2} + [f]_{1,1}^{2}(1 + |g|_{1}^{4} + ||f(\cdot)||_{0,1}^{4})].$$
(4.40)

Meanwhile, using the BDG inequality, we have

$$\begin{split} E_{x,\xi_{0}} \left[\sup_{t \in [0,\bar{\gamma}_{1}]} \left| \nabla Y_{t}^{\delta} - \nabla Y_{t}^{-\delta} \right|^{2} e^{4\beta t} \right] \\ \leq & E_{x,\xi_{0}} \left[\left| \nabla Y_{\bar{\gamma}_{1}}^{\delta} - \nabla Y_{\bar{\gamma}_{1}}^{-\delta} \right|^{2} e^{4\beta \bar{\gamma}_{1}} \right] + N_{1} \delta^{4} B_{3}^{\frac{1}{2}}(x,\xi_{0}) + \delta^{4} o(\delta,n,T) \\ & + \epsilon_{1} E_{x,\xi_{0}} \left[\sup_{t \in [0,\bar{\gamma}_{1}]} \left| \nabla Y_{t}^{\delta} - \nabla Y_{t}^{-\delta} \right|^{2} e^{4\beta t} \right] + E_{x,\xi_{0}} \int_{0}^{\bar{\gamma}_{1}} \frac{\epsilon_{2}}{c_{p}^{2}} \left\| \nabla Z_{s}^{\delta} - \nabla Z_{s}^{-\delta} \right\|^{2} e^{4\beta s} ds \\ & + \frac{1}{2} E_{x,\xi_{0}} \left[\sup_{t \in [0,\bar{\gamma}_{1}]} \left| \nabla Y_{t}^{\delta} - \nabla Y_{t}^{-\delta} \right|^{2} e^{4\beta t} \right] + 2c_{p}^{2} E_{x,\xi_{0}} \int_{0}^{\bar{\gamma}_{1}} \left\| \tilde{Z}_{s}^{\delta} - 2Z_{s} + \tilde{Z}_{s}^{-\delta} \right\|^{2} e^{4\beta s} ds \\ \leq & N E_{x,\xi_{0}} \left[\left| \nabla Y_{\bar{\gamma}_{1}}^{\delta} - \nabla Y_{\bar{\gamma}_{1}}^{-\delta} \right|^{2} e^{4\beta \bar{\gamma}_{1}} \right] + N_{1} \delta^{4} B_{3}^{\frac{1}{2}}(x,\xi_{0}) + \delta^{4} o(\delta,n,T) \\ & + (\epsilon_{1} + 2c_{p}^{2}\epsilon_{1}) E_{x,\xi_{0}} \left[\sup_{t \in [0,\bar{\gamma}_{1}]} \left| \nabla Y_{t}^{\delta} - \nabla Y_{t}^{-\delta} \right|^{2} e^{4\beta t} \right] \\ & + \left(\frac{\epsilon_{2}}{c_{p}^{2}} + 2\epsilon_{2} \right) E_{x,\xi_{0}} \int_{0}^{\bar{\gamma}_{1}} \left\| \nabla Z_{s}^{\delta} - \nabla Z_{s}^{-\delta} \right\|^{2} e^{4\beta s} ds, \quad \forall (x,\xi_{0}) \in D_{\delta_{1}}^{\lambda} \times \mathbb{R}^{d}. \end{split}$$

So, we have for $(x, \xi_0) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d$,

$$E_{x,\xi_{0}}\left[\left|\nabla Y_{t}^{\delta}-\nabla Y_{t}^{-\delta}\right|^{2}e^{4\beta t}\right] \leq NE_{x,\xi_{0}}\left[\left|\nabla Y_{\bar{\gamma}_{1}}^{\delta}-\nabla Y_{\bar{\gamma}_{1}}^{-\delta}\right|^{2}e^{4\beta\bar{\gamma}_{1}}\right]+N_{1}\delta^{4}B_{3}^{\frac{1}{2}}(x,\xi_{0}) +\epsilon_{3}E_{x,\xi_{0}}\int_{0}^{\bar{\gamma}_{1}}\left\|\nabla Z_{s}^{\delta}-\nabla Z_{s}^{-\delta}\right\|^{2}e^{4\beta s}ds+\delta^{4}o(\delta,n,T),$$

$$(4.41)$$

where $\epsilon_3 := \epsilon_2 c_p^{-2} + 2\epsilon_1 \epsilon_2 (c_p^{-2} + 2) [1 - 2\epsilon_1 (1 + 2c_p^2)]^{-1} < 1$. Then, we estimate the term $\|\nabla Z_s^{\delta} - \nabla Z_s^{-\delta}\|^2$. Set

$$II_5 := \|Z_t^{\delta} - \tilde{Z}_t^{\delta} - \tilde{Z}_t^{-\delta} + Z_t^{-\delta}\|^2$$

Through simple calculation, for sufficiently small $\epsilon_4 \in (0,1)$ satisfying $(1 + \epsilon_4)\epsilon_3 < 1$, we have

$$\left\|\nabla Z_{t}^{\delta} - \nabla Z_{t}^{-\delta}\right\|^{2} \le (1 + \epsilon_{4}) \left\|\tilde{Z}_{t}^{\delta} - 2Z_{t} + \tilde{Z}_{t}^{-\delta}\right\|^{2} + (1 + \frac{1}{\epsilon_{4}}) \Pi_{5}.$$

We continue to estimate the term II_5 . Write

$$\begin{split} \Pi_{5.1} &:= \delta^2 r_t^2 \| Z_t^{\delta} - Z_t^{-\delta} \|^2, \\ \Pi_{5.2} &:= \delta^4 (r_t^4 + \tilde{r}_t^2) [\| Z_t^{\delta} \|^2 + \| Z_t^{-\delta} \|^2], \\ \Pi_{5.3} &:= \delta^2 P_t^2 \| Z_t^{\delta} (1 + 2\delta r_t + \delta^2 \tilde{r}_t)^{\frac{1}{2}} - Z_t^{-\delta} (1 - 2\delta r_t + \delta^2 \tilde{r}_t)^{\frac{1}{2}} \|^2, \\ \Pi_{5.4} &:= \delta^4 P_t^4 [\| Z_t^{\delta} \|^2 (1 + 2\delta r_t + \delta^2 \tilde{r}_t) + \| Z_t^{-\delta} \|^2 (1 - 2\delta r_t + \delta^2 \tilde{r}_t)]. \end{split}$$

Using Taylor expansion as formula (4.31) to the terms $1 - (1 + 2\delta r_t + \delta^2 \tilde{r}_t)^{\frac{1}{2}} e^{\delta P_t}$ and $1 - (1 - 2\delta r_t + \delta^2 \tilde{r}_t)^{\frac{1}{2}} e^{-\delta P_t}$, we have

$$\begin{aligned} \mathrm{II}_{5} &= \|\delta r_{t}(-Z_{t}^{\delta}+Z_{t}^{-\delta})+(-\frac{1}{2}\delta^{2}\tilde{r}_{t}+\frac{1}{2}\delta^{2}r_{t}^{2})(Z_{t}^{\delta}+Z_{t}^{-\delta})\\ &+\delta P_{t}(-Z_{t}^{\delta}(1+2\delta r_{t}+\delta^{2}\tilde{r}_{t})^{\frac{1}{2}}+Z_{t}^{-\delta}(1-2\delta r_{t}+\delta^{2}\tilde{r}_{t})^{\frac{1}{2}})\\ &-\frac{1}{2}\delta^{2}P_{t}^{2}(Z_{t}^{\delta}(1+2\delta r_{t}+\delta^{2}\tilde{r}_{t})^{\frac{1}{2}}+Z_{t}^{-\delta}(1-2\delta r_{t}+\delta^{2}\tilde{r}_{t})^{\frac{1}{2}})\|^{2}+o(\delta^{5})\\ &\leq N(\mathrm{II}_{5.1}+\mathrm{II}_{5.2}+\mathrm{II}_{5.3}+\mathrm{II}_{5.4})+o(\delta^{5}).\end{aligned}$$

Then, in view of Assertion (iii) in Lemma 3.2 and formula (4.33), we have

$$E_{x,\xi_{0}} \int_{0}^{\bar{\gamma}_{1}} e^{4\beta t} \operatorname{II}_{5.1} dt$$

$$\leq \delta^{2} E_{x,\xi_{0}} \left[\sup_{0 \leq t \leq \bar{\gamma}_{1}} r_{t}^{4} e^{4\beta t} \right]^{\frac{1}{2}} \cdot E_{x,\xi_{0}} \left[\left(\int_{0}^{\bar{\gamma}_{1}} \left\| Z_{t}^{\delta} - Z_{t}^{-\delta} \right\|^{2} e^{2\beta t} dt \right)^{2} \right]^{\frac{1}{2}}$$

$$\leq N \delta^{4} B_{3}^{\frac{1}{2}}(x,\xi_{0}) (|g|_{1}^{2} + \|f(\cdot)\|_{0,1}^{2}) + \delta^{4} o(\delta, n, T), \quad \forall (x,\xi_{0}) \in D_{\delta_{1}}^{\lambda} \times \mathbb{R}^{d}$$

In view of Assertion (ii) in Lemma 3.5 and Proposition 4.3, we have

$$E_{x,\xi_{0}} \int_{0}^{\gamma_{1}} e^{4\beta t} \mathrm{II}_{5.2} dt$$

$$\leq \delta^{4} E_{x,\xi_{0}} \left\{ \sup_{0 \le t \le \bar{\gamma}_{1}} \left[(r_{t}^{4} + \tilde{r}_{t}^{2}) e^{4\beta t} \right] \cdot \int_{0}^{\bar{\gamma}_{1}} (\left\| Z_{t}^{\delta} \right\|^{2} + \left\| Z_{t}^{-\delta} \right\|^{2}) dt \right\}$$

$$\leq N \delta^{4} B_{3}^{\frac{1}{2}}(x,\xi_{0}) (\left| g \right|_{0}^{2} + \left| f(\cdot,0,0) \right|_{0}^{2}), \quad \forall (x,\xi_{0}) \in D_{\delta_{1}}^{\lambda} \times \mathbb{R}^{d}.$$

Terms $\mathrm{II}_{5.3}$ and $\mathrm{II}_{5.4}$ can be estimated in a similar way. In summary, we have

$$E_{x,\xi_0} \int_0^{\bar{\gamma}_1} e^{4\beta t} \mathrm{II}_5 dt \le N \delta^4 B_3^{\frac{1}{2}}(x,\xi_0) (|g|_1^2 + ||f(\cdot)||_{0,1}^2) + \delta^4 o(\delta,n,T), \quad \forall (x,\xi_0) \in D_{\delta_1}^{\lambda} \times \mathbb{R}^d.$$
(4.42)

So, by formulas (4.41) and (4.42), we have for $(x, \xi_0) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d$

$$E_{x,\xi_0}\left[\left|\nabla Y_t^{\delta} - \nabla Y_t^{-\delta}\right|^2 e^{4\beta t}\right] \le NE\left[\left|\nabla Y_{\bar{\gamma}_1}^{\delta} - \nabla Y_{\bar{\gamma}_1}^{-\delta}\right|^2 e^{4\beta \bar{\gamma}_1}\right] + N_1 \delta^4 B_3^{\frac{1}{2}}(x,\xi_0) + \delta^4 o(\delta,n,T). \quad (4.43)$$

Sixth, repeating **Step 2** in the proof of Theorem 4.2 and using Assertion (iii) in Lemma 3.5, we have for $(x, \xi_0) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d$ and $\eta_0 = 0$

$$\lim_{\delta \to 0} \frac{1}{\delta^{4}} E_{x,\xi_{0}} \left[\left| Y_{\bar{\gamma}_{1}}^{\delta} - 2Y_{\bar{\gamma}_{1}} + Y_{\bar{\gamma}_{1}}^{-\delta} \right|^{2} e^{4\beta\bar{\gamma}_{1}} \right] \\
\leq \lim_{\delta \to 0} E_{x,\xi_{0},0} \left[\left| \frac{u(X_{\bar{\gamma}_{1}}^{\delta}) - 2u(X_{\bar{\gamma}_{1}}) + u(X_{\bar{\gamma}_{1}}^{-\delta})}{\delta^{2}} - u_{(\eta\bar{\gamma}_{1})}(X_{\bar{\gamma}_{1}}) - u_{(\xi\bar{\gamma}_{1})(\xi\bar{\gamma}_{1})}(X_{\bar{\gamma}_{1}}) \right|^{2} e^{4\beta\bar{\gamma}_{1}} \right] \\
+ \lim_{\delta \to 0} E_{x,\xi_{0},0} \left[\left| u_{(\eta\bar{\gamma}_{1})}(X_{\bar{\gamma}_{1}}) + u_{(\xi\bar{\gamma}_{1})(\xi\bar{\gamma}_{1})}(X_{\bar{\gamma}_{1}}) \right|^{2} e^{4\beta\bar{\gamma}_{1}} \right] \\
\leq \lim_{\delta \to 0} \left[E_{x,\xi_{0}} \left| u_{(\xi\bar{\gamma}_{1})(\xi\bar{\gamma}_{1})}(X_{\bar{\gamma}_{1}}) \right|^{2} + \sup_{\substack{y \in \partial D_{\delta_{1}}^{\lambda}, \\ |\zeta|=1}} \left| u_{(\zeta)}(y) \right|^{2} \cdot E_{0} \left| \eta_{\bar{\gamma}_{1}} \right|^{2} \right] \\
\leq \sup_{\substack{y \in \partial D_{\delta_{1}}^{\lambda}, \\ 0 \neq \zeta \in \mathbb{R}^{d}}} \frac{\left| u_{(\zeta)}(\zeta)(y) \right|^{2}}{B_{3}^{\frac{1}{2}}(y,\zeta)} + \left[\sup_{\substack{y \in \partial D_{\delta_{1}}^{\lambda}, \\ |\zeta|=1}} \left| u_{(\zeta)}(y) \right|^{2} \right] NB_{3}^{\frac{1}{2}}(x,\xi_{0}). \tag{4.44}$$

For t = 0, divide δ^4 in both sides of (4.43). Let $\delta \to 0$, then $(T, n) \to (+\infty, +\infty)$. By (4.44), we have for $(x, \xi_0) \in D^{\lambda}_{\delta_1} \times \mathbb{R}^d \setminus \{0\}$ and $\eta_0 = 0$

$$\frac{\left|u_{(\xi_{0})(\xi_{0})}(x)\right|^{2}}{B_{3}^{\frac{1}{2}}(x,\xi_{0})} \leq \sup_{\substack{y \in \partial D_{\delta_{1}}^{\lambda}, \\ 0 \neq \zeta \in \mathbb{R}^{d}}} \frac{\left|u_{(\zeta)(\zeta)}(y)\right|^{2}}{B_{3}^{\frac{1}{2}}(y,\zeta)} + N \sup_{\substack{y \in \partial D_{\delta_{1}}^{\lambda}, \\ |\zeta|=1}} \left|u_{(\zeta)}(y)\right|^{2} + N_{1}$$

Step 2. Repeating the arguments in **Step 1** for $x \in D_{\lambda^2}$, we have for $(x, \xi_0) \in D_{\lambda^2} \times \mathbb{R}^d \setminus \{0\}$ and $\eta_0 = 0$,

$$\frac{\left|u_{(\xi_0)(\xi_0)}(x)\right|^2}{B_4^{\frac{1}{2}}(\xi_0)} \le \sup_{\substack{y \in \partial D_{\lambda^2}, \\ 0 \neq \zeta \in \mathbb{R}^d}} \frac{\left|u_{(\zeta)(\zeta)}(y)\right|^2}{B_4^{\frac{1}{2}}(\zeta)} + N_1.$$

Step 3. In view of Lemma 4.10 and Theorem 4.2, we have

$$\begin{split} &\lim_{\delta_{1}\to0}\sup_{\substack{x\in\partial D,\\|\zeta|=1}} |u_{(\zeta)}(x)|^{2} \\ &\leq \sup_{\substack{x\in\partial D,\\|l|=1,\ l\|\partial D}} |u_{(l)}(x)|^{2} + \sup_{\substack{x\in\partial D,\\|n|=1,\ n\perp\partial D}} |u_{(n)}(x)|^{2} + \sup_{\substack{\psi(x)=\lambda,\\|\zeta|=1}} |u_{(\zeta)}(x)|^{2} \\ &\leq \sup_{\substack{x\in\partial D,\\|l|=1,\ l\|\partial D}} |g_{(l)}(x)|^{2} + N(|g|_{2}^{2} + |f(\cdot,0,0)|_{0}^{2}) + N\left(1 + \frac{|\psi|_{1}}{\lambda^{\frac{3}{2}}}\right) (|g|_{1}^{2} + ||f(\cdot)||_{0,1}^{2}). \end{split}$$

With the analogues of Lemmas 4.6 and 4.7, we obtain

$$\left| u_{(\xi_0)(\xi_0)}(x) \right| \le N_2 \left(|\xi_0|^2 + \frac{\psi_{(\xi_0)}^2(x)}{\psi^{\frac{7}{4}}(x)} \right), \quad a.e. \ x \in D, \ \forall \xi_0 \in \mathbb{R}^d,$$
$$N_2 = N[|q|_{1,1} + ||f(\cdot)||_{0,1} + [f]_{1,1}(1 + |q|_1^2 + ||f(\cdot)||_{0,1}^2)]. \tag{4.45}$$

where

$$N_{2} = N[|g|_{1,1} + ||f(\cdot)||_{0,1} + [f]_{1,1}(1 + |g|_{1}^{2} + ||f(\cdot)||_{0,1}^{2})].$$
(4.45)
ete.

The proof is complete.

The proof of Theorem 2.9 remains to prove the existence and uniqueness of (2.16). Before that, we need the existence, uniqueness and probabilistic interpretation of weak solutions (in Sobolev sense) for the Dirichlet problem of systems of semi-linear degenerate elliptic PDEs, which was inspired by the work of Bally and Matoussi [1]. However, due to the length of the paper, we will not include all the arguments here. The outline of the proof is as follows: Firstly, since the coefficients b, σ are defined on the whole space, applying [1, Proposition 5.1] and choosing $\varphi := \varphi \cdot 1_D$, we get a norm equivalence result in a bounded domain. Secondly, according to the proof of [1, Theorem 2.1], using the approximation procedure (see [1, page 138]), we get a probabilistic interpretation for the solution of the linear elliptic system. Finally, for the semi-linear system, using the norm equivalence result and the probabilistic interpretation for the linear system mentioned above, following the proof of [1, Theorem 3.1], we have: under the assumptions (H1), (H4) - (H7), with the well-posedness of solutions to random horizon BSDEs (see Lemma 2.2), there is a unique weak solution $u \in L^2_{\rho}(\overline{D})$ of (2.16).

Proof of the existence and uniqueness of (2.16). Let u is given by (1.2). The Lipschitz continuity of the solution u up to the boundary can be proved by Lemma 4.10 for boundary normal derivative estimates and by $(H6)_1$ for boundary tangential derivative estimates. Moreover, since u is a weak solution of (2.16) in L^2_{ρ} , and has second derivatives almost everywhere by 4.9, then u given by (1.2) satisfies (2.16) almost everywhere in the space $C^{1,1}_{loc}(D) \cap C^{0,1}(\overline{D})$.

For the uniqueness, if PDE (2.16) has a weak solution u, then we have $u(X_t) = Y_t$ and $(\nabla u \sigma)(X_t) = Z_t$, where (Y_t, Z_t) is the solution of BSDE (1.3). Then the uniqueness of the weak solutions of PDEs follows from the uniqueness of solutions of FBSDEs. The proof is complete.

Remark 4.11. Consider the case of k = 1. Let U be a separable metric space. By \mathcal{U} , we denote the set of progressively measurable processes α_t taking values in U. In the above proof, if we replace $(\sigma(x), b(x), f(x, y, z) \text{ and } g(x) \text{ in } (1.3) \text{ and } (1.4) \text{ with } \sigma(\alpha, x), b(\alpha, x), f(\alpha, x, y, z) \text{ and } g(\alpha, x), \text{ where } \alpha \in \mathcal{U}$ is the control variable, under appropriate measurable assumptions, the gradient and Hessian estimates (2.15) and (2.9) are still true. In this way, we can get the interior regularity estimates for the solution of the so-called HJB equations, which is nonlinear and degenerate elliptic PDEs in a domain.

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