A Note on I^K and I^{K^*}-convergence in topological spaces

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Abstract

In this paper we have studied some important topological properties and characterization of I^K-convergence of functions which is a common generalization of I^*-convergence of functions. We also introduce the idea of I^{K^*}-convergence and I^K-limit points of functions.

Keywords : I^K-convergence, I^{K^*}-convergence, AP(I, K)-condition, P-ideals, I^K-limit points.

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1 Introduction and Background

The aim of this paper is to study the notion of I^K and I^{K^*}-convergence of functions which are the common generalization of various type of I and I^*-convergence of functions in some restriction. Let us start with brief discussion on two types of ideal convergence.

The concept of usual convergence of a real sequence has been extended to statistical convergence by H. Fast[11] and then H. Steinhaus[24] in the year 1951. Now we recall natural density of a set $K \subset \mathbb{N}$ where $\mathbb{N}$ denotes the set of natural numbers. Let K_n denote the set $\{k \in K : k \leq n\}$ and $|K_n|$ stands for the cardinality of K_n.The natural density of K is defined by

$$d(K) = \lim_n \frac{|K_n|}{n}$$

if the limit exits. A real sequence $\{x_n\}$ is said to be statistically convergent to l if for every $\epsilon > 0$ the set $K(\epsilon) = \{k \in \mathbb{N} : |x_k - l| \geq \epsilon\}$ has natural density zero[11, 13, 24]. Ordinary convergence always implies statistical convergence[19, 22, 23]. Later it was developed by many authors and after long 50 years, the concept of statistical convergence has been extended to I and I^*-convergence which depends on the structure of ideals of subsets of the natural numbers by P.Kostyrko et al[16, 17, 18]. The concept of I^*-convergence which is closely related to that of I-convergence and which arises from a particular result on statistical convergence of real sequence was introduced by P.Kostyrko et al. The result is as follows:

A real sequence $\{x_n\}$ is statistically convergent to ξ if and only if there exist a set $M = \{m_1 < m_2 < m_3 < ... < m_k < ...\}$ such that $d(M) = 1$ and $\lim_k x_{m_k} = \xi$. [16, 17]

If I is an admissible ideal, I^*-convergence implies I-convergence. But converse may not be true. Moreover a statistical convergent sequence and I and I^*- convergent sequence need not even be bounded[15, 23]. I and I^*-convergence coincide for every admissible ideal I if the space is discrete or if I satisfies AP(I,Fin)-condition.[9, 16]. B.K.Lahiri and Pratulananda Das in the year 2005, extended the concept of I and I^*-convergence in a topological space and they observed that the basic properties are preserved also in a topological space[15]. Later many works on I-convergence were done in topological spaces[2, 3, 4, 6, 7, 8].

In the year 2010, M. Macaj and M. Sleziak[20] defined I^K-convergence and shew that this type of convergence is a common generalization for all types of I and I^*-convergence we have mentioned so far. They also gave the condition AP(I, K) modifying condition AP from [9, 17]. Later in the year 2014, I^K-Cauchy and I^K-Cauchy net have been studied in [10, 21].

In this paper we have studied further some basic properties of I^K-convergence of functions in topological

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spaces which were not studied before. Also we have defined the notion of I^{K^*}-convergence and have found out the relation between I, I^*, K^*, I^{K^*} and I^K-convergence of functions. While studying the convergence of functions, several closely related notions occur quite naturally such as limit points, cluster points etc. In the last section we have introduced I^K-limit points and examined some important topological properties like characterization of compactness in terms of I^K-limit points.

2 Basic Definition and Notation

Definition 2.1. *Let S be a non-void set then a family of sets $I \subset 2^S$ is said to be an ideal if*

(i) $A, B \in I \Rightarrow A \cup B \in I$

(ii) $A \in I, B \subset A \Rightarrow B \in I$

I is called nontrivial ideal if $S \notin I$ and $I \neq \{\phi\}$. In view of condition (ii) $\phi \in I$ i.e. an ideal is a non-void system of sets I hereditary with respect to additive and inclusion. If $I \subsetneq 2^S$ we say that I is proper ideal on S. Several examples of non-trivial ideals are seen in [17]. A nontrivial ideal I is called admissible if it contains all the singleton of $\mathbb{N}$. A nontrivial ideal I is called non-admissible if it is not admissible. An example of an admissible ideal on a set S is the ideal of all finite subsets of S which we shall denote by Fin(S). If $S = \mathbb{N}$ then we write Fin instead of Fin$(\mathbb{N})$ for short.

Example 2.1. *Let I be the class of all $A \subset \mathbb{N}$ with $d(A) = 0$. Then I is an admissible ideal of $\mathbb{N}$, since singleton sets has density zero. For any proper subset $M \subset \mathbb{N}$, $I = 2^M$ is an non-admissible ideal of $\mathbb{N}$.*

Note 2.1. *The dual notion to the ideal is the notion of the filter i.e. a filter on S is non-void system of subsets of S, which is closed under finite intersection and super sets. If I is a non-trivial ideal on X then $F = F(I) = \{A \subset X : X \setminus A \in I\}$ is clearly a filter on X and conversely. $F(I)$ is called associated filter with respect to ideal I.*

Now we will give the definition of I-convergence using function instead of sequence.

Definition 2.2. *[20] Let I be an ideal on a non-void set S and X be a topological space. A function $f : S \to X$ is said to be I-convergent to $x \in X$, written as I-$\lim f = x$ if*

$$f^{-1}(U) = \{s \in S : f(s) \in U\} \in F(I)$$

for every neighborhood U of the point x. i.e. if $f^{-1}(X \setminus U) = \{s \in S : f(s) \notin U\} \in I$ for every neighborhood U of x.

If $S = \mathbb{N}$ we obtain the usual definition of I-convergence of sequence.

Definition 2.3. *[20] Let I be an ideal on a set S and let $f : S \to X$ be a function to a topological space X. The function f is called I^*-convergent to the point x of X if there exists a set $M \in F(I)$ such that the function $g : S \to X$ defined by*

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ x & \text{if } s \notin M \end{cases}$$

is Fin(S)-convergent to x.

If f is I^*-convergent to x, then we write I^*-$\lim f = x$. The usual notion of I^*-convergence of sequence is a special case when $S = \mathbb{N}$. I^K-convergence as a common generalization of all types of I^*-convergence of sequences or functions from S into X. Here we will work with functions from a non-void arbitrary set S to a topological space X. One of the reasons is that using functions sometimes helps to simplify notation.

Definition 2.4. *[20] Let K and I be an ideal on a non-void set S, X be a topological space and let x be an element of X. A function $f : S \to X$ is called I^K-convergent to the point x if there exists a set $M \in F(I)$ such that the function $g : S \to X$ given by*

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ x & \text{if } s \notin M \end{cases}$$

is K-convergent to x.

If f is I^K-convergent to x, then we write $I^K\text{-}\lim f = x$. As usual, notion of I^K-convergence of sequence is a special case for $S = \mathbb{N}$. Similarly as for I-convergence of sequences. We write $I^K\text{-}\lim x_n = x$.

Lemma 2.1. *[20] If I and K are ideals on a set S and $f : S \to X$ is a function such that $K\text{-}\lim f = x$, then $I^K\text{-}\lim f = x$.*

Theorem 2.1. *[20] Let I,K be ideals on a set S, X be a topological space and let f be a function from S to X then $I^K\text{-}\lim f = x \Rightarrow I\text{-}\lim f = x$ if and only if $K \subset I$.*

Proposition 2.1. *[20] Let I, I_1, I_2, K, K_1 and K_2 be ideals on a set S such that $I_1 \subset I_2$ and $K_1 \subset K_2$ and let X be a topological space. Then for any function $f : S \to X$, we have $I_1^K\text{-}\lim f = x \quad \Rightarrow \quad I_2^K\text{-}\lim f = x$ and $I^{K_1}\text{-}\lim f = x \quad \Rightarrow \quad I^{K_2}\text{-}\lim f = x$.*

3 Basic Properties of I^K-Convergence in Topological Spaces

Throughout the paper X stands for a topological space (X, τ) and I, K are non-trivial ideals of a non empty set S unless otherwise stated. First we introduce a construction regarding double ideal. For any two ideals I, K on a non-void set S we have the ideal

$$I \vee K = \{A \cup B : A \in I, B \in K\}$$

which is the smallest ideal containing both I and K on S i.e. $I, K \subseteq I \vee K$. It is clear that if $I \vee K$ is non-trivial and I and K are both proper subset of $I \vee K$ then I and K both are non-trivial. But converse part may not be true. To support this following examples are given.

Example 3.1. *Consider the two sets $N_1 = \{4n : n \in \mathbb{N}\}$ and $N_2 = \{4n - 1 : n \in \mathbb{N}\}$ now it is clear that 2^{N_1}, 2^{N_2} and $2^{N_1} \vee 2^{N_2}$ all are non-trivial ideal on $\mathbb{N}$.*

Example 3.2. *Now let N_1 be set of all odd integers and N_2 be set of all even integers. Then it is clear that $I = 2^{N_1}$, $K = 2^{N_2}$ both are non-trivial ideals on $\mathbb{N}$ but $I \vee K$ is a trivial ideal on $\mathbb{N}$.*

If $I \vee K$ is a non-trivial on X then the dual filter is $F(I \vee K) = \{G \cap H : G \in F(I), H \in F(K)\}$.

Theorem 3.1. *Let $I \vee K$ is non-trivial on set S. If X is Hausdorff and a function $f : S \to X$ is I^K-convergent then f has a unique I^K-limit.*

Proof. If possible let us consider that the function f has two distinct I^K-limits say x and y. Since X is Hausdorff then there exists $U, V \in \tau$ such that $x \in U$ and $y \in V$ and $U \cap V = \phi$. Since f has I^K-limit x, so from the definition of I^K-limit, there exists a set $A_1 \in F(I)$ such that the function $g : S \to X$ given by

$$g(s) = \begin{cases} f(s) & \text{if } s \in A_1 \\ x & \text{if } s \notin A_1 \end{cases}$$

is K-convergent to x. So, $g^{-1}(U) = \{s \in S : g(s) \in U\} = \{s \in A_1 : g(s) \in U\} \cup \{s \in S \setminus A_1 : g(s) \in U\} = (S \setminus A_1) \cup f^{-1}(U) = S \setminus (A_1 \setminus f^{-1}(U)) \in F(K)$ i.e. $A_1 \setminus f^{-1}(U) \in K$ or $A_1 \setminus B_1 \in K$

where $B_1 = f^{-1}(U)$. Similarly, f has I^K-limit y so there exists a set $A_2 \in F(I)$ s.t. $A_2 \setminus f^{-1}(V) \in K$ or $A_2 \setminus B_2 \in K$ where $B_2 = f^{-1}(V)$. So,

$$(A_1 \setminus B_1) \cup (A_2 \setminus B_2) \in K \tag{3.1}$$

Now let $x \in (A_1 \cap A_2) \cap (B_1 \cap B_2)^c = (A_1 \cap A_2) \cap (B_1^c \cup B_2^c) = ((A_1 \cap A_2) \cap B_1^c) \cup ((A_1 \cap A_2) \cap B_2^c)$ i.e. either $x \in (A_1 \cap A_2) \cap B_1^c \subset A_1 \cap B_1^c$ or $x \in ((A_1 \cap A_2) \cap B_2^c) \subset A_2 \cap B_2^c$ i.e. $x \in (A_1 \cap B_1^c) \cup (A_2 \cap B_2^c)$. So, $(A_1 \cap A_2) \cap (B_1 \cap B_2)^c \subset (A_1 \cap B_1^c) \cup (A_2 \cap B_2^c) \in K$ (from the equation (3.1)). Thus $(A_1 \cap A_2) \cap (B_1 \cap B_2)^c \in K$ i.e. $(A_1 \cap A_2) \setminus (f^{-1}(U) \cap f^{-1}(V)) \in K$ i.e. $(A_1 \cup A_2) \setminus (f^{-1}(U \cup V)) \in K$. Since $U \cap V = \phi$, then $f^{-1}(U \cap V) = \phi$ so $A_1 \cap A_2 \in K$ i.e.

$$S \setminus (A_1 \cap A_2) \in F(K) \tag{3.2}$$

Since $A_1, A_2 \in F(I)$,

$$A_1 \cap A_2 \in F(I) \tag{3.3}$$

Since $I \vee K$ is non-trivial so the dual filter is $F(I \vee K) = \{G \cap H : G \in F(I), H \in F(K)\}$. Now using this from 3.2 and 3.3 we get $\phi \in F(I \vee K)$, which is a contradiction. Hence the I^K-limit is unique. □

Theorem 3.2. *If I and K be two admissible ideal and if there exists an injective function $f : S \to E \subset X$ which is I^K-convergent to $x_0 \in X$ then x_0 is a limit point of E*

Proof. The function f has I^K-limit x_0, so I^K-limit there exists a set $M \in F(I)$ such that the function $g : S \to X$ given by

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ x_0 & \text{if } s \notin M \end{cases}$$

is K-convergent to x_0. Let U be an arbitrary open set containing x_0. Then $g^{-1}(U) = \{s : g(s) \in U\} \in F(K)$. So $\{s : g(s) \in U\} \notin K$ i.e. $\{s : g(s) \in U\}$ is an infinite set, as K is an admissible ideal. Choose $k_0 \in \{s : g(s) \in U\}$ such that $g(k_0) \neq x_0$ then $g(k_0) \in U \cap (E \setminus \{x_0\})$. Thus x_0 is a limit point of E. □

Theorem 3.3. *A Continuous function $h : X \to X$ preserves I^K-convergence.*

Proof. Let the function f has I^K-limit x, so there exists a set $M \subset S \in F(I)$ such that the function $g : S \to X$ given by

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ x & \text{if } s \notin M \end{cases}$$

is K-convergent to x. Let U be an arbitrary open set containing x. Then $g^{-1}(U) = S \setminus (M \setminus f^{-1}(U)) \in F(K)$ i.e. $M \setminus f^{-1}(U) \in K$. So to prove the theorem we have to show that $I^K\text{-}\lim h(f(x)) = h(x)$ i.e. it suffices to show that the function $g_1 : S \to X$ given by

$$g_1(s) = \begin{cases} (h \circ f)(s) & \text{if } s \in M \\ h(x) & \text{if } s \notin M \end{cases}$$

is K-convergent to $h(x)$. Let V be an open set containing $h(x)$. Since h is continuous so there exists an open set U containing x such that $h(U) \subset V$. Clearly $\{x : h(f(x)) \notin V\} \subset \{x : f(x) \notin U\}$ which implies that $\{x : f(x) \in U\} \subset \{x : h \circ f(x) \in V\}$ i.e. $f^{-1}(U) \subset (h \circ f)^{-1}(V)$. So $M \setminus (h \circ f)^{-1}(V) \subset M \setminus f^{-1}(U)$. Then $M \setminus (h \circ f)^{-1}(V) \in K$ as $M \setminus f^{-1}(U) \in K$. So its complement $g_1^{-1}(V) \in F(K)$, as required. Hence $I^K\text{-}\lim(h \circ f)(x) = h(x)$. □

Theorem 3.4. *If X is a discrete space then I-convergence implies I^K-convergence, where I and K are two admissible ideals.*

Proof. Let $f : S \to X$ be a function such that $I\text{-}\lim f = x_0$. Since X is a discrete space so it has no limit point then $U = \{x_0\}$ is open. Thus we have $f^{-1}(X \setminus U) = \{s \in S : f(s) \notin U\} \in I$. Let the set $M = f^{-1}(U) =$

$\{s \in S : f(s) \in U\} \in F(I)$. Thus there exists a set $M = \{s : f(s) \in U\} = \{s : f(s) = x_0\} \in F(I)$ such that the function $g : S \to X$ defined by

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ x_0 & \text{if } s \notin M \end{cases}$$

is K-convergent to x_0, since for any open set U containing x_0 the set $g^{-1}(U) = S \in F(K)$. Hence I^K-$\lim f = x_0$ □

Note 3.1. *Converse of above theorem may not be true. Let I and K be two ideals on a set S. Consider a set $A \in K \setminus I$. Let $y_0 \in X \setminus \{x_0\}$ be a fixed element and define a function $f : S \to X$ by*

$$f(s) = \begin{cases} x_0 & \text{if } s \in S \setminus A \\ y_0 & \text{otherwise} \end{cases}$$

Now if V is any open set containing x_0 then $f^{-1}(V) = S \setminus A$ if $y_0 \notin V$ and $f^{-1}(V) = S$ if $y_0 \in V$. So in both case $f^{-1}(V) \in F(K)$. Hence K-$\lim f = x_0$ then by lemma (2.1) we get I^K-$\lim f = x_0$. But $U = \{x_0\}$ is an open set containing x_0 since X is a discrete space and $f^{-1}(X \setminus U) = A \notin I$. Hence f is not I-convergent to x_0.

Theorem 3.5. *Let (X, τ) be a topological space and let $f : S \to X$ be a function, where S is a non-empty set, such that $x \in X$ is an I^K-limit of the function f, for some non-trivial ideals I and K of S. Then there exists a filter F on X such that x is also a limit of the filter F.*

Proof. Let I & K be two non-trivial ideals on non-empty set S. Also let x is I^K-limit of the function $f : S \to X$. Then from the definition of I^K-convergence then there exists a set $M_1 \in F(I)$ such that the function $g : S \to X$ given by

$$g(s) = \begin{cases} f(s) & \text{if } s \in M_1 \\ x & \text{if } s \notin M_1 \end{cases}$$

is K-convergent to x. So for every open set U containing x, the set

$$M = g^{-1}(U) = \{s \in S : g(s) \in U\} \in F(K) \tag{3.4}$$

Let us construct for each $M \in F(K)$ the set $A_M = \{g(n) : n \in M\}$ and $\mathcal{B} = \{A_M : M \in F(K)\}$. Then the family $\mathcal{B}$ forms a filter base on X. In fact, (i)We observe that each A_M is non-empty. Since M is non-empty so $\mathcal{B}$ is non-empty. (ii)Since $F(K)$ is filter, $\phi \notin F(K)$ and so $A_M \neq \phi$ for all $M \in F(K)$ and $\phi \notin \mathcal{B}$. (iii) Let us take any two members $A_M, A_R \in$ where $M, R \in F(K)$. $M \cap R \in F(K)$ since $F(K)$ is filter on S. So $A_{M \cap R} \in \mathcal{B}$. Also $A_{M \cap R} \subset A_M \cap A_R$. So $\mathcal{B}$ is a filter base. Let F be the filter generated by this filter base. Now we will show that x be the limit of filter F. Let V be any open set of x. Then from (3.4) the set $M = \{s \in S : g(s) \in V\} \in F(K)$. So by our construction of A_M, we get $A_M = \{g(n) : n \in M\} \subset V$. Since $A_M \in \mathcal{B}$ we get $V \in F$. So we conclude that $V \in F$ for all open set V of x. Hence x becomes limit of the filter F. □

Theorem 3.6. *Let (X, τ) be a topological space and $x \in X$. Then for every function $f : S \to X$ there exists a filter F on X such that if x is limit of filter F then x is also I^K-limit of the function f.*

Proof. Let $f : S \to X$ be a function and I, K be two non-trivial ideals of S. For each $M \in F(K)$ let $A_M = \{f(n) : n \in M\}$ and $\mathcal{B} = \{A_M : M \in F(K)\}$. Then the family $\mathcal{B}$ forms a filter base on X. Let F be the filter generated by this filter base. Let x be the limit of filter F. Then $\eta_x \subset F$ where η_x is the neighborhood filter of the point x. Let $U \in \eta_x$ be arbitrary. Then $U \in F$ and so $A_M \subset U$ for some $M \in F(K)$. This implies that $M \subset \{n \in S : f(n) \in U\}$ which further implies that $\{n \in S : f(n) \in U\} \in F(K)$ since $M \in F(K)$. Now U is arbitrary so the function f is K-convergent to x. Hence from the lemma (2.1) we get f is I^K-convergent to x. □

4 I^{K^*}-Convergence in Topological Spaces

I^{K^*}-convergence is also a common generalization of all types of I^* and K^*-convergence. It is interesting to find the relation between I, I^*, K^*, I^{K^*} and I^K-convergence.

Definition 4.1. *Let X be a topological space and $x \in X$ and let I, K be two ideals on a non-void set S. A function $f : S \to X$ is called I^{K^*}-convergent to the point x if there exists a set $M \in F(I)$ and $M_1 \in F(K)$ such that the function $g : S \to X$ given by*

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \cap M_1 \\ x & \text{if } s \notin M \cap M_1 \end{cases}$$

is Fin(S)-convergent to x.

If f is I^{K^*}-convergent to x then we write $I^{K^*}\text{-}\lim f = x$.

Note 4.1. *It follows from the definition that f is I^{K^*}-convergent to x if and only if there exist a set $M \in F(I)$ such that the function $g : S \to X$ given by*

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ x & \text{if } s \notin M \end{cases}$$

is K^*-convergent to x.

Lemma 4.1. *If I and K are two ideals on a set S and if $f : S \to X$ is a function such that $K^*\text{-}\lim f = x$ then $I^{K^*}\text{-}\lim f = x$.*

Proof. Follows from the lemma 2.1. □

Lemma 4.2. *If I and K be two admissible ideals on a set S and $f : S \to X$ is a function such that $I^{K^*}\text{-}\lim f = x$ then $I^K\text{-}\lim f = x$.*

Proof. The proof follows from the note (4.1) and since K^*-convergence implies K-convergence of the function g. □

Theorem 4.1. *If X is a discrete space then I^K and I^{K^*}-convergence coincide for every admissible ideal I and K.*

Proof. Let X be a discrete topological space then it has no limit point and $x \in X$. Let I and K be two admissible ideals on a set S and $f : S \to X$ is a function such that $I^K\text{-}\lim f = x$. Because of previous lemma (4.2) we have only to show that $I^{K^*}\text{-}\lim f = x$. Now from the definition of I^K-convergence there exists a set $M \in F(I)$ such that the function $g : S \to X$ defined by

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ x & \text{if } s \notin M \end{cases}$$

is K-convergent to x i.e. $K\text{-}\lim g(x) = x$. Since X has no limit point so $U = \{x\}$ is open. So we have $\{s : g(s) \notin U\} \in K$. Hence the set $M_1 = \{s : g(s) \in U\} = \{s : g(s) = x\} \in F(K)$. So there exist $M_1 \in F(I)$ such that the function $g_1 : S \to X$ defined by

$$g_1(s) = \begin{cases} f(s) & \text{if } s \in M_1 \\ x & \text{if } s \notin M_1 \end{cases}$$

is Fin(S)-convergent to x, since for any open set U containing x, $g^{-1}(X \setminus U) = \phi$ is a finite set. Thus $K^*\text{-}\lim g(x) = x$. So $I^{K^*}\text{-}\lim f = x$. □

Theorem 4.2. *Let I and K be two admissible ideals on a non-empty set S and let $f : S \to X$ be a function where X is a topological space. Then I^{K^*}-convergence implies I-convergence if $K \subseteq I$.*

Proof. Suppose that the function $f : S \to X$ is I^{K^*}-convergent to $x \in X$. So there exists sets $M \in F(I)$ and $M_1 \in F(K)$ such that the function $g : S \to X$ given by

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \cap M_1 \\ x & \text{if } s \notin M \cap M_1 \end{cases}$$

is Fin(S)-convergent to x i.e. $g^{-1}(X \setminus U) = \{s \in S : g(s) \notin U\}$ is a finite set for each open set U containing the point x. Now the set C (say)$= f^{-1}(X \setminus U) \cap (M \cap M_1) \subset g^{-1}(X \setminus U)$ i.e. C is finite. So $C \in I$. Now,

$$f^{-1}(X \setminus U) \subseteq (S \setminus (M \cap M_1)) \cup C \tag{4.1}$$

and $F(K) \subset F(I)$, since $K \subseteq I$. Therefore $M \cap M_1 \in F(I)$. So $S \setminus (M \cap M_1) \in I$. So from (4.1) we get $f^{-1}(X \setminus U) \in I$. Therefore f is I-convergent to x. i.e.I-$\lim f = x$ □

Lemma 4.3. *If I and K be two admissible ideals on a set S and f be a function from S to X, where X be a topological space. Then I^{K^*}-convergence implies K-convergence if $I \subseteq K$.*

Proof. The proof is similar to the proof of Theorem (4.2) and so omitted. □

Theorem 4.3. *I^*-convergence implies I^{K^*}-convergence.*

Proof. Let I and K be two ideals on a non-void set S and $f : S \to X$ be a function such that f is I^*-convergence to x of X. So $\exists$ a set $M \in F(I)$ such that the function $g : S \to X$ defined by

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ x & \text{if } s \notin M \end{cases}$$

is Fin(S)-convergent to x. Since Fin-convergent always implies K^*-convergent then the function g is K^*-convergent to x. and so f is I^{K^*}-convergent to x by the Note(4.1). □

Lemma 4.4. *K^*-convergence implies I^{K^*}-convergence.*

4.1 Additive Property with $I^K \& I^{K^*}$-Convergence

We now study the relationship between $I, I^{K^*} \& I^K$-convergence. The following definition is important in this regard.

Definition 4.2. *[10] Let I, K be ideals on the non-empty set S. We say that I has additive property with respect to K or that the condition AP(I, K) holds if for every sequence of pairwise disjoint sets $A_n \in I$, there exists a sequence $B_n \in I$ such that $A_n \triangle B_n \in K$ for each n and $\cup_{n \in \mathbb{N}} B_n \in I$*

Another formulation of condition AP(I, K) are given in [20]. Before giving this definition we need to state definition of K-pseudo-intersection of a system.

Definition 4.3. *[20] Let K be an ideal on a set S. We write $A \subset_K B$ whenever $A \setminus B \in K$. If $A \subset_K B$ and $B \subset_K A$ then we write $A \sim_K B$. Clearly $A \sim_K B \Leftrightarrow A \triangle B \in K$*
We say that a set A is K-pseudo-intersection of a system $\{A_n : n \in \mathbb{N}\}$ if $A \subset_K A_n$ holds for each $n \in \mathbb{N}$

Definition 4.4. *[20] Let I, K be ideals on the set S. We say that I has additive property with respect to K or that the condition AP(I, K) holds if any of the equivalent condition of following holds:*

(i) For every sequence $(A_n)_{n \in \mathbb{N}}$ of sets from I there is $A \in I$ such that $A_n \subset_K A$ for all n's.

(ii) Any sequence $(F_n)_{n \in \mathbb{N}}$ of sets from $F(I)$ has K-pseudo-intersection in $F(I)$.

(iii) For every sequence $(A_n)_{n\in\mathbb{N}}$ of sets from I there exists a sequence $(B_n)_{n\in\mathbb{N}} \in I$ such that $A_j \sim_K B_j$ for $j \in \mathbb{N}$ and $B = \cup_{j\in\mathbb{N}} B_j \in I$.

(iv) For every sequence of mutually disjoint sets $(A_n)_{n\in\mathbb{N}} \in I$ there exists a sequence $(B_n)_{n\in\mathbb{N}} \in I$ such that $A_j \sim_K B_j$ for $j \in \mathbb{N}$ and $B = \cup_{j\in\mathbb{N}} B_j \in I$.

(v) For every non-decreasing sequence $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \cdots$ of sets from I $\exists$ a sequence $(B_n)_{n\in\mathbb{N}} \in I$ such that $A_j \sim_K B_j$ for $j \in \mathbb{N}$ and $B = \cup_{j\in\mathbb{N}} B_j \in I$.

(vi) In the Boolean algebra $2^S/K$ the ideal I corresponds to a σ-directed subset,i.e. every countable subset has an upper bound.

In the case $S = \mathbb{N}$ and $K =$ Fin we get the condition AP from [17] which characterize ideal such that I^*-convergence implies I-convergence. The condition AP(I, K) is more generalization of condition AP from[9][17] . Ideals which fulfill the condition AP(I,Fin) are sometimes called P-ideals.(see for examples [1][12])
In the paper [20] the author showed that I-convergence implies I^K-convergence if AP(I, K) holds. Here we will introduce a new theorem regarding I and I^{K^*}-convergence.

Theorem 4.4. *Let I and K be two ideals on a set S and X be a first countable topological space. If the ideal I has the additive property with respect to P-ideal K then I-convergence implies I^{K^*}-convergence.*

Proof. Let $f : S \to X$ be a function such that I-$\lim f = x_0$. Let $\mathcal{B} = \{U_n : n \in \mathbb{N}\}$ be a countable base for X at the point x_0. Now from the definition of I-convergence we have $f^{-1}(U_n) \in F(I)$ for each n. Thus there exists $A \in F(I)$ with $A \subset_K f^{-1}(U_n)$ for each n i.e. $A \setminus f^{-1}(U_n) \in K$. Now it suffices to show that the function the $g : S \to X$ defined by

$$g(n) = \begin{cases} f(n) & \text{if } n \in A \\ x_0 & \text{if } n \notin A \end{cases}$$

is K^*-convergent to x_0. For $U_n \in B$, we have $g^{-1}(U_n) = (S \setminus A) \cup f^{-1}(U_n) = S \setminus (A \setminus f^{-1}(U_n))$ and since the set $A \setminus f^{-1}(U_n) \in K$ so $S \setminus (A \setminus f^{-1}(U_n)) \in F(K)$ i.e. $g^{-1}(U_n) \in F(K)$. Therefore g is K-convergent to x_0. Since K is P-ideal so g is also K^*-convergent to x_0. ☐

5 I^K-Limit Points

We modify the definition of I-limit points in the following way:

Definition 5.1. *Let $f : S \to X$ be a function and I be non-trivial ideal of S. Then $y \in X$ is called an I-limit point of f if there exists a set $M \subset S$ such that $M \notin I$ and the function $g : S \to X$ defined by*

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ y & \text{if } s \notin M \end{cases}$$

is Fin(S)-convergent to y.

Definition 5.2. *Let $f : S \to X$ be a function and I, K be two non-trivial ideals of S. Then $y \in X$ is called an I^K-limit point of f if there exists a set $M \subset S$ such that $M \notin I, K$ and the function $g : S \to X$ defined by*

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ y & \text{if } s \notin M \end{cases}$$

is K-convergent to y.

We denote respectively by $I(L_f)$ and $I^K(L_f)$ the collection of all I and I^K-limit points of f.

Theorem 5.1. *If K is an admissible ideal and $K \subset I$ then $I(L_f) \subset I^K(L_f)$*

Proof. Let $y \in I(L_f)$. Since y is an I-limit point of the function $f : S \to X$, then there exists a set $M \notin I$ such that and the function $g : S \to X$ defined by

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ y & \text{if } s \notin M \end{cases}$$

is Fin(S)-convergent to y. So for any open set U containing x the set $\{s : g(s) \notin U\} \in$ Fin. i.e. $\{s : g(s) \notin U\}$ is a finite set. So $\{s : g(s) \notin U\} \in K$, as K is an admissible ideal. Therefore g is K-convergent function. Again $M \notin I$ and $K \subset I$ so $M \notin I, K$. Thus y is I^K-limit point of f i.e. $y \in I^K(L_f)$. Hence the theorem is proved. □

Note 5.1. *If I is an admissible ideal and $I \subset K$ then $K(L_f) \subset I^K(L_f)$*

Theorem 5.2. *If every function $f : S \to X$ has an I^K-limit point then every infinite set A in X has an ω-accumulation point where cardinality of S is less or equal to cardinality of A.*

Proof. Let A be an infinite set. Define an injective function $f : S \to A \subset X$. Then f has an I^K-limit point say y. Then $\exists$ a set $M \subset S$ such that $M \notin I, K$ and the function $g : S \to X$ given by

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ y & \text{if } s \notin M \end{cases}$$

is K-convergent to y. Let U be open set containing y then $g^{-1}(U) = (S \setminus M) \cup f^{-1}(U) = S \setminus (M \setminus f^{-1}(U)) \in F(K)$ i.e. $M \setminus f^{-1}(U) \in K$. So $f^{-1}(U) \notin K$.(For if $f^{-1}(U) \in K$ then we get $M \in K$, which is a contradiction.) So $\{s : f(s) \in U\}$ is an infinite set. Consequently U contains infinitely many points of the function $f(s)$ in X. So U contains infinitely many elements of A. Thus y becomes ω-accumulation point of A. □

Theorem 5.3. *If X, τ is a Lindelof space such that every function $f : \mathbb{N} \to X$ has an I^K-limit point then (X, τ) is compact.*

Proof. Let (X, τ) be a Lindelof space such that every $f : N \to X$ has an I^K-limit point. We have to show that any open cover of space X has a finite subcover. Let $\{A_\alpha : \alpha \in \wedge\}$ be an open cover of the space X, where $\wedge$ is an index set. Since (X, τ) is a Lindelof space so this open cover admits a countable sub-cover say $\{A_1, A_2, \cdots, A_n, \cdots\}$. Proceeding inductively let $B_1 = A_1$ and for each $m > 1$, let B_m be the first member of the sequence of A's which is not covered by $B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_{m-1}$. If this choice becomes impossible at any stage then the sets already selected becomes a required finite sub-cover. Otherwise it is possible to select a point b_n in B_n for each positive integer n such that $b_n \notin B_r, r < n$.
Let $f : \mathbb{N} \to X$ be a function defined by $f(n) = b_n$. Now let x be an I^K-limit point of the function f. Then $x \in B_p$ for some p. Now from the definition of I^K-limit point we get $g^{-1}(B_p) = (\mathbb{N} \setminus M) \cup f^{-1}(B_p) = \mathbb{N} \setminus (M \setminus f^{-1}(B_p)) \in F(K)$ i.e. $M \setminus f^{-1}(B_p) \in K$. So the set $S = f^{-1}(B_p) = \{n \in \mathbb{N} : f(x_n) \in B_p\} \notin K$. Hence S must be an infinite subset of $\mathbb{N}$. So there is some $q > p$ such that $q \in S$ i.e. there exists some $q > p$ such that $f(x_q) \in B_p$ which leads to a contradiction. Thus the result follows. □

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