

# SOME PROPERTIES OF MULTISYMPLECTIC MANIFOLDS

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## Abstract

This lecture is devoted to review some of the main properties of multisymplectic geometry. In particular, after reminding the standard definition of multisymplectic manifold, we introduce its characteristic submanifolds, the canonical models, and other relevant kinds of multisymplectic manifolds, such as those where the existence of Darboux-type coordinates is assured. The Hamiltonian structures that can be defined in these manifolds are also studied, as well as other important properties, such as their invariant forms and the characterization by automorphisms.

**Key words:** *Multisymplectic forms, Bundles of forms, Hamiltonian structures, Invariant forms, Multi-vector fields.*

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## 1 Introduction

Although there are several geometrical models for describing classical field theories, namely, *polysymplectic*, *k-symplectic* and *k-cosymplectic manifolds* [12, 17, 23, 29, 30]; *multisymplectic manifolds* are the most general and complete tool for describing geometrically (covariant) first and higher-order field theories (see, for instance, [1, 4, 8, 14, 16, 18, 20, 24, 28, 31, 33] and the references quoted on them). All of these kinds of manifolds are generalizations of the concept of *symplectic manifold*, which are used to describe geometrically mechanical (autonomous) systems.

This talk is devoted to review some of the main properties of multisymplectic geometry and is mainly based on the results presented in [5, 6, 9, 13, 22]. In particular we discuss the following topics: the basic definition of *multisymplectic manifold* (in Section 2) and the *Hamiltonian structures* associated to a multisymplectic form (Section 3), the characteristic submanifolds of multisymplectic manifolds (Section 4), the canonical models and the existence of Darboux-type coordinates (Section 5), other kinds of relevant multisymplectic manifolds (Section 6) and, finally, some interesting theorems of invariance and characterization by automorphisms (Section 7).

All the manifolds are real, second countable and  $C^\infty$ . The maps and the structures are  $C^\infty$ . Sum over repeated indices is understood.

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## 2 Multisymplectic manifolds

(See [5, 6, 13] for more details).

**Definition 1.** Let  $M$  be a differentiable manifold, with  $\dim M = n$ , and  $\Omega \in \Omega^k(M)$  ( $\Omega^k(M)$  denotes the set of differentiable  $k$ -forms in  $M$ ), with  $k \leq n$ .

- The form  $\Omega$  is 1-nondegenerate if, for every  $p \in M$  and  $X_p \in T_pM$ ,

$$i(X_p)\Omega_p = 0 \iff X_p = 0 .$$

- The form  $\Omega$  is a multisymplectic form if it is closed and 1-nondegenerate.
- A multisymplectic manifold of degree  $k$  is a couple  $(M, \Omega)$ , where  $\Omega \in \Omega^k(M)$  is a multisymplectic form.

If  $\Omega$  is only closed then it is called a pre-multisymplectic form.

If  $\Omega$  is only 1-nondegenerate then it is an almost-multisymplectic form.

If  $\dim M \geq 2$ , then a multisymplectic  $k$ -form must have degree  $k \geq 2$ .

The property of 1-nondegeneracy can be characterized equivalently as follows: a differentiable  $k$ -form  $\Omega$  is 1-nondegenerate if, and only if, the vector bundle morphism

$$\begin{aligned} \Omega^\flat: TM &\rightarrow \Lambda^{k-1}T^*M \\ X_p &\mapsto i(X_p)\Omega_p \end{aligned}$$

and thus the corresponding morphism of  $C^\infty(M)$ -modules

$$\begin{aligned} \Omega^\flat: \mathfrak{X}(M) &\rightarrow \Omega^{k-1}(M) \\ X &\mapsto i(X)\Omega \end{aligned}$$

are injective.

Some examples of multisymplectic manifolds are the following: Multisymplectic manifolds of degree 2 are just *symplectic manifolds*. Multisymplectic manifolds of degree  $n$  are *orientable manifolds* and the multisymplectic forms are *volume forms*. Bundles of  $k$ -forms ( $k$ -multicotangent bundles) endowed with their *canonical*  $(k+1)$ -forms are multisymplectic manifolds of degree  $k+1$ . Jet bundles (over  $m$ -dimensional manifolds) endowed with the *Poincaré-Cartan*  $(m+1)$ -forms associated with (singular)*Lagrangian densities* are (pre)multisymplectic manifolds of degree  $m+1$ .

## 3 Hamiltonian structures in multisymplectic manifolds

(See [5, 6, 13] for more details).

**Definition 2.** A  $m$ -vector field (or a multivector field of degree  $m$ ) in a manifold  $M$  (with  $m \leq n = \dim M$ ) is any section of the bundle  $\Lambda^m(TM) \rightarrow M$  (that is, a contravariant, skewsymmetric tensor field of degree  $m$  in  $M$ ). The set of  $m$ -vector fields in  $M$  is denoted by  $\mathfrak{X}^m(M)$ .

The local description of multivector fields of degree  $m$  is the following: for every  $p \in M$ , there are a neighbourhood  $U_p \subset M$  and local vector fields  $X_1, \dots, X_r \in \mathfrak{X}(U_p)$ , with  $m \leq r \leq \dim M$ , such that

$$\mathbf{X}|_{U_p} = \sum_{1 \leq i_1 < \dots < i_m \leq r} f^{i_1 \dots i_m} X_{i_1} \wedge \dots \wedge X_{i_m} ; \text{ with } f^{i_1 \dots i_m} \in C^\infty(U_p) . \quad (1)$$

**Definition 3.** Let  $\mathbf{X} \in \mathfrak{X}^m(M)$  be a multivector field.

$\mathbf{X}$  is homogeneous (or decomposable) if there are  $X_1, \dots, X_m \in \mathfrak{X}(M)$  such that  $\mathbf{X} = X_1 \wedge \dots \wedge X_m$ .

$\mathbf{X}$  is locally homogeneous (decomposable) if, for every  $p \in M$ , there exist  $U_p \subset M$  and  $X_1, \dots, X_m \in \mathfrak{X}(U_p)$  such that  $\mathbf{X}|_{U_p} = X_1 \wedge \dots \wedge X_m$ .

**Remark 1.** Locally decomposable  $m$ -multivector fields  $\mathbf{X} \in \mathfrak{X}^m(M)$  are locally associated with  $m$ -dimensional distributions  $D \subset \text{TM}$ .

Every multivector field  $\mathbf{X} \in \mathfrak{X}^m(M)$  defines a contraction with differential forms  $\Omega \in \Omega^k(M)$ , which is the natural contraction between tensor fields. In particular, if  $\mathbf{X}$  is expressed as in (1), we have

$$\begin{aligned} i(\mathbf{X})\Omega|_{U_p} &= \sum_{1 \leq i_1 < \dots < i_m \leq r} f^{i_1 \dots i_m} i(X_{i_1} \wedge \dots \wedge X_{i_m})\Omega \\ &= \sum_{1 \leq i_1 < \dots < i_m \leq r} f^{i_1 \dots i_m} i(X_{i_1}) \dots i(X_{i_m})\Omega. \end{aligned}$$

Then, the  $k$ -form  $\Omega$  is said to be  $j$ -nondegenerate (for  $1 \leq j \leq k - 1$ ) if, for every  $p \in E$  and  $\mathbf{Y} \in \mathfrak{X}^j(M)$ , we have that  $i(\mathbf{Y}_p)\Omega_p = 0$  if, and only if,  $\mathbf{Y}_p = 0$ .

Then, for every form  $\Omega \in \Omega^k(M)$  ( $k \geq m$ ) we have the morphisms

$$\begin{array}{ccc} \Omega^b : \Lambda^m(\text{TM}) & \longrightarrow & \Lambda^{k-m}(\text{T}^*M) & ; & \Omega^b : \mathfrak{X}^m(M) & \longrightarrow & \Omega^{k-m}(M) \\ \mathbf{X}_p & \mapsto & i(\mathbf{X}_p)\Omega_p & & \mathbf{X} & \mapsto & i(\mathbf{X})\Omega \end{array} .$$

In addition, if  $\mathbf{X} \in \mathfrak{X}^m(M)$ , the Lie derivative of  $\Omega \in \Omega^k(M)$  is

$$\mathbb{L}(\mathbf{X})\Omega := [\text{d}, i(\mathbf{X})]\Omega = \text{d}i(\mathbf{X})\Omega - (-1)^m i(\mathbf{X})\text{d}\Omega .$$

**Definition 4.** Let  $(M, \Omega)$  be a multisymplectic manifold of degree  $k$ . A diffeomorphism  $\varphi : M \rightarrow M$  is a multisymplectomorphism if  $\varphi^*\Omega = \Omega$ .

**Definition 5.** Let  $(M, \Omega)$  be a multisymplectic manifold of degree  $k$ .

1. A vector field  $X \in \mathfrak{X}(M)$  is a locally Hamiltonian vector field if its flow consists of multisymplectic diffeomorphisms. It is equivalent to demand that  $\mathbb{L}(X)\Omega = 0$ , or equivalently,  $i(X)\Omega \in \Omega^{k-1}(M)$  is a closed form.

2. A multivector field  $\mathbf{X} \in \mathfrak{X}^m(M)$  ( $m < k$ ) is a locally Hamiltonian multivector field if  $\mathbb{L}(\mathbf{X})\Omega = 0$  or, what is equivalent,  $i(\mathbf{X})\Omega \in \Omega^{k-m}(M)$  is a closed form. Then, for every  $p \in M$ , exist  $U \subset M$  and  $\zeta \in \Omega^{k-m-1}(U)$  such that  $i(\mathbf{X})\Omega = \text{d}\zeta$  (on  $U$ ).

In this case  $\zeta \in \Omega^{k-m-1}(U)$  is said to be a locally Hamiltonian form for  $\mathbf{X}$ .

3.  $\mathbf{X} \in \mathfrak{X}^m(M)$  is a Hamiltonian multivector field if  $i(\mathbf{X})\Omega \in \Omega^{k-m}(M)$  is an exact form; that is, there exists  $\zeta \in \Omega^{k-m-1}(M)$  such that  $i(\mathbf{X})\Omega = \text{d}\zeta$ .

In this case  $\zeta \in \Omega^{k-m-1}(M)$  is said to be a Hamiltonian form for  $\mathbf{X}$ .

## 4 Characteristic submanifolds of multisymplectic manifolds

(See [6, 9] for more details and proofs).

**Definition 6.** Let  $(M, \Omega)$  be a multisymplectic manifold of degree  $k$ , and  $\mathcal{W}$  a distribution in  $M$ .  $\forall p \in M$  and  $1 \leq r \leq k - 1$ , the  $r$ -orthogonal multisymplectic vector space at  $p$  is

$$\mathcal{W}_p^{\perp, r} = \{v \in \mathbb{T}_p M \mid i(v \wedge w_1 \wedge \dots \wedge w_r)\Omega_p = 0, \forall w_1, \dots, w_r \in \mathcal{W}_p\},$$

the  $r$ -orthogonal multisymplectic complement of  $\mathcal{W}$  is the distribution  $\mathcal{W}^{\perp, r} := \cup_{p \in M} \mathcal{W}_p^{\perp, r}$ .

1.  $\mathcal{W}$  is an  $r$ -coisotropic distribution if  $\mathcal{W}^{\perp, r} \subset \mathcal{W}$ .
2.  $\mathcal{W}$  is an  $r$ -isotropic distribution if  $\mathcal{W} \subset \mathcal{W}^{\perp, r}$ .
3.  $\mathcal{W}$  is an  $r$ -Lagrangian distribution if  $\mathcal{W} = \mathcal{W}^{\perp, r}$ .
4.  $\mathcal{W}$  is a multisymplectic distribution if  $\mathcal{W} \cap \mathcal{W}^{\perp, k-1} = \{0\}$ .

**Remark 2.** For every distribution  $\mathcal{W}$ , we have that  $\mathcal{W}^{\perp, r} \subset \mathcal{W}^{\perp, r+1}$ . As a consequence, every  $r$ -isotropic distribution is  $(r + 1)$ -isotropic, and every  $r$ -coisotropic distribution is  $(r - 1)$ -coisotropic.

As a particular situation, if we have a submanifold  $N$  of multisymplectic manifold  $M$ , we can take as distribution in  $TM$  the tangent bundle  $TN$  and this allows us to establish a classification of these submanifolds as follows:

**Definition 7.** Let  $(M, \Omega)$  be a multisymplectic manifold of degree  $k$ , and  $N$  a submanifold of  $M$ . If  $0 \leq r \leq k - 1$ , then:

1.  $N$  is an  $r$ -coisotropic submanifold of  $M$  if  $TN^{\perp, r} \subset TN$ .
2.  $N$  is an  $r$ -isotropic submanifold of  $M$  if  $TN \subset TN^{\perp, r}$ .
3.  $N$  is an  $r$ -Lagrangian submanifold of  $M$  if  $TN = TN^{\perp, r}$ .
4.  $N$  is a multisymplectic submanifold of  $M$  if  $TN \cap TN^{\perp, k-1} = \{0\}$ .

And, in particular we have:

**Proposition 1.** A submanifold  $N$  of  $M$  is  $r$ -Lagrangian if, and only if, it is  $r$ -isotropic and maximal.

## 5 Canonical models for multisymplectic manifolds. Darboux-type coordinates

(See [9] for more details).

In the same way as the tangent bundle of a manifold is the canonical model for symplectic manifolds, the canonical models of multisymplectic manifolds are the *bundles of forms*. These canonical models are constructed as follows:

- If  $Q$  is a manifold, the bundle  $\rho: \Lambda^k(\mathbb{T}^*Q) \rightarrow Q$  is the *bundle of  $k$ -forms* in  $Q$ .

The *tautological form* (or *canonical form*)  $\Theta_Q \in \Omega^k(\Lambda^k(\mathbb{T}^*Q))$  is defined as follows: if  $\alpha \in \Lambda^k(\mathbb{T}^*Q)$ , and  $V_1, \dots, V_k \in \mathbb{T}_\alpha(\Lambda^k(\mathbb{T}^*Q))$ , then

$$\Theta_{Q\alpha}(V_1, \dots, V_k) = i(\rho_* V_k \wedge \dots \wedge \rho_* V_1)\alpha.$$

We have that,  $\Omega_Q = d\Theta_Q \in \Omega^{k+1}(\Lambda^k(\mathbb{T}^*Q))$  is a 1-nondegenerate form and then  $(\Lambda^k(\mathbb{T}^*Q), \Omega_Q)$  is a multisymplectic manifold of degree  $k + 1$ .

If  $(x^i, p_{i_1 \dots i_k})$  is a system of *natural coordinates* in  $U \subset \Lambda^k(\mathbb{T}^*Q)$ , then the local expressions of these canonical forms are

$$\Theta_Q|_U = p_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \quad , \quad \Omega_Q|_U = dp_{i_1 \dots i_k} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} .$$

These are called *Darboux coordinates* in  $\Lambda^k(\mathbb{T}^*Q)$ .

- If  $\pi: Q \rightarrow E$  is a fibration, let  $\rho_r: \Lambda_r^k(\mathbb{T}^*Q) \rightarrow Q$  be the subbundle of  $\Lambda^k(\mathbb{T}^*Q)$  made of the  $r$ -horizontal  $k$ -forms in  $Q$  with respect to the projection  $\pi$  (that is, the  $k$ -forms in  $Q$  vanishing when applied to  $r$   $\pi$ -vertical vector fields in  $Q$ ).

Let  $\Theta_Q^r \in \Omega^k(\Lambda_r^k(\mathbb{T}^*Q))$  be the pull-back of  $\Theta_Q$  to  $\Lambda_r^k(\mathbb{T}^*Q)$ . This is the tautological  $k$ -form in  $\Lambda_r^k(\mathbb{T}^*Q)$ , and then, if we construct  $\Omega_Q^r = d\Theta_Q^r \in \Omega^{k+1}(\Lambda_r^k(\mathbb{T}^*Q))$ , we have that  $(\Lambda_r^k(\mathbb{T}^*Q), \Omega_Q^r)$  is also a multisymplectic manifold of degree  $k+1$ .

In the same way, there are also charts of Darboux coordinates in  $\Lambda_r^k(\mathbb{T}^*Q)$  on which these canonical forms have a local expressions similar to the above ones.

Nevertheless, unlike symplectic manifolds, multisymplectic manifolds  $(M, \Omega)$  in general are not (locally) diffeomorphic to their canonical models, and additional properties are needed in order to have a Darboux theorem which assures the existence of Darboux-type coordinates [27]. In particular:

**Definition 8.** *A special multisymplectic manifold is a multisymplectic manifold  $(M, \Omega)$  of degree  $k$  such that:*

1.  $\Omega = d\Theta$ , for some  $\Theta \in \Omega^{k-1}(M)$ .
2. *There is a diffeomorphism  $\phi: M \rightarrow \Lambda^{k-1}(\mathbb{T}^*Q)$ ,  $\dim Q = n \geq k-1$ , (or  $\phi: M \rightarrow \Lambda_r^{k-1}(\mathbb{T}^*Q)$ ), and a fibration  $\pi: M \rightarrow Q$  such that  $\rho \circ \phi = \pi$  (resp.  $\rho_r \circ \phi = \pi$ ), and  $\phi^*\Theta_Q = \Theta$  (resp.  $\phi^*\Theta_Q^r = \Theta$ ).*

*(It is said that  $(M, \Omega)$  is multisymplectomorphic to a bundle of forms).*

In order to have multisymplectic manifolds which locally behave as the canonical models, it is necessary to endow them with additional structures; in particular, a 1-isotropic distribution  $\mathcal{W}$  satisfying some dimensionality conditions, and a “generalized distribution”  $\mathcal{E}$  defined on the space of leaves determined by  $\mathcal{W}$ . In fact, the existence of distributions satisfying certain properties is a necessary condition in order to establish Darboux-type theorems for different kinds of geometrical structures (presymplectic, cosymplectic,  $k$ -(pre)symplectic, and  $k$ -(pre)cosymplectic) [2, 7, 10, 11, 21]. Thus:

**Definition 9.** *Let  $(M, \Omega)$  be a multisymplectic manifold of degree  $k$ , and  $\mathcal{W}$  a regular 1-isotropic involutive distribution in  $(M, \Omega)$ .*

1. *A multisymplectic manifold of type  $(k, 0)$  is a triple  $(M, \Omega, \mathcal{W})$  such that, for every  $p \in M$ , we have that:*

- (a)  $\dim \mathcal{W}(p) = \dim \Lambda^{k-1}(\mathbb{T}_p M / \mathcal{W}(p))^*$ .
- (b)  $\dim(\mathbb{T}_p M / \mathcal{W}(p)) > k - 1$ .

2. *A multisymplectic manifold of type  $(k, r)$  ( $1 \leq r \leq k - 1$ ) is a quadruple  $(M, \Omega, \mathcal{W}, \mathcal{E})$ , where  $\mathcal{E}$  is a “generalized distribution” on  $M$  (this means that, for every  $p \in M$ ,  $\mathcal{E}(p)$  is a vector subspace of  $\mathbb{T}_p M / \mathcal{W}(p)$ ) and, denoting by  $\pi_p: \mathbb{T}_p M \rightarrow \mathbb{T}_p M / \mathcal{W}(p)$  the canonical projection, we have that:*

- (a)  $i(v_1 \wedge \dots \wedge v_r)\Omega_p = 0$ , for every  $v_i \in \mathbb{T}_p M$  such that  $\pi_p(v_i) \in \mathcal{E}(p)$  ( $i = 1, \dots, r$ ).

- (b)  $\dim \mathcal{W}(p) = \dim \Lambda_r^{k-1}(\mathbb{T}_p M / \mathcal{W}(p))^*$ , where the horizontal forms are considered with respect to the subspace  $\mathcal{E}(p)$ .
- (c)  $\dim(\mathbb{T}_p M / \mathcal{W}(p)) > k - 1$ .

And the fundamental result is the following:

**Proposition 2.** *Every multisymplectic manifold  $(M, \Omega)$  of type  $(k, 0)$  (resp. of type  $(k, r)$ ) is locally multisymplectomorphic to a bundle of  $(k - 1)$ -forms  $\Lambda^{k-1}(\mathbb{T}^*Q)$  (resp.  $\Lambda_r^{k-1}(\mathbb{T}^*Q)$ ), for some manifold  $Q$ ; that is, to a canonical multisymplectic manifold.*

*Therefore, there is a local chart of Darboux coordinates around every point  $p \in M$ .*

(Proof): The proof of this Theorem is very long and can be found in [9] (where, in particular, the relation with the canonical models is shown). □

Then we define:

**Definition 10.** *Multisymplectic manifolds which are locally multisymplectomorphic to bundles of forms are called locally special multisymplectic manifolds.*

Of course, every special multisymplectic manifold is a locally special multisymplectic manifold and hence has charts of *Darboux coordinates* at every point.

As an interesting example, if  $\pi: E \rightarrow M$  is a fiber bundle (where  $M$  is an  $m$ -dimensional oriented manifold),  $J^1\pi$  is the corresponding first-order jet bundle, and  $\mathcal{L}$  is a first-order hyperregular Lagrangian density, then the Poincaré-Cartan form  $\Omega_{\mathcal{L}} \in \Omega^{m+1}(J^1\pi)$  is a multisymplectic form and  $(J^1\pi, \Omega_{\mathcal{L}})$  is a special multisymplectic manifold [4, 15, 31].

## 6 Other kinds of multisymplectic manifolds

(See [13] for more details).

It is a well-known property of symplectic manifolds that the set of local Hamiltonian vector fields span locally the tangent bundle of the manifold and, hence, the action of the group of multisymplectic diffeomorphisms on  $M$  is transitive (in fact, these properties are a consequence of the existence of Darboux coordinates). Nevertheless, in general, these properties do not hold for multisymplectic manifolds and so locally Hamiltonian vector fields in a multisymplectic manifold  $(M, \Omega)$  do not span the tangent bundle of this manifold, and the group of multisymplectic diffeomorphisms does not act transitively on  $M$ . In order to achieve this we need to introduce additional conditions. Hence, we define:

**Definition 11.** *Let  $M$  be a differentiable manifold,  $p \in M$  and a compact set  $K$  with  $p \in \overset{\circ}{K}$ . A local Liouville or local Euler-like vector field at  $p$  with respect to  $K$  is a vector field  $\Delta^p \in \mathfrak{X}(M)$  such that:*

1.  $\text{supp } \Delta^p := \overline{\{x \in M \mid \Delta^p(x) \neq 0\}} \subset K$ ,
2. *there exists a diffeomorphism  $\varphi: \overset{\circ}{\text{supp } \Delta^p} \rightarrow \mathbb{R}^n$  such that  $\varphi_* \Delta^p = \Delta$ , where  $\Delta = x^i \frac{\partial}{\partial x^i}$  is the standard Liouville or dilation vector field in  $\mathbb{R}^n$ .*

**Definition 12.** *A form  $\Omega \in \Omega^k(M)$  is said to be locally homogeneous at  $p \in M$  if, for every open set  $U \subset M$  containing  $p$ , there exists a local Euler-like vector field  $\Delta^p$  at  $p$  with respect to a compact set  $K \subset U$  such that*

$$L(\Delta^p)\Omega = f\Omega; \quad f \in C^\infty(U).$$

$\Omega$  is locally homogeneous if it is locally homogeneous for all  $p \in M$ .

A locally homogeneous manifold is a couple  $(M, \Omega)$ , where  $M$  is a manifold and  $\Omega \in \Omega^k(M)$  is locally homogeneous.

Therefore we have that:

**Proposition 3.** *Let  $(M, \Omega)$  be a locally homogeneous multisymplectic manifold. Then the family of locally Hamiltonian vector fields span locally the tangent bundle of  $M$ ; that is,  $\forall p \in M, T_p M = \text{span}\{X_p \mid X \in \mathfrak{X}(M), \mathbb{L}(X)\Omega = 0\}$ .*

(Outline of the proof): The proof is very technical (see [13] for all the details). First, the existence of local Euler-like vector fields and their properties allows us to prove a previous result known as the *localization Lemma* which states that, if  $X$  is a locally Hamiltonian vector field, and  $x_0 \in M$ , then for each open set  $U \ni x_0$ , there exists an open neighbourhood  $V$  of  $x_0$  such that  $V \subset \bar{V} \subset U$ , with  $\bar{V}$  compact, and a locally Hamiltonian vector field  $X'$  such that  $X'$  coincides with  $X$  in  $V$  and vanishes identically outside of  $U$ . Then, the proof of this Proposition follows from the application of this Lemma and using again Euler-like vector fields.  $\square$

**Theorem 1.** *The group of multisymplectic diffeomorphisms  $G(M, \Omega)$  of a locally homogeneous multisymplectic manifold  $(M, \Omega)$  acts transitively on  $M$ .*

(Outline of the proof [13]): The proof is based on the application of Proposition 3 and the above mentioned localization Lemma.  $\square$

**Remark 3.** Locally special multisymplectic manifolds have local Euler-like vector fields; in particular, the local vector fields  $\left\{ x^i \frac{\partial}{\partial x^i} + p_{i_1 \dots i_k} \frac{\partial}{\partial p_{i_1 \dots i_k}} \right\}$ . Then, the corresponding multisymplectic forms are locally homogeneous.

As a consequence, if  $(M, \Omega)$  is a locally special multisymplectic manifold, then the family of locally Hamiltonian vector fields span locally the tangent bundle of  $M$  and the group of multisymplectic diffeomorphisms acts transitively on  $M$ . In fact, the local vector fields  $\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_{i_1 \dots i_k}} \right\}$  are locally Hamiltonian.

## 7 Invariance theorems

(See [13, 22] for more details).

As final remarks, in this Section we generalize some classical theorems of symplectic geometry in the field of multisymplectic manifolds.

The first one is a partial generalization of *Lee Hwa Chung's Theorem* for symplectic manifolds, which characterizes all the differential forms which are invariant under infinitesimal symplectomorphisms [25, 26, 19]:

**Theorem 2.** *Let  $(M, \Omega)$  be a locally homogeneous multisymplectic manifold of degree  $k$  and  $\alpha \in \Omega^p(M)$ , with  $p = k - 1, k$ , such that:*

- (i) *The form  $\alpha$  is invariant by the set of locally Hamiltonian  $(k - 1)$ -vector fields; that is,  $\mathbb{L}(\mathbf{X})\alpha = 0$ , for every  $\mathbf{X} \in \mathfrak{X}_{lh}^{k-1}(M)$ .*
- (ii) *The form  $\alpha$  is invariant by the set of locally Hamiltonian vector fields; that is,  $\mathbb{L}(Z)\alpha = 0$ , for every  $Z \in \mathfrak{X}_{lh}(M)$ .*

Therefore:

1. If  $p = k$  then  $\alpha = c\Omega$ , with  $c \in \mathbb{R}$ .
2. If  $p = k - 1$  then  $\alpha = 0$ .

(Outline of the proof [13]): It is an adaptation of the proofs given in [19, 26] for presymplectic and symplectic manifolds. From the hypothesis of the Theorem and bearing in mind the properties stated in Section 3, it can be proved that, for every  $X, Y \in \mathfrak{X}_{lh}^{k-1}(M)$ , the following relation holds:  $i(X)\Omega \wedge i(Y)\alpha + i(Y)\Omega \wedge i(X)\alpha = 0$ ; and taking  $X \in \ker^{k-1}\Omega$ , from here you get to  $i(X)\alpha = 0$ . Then it is proved that, if  $p = k - 1$  then  $\alpha = 0$ ; but, if  $p = k$ , then there exists a unique  $\alpha' \in C^\infty(M)$  such that  $i(X)\alpha = \alpha' i(X)\Omega$ , for every  $X \in \mathfrak{X}_{lh}^{k-1}(M)$ . Therefore, using some local properties of the locally Hamiltonian  $(k-1)$ -multivector fields, it is concluded that  $\alpha'$  is constant and the final conclusion follows straightforwardly from the last results and Proposition 3.  $\square$

The second one is a generalization of some *Theorems of Banyaga* for symplectic and other orientable manifolds [3]:

**Theorem 3.** *Let  $(M_i, \Omega_i)$ ,  $i = 1, 2$ , be local homogeneous multisymplectic manifolds of degree  $k$  and  $G(M_i, \Omega_i)$  their groups of multisymplectic automorphisms. Let  $\Phi: G(M_1, \Omega_1) \rightarrow G(M_2, \Omega_2)$  be a group isomorphism (which is a homeomorphism when  $G(M_i, \Omega_i)$  are endowed with the point-open topology). Then, there exists a diffeomorphism  $\varphi: M_1 \rightarrow M_2$ , such that :*

1.  $\Phi(\psi) = \varphi \circ \psi \circ \varphi^{-1}$ , for every  $\psi \in G(M_1, \Omega_1)$ .
2. The map  $\varphi_*$  maps locally Hamiltonian vector fields of  $(M_1, \Omega_1)$  into locally Hamiltonian vector fields of  $(M_2, \Omega_2)$ .
3. In addition, if  $\varphi_*$  maps locally Hamiltonian multivector fields of  $(M_1, \Omega_1)$  into locally Hamiltonian multivector fields of  $(M_2, \Omega_2)$ , then there is a constant  $c$  such that  $\varphi^*\Omega_2 = c\Omega_1$ .

(Outline of the proof [13]): By Theorem 1,  $G(M_i, \Omega_i)$  acts transitively on  $M_i$  and, by the main theorem in [34], there exists a bijective map  $\varphi: M_1 \rightarrow M_2$  such that  $\Phi(\psi) = \varphi \circ \psi \circ \varphi^{-1}$ . Then it is proved that  $\varphi$  is a homeomorphism and, adapting the proof in [3] to our setting, that it is also a smooth diffeomorphism. Therefore, as a consequence of this proof, we conclude that  $\varphi_*$  maps locally Hamiltonian vector fields into locally Hamiltonian vector fields. Finally, assuming the hypothesis of the third item, using Theorem 2 we have that  $\varphi^*\Omega_2 = c\Omega_1$ .  $\square$

## 8 Conclusions and discussion

Some of the main properties and characteristics of multisymplectic manifolds have been reviewed in this dissertation. Although most of them are generalizations of other well-known results for symplectic geometry, in the multisymplectic case, they are more elaborated and richer than for symplectic manifolds, in general; and it is for this reason that this is a topic of active research [32].

In particular, other interesting properties of multisymplectic manifolds which have not been analyzed here are, for instance: the *graded Lie algebra structure* of the sets of *Hamiltonian forms* and *Hamiltonian multivector fields* [5, 13], *polarized multisymplectic manifold* and its general structure theorem [6], as well as other properties and relevance of *r-coisotropic*, *r-isotropic* and, especially, of *r-Lagrangian distributions* and *submanifolds* [6, 9], and the characterizations of multisymplectic transformations [13].



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## References

- [1] V. Aldaya, J.A. de Azcárraga, “Geometric formulation of classical mechanics and field theory”, *Riv. Nuovo Cimento* **3**(10) (1980) 1–66. (doi: 10.1063/1.523904).
- [2] A. Awane, “ $k$ -symplectic structures”. *J. Math. Phys.* **33**(12) (1992) 4046–4052. (doi: 10.1063/1.529855).
- [3] A. Banyaga, “On isomorphic classical diffeomorphism groups. I”, *Proc. Am. Math. Soc.* **98**(1) (1986) 113–118. (doi: 10.1090/S0002-9939-1986-0848887-5).
- [4] J.F. Cariñena, M. Crampin, L.A. Ibort, “On the multisymplectic formalism for first order field theories”, *Diff. Geom. Appl.* **1**(4) (1991) 345–374. (doi: 10.1016/0926-2245(91)90013-Y).
- [5] F. Cantrijn, A. Ibort, M. de León, “Hamiltonian structures on multisymplectic manifolds”, *Rend. Sem. Mat. Univ. Pol. Torino* **54**(3) (1996) 225–236.
- [6] F. Cantrijn, A. Ibort, M. de León, “On the geometry of multisymplectic manifolds”. *J. Austral. Math. Soc. Ser.* **66** (1999) 303–330. (doi: 10.1017/S1446788700036636).
- [7] M. Crampin, F.A.E. Pirani, *Applicable Differential Geometry*. London Mathematical Society, *Lecture Notes Series* **59**. Cambridge University Press, Cambridge 1986. (doi: 10.1017/CBO9780511623905).
- [8] M. de León, J. Marín-Solano, J.C. Marrero, “A Geometrical approach to Classical Field Theories: A constraint algorithm for singular theories”, *Proc. New Develops. Dif. Geom.*, L. Tamassi-J. Szenthe eds., Kluwer Acad. Press, (1996) 291–312. (doi: 10.1007/978-94-009-0149-0\_22).
- [9] M. de León D. Martín de Diego, A. Santamaría-Merino, “Tulczyjew triples and Lagrangian submanifolds in classical field theories”, in *Applied Differential Geometry and Mechanics* (eds. W. Sarlet, F. Cantrijn), Univ. Gent, Gent, Academia Press, (2003), 21–47.
- [10] M. de León, I. Méndez, M. Salgado, “Regular  $p$ -almost cotangent structures”. *J. Korean Math. Soc.* **25**(2) (1988) 273–287.
- [11] M. de León, E. Merino, J. A. Oubiña, P. R. Rodrigues, M. Salgado, “Hamiltonian systems on  $k$ -cosymplectic manifolds”, *J. Math. Phys.* **39**(2) (1998) 876–893. (doi: 10.1063/1.532358).
- [12] M. de León, M. Salgado, S. Vilariño, *Methods of differential geometry in classical field theories:  $k$ -symplectic and  $k$ -cosymplectic approaches*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ (2016). (ISBN: 978-981-4699-75-4; 978-981-4699-77-8).
- [13] A. Echeverría-Enríquez, A. Ibort, M.C. Muñoz-Lecanda, N. Román-Roy, “Invariant Forms and Automorphisms of Locally Homogeneous Multisymplectic Manifolds”, *J. Geom. Mech.* **4**(4) (2012) 397–419. (doi: 10.3934/jgm.2012.4.397).
- [14] A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy, “Geometry of Lagrangian first-order classical field theories”. *Forts. Phys.* **44**(3) (1996) 235–280. (doi: 10.1002/prop.2190440304).
- [15] A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy, “On the multimomentum bundles and the Legendre Maps in field theories”, *Rep. Math. Phys.* **45**(1) (2000) 85–105. (doi: 10.1016/S0034-4877(00)88873-4).
- [16] P.L. García, “The Poincaré-Cartan invariant in the calculus of variations”, *Symp. Math.* **14** (1973) 219–246.
- [17] G. Giachetta, L. Mangiarotti, G. Sardanashevily, *New Lagrangian and Hamiltonian methods in field theory*, World Scientific Publishing Co., Inc., River Edge, NJ, 1997. (ISBN: 981-02-1587-8).

- [18] H. Goldschmidt, S. Sternberg, “The Hamilton-Cartan formalism in the calculus of variations”, *Ann. Inst. Fourier Grenoble* **23**(1) (1973) 203–267. (doi : 10.5802/aif.451).
- [19] J. Gomis, J. Llosa, N. Román-Roy, “Lee Hwa Chung theorem for presymplectic manifolds. Canonical transformations for constrained systems”, *J. Math. Phys.* **25**(5) (1984) 1348–1355. (doi: 10.1063/1.526303).
- [20] M.J. Gotay, J. Isenberg, J.E. Marsden, “Momentum maps and classical relativistic fields I: Covariant field theory”. arXiv:physics/9801019 (2004).
- [21] X. Gràcia, X. Rivas, N. Román-Roy, “Darboux theorem for  $k$ -presymplectic and  $k$ -precosymplectic manifolds” (2018) In preparation.
- [22] L. A. Ibort, “Multisymplectic geometry: Generic and exceptional”, in *IX Fall Workshop on Geometry and Physics*, Vilanova i la Geltrú, Spain (2000). (eds. X. Gràcia, J. Marín-Solano, M. C. Muñoz-Lecanda and N. Román-Roy), UPC Eds., (2001) 79–88.
- [23] I.V Kanatchikov, “Canonical structure of classical field theory in the polymomentum phase space”, *Rep. Math. Phys.* **41**(1) (1998) 49–90. (doi: 10.1016/S0034-4877(98)80182-1).
- [24] J. Kijowski, W.M. Tulczyjew, *A Symplectic Framework for Field Theories*, *Lect. Notes Phys.* **170**, Springer-Verlag, Berlin (1979). (ISBN: 978-3-540-09538-5).
- [25] L. Hwa Chung, “The universal integral invariants of Hamiltonian systems and applications to the theory of canonical transformations”, *Proc. Roy. Soc.* **62**(3) (1948) 237–246. (doi: 10.1017/S0080454100006646).
- [26] J. Llosa, N. Román-Roy, “Invariant forms and Hamiltonian systems: A geometrical setting”, *Int. J. Theor. Phys.* **27**(12) (1988) 1533–1543. (doi: 10.1007/BF00669290).
- [27] G. Martin, “A Darboux theorem for multi-symplectic manifolds”, *Lett. Math. Phys.* **16**(2) (1988) 133–138. (doi: 10.1007/BF00402020).
- [28] P.D. Prieto Martínez, N. Román-Roy, “A new multisymplectic unified formalism for second-order classical field theories”, *J. Geom. Mech.* **7**(2) (2015) 203–253. (doi: 10.3934/jgm.2015.7.203).
- [29] A.M. Rey, N. Román-Roy, M. Salgado, “Gunther’s formalism  $k$ -symplectic formalism in classical field theory: Skinner-Rusk approach and the evolution operator”, *J. Math. Phys.* **46**(5) (2005) 052901. (doi: 10.1063/1.1876872).
- [30] A.M. Rey, N. Román-Roy, M. Salgado, S. Vilariño, “ $k$ -cosymplectic classical field theories: Tulczyjew and Skinner-Rusk formulations”, *Math. Phys. Anal. Geom.* **15**(2) (2011) 1–35. (doi: 10.1007/s11040-012-9104-z).
- [31] N. Román-Roy, “Multisymplectic Lagrangian and Hamiltonian formalisms of classical field theories”, *Symm. Integ. Geom. Methods and Appl. (SIGMA)* **5** (2009) 100, 25pp. (doi: 10.3842/SIGMA.2009.100).
- [32] L. Ryvkin, T. Wurzbacher, “An invitation to multisymplectic geometry”, arXiv:1804.02553 [math.DG] (2018).
- [33] D.J. Saunders, *The geometry of jet bundles*, London Mathematical Society, *Lecture notes series* **142**, Cambridge University Press, Cambridge, New York 1989. (ISBN-13: 978-0521369480).
- [34] M. Wechsler, “Homeomorphism groups of certain topological spaces”, *Ann. Math.* **62**(3) (1954) 360–373. (doi: 10.2307/1970069).