Maximal displacement and population growth for branching Brownian motions

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Abstract

We study the maximal displacement and related population for a branching Brownian motion in Euclidean space in terms of the principal eigenvalue of an associated Schrödinger type operator. We first determine their growth rates on the survival event. We then establish the upper deviation for the maximal displacement under the possibility of extinction. Under the non-extinction condition, we further discuss the decay rate of the upper deviation probability and the population growth at the critical phase.

1 Introduction

We are concerned with the population growth rate related to the maximal displacement for a spatially inhomogeneous branching Brownian motion in Euclidean space \mathbb{R}^d . We proved in [35] that under the non-extinction condition, this rate is given in terms of the principal eigenvalue of an associated Schrödinger type operator. This result implies the existence of the phase transition for the growth rate. As its corollary, we determined the linear growth rate of the maximal displacement. We further established the upper deviation for the maximal displacement. In this paper, we first remove the non-extinction condition in [35] (Theorem 3.2, Corollary 3.3 and Theorem 3.6). We next discuss the decay rate of the tail probability of the maximal displacement as a refinement of the upper deviation under the non-extinction condition (Theorem 3.7). We finally prove that for $d \geq 3$, the population growth rate as mentioned before is polynomial at the critical phase under the same condition (Theorem 3.9).

The maximal displacement is one of the important research objects for branching Brownian motions because it reflects quantitatively the interplay between the randomness of branching and that of particle motions. The distribution of the maximal displacement is also related to the so called Fisher-Kolmogorov-Petrovskii-Piskunov equation (see, e.g., [11, 17, 28, 29, 35] and references therein). We would like to mention some of the results, which are related to the problems in this paper, for a one dimensional binary branching

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Brownian motion such that the splitting time is exponentially distributed with rate c > 0. As is well known, the maximal displacement R_t at time t satisfies the law of large numbers $R_t/t \to \sqrt{2c}$ $(t \to \infty)$ a.s. (see, e.g., Bramson [7] and Roberts [32] for more detailed properties). Chauvin and Rouault [11] determined the decay rate of the probability of the upper deviation type for the maximal displacement. Biggins [2, 3] further obtained the growth rate of the population right to the point δt at time t, where δ is a positive constant such that $\delta \neq \sqrt{2c}$. We note that the law of large numbers for the maximal displacement is valid also for $d \geq 2$ and the offspring distribution is more general so that extinction may occur; however, that distribution is assumed to be spatially independent (see, e.g., [23], [27], [30]). Biggins [2, 3] also mentioned that his result is valid under a setting similar to that as above.

Our interest here is how the spatial inhomogeneity of the branching structure affects the behavior of the population growth related to the maximal displacement. By the spatial inhomogeneity, we mean that the distributions of the splitting time and offspring depend on the trajectory of each particle and branching site, respectively (see Subsection 2.2 below for details). As for the population size, the long time behavior is characterized in terms of the principal eigenvalue of a Schrödinger type operator associated with the branching structure (see, e.g., [13, 14, 16, 22, 42]). This characterization also applies to the maximal displacement. In fact, when d = 1 and non-extinction occurs, Erickson [17] proved that even if the branching intensity is small at infinity, the maximal displacement grows linearly and its rate is determined by the same principal eigenvalue as mentioned before. This result is valid also for d > 2 if the branching intensity is spherically symmetric. We can further obtain the exponential growth rate of the population outside balls with time dependent radius for $d \ge 1$ under the setting similar to that in [17] (see [4], [22], [35]). In particular, we can allow the spherical asymmetry of the branching intensity. This result is regarded as a spatially inhomogeneous counterpart of Biggins [2, 3]. We note that the results of [17] and [4] are also extended by Lalley and Sellke [24] and Bocharov and Wang [5], respectively, to the model in which the branching intensity is inhomogeneous and not small at infinity.

In connection with the extinction problem, it is natural to allow the possibility of extinction for the spatially inhomogeneous model. More precisely, we would like to see the behavior of the maximal displacement under the *survival event*. Our results (Theorem 3.2, Corollary 3.3 and Theorem 3.6) say that the previous results in [35] remain true, and the effect of the possibility of extinction appears in the principal eigenvalue of the Schrödinger type operator as mentioned before. Our approach is similar to that of [35], which is an extension of [4] to the multidimensional branching Brownian motions with singular branching intensity; however, we overcome several difficulties arising from the fact that the total population is not increasing in time (see comments just after Lemmas 5.1 and 5.2). We also reveal the long time behavior of the expected Feynman-Kac functional associated with a *signed* measure (see (2.5) and comment just after Theorem 4.1 below).

Corollary 3.3 is partially regarded as a continuous (time-)space counterpart of Carmona and Hu [9] and Bulinskaya [8]. They studied the growth rate of the maximal displacement for a branching random walk on the integer lattice such that each particle moves as a general irreducible (non-symmetric) random walk and branching occurs only on finite points. They also allow the possibility of extinction. As for our model, even though we assume that each particle moves as a Brownian motion, branching can occur on a non-compact set.

Our refinement on the upper deviation type probability of the maximal displacement (Theorem 3.7) is regarded as a spatially inhomogeneous counterpart of Chauvin and Rouault [11]. In particular, we determine the exponential decay rate of this probability more precisely than [35], and bound the polynomial order. Our argument is also similar to that of [11]. For the lower bound of the probability especially, we utilize its Feynman-Kac expression originating from McKean [28, 29] (see (6.8) and (6.13)). Here we impose the non-extinction condition on the branching structure because of the inequality (6.16) below. We do not know if this condition can be dropped.

Theorem 3.9 provides an information about the long time behavior of the population around the forefront. In particular, we see that for $d \geq 3$, such population grows polynomially with dimension dependent growth rate. Our approach for Theorem 3.9 is a refinement of that applied to the non-critical case in [35]. To derive the polynomial growth, we make use of the long time behavior of the Feynman-Kac functional associated with a *positive* measure (see (4.2) below). This also imposes the non-extinction condition on the branching structure. To the best of the author's knowledge, there are no references on the population growth around the forefront.

The rest of this paper is organized as follows. In Section 2, we first introduce the Kato class measure and Feynman-Kac semigroups. We then introduce the model of branching Brownian motions. In Section 3, we present our results in this paper and their applications to some concrete models. In Section 4, we derive the exponential growth rate of the expectation of the Feynman-Kac functional associated with a signed measure. The subsequent sections are devoted to the proofs of the results presented in Section 3. In Appendix A.1, we show a convergence result for the expectation of the Feynman-Kac functional associated with a signed measure functional associated with a signed measure (see (2.3) below). We follow the argument of Carmona [10] and Takeda [39, Theorem 5.2]. In Appendix A.2, we discuss the relation between the regular growth and survival in order for the validity of the consequence of Theorem 3.2 and Corollary 3.3 on the survival event (see Remark 3.5). In Appendix A.3, we give a part on the elementary calculation in Section 4.

Throughout this paper, the letters c and C (with subscript) denote finite positive constants which may vary from place to place. For positive functions f(t) and g(t) on $(0,\infty)$, we write $f(t) \approx g(t)$ $(t \to \infty)$ if there exist positive constants T, c_1 and c_2 such that $c_1g(t) \leq f(t) \leq c_2g(t)$ for all $t \geq T$. We also write $f(t) \leq g(t)$ $(t \to \infty)$ if there exist positive constants T and c such that $f(t) \leq cg(t)$ for all $t \geq T$.

2 Preliminaries

2.1 Kato class measures and Feynman-Kac semigroups

Let $\mathbf{M} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \ge 0}, \{B_t\}_{t \ge 0}, \{P_x\}_{x \in \mathbb{R}^d}, \{\theta_t\}_{t \ge 0})$ be the Brownian motion on \mathbb{R}^d , where $\{\mathcal{F}_t\}_{t \ge 0}$ is the minimal admissible filtration and $\{\theta_t\}_{t \ge 0}$ is the time shift operator of the

paths such that $B_s \circ \theta_t = B_{s+t}$ identically for $s, t \ge 0$. Let

$$p_t(x,y) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x-y|^2}{2t}\right) \quad (x,y \in \mathbb{R}^d, t > 0).$$

Then $p_t(x, y)$ is the density of the transition function of **M**, that is,

$$P_x(B_t \in A) = \int_A p_t(x, y) \, \mathrm{d}y, \quad A \in \mathcal{B}(\mathbb{R}^d).$$

Here $\mathcal{B}(\mathbb{R}^d)$ is the totality of Borel subsets of \mathbb{R}^d . For $\alpha > 0$, let $G_{\alpha}(x, y)$ be the α -resolvent density of **M**:

$$G_{\alpha}(x,y) = \int_0^\infty e^{-\alpha t} p_t(x,y) \,\mathrm{d}t.$$

Then for any $\alpha > 0$,

$$G_{\alpha}(x,y) \sim c \frac{e^{-\sqrt{2\alpha}|x-y|}}{|x-y|^{(d-1)/2}} \quad (|x-y| \to \infty)$$
 (2.1)

(see, e.g., [35, (2.1)]). For $d \ge 3$, we denote by G(x, y) the Green function of M:

$$G(x,y) = \int_0^\infty p_t(x,y) \, \mathrm{d}t = \frac{\Gamma(d/2-1)}{2\pi^{d/2}} \frac{1}{|x-y|^{d-2}}.$$

We also define $G_0(x, y) := G(x, y)$.

According to [1, 12, 39], we first introduce two classes of measures:

Definition 2.1. (1) Let μ be a positive Radon measure on \mathbb{R}^d . Then μ belongs to the Kato class ($\mu \in \mathcal{K}$ in notation) if one of the following holds:

(i) d = 1 and

$$\sup_{x\in\mathbb{R}}\int_{|x-y|\leq 1}\mu(\mathrm{d} y)<\infty;$$

(ii) d = 2 and

$$\lim_{R \to +0} \sup_{x \in \mathbb{R}^2} \int_{|x-y| \le R} \log\left(\frac{1}{|x-y|}\right) \, \mu(\mathrm{d}y) = 0;$$

(iii) $d \geq 3$ and

$$\lim_{R \to +0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \le R} G(x, y) \mu(\mathrm{d}y) = 0.$$

(2) For $\beta > 0$, $\mu \in \mathcal{K}$ is β -Green tight ($\mu \in \mathcal{K}_{\infty}(\beta)$ in notation) if

$$\lim_{R \to \infty} \sup_{x \in \mathbb{R}^d} \int_{|y| \ge R} G_\beta(x, y) \,\mu(\mathrm{d}y) = 0.$$

When $d \geq 3$, $\mu \in \mathcal{K}$ is Green tight if the equality above is valid for $\beta = 0$.

We know by [39] and [40, Corollary 4.2 and Lemma 4.2] that $\mathcal{K}_{\infty}(\beta)$ ($\beta > 0$) is independent of β and $\mathcal{K}_{\infty}(0) \subsetneq \mathcal{K}_{\infty}(1)$. Define

$$\mathcal{K}_{\infty} = \begin{cases} \mathcal{K}_{\infty}(1) & (d = 1, 2), \\ \mathcal{K}_{\infty}(0) & (d \ge 3). \end{cases}$$

If μ is a Kato class measure with compact support in \mathbb{R}^d , then $\mu \in \mathcal{K}_{\infty}$ by definition. For examples of measures in \mathcal{K}_{∞} , see [35, Subsection 2.1] and references therein.

We next introduce the notion of positive continuous additive functionals. Let $A = \{A_t\}_{t\geq 0}$ be a $[0,\infty]$ -valued stochastic process on (Ω, \mathcal{F}) . We say that A is a positive continuous additive functional (in the strict sense) (PCAF in short) of **M** if

- (i) A_t is \mathcal{F}_t -measurable for any $t \ge 0$;
- (ii) There exists an event $\Lambda \in \mathcal{F}_{\infty}$, which satisfies $P_x(\Lambda) = 1$ for any $x \in \mathbb{R}^d$ and $\theta_t \Lambda \subset \Lambda$ for any t > 0, such that for any $\omega \in \Lambda$,
 - $A_0(\omega) = 0;$
 - $A_t(\omega)$ is finite and continuous in $t \in [0, \infty)$;
 - $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega)$ for any $s, t \ge 0$

(see, e.g., [1] and [18, p.401]). For each $\mu \in \mathcal{K}$, there exists a unique PCAF (A^{μ} in notation) such that for any nonnegative Borel function f,

$$\lim_{t \to 0} \frac{1}{t} \int_{\mathbb{R}^d} E_x \left[\int_0^t f(B_s) \, \mathrm{d}A_s^\mu \right] \, \mathrm{d}x = \int_{\mathbb{R}^d} f(x) \, \mu(\mathrm{d}x)$$

([1, Proposition 3.8] and [18, Theorems 5.1.3 and 5.1.7]). We note that if $d \ge 3$, then by [12, Proposition 2.2], any measure $\mu \in \mathcal{K}_{\infty}$ is *Green-bounded*:

$$\sup_{x \in \mathbb{R}^d} E_x \left[A^{\mu}_{\infty} \right] = \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} G(x, y) \, \mu(\mathrm{d}y) < \infty.$$
(2.2)

Let μ be a signed measure on \mathbb{R}^d such that $\mu = \mu^+ - \mu^-$ for some $\mu^+, \mu^- \in \mathcal{K}$ and define $A_t^{\mu} = A_t^{\mu^+} - A_t^{\mu^-}$. Then the multiplicative functional $e^{A_t^{\mu}}$ is called the *Feynman-Kac* functional. Using it, we define the *Feynman-Kac semigroup* $\{p_t^{\mu}\}_{t\geq 0}$ by

$$p_t^{\mu}f(x) := E_x \left[e^{A_t^{\mu}} f(B_t) \right], \quad f \in L^2(\mathbb{R}^d) \cap \mathcal{B}_b(\mathbb{R}^d),$$

where $\mathcal{B}_b(\mathbb{R}^d)$ stands for the set of bounded Borel functions on \mathbb{R}^d . Then $\{p_t^{\mu}\}_{t\geq 0}$ forms a strongly continuous symmetric semigroup on $L^2(\mathbb{R}^d)$ such that its L^2 -generator is formally expressed as the Schrödinger type operator $\mathcal{H}^{\mu} := -\Delta/2 - \mu$. We can further extend $\{p_t^{\mu}\}_{t\geq 0}$ to $L^p(\mathbb{R}^d)$ for any $p \in [1, \infty]$ ([1, Theorem 6.1 (i)]). For simplicity, we use the same notation for the extended semigroup. Let $\|\cdot\|_{p,q}$ denote the operator norm from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$. We then have **Theorem 2.2.** ([1, Theorems 6.1 (iii) and 7.1 (ii)]) Let μ be a signed measure on \mathbb{R}^d such that $\mu = \mu^+ - \mu^-$ for some $\mu^+, \mu^- \in \mathcal{K}$.

- (i) For any t > 0, $||p_t^{\mu}||_{p,q} < \infty$ for any $1 \le p \le q \le \infty$.
- (ii) For any $f \in \mathcal{B}_b(\mathbb{R}^d)$ and t > 0, $p_t^{\mu} f$ is a bounded continuous function on \mathbb{R}^d .

Assume that $\mu = \mu^+ - \mu^-$ for some $\mu^+, \mu^- \in \mathcal{K}_{\infty}(1)$. Define

$$\lambda(\mu) := \inf \left\{ \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, \mathrm{d}x - \int_{\mathbb{R}^d} u^2 \, \mathrm{d}\mu \mid u \in C_0^\infty(\mathbb{R}^d), \int_{\mathbb{R}^d} u^2 \, \mathrm{d}x = 1 \right\},$$

where $C_0^{\infty}(\mathbb{R}^d)$ stands for the set of smooth functions on \mathbb{R}^d with compact support. Then $\lambda(\mu)$ is the bottom of the L^2 -spectrum of \mathcal{H}^{μ} . In particular, if $\lambda(\mu) < 0$, then $\lambda(\mu)$ is the principal eigenvalue of \mathcal{H}^{μ} (see [38, Lemma 4.3] or [39, Theorem 2.8]) and the corresponding eigenfunction h has a bounded, continuous and strictly positive version by Theorem 2.2 (see, e.g., [39, Section 4]).

In what follows, we assume that $\lambda := \lambda(\mu) < 0$ and the eigenfunction h is bounded, continuous and strictly positive on \mathbb{R}^d such that $\int_{\mathbb{R}^d} h(x)^2 dx = 1$. Then by the proof of [39, Theorem 5.2] (see also Subsection A.1), we have for any $f \in \mathcal{B}_b(\mathbb{R}^d)$,

$$\lim_{t \to \infty} e^{\lambda t} E_x \left[e^{A_t^{\mu}} f(B_t) \right] = h(x) \int_{\mathbb{R}^d} f(y) h(y) \, \mathrm{d}y \quad (x \in \mathbb{R}^d).$$
(2.3)

2.2 Branching Brownian motions

In this subsection, we introduce the model of branching Brownian motions by following [19, 20, 21]. For $x \in \mathbb{R}^d$, let $\{p_n(x)\}_{n>0}$ be a sequence such that

$$0 \le p_n(x) \le 1$$
 $(n \ge 0)$ and $\sum_{n=0}^{\infty} p_n(x) = 1.$

Let τ be the nonnegative random variable defined on $(\Omega, \mathcal{F}, P_x)$, which is independent of the Brownian motion, of exponential distribution with rate 1; $P(\tau > t) = e^{-t}$ for any t > 0. Let μ be a Kato class measure on \mathbb{R}^d . We define

$$Z := \inf \left\{ t > 0 \mid A_t^{\mu} \ge \tau \right\}$$

so that

$$P_x(Z > t \mid \mathcal{F}_\infty) = e^{-A_t^\mu}.$$

We can describe the branching Brownian motion as follows: a Brownian particle $\{B_t\}_{t\geq 0}$ starts at $x \in \mathbb{R}^d$ according to the law P_x . At time Z, this particle splits into n particles with probability $p_n(B_Z)$. These particles then start at B_Z independently according to the law P_{B_Z} , and each of them continues the same procedure.

Let $(\mathbb{R}^d)^{(0)} = \{\Delta\}$ and $(\mathbb{R}^d)^{(1)} = \mathbb{R}^d$. For $n \ge 2$, we define the equivalent relation \sim on $(\mathbb{R}^d)^n = \underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_n$ as follows: for $\mathbf{x}^n = (x^1, \dots, x^n)$ and $\mathbf{y}^n = (y^1, \dots, y^n) \in (\mathbb{R}^d)^n$, we write $\mathbf{x} \sim \mathbf{y}$ if there exists a permutation σ on $\{1, 2, ..., n\}$ such that $y^i = x^{\sigma(i)}$ for any $i \in \{1, 2, ..., n\}$. If we define $(\mathbb{R}^d)^{(n)} = (\mathbb{R}^d)^n / \sim$ for $n \geq 2$ and $\mathbf{X} = \bigcup_{n=0}^{\infty} (\mathbb{R}^d)^{(n)}$, then n points in \mathbb{R}^d determine a point in $(\mathbb{R}^d)^{(n)}$. Hence we can define a branching Brownian motion $\overline{\mathbf{M}} = (\{\mathbf{B}_t\}_{t\geq 0}, \{\mathbf{P}_{\mathbf{x}}\}_{\mathbf{x}\in\mathbf{X}})$ on \mathbf{X} with branching rate μ and branching mechanism $\{p_n(x)\}_{n\geq 0}$.

Let T be the first splitting time of $\overline{\mathbf{M}}$ given by

$$\mathbf{P}_{x}(T > t \mid \sigma(B)) = P_{x}(Z > t \mid \mathcal{F}_{\infty}) = e^{-A_{t}^{\mu}} \quad (t > 0).$$
(2.4)

By definition, the first splitting time becomes small if the particle moves on the support of μ often. Let

$$Q(x) := \sum_{n=0}^{\infty} n p_n(x)$$

be the expected offspring number at branching site $x \in \mathbb{R}^d$. Denote by Z_t the total number of particles at time t, that is,

$$Z_t = n$$
 if $\mathbf{B}_t = (\mathbf{B}_t^1, \dots, \mathbf{B}_t^n) \in (\mathbb{R}^d)^{(n)}$.

Let

$$e_0 = \inf \{ t > 0 \mid Z_t = 0 \}$$

be the extinction time of $\overline{\mathbf{M}}$ and $u_e(x) = \mathbf{P}_x(e_0 < \infty)$. We say that $\overline{\mathbf{M}}$ becomes extinct if $u_e \equiv 1$.

We define for $f \in \mathcal{B}_b(\mathbb{R}^d)$,

$$Z_t(f) := \sum_{k=1}^{Z_t} f(\mathbf{B}_t^k).$$

For $A \in \mathcal{B}(\mathbb{R}^d)$, let $Z_t(A) := Z_t(\mathbf{1}_A)$ be the number of particles on the set A at time t. If the measure

$$\nu(\mathrm{d}x) := (Q(x) - 1)\mu(\mathrm{d}x)$$

is written as $\nu = \nu^+ - \nu^-$ for some $\nu^+, \nu^- \in \mathcal{K}$, then by the same way as in [34, Lemma 3.3], we have

$$\mathbf{E}_x \left[Z_t(f) \right] = E_x \left[e^{A_t^{\nu}} f(B_t) \right].$$
(2.5)

Assume that $\nu^+, \nu^- \in \mathcal{K}_{\infty}(1)$ and $\lambda := \lambda(\nu) < 0$. Let *h* be the eigenfunction of $\mathcal{H}^{(Q-1)\mu}$ corresponding to λ and

$$M_t := e^{\lambda t} Z_t(h) \quad (t \ge 0).$$

Since M_t is a nonnegative \mathbf{P}_x -martingale, the limit $M_{\infty} := \lim_{t\to\infty} M_t \in [0,\infty)$ exists \mathbf{P}_x -a.s. Furthermore, by [14, Theorem 3.7], there exists an event of \mathbf{P}_x -full probability measure such that we have on this event,

$$\lim_{t \to \infty} e^{\lambda t} Z_t(A) = M_\infty \int_A h(y) \,\mathrm{d}y \tag{2.6}$$

for any $A \in \mathcal{B}(\mathbb{R}^d)$ such that its boundary has zero Lebesgue measure.

3 Results

In this section, we state the results in this paper. Let $\overline{\mathbf{M}} = (\{\mathbf{B}_t\}_{t\geq 0}, \{P_{\mathbf{x}}\}_{\mathbf{x}\in\mathbf{X}})$ be a branching Brownian motion on \mathbf{X} with branching rate μ and branching mechanism $\{p_n(x)\}_{n\geq 0}$. We impose the next assumption on the branching rate and mechanism.

Assumption 3.1. Let $\nu(dx) = (Q(x) - 1)\mu(dx)$ and $\lambda = \lambda(\nu)$.

- (i) $\lambda < 0$.
- (ii) The measure ν is written as $\nu = \nu^+ \nu^-$ for some ν^+, ν^- such that for any $\beta > 0$, $\nu^+_{\beta}(\mathrm{d}x) := e^{\beta|x|} \nu^+(\mathrm{d}x)$ and $\nu^-_{\beta}(\mathrm{d}x) := e^{\beta|x|} \nu^-(\mathrm{d}x)$ belong to $\mathcal{K}_{\infty}(1)$.

The condition (ii) says that the measure ν is small enough at infinity. In particular, this condition implies that $\nu^{\pm}(\mathbb{R}^d) < \infty$ because $\nu^{\pm}(K) < \infty$ for any compact set $K \subset \mathbb{R}^d$ and there exist $\beta > 0$, c > 0, and R > 0 by (2.1) and the definition of $\mathcal{K}_{\infty}(1)$ such that

$$\int_{|y|\geq R} \nu^{\pm}(\mathrm{d}y) = \int_{|y|\geq R} \frac{e^{-\sqrt{2}|y|}}{|y|^{(d-1)/2}} e^{\sqrt{2}|y|} |y|^{(d-1)/2} \nu^{\pm}(\mathrm{d}y) \le c \sup_{x\in\mathbb{R}^d} \int_{|y|\geq R} G_1(x,y) \nu_{\beta}^{\pm}(\mathrm{d}y) < \infty.$$

3.1 Population growth and spread rate on the survival event

We first show that the results in [35, Theorem 2.8 and Corollary 2.9] are valid even if $p_0 \neq 0$. For R > 0, let $B_R := \{x \in \mathbb{R}^d \mid |x| < R\}$ and $Z_t^R := Z_t(B_R^c)$. Define for $\delta > 0$,

$$\Lambda_{\delta} := \begin{cases} \lambda + \sqrt{-2\lambda}\delta & (\delta \leq \sqrt{-2\lambda}), \\ \frac{\delta^2}{2} & (\delta > \sqrt{-2\lambda}). \end{cases}$$

Theorem 3.2. Under Assumption 3.1, the next assertions hold.

- (i) If $\delta > \sqrt{-\lambda/2}$, then $\lim_{t \to \infty} Z_t^{\delta t} = 0, \quad \mathbf{P}_x\text{-}a.s.$
- (ii) If $\mathbf{P}_x(M_{\infty} > 0) > 0$, then for any $\delta \in [0, \sqrt{-\lambda/2})$, $\lim_{t \to \infty} \frac{1}{t} \log Z_t^{\delta t} = -\Lambda_{\delta}, \quad \mathbf{P}_x(\cdot \mid M_{\infty} > 0) \text{-}a.s.$

This result says that for
$$\delta > \sqrt{-\lambda/2}$$
, all the particles at time t will be inside the ball $B_{\delta t}$ for all sufficiently large time $t > 0$. On the other hand, for $\delta < \sqrt{-\lambda/2}$, the population outside the ball $B_{\delta t}$ at time t grows exponentially with rate $-\Lambda_{\delta}$.

Let L_t be the maximal norm of the particles alive at time t:

$$L_t := \begin{cases} \max_{1 \le k \le Z_t} |\mathbf{B}_t^k| & (t < e_0), \\ 0 & (t \ge e_0). \end{cases}$$

By the same way as for the proof of [35, Corollary 3.4], the next corollary follows from Theorem 3.2.

Corollary 3.3. Under Assumption 3.1, if $\mathbf{P}_x(M_{\infty} > 0) > 0$, then

$$\lim_{t \to \infty} \frac{L_t}{t} = \sqrt{-\frac{\lambda}{2}}, \quad \mathbf{P}_x(\cdot \mid M_\infty > 0) \text{-} a.s.$$

Remark 3.4. By the same way as for the proofs of Theorem 3.2 and Corollary 3.3, we can show that the population growth and spread rate are uniform in direction. Let $\langle \cdot, \cdot \rangle$ be the standard inner product in \mathbb{R}^d . For a unit vector $r \in \mathbb{R}^d$, define $B_R^r := \{x \in \mathbb{R}^d \mid \langle x, r \rangle < R\}$ and $Z_t^{\delta t, r} := Z_t((B_{\delta t}^r)^c)$. Let

$$L_t^r := \begin{cases} \max_{1 \le k \le Z_t} \langle \mathbf{B}_t^k, r \rangle & (t < e_o), \\ 0 & (t \ge e_o) \end{cases}$$

be the maximal displacement in direction r of particles alive at time t. For $t < e_0$, we denote by $K_r(t)$ the index of a particle at time t such that $L_t^r = \langle \mathbf{B}_t^{K_r(t)}, r \rangle$. Then

- Theorem 3.2 holds by replacing $Z_t^{\delta t}$ with $Z_t^{\delta t,r}$;
- Corollary 3.3 holds by replacing L_t with L_t^r and

$$\lim_{t \to \infty} \frac{\mathbf{B}_t^{K_r(t)}}{t} = \sqrt{-\frac{\lambda}{2}}r, \quad \mathbf{P}_x(\cdot \mid M_\infty > 0)\text{-a.s.}$$

We omit the proof of these assertions because the argument is similar to that of [35, Theorem 2.10] by using Remark 4.2 and Lemma 5.2 below.

Remark 3.5. (i) If $\mu \in \mathcal{K}_{\infty}$ and

$$\sup_{x \in \mathbb{R}^d} \sum_{n=0}^{\infty} n^2 p_n(x) < \infty,$$

then $M_t \in L^2(\mathbf{P}_x)$ by [34, Lemma 3.4] and thus $\mathbf{P}_x(M_\infty > 0) > 0$. (ii) Suppose that $\mu \in \mathcal{K}_\infty$ and $\mathbf{P}_x(M_\infty > 0) > 0$. If d = 1, 2, then by Proposition A.4 below, we have for any $\delta > 0$,

$$\lim_{t \to \infty} \frac{1}{t} \log Z_t^{\delta t} = -\Lambda_{\delta}, \quad \mathbf{P}_x(\cdot \mid e_0 = \infty) \text{-a.s.}$$

and

$$\lim_{t \to \infty} \frac{L_t}{t} = \sqrt{-\frac{\lambda}{2}}, \quad \mathbf{P}_x(\cdot \mid e_0 = \infty)\text{-a.s.}$$

On the other hand, if $d \ge 3$, then $\mathbf{P}_x(M_\infty = 0) > 0$ as in [35, Remark 2.7]. Moreover, since branching occurs only finite times on the event $\{M_\infty = 0\}$ by Proposition A.4, Z_t becomes a random positive constant eventually on the event $\{e_0 = \infty\} \cap \{M_\infty = 0\}$. Therefore,

$$\limsup_{t \to \infty} \frac{L_t}{\sqrt{2t \log \log t}} = 1, \quad \mathbf{P}_x(\cdot \mid \{e_0 = \infty\} \cap \{M_\infty = 0\}) \text{-a.s.}$$

The remarks here apply to $Z_t^{\delta t,r}$ and L_t^r in Remark 3.4.

3.2 Upper deviation for the maximal position

We next show that the upper deviation of L_t in [35, Lemma 3.10] is true local uniformly with respect to the initial point and even if we allow the possibility of extinction.

Theorem 3.6. Under Assumption 3.1, the next assertions hold.

(i) If
$$\delta \ge \sqrt{-2\lambda}$$
, then for any compact set $K \subset \mathbb{R}^d$,
$$\lim_{t \to \infty} \frac{1}{t} \log \inf_{x \in K} \mathbf{P}_x \left(\frac{L_t}{t} \ge \delta\right) = \lim_{t \to \infty} \frac{1}{t} \log \sup_{x \in K} \mathbf{P}_x \left(\frac{L_t}{t} \ge \delta\right) = -\frac{\delta^2}{2}.$$
(3.1)

(ii) If $\sqrt{-\lambda/2} < \delta < \sqrt{-2\lambda}$ and $\mathbf{P}_x(M_\infty > 0) > 0$, then for any $x \in \mathbb{R}^d$ and for any compact set $K \subset \mathbb{R}^d$,

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbf{P}_x \left(\frac{L_t}{t} \ge \delta \right) = \lim_{t \to \infty} \frac{1}{t} \log \sup_{y \in K} \mathbf{P}_y \left(\frac{L_t}{t} \ge \delta \right) = -\lambda - \sqrt{-2\lambda}\delta.$$
(3.2)

Under restricted conditions, we can get the decay rate of $\mathbf{P}_x(L_t/t \ge \delta)$ as $t \to \infty$ more precisely.

Theorem 3.7. Assume that $p_0 \equiv 0$ and μ is a Kato class measure with compact support in \mathbb{R}^d . If $\lambda < 0$, then the next assertions hold.

(i) If $\delta \ge \sqrt{-2\lambda}$, then for any $x \in \mathbb{R}^d$, there exist positive constants C_1 , C_2 and T such that for all $t \ge T$,

$$C_1 e^{-\delta^2 t/2} t^{(d-2)/2} \le \mathbf{P}_x \left(\frac{L_t}{t} \ge \delta\right) \le C_2 e^{-\delta^2 t/2} t^{(d-2)/2}.$$
 (3.3)

(ii) If $\sqrt{-\lambda/2} < \delta < \sqrt{-2\lambda}$ and $\sup_{x \in \mathbb{R}^d} \sum_{n=1}^{\infty} n^2 p_n(x) < \infty$, then for any $x \in \mathbb{R}^d$, there exist positive constants C_3 , C_4 and T such that for all $t \ge T$,

$$C_3 e^{(-\lambda - \sqrt{-2\lambda}\delta)t} t^{(d-2)/2} \le \mathbf{P}_x \left(\frac{L_t}{t} \ge \delta\right) \le C_4 e^{(-\lambda - \sqrt{-2\lambda}\delta)t} t^{(d-1)/2}.$$
 (3.4)

The lower bound of (3.3) is valid even if $p_0 \neq 0$.

Remark 3.8. If $\delta \geq \sqrt{-2\lambda}$, then by (3.3), (2.5) and Remark 4.2 below,

$$\mathbf{P}_{x}\left(\frac{L_{t}}{t} \ge \delta\right) = \mathbf{P}_{x}(Z_{t}^{\delta t} \ge 1) \asymp \mathbf{E}_{x}\left[Z_{t}^{\delta t}\right] \quad (t \to \infty).$$
(3.5)

However, we do not know if (3.5) holds for $\sqrt{-\lambda/2} < \delta < \sqrt{-2\lambda}$.

Chauvin and Rouault [11, Theorems 2 and 3] established a precise asymptotic behavior of the form like (3.5) and a Yaglom type theorem for branching Brownian motions on \mathbb{R} with constant branching rate and mechanism.

3.3 Population growth at the critical phase

According to Theorem 3.2 and [35, Theorem 2.8], the growth order of $Z_t^{\delta t}$ undergoes the phase transition at $\delta = \sqrt{-\lambda/2}$. We finally discuss this order at the critical phase under some restricted condition. For $\varepsilon > 0$, let $R_{\varepsilon}(t) = t^{(d+3)/2} (\log t) (\log \log t)^{1+\varepsilon}$ and $r_{\varepsilon}(t) = t^{(d-2)/2} / (\log \log t)^{\varepsilon}$.

Theorem 3.9. Assume that $p_0 \equiv 0$ and $\lambda < 0$. Let $\delta = \sqrt{-\lambda/2}$.

(i) If μ is compactly supported in \mathbb{R}^d , then for any $\varepsilon > 0$,

 \mathbf{P}_x (there exists T > 0 such that $Z_t^{\delta t} \leq R_{\varepsilon}(t)$ for all $t \geq T$) = 1.

(ii) If $d \ge 3$ and $\mathbf{P}_x(M_\infty > 0) > 0$, then for any $\varepsilon > 0$,

 \mathbf{P}_x (there exists T > 0 such that $Z_t^{\delta t} \ge r_{\varepsilon}(t)$ for all $t \ge T \mid M_{\infty} > 0$) = 1.

If $d \geq 3$, then under the full conditions of Theorem 3.9 (i) and (ii), $Z_t^{\delta t}$ grows polynomially $\mathbf{P}_x(\cdot \mid M_{\infty} > 0)$ -a.s. at $\delta = \sqrt{-\lambda/2}$ and the growth rate depends on the spatial dimension d. However, the exact growth rate remains unknown.

3.4 Examples

We apply the results in this paper to some concrete branching Brownian motions on \mathbb{R}^d .

Example 3.10. Assume that d = 1. Let δ_a be the Dirac measure at $a \in \mathbb{R}$. For $\gamma > 0$, let $G_{\alpha}^{\gamma \delta_0}(x, y)$ be the α -resolvent of the one dimensional Brownian motion killed by $\gamma \delta_0$:

$$G_{\alpha}^{\gamma\delta_0}(x,y) = \frac{1}{\sqrt{2\alpha}} \left(e^{-\sqrt{2\alpha}|x-y|} - \frac{\gamma}{\sqrt{2\alpha} + \gamma} e^{-\sqrt{2\alpha}(|x|+|y|)} \right)$$

(see, e.g., [6, p.123, 7]).

For a > 0, let $\mu = \gamma \delta_0 - \beta \delta_a$ for some $\beta > 0$ and $\gamma > 0$, and let $\lambda = \lambda(\mu)$. By the same way as in [34, Example 4.1], we have

$$1 = \beta G^{\gamma \delta_0}_{-\lambda}(a, a) = \frac{\beta}{\sqrt{-2\lambda}} \left(1 - \frac{\gamma}{\sqrt{-2\lambda} + \gamma} e^{-2a\sqrt{-2\lambda}} \right)$$

If we let $A = \sqrt{-2\lambda}$, then the equality above becomes

$$A^{2} - (\beta - \gamma)A = \beta\gamma(1 - e^{-2aA}),$$

This equation has a positive solution if and only if $\beta > \gamma/(1 + 2a\gamma)$. Note that this condition is derived by Takeda [37, Example 3.10].

Let $\overline{\mathbf{M}}$ be a branching Brownian motion on \mathbb{R} with branching rate $\mu = \delta_0 + \delta_a$ and branching mechanism $\{p_n(x)\}_{n\geq 0}$ such that $p_0(x) + p_2(x) \equiv 1$. We let $p = p_2(0)$ and $q = p_2(a)$ so that Q(0) = 2p and Q(a) = 2q. Assume that $q \geq p$ for simplicity. Then $\lambda((Q-1)\mu) < 0$ if one of the next conditions hold:

- $p \ge 1/2$ and q > 1/2;
- p < 1/2, q > 1/2 and

$$2q - 1 > \frac{1 - 2p}{1 + 2a(1 - 2p)}$$

In particular, Theorem 3.2, Corollary 3.3, Remark 3.5 and Theorem 3.6 hold under one of these conditions.

Let $\overline{\mathbf{M}}$ be a branching Brownian motion on \mathbb{R} with branching rate $\mu = c\delta_0$ for some c > 0 and branching mechanism $\{p_n(x)\}_{n\geq 0}$ such that $p_2(0) = 1$. Then Theorems 3.7 and 3.9 (i) are valid \mathbf{P}_x -a.s. with $\lambda = -c^2/2$ by Remark 3.5. Theorem 3.2 and Corollary 3.3 are proved by Bocharov and Harris [4]. Corollary 3.3 also follows from [17].

Example 3.11. Assume that $d \ge 2$. For r > 0, let δ_r be the surface measure on the sphere $\{x \in \mathbb{R}^d \mid |x| = r\}$. Let $\mu = \gamma \delta_r - \beta \delta_R$ for $\beta > 0$ and $\gamma > 0$, and let $\lambda = \lambda(\mu)$. Define

$$\check{\lambda} = \inf\left\{\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x + \beta \int_{\mathbb{R}^d} u^2 \,\mathrm{d}\delta_r \mid u \in C_0^\infty(\mathbb{R}^d), \gamma \int_{\mathbb{R}^d} u^2 \,\mathrm{d}\delta_R = 1\right\}.$$

Then by [41, Lemma 2.2], $\lambda < 0$ if and only if $\check{\lambda} < 1$.

(i) Assume first that r < R. Then by [37, Example 3.10],

$$\check{\lambda} = \begin{cases} \frac{\beta(r/R)}{\gamma(1+2\beta r \log(R/r))} & (d=2), \\ \frac{d-2}{\gamma} \left[\frac{\beta(r/R)^{d-1}}{d-2+2\beta r(1-(r/R)^{d-2})} + \frac{1}{2R} \right] & (d \ge 3). \end{cases}$$

In particular, $\lambda < 0$ if and only if $\lambda < 1$, that is,

•
$$\gamma > \frac{\beta(r/R)}{1 + 2\beta r \log(R/r)}$$
 $(d = 2),$
• $\gamma > (d - 2) \left[\frac{\beta(r/R)^{d-1}}{d - 2 + 2\beta r (1 - (r/R)^{d-2})} + \frac{1}{2R} \right]$ $(d \ge 3).$

(ii) Assume next that r > R. Then by the same way as in [37, Example 3.10] together with [6, 2.3.1 in p.398–399 and 2.3.1 in p.507], we have

$$\check{\lambda} = \begin{cases} \frac{\beta(r/R)}{\gamma(1 + 2\beta r \log(r/R))} & (d = 2), \\ \frac{(d - 2)(d - 2 + 2\beta r)}{2\gamma R(d - 2 + 2\beta r(1 - (R/r)^{d-2}))} & (d \ge 3). \end{cases}$$

Therefore, $\lambda < 0$ if and only if $\check{\lambda} < 1$, that is,

• $\gamma > \frac{\beta(r/R)}{1+2\beta r \log(r/R)}$ (d=2),

•
$$\gamma > \frac{(d-2)(d-2+2\beta r)}{2R(d-2+2\beta r(1-(R/r)^{d-2}))} \quad (d \ge 3).$$

Let $\overline{\mathbf{M}}$ be a branching Brownian motion on \mathbb{R}^d with branching rate $\mu = \delta_r + \delta_R$ and spherically symmetric branching mechanism $\{p_n(x)\}_{n=0}^{\infty}$ such that $p_0(x) + p_2(x) \equiv 1$. We use the notation $p_n(x) = p_n(|x|)$. Let $p = p_2(r)$ and $q = p_2(R)$. If p < 1/2 and q > 1/2, then by using (i) and (ii) with $\beta = 1 - 2p$ and $\gamma = 2q - 1$, we can give a necessary and sufficient condition for $\lambda((Q-1)\mu) < 0$ in terms of p and q. Theorem 3.2, Corollary 3.3, Remark 3.5 and Theorem 3.6 are valid under this condition.

Let $\overline{\mathbf{M}}$ be a binary branching Brownian motion with branching rate $\mu = c\delta_R$ for some c > 0. Then $\lambda < 0$ if and only if c > (d-2)/2. Theorems 3.7 and 3.9 are valid under this condition.

4 Growth of Feynman-Kac functionals

To prove results in Section 3, we reveal the growth rate of the expectation of $Z_t^{\delta t}$. By (2.5),

$$\mathbf{E}_{x}\left[Z_{t}^{\delta t}\right] = E_{x}\left[e^{A_{t}^{\nu}}; |B_{t}| \ge \delta t\right].$$

$$(4.1)$$

Then ν is a signed measure in general because we allow $p_0 \neq 0$. In what follows, we discuss the growth rate of the expectation similar to that at the right hand side of (4.1).

Let μ^+ and μ^- be positive Radon measures on \mathbb{R}^d in $\mathcal{K}_{\infty}(1)$. Let $\mu = \mu^+ - \mu^-$ and $\lambda := \lambda(\mu)$. For $\delta > 0$, we define

$$\Lambda_{\delta} := \begin{cases} \lambda + \sqrt{-2\lambda}\delta & (\delta \le \sqrt{-2\lambda}), \\ \frac{\delta^2}{2} & (\delta > \sqrt{-2\lambda}). \end{cases}$$

Let a(t) be a function on $(0, \infty)$ such that a(t) = o(t) $(t \to \infty)$ and $R(t) := \delta t + a(t)$.

Theorem 4.1. If Assumption 3.1 is satisfied by replacing ν with μ , then for any $\delta > 0$, $x \in \mathbb{R}^d$ and for any compact set $K \subset \mathbb{R}^d$,

$$\lim_{t \to \infty} \frac{1}{t} \log \sup_{y \in K} E_y \left[e^{A_t^{\mu}}; |B_t| \ge R(t) \right] = \lim_{t \to \infty} \frac{1}{t} \log E_x \left[e^{A_t^{\mu}}; |B_t| \ge R(t) \right] = -\Lambda_{\delta}.$$

In [35, Proposition 3.3], we proved Theorem 4.1 under the condition that $\mu^- = 0$. The proof relied on the L^p -independence of the spectral bounds of the Schördinger type operator ([36, 38, 39]) together with the fact that A_t^{μ} is nondecreasing for $\mu^- = 0$. Instead of these properties, we make use of the gaugeability for Feynman-Kac semigroups developed by [12, 37].

Remark 4.2. (i) Let r be a unit vector in \mathbb{R}^d . Then the assertion in Theorem 4.1 is true even if we replace $|B_t|$ by $\langle B_t, r \rangle$. The proof is similar to that of Theorem 4.1 by noting that $\{\langle B_t, r \rangle\}_{t>0}$ is the one dimensional Brownian motion.

(ii) Suppose that μ^+ is a Kato class measure with compact support in \mathbb{R}^d and $\mu^- = 0$. Then for any $x \in \mathbb{R}^d$, we have as $t \to \infty$,

$$E_x\left[e^{A_t^{\mu}}; |B_t| \ge R(t)\right] \asymp \begin{cases} e^{(-\lambda t - \sqrt{-2\lambda}R(t))} t^{(d-1)/2} & (\delta < \sqrt{-2\lambda}), \\ e^{-R(t)^2/(2t)} t^{(d-2)/2} & (\delta \ge \sqrt{-2\lambda}). \end{cases}$$
(4.2)

In [35, Proposition 3.1], we proved this result under the condition that $a(t) \equiv 0$, but the proof still works for $a(t) \neq 0$. If we replace $|B_t|$ by $\langle B_t, r \rangle$ in (4.2), then the consequence is valid with d = 1.

We split the proof of Theorem 4.1 into the following three lemmas.

Lemma 4.3. If $\lambda < 0$, then there exists $p_* > 1$ for any $\varepsilon > 0$ such that for all $p \in (1, p_*)$,

$$\sup_{x\in\mathbb{R}^d} E_x \left[\sup_{t\geq 0} \left(e^{p(\lambda-\varepsilon)t + A_t^{p\mu}} \right) \right] < \infty.$$

Proof. Let $\varepsilon > 0$ and p > 1. Since $\lambda - \varepsilon < 0$, we have for any $t \ge 0$,

$$e^{p(\lambda-\varepsilon)t+A_t^{p\mu}} = e^{p(\lambda-\varepsilon)t} \left(1 + \int_0^t e^{A_s^{p\mu}} \, \mathrm{d}A_s^{p\mu}\right) \le e^{p(\lambda-\varepsilon)t} \left(1 + \int_0^t e^{A_s^{p\mu}} \, \mathrm{d}A_s^{p\mu^+}\right) \le 1 + \int_0^\infty e^{p(\lambda-\varepsilon)u} e^{A_s^{p\mu}} \, \mathrm{d}A_s^{p\mu^+},$$
(4.3)

which implies that

$$E_x \left[\sup_{t \ge 0} \left(e^{p(\lambda - \varepsilon)t + A_t^{p\mu}} \right) \right] \le 1 + E_x \left[\int_0^\infty e^{p(\lambda - \varepsilon)s} e^{A_s^{p\mu}} \, \mathrm{d}A_s^{p\mu^+} \right]$$

$$= 1 + \hat{E}_x \left[\int_0^\zeta e^{A_s^{p\mu}} \, \mathrm{d}A_s^{p\mu^+} \right].$$
(4.4)

Here \hat{P}_x and ζ are the law and life time, respectively, of the killed process of **M** by the exponential distribution with rate $p(-\lambda + \varepsilon)$.

Since

$$\inf\left\{\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x + p(-\lambda + \varepsilon) \int_{\mathbb{R}^d} u^2 \,\mathrm{d}x - p \int_{\mathbb{R}^d} u^2 \,\mathrm{d}\mu \mid u \in C_0^\infty(\mathbb{R}^d), \int_{\mathbb{R}^d} u^2 \,\mathrm{d}x = 1\right\}$$
$$= \lambda(p\mu) + p(-\lambda + \varepsilon) \to \varepsilon > 0 \ (p \to 1 + 0),$$

we see by (4.4), [37, Lemma 3.5] and [12, Corollary 2.9 and Theorem 5.2] that there exists $p_* > 1$ for any $\varepsilon > 0$ such that for any $p \in (1, p_*)$,

$$\sup_{x \in \mathbb{R}^d} E_x \left[\sup_{t \ge 0} \left(e^{p(\lambda - \varepsilon)t + A_t^{p\mu}} \right) \right] \le 1 + \sup_{x \in \mathbb{R}^d} \hat{E}_x \left[\int_0^{\zeta} e^{A_s^{p\mu}} \, \mathrm{d}A_s^{p\mu^+} \right] < \infty.$$
(4.5)

This completes the proof.

Lemma 4.4. Under the same setting as in Theorem 4.1, for any compact set $K \subset \mathbb{R}^d$,

$$\limsup_{t \to \infty} \frac{1}{t} \log \sup_{y \in K} E_y \left[e^{A_t^{\mu}}; |B_t| \ge R(t) \right] \le -\Lambda_{\delta}.$$
(4.6)

Proof. Since

$$e^{A_t^{\mu}} = 1 + \int_0^t e^{A_s^{\mu}} \, \mathrm{d}A_s^{\mu} \le 1 + \int_0^t e^{A_s^{\mu}} \, \mathrm{d}A_s^{\mu^+},$$

we have by the Markov property (see [31, p.186, Exercise 1.13] and [35, (3.8)]),

$$E_{x}[e^{A_{t}^{\mu}};|B_{t}| \geq R(t)] \leq E_{x}\left[1 + \int_{0}^{t} e^{A_{s}^{\mu}} dA_{s}^{\mu^{+}};|B_{t}| \geq R(t)\right]$$

$$= P_{x}(|B_{t}| \geq R(t)) + E_{x}\left[\int_{0}^{t} e^{A_{s}^{\mu}} \mathbf{1}_{\{|B_{t}| \geq R(t)\}} dA_{s}^{\mu^{+}}\right]$$

$$= P_{x}(|B_{t}| \geq R(t)) + E_{x}\left[\int_{0}^{t} e^{A_{s}^{\mu}} P_{B_{s}}(|B_{t-s}| \geq R(t)) dA_{s}^{\mu^{+}}\right] = (I) + (II).$$

(4.7)

Let $K \subset \mathbb{R}^d$ be a compact set. Then there exist positive constants T and $c = c_{\delta,K}$ such that for all $x \in K$ and $t \geq T$, we have |x| < R(t) and

$$(\mathbf{I}) \leq P_x(|B_t - x| \geq R(t) - |x|) = \frac{\omega_d}{(2\pi)^{d/2}} \int_{(R(t) - |x|)/\sqrt{t}}^{\infty} e^{-r^2/2} r^{d-1} dr$$

$$\leq c e^{-R(t)^2/(2t)} \left(\frac{R(t)}{\sqrt{t}}\right)^{(d-2)/2},$$
(4.8)

where ω_d is the area of the unit surface in \mathbb{R}^d .

For any $\varepsilon_1 \in (0, \delta)$, we let

$$(\mathrm{II}) = E_x \left[\int_0^t e^{A_s^{\mu}} P_{B_s}(|B_{t-s}| \ge R(t)) \mathbf{1}_{\{|B_s| < \varepsilon_1 t\}} \, \mathrm{d}A_s^{\mu^+} \right] + E_x \left[\int_0^t e^{A_s^{\mu}} P_{B_s}(|B_{t-s}| \ge R(t)) \mathbf{1}_{\{|B_s| \ge \varepsilon_1 t\}} \, \mathrm{d}A_s^{\mu^+} \right] = (\mathrm{II})_1 + (\mathrm{II})_2.$$

$$(4.9)$$

Then by the same way as in (4.8),

$$(\mathrm{II})_{1} \leq \frac{\omega_{d}}{(2\pi)^{d/2}} E_{x} \left[\int_{0}^{t} e^{A_{s}^{\mu}} \left(\int_{(R(t)-|B_{s}|)/\sqrt{t-s}}^{\infty} e^{-r^{2}/2} r^{d-1} \,\mathrm{d}r \right) \mathbf{1}_{\{|B_{s}|<\varepsilon_{1}t\}} \,\mathrm{d}A_{s}^{\mu^{+}} \right] \\ \leq \frac{\omega_{d}}{(2\pi)^{d/2}} E_{x} \left[\int_{0}^{t} e^{A_{s}^{\mu}} \left(\int_{(R(t)-\varepsilon_{1}t)/\sqrt{t-s}}^{\infty} e^{-r^{2}/2} r^{d-1} \,\mathrm{d}r \right) \,\mathrm{d}A_{s}^{\mu^{+}} \right].$$

$$(4.10)$$

For any $\varepsilon_2 > 0$,

$$\int_{0}^{t} e^{A_{s}^{\mu}} \left(\int_{(R(t)-\varepsilon_{1}t)/\sqrt{t-s}}^{\infty} e^{-r^{2}/2} r^{d-1} \, \mathrm{d}r \right) \, \mathrm{d}A_{s}^{\mu^{+}} \\
\leq \sup_{0 \leq s \leq t} \left(e^{(\lambda-\varepsilon_{2})s+A_{s}^{\mu}} \right) \int_{0}^{t} e^{(-\lambda+\varepsilon_{2})s} \left(\int_{(R(t)-\varepsilon_{1}t)/\sqrt{t-s}}^{\infty} e^{-r^{2}/2} r^{d-1} \, \mathrm{d}r \right) \, \mathrm{d}A_{s}^{\mu^{+}}.$$
(4.11)

Since $-\lambda + \varepsilon_2 > 0$, we have

$$\begin{split} &\frac{\partial}{\partial s} \left[e^{(-\lambda+\varepsilon_2)s} \left(\int_{(R(t)-\varepsilon_1 t)/\sqrt{t-s}}^{\infty} e^{-r^2/2} r^{d-1} \, \mathrm{d}r \right) \right] \\ &= (-\lambda+\varepsilon_2) e^{(-\lambda+\varepsilon_2)s} \left(\int_{(R(t)-\varepsilon_1 t)/\sqrt{t-s}}^{\infty} e^{-r^2/2} r^{d-1} \, \mathrm{d}r \right) \\ &- e^{(-\lambda+\varepsilon_2)s} \frac{(R(t)-\varepsilon_1 t)}{2(t-s)^{3/2}} \exp\left(-\frac{(R(t)-\varepsilon_1 t)^2}{2(t-s)} \right) \left(\frac{R(t)-\varepsilon_1 t}{\sqrt{t-s}} \right)^{d-1} \\ &\geq -e^{(-\lambda+\varepsilon_2)s} \frac{(R(t)-\varepsilon_1 t)^2}{2(t-s)^{3/2}} \exp\left(-\frac{(R(t)-\varepsilon_1 t)^2}{2(t-s)} \right) \left(\frac{R(t)-\varepsilon_1 t}{\sqrt{t-s}} \right)^{d-1}. \end{split}$$

Hence by the integration by parts formula,

$$\begin{split} &\int_0^t e^{(-\lambda+\varepsilon_2)s} \left(\int_{(R(t)-\varepsilon_1 t)/\sqrt{t-s}}^\infty e^{-r^2/2} r^{d-1} \,\mathrm{d}r \right) \,\mathrm{d}A_s^{\mu^+} \\ &= -\int_0^t A_s^{\mu^+} \frac{\partial}{\partial s} \left[e^{(-\lambda+\varepsilon_2)s} \left(\int_{(R(t)-\varepsilon_1 t)/\sqrt{t-s}}^\infty e^{-r^2/2} r^{d-1} \,\mathrm{d}r \right) \right] \,\mathrm{d}s \\ &\leq \int_0^t A_s^{\mu^+} e^{(-\lambda+\varepsilon_2)s} \frac{(R(t)-\varepsilon_1 t)}{2(t-s)^{3/2}} \exp\left(-\frac{(R(t)-\varepsilon_1 t)^2}{2(t-s)} \right) \left(\frac{R(t)-\varepsilon_1 t}{\sqrt{t-s}} \right)^{d-1} \,\mathrm{d}s \\ &\leq \frac{(R(t)-\varepsilon_1 t)^d}{2} A_t^{\mu^+} \int_0^t \frac{e^{(-\lambda+\varepsilon_2)s}}{(t-s)^{(d+2)/2}} \exp\left(-\frac{(R(t)-\varepsilon_1 t)^2}{2(t-s)} \right) \,\mathrm{d}s. \end{split}$$

Combining this with (4.10) and (4.11), we get

$$(\mathrm{II})_{1} \leq \frac{\omega_{d}}{2(2\pi)^{d/2}} E_{x} \left[\sup_{0 \leq s \leq t} \left(e^{(\lambda - \varepsilon_{2})s + A_{s}^{\mu}} \right) A_{t}^{\mu^{+}} \right] \times (R(t) - \varepsilon_{1}t)^{d} \int_{0}^{t} \frac{e^{(-\lambda + \varepsilon_{2})s}}{(t-s)^{(d+2)/2}} \exp\left(-\frac{(R(t) - \varepsilon_{1}t)^{2}}{2(t-s)}\right) \mathrm{d}s.$$

$$(4.12)$$

We will evaluate the integral in the right hand side of (4.12) in Appendix A.3 below. Here we evaluate the expectation in the right hand side of (4.12). For any p > 1 and q > 1 with 1/p + 1/q = 1, we have by the Cauchy-Schwartz inequality,

$$E_x \left[\sup_{0 \le s \le t} \left(e^{(\lambda - \varepsilon_2)s + A_s^{\mu}} \right) A_t^{\mu^+} \right] \le E_x \left[\sup_{0 \le s \le t} \left(e^{p(\lambda - \varepsilon_2)s + A_s^{p\mu}} \right) \right]^{1/p} E_x \left[(A_t^{\mu^+})^q \right]^{1/q}.$$
(4.13)

Then by Lemma 4.3, there exists $p_* \in (1, \infty)$ for $\varepsilon_2 > 0$ such that for all $p \in (1, p_*)$,

$$E_x \left[\sup_{0 \le s \le t} \left(e^{p(\lambda - \varepsilon_2)s + A_s^{p\mu}} \right) \right] \le \sup_{x \in \mathbb{R}^d} E_x \left[\sup_{s \ge 0} \left(e^{p(\lambda - \varepsilon_2)s + A_s^{p\mu}} \right) \right] < \infty.$$

If we take $p \in (1, p_*)$ so that q > 1 is a positive integer, then by [15, p.73, Corollary to Proposition 3.8], there exist positive constants c_1 and c_2 such that for all $t \ge 0$,

$$E_x \left[(A_t^{\mu^+})^q \right]^{1/q} \le (q!)^{1/q} \sup_{y \in \mathbb{R}^d} E_y \left[A_t^{\mu^+} \right] \le c_1 + c_2 t.$$

Hence by (4.13),

$$E_x \left[\sup_{0 \le s \le t} \left(e^{(\lambda - \varepsilon_2)s + A_s^{\mu}} \right) A_t^{\mu^+} \right] \le c(\varepsilon_2)(c_1 + c_2 t).$$
(4.14)

Then by (4.12), (A.13), and (4.14), we have as $t \to \infty$,

$$(II)_{1} \lesssim \begin{cases} e^{-(\delta-\varepsilon_{1})^{2}t/2}t^{d/2} & (\delta > \sqrt{-2\lambda}), \\ e^{(-\lambda+\varepsilon_{2})t}e^{-\sqrt{2(-\lambda+\varepsilon_{2})}(R(t)-\varepsilon_{1}t)}t^{(d+1)/2} & (\delta \le \sqrt{-2\lambda}). \end{cases}$$
(4.15)

Define $\nu_c(\mathrm{d}x) = e^{2c|x|/\varepsilon_1}\mu^+(\mathrm{d}x)$ for c > 0. Since $\nu_c \in \mathcal{K}_{\infty}(1)$ by assumption, it follows by [37, Lemma 3.5] and [12, Corollary 2.9 and Theorem 5.2] again that for any $c > -\lambda$,

$$\sup_{x \in \mathbb{R}^d} E_x \left[\int_0^\infty e^{-cs} e^{A_s^{\mu}} \, \mathrm{d}A_s^{\nu_c^+} \right] < \infty$$

and thus

$$(II)_{2} \leq E_{x} \left[\int_{0}^{t} e^{A_{s}^{\mu}} \mathbf{1}_{\{|B_{s}| > \varepsilon_{1}t\}} \, \mathrm{d}A_{s}^{\mu^{+}} \right] \leq e^{-ct} E_{x} \left[\int_{0}^{\infty} e^{-cs} e^{A_{s}^{\mu}} e^{2c|B_{s}|/\varepsilon_{1}} \, \mathrm{d}A_{s}^{\mu^{+}} \right] \\ \leq e^{-ct} \sup_{x \in \mathbb{R}^{d}} E_{x} \left[\int_{0}^{\infty} e^{-cs} e^{A_{s}^{\mu}} \, \mathrm{d}A_{s}^{\nu_{c}^{+}} \right] = c_{3}e^{-ct}.$$
(4.16)

By taking c > 0 large enough, we see by (4.7), (4.8), (4.9), (4.15) and (4.16) that

$$\limsup_{t \to \infty} \frac{1}{t} \log \sup_{x \in K} E_x \left[e^{A_t^{\mu}}; |B_t| \ge R(t) \right]$$

$$\leq \begin{cases} -\frac{(\delta - \varepsilon_1)^2}{2} & (\delta > \sqrt{-2\lambda}), \\ \left(-\lambda + \varepsilon_2 - \sqrt{2(-\lambda + \varepsilon_2)}(\delta - \varepsilon_1) \right) \lor \left(-\frac{\delta^2}{2} \right) & (\delta \le \sqrt{-2\lambda}). \end{cases}$$

Letting $\varepsilon_2 \to +0$ and then $\varepsilon_1 \to +0$, we arrive at (4.6).

Lemma 4.5. Under the same setting as in Theorem 4.1, if $\delta \geq \sqrt{-2\lambda}$, then for any compact set $K \subset \mathbb{R}^d$,

$$\liminf_{t \to \infty} \frac{1}{t} \log \inf_{x \in K} E_x \left[e^{A_t^{\mu}}; |B_t| \ge R(t) \right] \ge -\frac{\delta^2}{2}.$$
(4.17)

On the other hand, if $\delta < \sqrt{-2\lambda}$, then for any $x \in \mathbb{R}^d$,

$$\liminf_{t \to \infty} \frac{1}{t} \log E_x \left[e^{A_t^{\mu}}; |B_t| \ge R(t) \right] \ge -\lambda - \sqrt{-2\lambda}\delta.$$
(4.18)

Proof. We first assume that $\delta \geq \sqrt{-2\lambda}$. For any p > 1, we have by the Cauchy-Schwarz inequality,

$$E_x\left[e^{A_t^{\mu}}; |B_t| \ge R(t)\right] \ge E_x\left[e^{-A_t^{\mu^-}}; |B_t| \ge R(t)\right] \ge \frac{P_x(|B_t| \ge R(t))^p}{E_x\left[e^{A_t^{\mu^-}/(p-1)}; |B_t| \ge R(t)\right]^{p-1}}.$$
(4.19)

Let

$$\alpha_p = \inf\left\{\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x - \frac{1}{p-1}\int_{\mathbb{R}^d} u^2 \,\mathrm{d}\mu^- \mid u \in C_0^\infty(\mathbb{R}^d), \int_{\mathbb{R}^d} u^2 \,\mathrm{d}x = 1\right\}.$$

Since there exists $p_* > 1$ such that $\sqrt{-2\lambda} \ge \sqrt{-2\alpha_{p_*}} > 0$, it follows by Lemma 4.4 that for any compact set $K \subset \mathbb{R}^d$,

$$\limsup_{t \to \infty} \frac{1}{t} \log \sup_{x \in K} E_x \left[e^{A_t^{\mu^-} / (p_* - 1)}; |B_t| \ge R(t) \right] \le -\frac{\delta^2}{2}.$$
 (4.20)

By [35, Appendix A], we also have as $t \to \infty$,

$$P_x(|B_t| \ge R(t)) \ge P_0(|B_t| \ge R(t)) \sim \frac{\omega_d}{(2\pi)^{d/2}} e^{-R(t)^2/(2t)} \left(\frac{R(t)}{\sqrt{t}}\right)^{d-2} = \frac{\omega_d}{(2\pi)^{d/2}} e^{-\delta^2 t/2 - \delta a(t) - a(t)^2/(2t)} \left(\delta\sqrt{t} + \frac{a(t)}{\sqrt{t}}\right)^{d-2}.$$
(4.21)

By taking $p = p_*$ in (4.19), we have (4.17) by (4.20) and (4.21).

We next assume that $\delta < \sqrt{-2\lambda}$. Fix $p \in (0, 1)$ with $\delta/(1-p) \ge \sqrt{-2\lambda}$. Let G be a relatively compact open subset in \mathbb{R}^d and $K = \overline{G}$. Then by the Markov property,

$$E_{x}\left[e^{A_{t}^{\mu}};|B_{t}|\geq R(t)\right] = E_{x}\left[e^{A_{pt}^{\mu}}E_{B_{pt}}\left[e^{A_{(1-p)t}^{\mu}};|B_{(1-p)t}|\geq R(t)\right]\right]$$

$$\geq E_{x}\left[e^{A_{pt}^{\mu}}E_{B_{pt}}\left[e^{A_{(1-p)t}^{\mu}};|B_{(1-p)t}|\geq R(t)\right];B_{pt}\in G\right]$$

$$\geq E_{x}\left[e^{A_{pt}^{\mu}};B_{pt}\in G\right]\inf_{y\in K}E_{y}\left[e^{A_{(1-p)t}^{\mu}};|B_{(1-p)t}|\geq R(t)\right].$$
(4.22)

Hence by (4.17) and (2.3),

$$\liminf_{t \to \infty} \frac{1}{t} \log E_x \left[e^{A_t^{\mu}}; |B_t| \ge R(t) \right] \ge -\lambda p - \frac{\delta^2}{2(1-p)}$$

Since the right hand side above attains the maximum value $-\lambda - \sqrt{-2\lambda}\delta$ at $p = 1 - \delta/\sqrt{-2\lambda} \in (0, 1)$, we obtain (4.18).

Theorem 4.1 is a consequence of Lemmas 4.4 and 4.5.

Remark 4.6. Suppose that μ^- is compactly supported in \mathbb{R}^d and $\delta \ge \sqrt{-2\lambda}$. As in the proof of Lemma 4.5, we take $p_* > 1$ such that $\sqrt{-2\lambda} \ge \sqrt{-2\alpha_{p_*}} > 0$. If $R(t) = \delta t + a(t)$ for a(t) = O(1) $(t \to \infty)$, then by the same calculation as in (4.19) and Remark 4.2, there exists c > 0 such that for each $x \in \mathbb{R}^d$ and for all sufficiently large t > 0,

$$E_x\left[e^{A_t^{\mu}}; |B_t| \ge R(t)\right] \ge \frac{P_x(|B_t| \ge R(t))^{p_*}}{E_x\left[e^{A_t^{\mu^-}/(p_*-1)}; |B_t| \ge R(t)\right]^{p_*-1}} \ge ce^{-R(t)^2/(2t)} t^{(d-2)/2}$$

5 Proof of Theorem 3.2

We first discuss the upper bound of $Z_t^{\delta t}$.

Lemma 5.1. Under Assumption 3.1, for any $\delta \geq 0$,

$$\limsup_{t \to \infty} \frac{1}{t} \log Z_t^{\delta t} \le -\Lambda_{\delta}, \quad \mathbf{P}_x \text{-} a.s.$$

Under the additional condition that $p_0 \equiv 0$, we proved Lemma 5.1 as [35, Lemma 3.8] In the proof, we used the fact that Z_t is nondecreasing, but this property fails for $p_0 \neq 0$. To avoid the use of this property, we modify the proof of [35, Lemma 3.8] by Theorem 4.1 together with the introduction of another branching Brownian motion such that it does not become extinct and its population distribution is comparable to the original one. This approach is similar to that of [26, Subsection 4.2] (see also [17, Section 3] for a similar discussion).

Proof. Assume that $\delta > 0$. For i = 1, 2, ... and for any $\varepsilon > 0$, we have by the Chebyshev inequality,

$$\mathbf{P}_{x}\left(\max_{i\leq s\leq i+1}Z_{s}^{\delta i}\geq e^{(-\Lambda_{\delta}+\varepsilon)i}\right)\leq e^{-(-\Lambda_{\delta}+\varepsilon)i}\mathbf{E}_{x}\left[\max_{i\leq s\leq i+1}Z_{s}^{\delta i}\right].$$
(5.1)

Then by the Markov property,

$$\mathbf{E}_{x}\left[\max_{i\leq s\leq i+1}Z_{s}^{\delta i}\right] = \mathbf{E}_{x}\left[\mathbf{E}_{\mathbf{B}_{i}}\left[\max_{0\leq s\leq 1}Z_{s}^{\delta i}\right]\right] \leq \mathbf{E}_{x}\left[\sum_{k=1}^{Z_{i}}\mathbf{E}_{\mathbf{B}_{i}^{k}}\left[\max_{0\leq s\leq 1}Z_{s}^{\delta i}\right]\right] = E_{x}\left[e^{A_{i}^{(Q-1)\mu}}\mathbf{E}_{B_{i}}\left[\max_{0\leq s\leq 1}Z_{s}^{\delta i}\right]\right].$$
(5.2)

Let $\overline{\mathbf{M}} = (\{\mathbf{\tilde{B}}_t\}_{t \ge 0}, \{\mathbf{\tilde{P}}_{\mathbf{x}}\}_{\mathbf{x} \in \mathbf{X}})$ be a branching Brownian motion on \mathbb{R}^d with branching rate μ and branching mechanism $\{q_n\}_{n=1}^{\infty}$ given by

$$q_1 = p_0 + p_1, \quad q_n = p_n \ (n = 2, 3, \dots).$$
 (5.3)

Namely, for the process $\overline{\mathbf{M}}$, if a particle has no child at the splitting time, then we add one branching Brownian particle at the branching site. Hence if $\tilde{Z}_t(A)$ denotes the number of particles on a set $A \in \mathcal{B}(\mathbb{R}^d)$ at time t for $\overline{\overline{\mathbf{M}}}$, then for any $\mathbf{x} \in \mathbf{X}$,

$$\mathbf{P}_{\mathbf{x}}\left(\max_{0\leq s\leq t} Z_s(A)\geq k\right)\leq \tilde{\mathbf{P}}_{\mathbf{x}}\left(\max_{0\leq s\leq t} \tilde{Z}_s(A)\geq k\right) \quad (t\geq 0, \ k=0,1,2,\dots).$$
(5.4)

Let $\tilde{Z}_t = \tilde{Z}_t(\mathbb{R}^d)$ and $\tilde{Z}_t^R = \tilde{Z}_t((B_R)^c)$. For $s \leq t$, let $\tilde{\mathbf{B}}_s^{(t),k}$ be the position at time s of the kth particle alive at time t for $\overline{\mathbf{M}}$. Since \tilde{Z}_t is nondecreasing, we have

$$\max_{0 \le s \le 1} \tilde{Z}_s^{\delta i} \le \sum_{k=1}^{\tilde{Z}_1} \mathbf{1}_{\{\sup_{0 \le s \le 1} |\tilde{\mathbf{B}}_s^{(1),k}| \ge \delta i\}}.$$

Then by (5.4) and [35, Lemma 3.6],

$$\mathbf{E}_{x}\left[\max_{0\leq s\leq 1}Z_{s}^{\delta i}\right] \leq \tilde{\mathbf{E}}_{x}\left[\max_{0\leq s\leq 1}\tilde{Z}_{s}^{\delta i}\right] \leq \tilde{\mathbf{E}}_{x}\left[\sum_{k=1}^{\tilde{Z}_{1}}\mathbf{1}_{\{\sup_{0\leq s\leq 1}|\tilde{\mathbf{B}}_{s}^{(1),k}|\geq \delta i\}}\right]$$
$$= E_{x}\left[e^{A_{1}^{(\tilde{Q}-1)\mu}};\sup_{0\leq s\leq 1}|B_{s}|\geq \delta i\right]$$

for

$$\tilde{Q}(x) = \sum_{n=1}^{\infty} nq_n(x) = Q(x) + p_0(x) = 1 + \sum_{n=1}^{\infty} (n-1)p_n(x)$$

Therefore, for any $\alpha \in (0, \delta)$,

$$E_{x}\left[e^{A_{i}^{(Q-1)\mu}}\mathbf{E}_{B_{i}}\left[\max_{0\leq s\leq 1}Z_{s}^{\delta i}\right]\right] \leq E_{x}\left[e^{A_{i}^{(Q-1)\mu}}E_{B_{i}}\left[e^{A_{1}^{(\bar{Q}-1)\mu}};\sup_{0\leq s\leq 1}|B_{s}|\geq \delta i\right]\right]$$

$$=E_{x}\left[e^{A_{i}^{(Q-1)\mu}}E_{B_{i}}\left[e^{A_{1}^{(\bar{Q}-1)\mu}};\sup_{0\leq s\leq 1}|B_{s}|\geq \delta i\right];|B_{i}|\geq (\delta-\alpha)i\right]$$

$$+E_{x}\left[e^{A_{i}^{(Q-1)\mu}}E_{B_{i}}\left[e^{A_{1}^{(\bar{Q}-1)\mu}};\sup_{0\leq s\leq 1}|B_{s}|\geq \delta i\right];|B_{i}|<(\delta-\alpha)i\right]=(\mathbf{I})+(\mathbf{II}).$$
(5.5)

By Theorem 4.1, there exists $N \ge 1$ for any $x \in \mathbb{R}^d$ and $\varepsilon > 0$ such that for all $i \ge N$,

$$(\mathbf{I}) \le E_x \left[e^{A_i^{(Q-1)\mu}}; |B_i| \ge (\delta - \alpha)i \right] \sup_{x \in \mathbb{R}^d} E_x \left[e^{A_1^{(\tilde{Q}-1)\mu}} \right] \le c e^{(-\Lambda_{\delta - \alpha} + \varepsilon/2)i}.$$
(5.6)

For any $x \in \mathbb{R}^d$ and $\theta > 0$, since

$$P_x\left(\sup_{0\le s\le 1}|B_s-B_0|\ge \alpha i\right) = P_0\left(\sup_{0\le s\le 1}|B_s|\ge \alpha i\right) \le e^{-\theta\alpha i}E_0\left[e^{\theta\sup_{0\le s\le 1}|B_s|}\right],$$

we have

$$E_x \left[e^{A_1^{(\tilde{Q}-1)\mu}}; \sup_{0 \le s \le 1} |B_s - B_0| \ge \alpha i \right] \le \sup_{x \in \mathbb{R}^d} E_x \left[e^{2A_1^{(\tilde{Q}-1)\mu}} \right]^{1/2} P_x \left(\sup_{0 \le s \le 1} |B_s - B_0| \ge \alpha i \right)^{1/2} \\ \le e^{-\theta \alpha i/2} E_0 \left[e^{\theta \sup_{0 \le s \le 1} |B_s|} \right]^{1/2} \sup_{x \in \mathbb{R}^d} E_x \left[e^{2A_1^{(\tilde{Q}-1)\mu}} \right]^{1/2} \\ = c_1(\theta) e^{-\theta \alpha i/2}.$$

By (2.3), there exist c > 0 and $N' \ge 1$ for any $x \in \mathbb{R}^d$ such that for any $i \ge N'$,

$$E_x\left[e^{A_i^{(Q-1)\mu}}\right] \le ce^{-\lambda i}.$$
(5.7)

Then for any $x \in \mathbb{R}^d$ and $i \ge N \lor N'$,

(II)
$$\leq E_x \left[e^{A_i^{(Q-1)\mu}} E_{B_i} \left[e^{A_1^{(\tilde{Q}-1)\mu}}; \sup_{0 \leq s \leq 1} |B_s - B_0| \geq \alpha i \right] \right] \leq c_1(\theta) e^{-\theta \alpha i/2} E_x \left[e^{A_i^{(Q-1)\mu}} \right]$$

 $\leq c_2(\theta) e^{(-\theta \alpha/2 - \lambda)i}.$

Combining this with (5.6), we see by (5.1), (5.2) and (5.5) that for any $i \ge N \lor N'$,

$$\mathbf{P}_{x}\left(\max_{i\leq s\leq i+1}Z_{s}^{\delta i}\geq e^{(-\Lambda_{\delta}+\varepsilon)i}\right)\leq e^{-(-\Lambda_{\delta}+\varepsilon)i}\mathbf{E}_{x}\left[\max_{i\leq s\leq i+1}Z_{s}^{\delta i}\right]\leq e^{-(-\Lambda_{\delta}+\varepsilon)i}((\mathbf{I})+(\mathbf{II}))$$

$$\leq ce^{-\kappa_{1}i}+c_{2}(\theta)e^{-\kappa_{2}i}$$
(5.8)

for

$$\kappa_1 = -\Lambda_{\delta} + \Lambda_{\delta-\alpha} + \frac{\varepsilon}{2}, \quad \kappa_2 = -\Lambda_{\delta} + \varepsilon + \frac{\theta\alpha}{2} + \lambda.$$

We take $\alpha > 0$ so small that $\kappa_1 > 0$, and then take $\theta > 0$ so large that $\kappa_2 > 0$. Then, since it follows by (5.8) that

$$\sum_{i=1}^{\infty} \mathbf{P}_x \left(\max_{i \le s \le i+1} Z_s^{\delta i} \ge e^{(-\Lambda_{\delta} + \varepsilon)i} \right) < \infty,$$

we have by the Borel-Cantelli lemma,

$$\mathbf{P}_x \left(\max_{i \le s \le i+1} Z_s^{\delta i} \le e^{(-\Lambda_{\delta} + \varepsilon)i} \text{ for all sufficiently large } i \ge 1 \right) = 1.$$

Therefore, for all sufficiently large $i \ge 1$ and for all t > 0 with $i \le t \le i + 1$,

$$Z_t^{\delta t} \le \max_{i \le s \le i+1} Z_s^{\delta i} \le e^{(-\Lambda_{\delta} + \varepsilon)i} \le (1 \lor e^{\Lambda_{\delta} - \varepsilon})e^{(-\Lambda_{\delta} + \varepsilon)t},$$

which yields that

$$\limsup_{t \to \infty} \frac{1}{t} \log Z_t^{\delta t} \le -\Lambda_{\delta} + \varepsilon \to -\Lambda_{\delta} \quad (\varepsilon \to +0).$$

For $\delta = 0$, we can show the same assertion by using (5.1), (5.2), the first inequality in (5.5) and (5.7).

We next discuss the lower bound of $Z_t^{\delta t}$.

Lemma 5.2. Under Assumption 3.1, if $\mathbf{P}_x(M_{\infty} > 0) > 0$, then for any $\delta \in [0, \sqrt{-\lambda/2})$,

$$\liminf_{t \to \infty} \frac{1}{t} \log Z_t^{\delta t} \ge -\Lambda_{\delta}, \quad \mathbf{P}_x(\cdot \mid M_{\infty} > 0) \text{-}a.s.$$

Under the condition that $p_0 \equiv 0$, we proved Lemma 5.2 as [35, Lemma 3.9]. In the proof, we gave an asymptotic lower bound of the number of particles which are located outside the increasing balls over some time interval. If branching occurs during this time interval, then we choose one of the offspring and chase its trajectory. However, if $p_0 \neq 0$, then particles may vanish at the splitting time. Here we will give an asymptotic lower bound of the number of particles as we mentioned before under the condition that no branching occurs during the time interval. In order to do so, we derive the locally uniform lower bound of the expectation in (5.13) below.

Proof. For $\delta = 0$, the proof is complete by the inequality $M_t \leq ||h||_{\infty} e^{\lambda t} Z_t$.

In what follows, we assume that $\delta \in (0, \sqrt{-\lambda/2})$. Recall that \mathbf{B}_t^k is the position of the *k*th particle alive at time *t*. At the splitting time of this particle, we choose one of its children and follow its trajectory. Repeating this procedure inductively, we can construct a trajectory starting from \mathbf{B}_t^k . We denote by $\mathbf{B}_s^{t,k}$ the position of such trajectory at time $s \ (s \ge t)$.

For $t > s \ge 0$, let $D_{s,t}$ be the event defined by

 $D_{s,t} := \{ \text{no branching occurs during the time interval } [s,t] \}.$

Fix a constant $p \in (0, 1)$ and a compact set $K \subset \mathbb{R}^d$. Then for each index k,

$$\left\{ \mathbf{B}_{np}^{np,k} \in K, \, |\mathbf{B}_{s}^{np,k}| > \delta s \text{ for all } s \in [n, n+1] \right\} \cap D_{np,n+1} \\ \supset \left\{ \mathbf{B}_{np}^{np,k} \in K, \, |\mathbf{B}_{n}^{np,k}| > |\mathbf{B}_{n}^{np,k} - \mathbf{B}_{np}^{np,k}| > \delta(n+1) + 1, \\ \sup_{n \leq s \leq n+1} |\mathbf{B}_{s}^{np,k} - \mathbf{B}_{n}^{np,k}| < 1 \right\} \cap D_{np,n+1} =: E_{n}^{k}.$$

Let $x \in K$. Then for any $\varepsilon > 0$ and $\alpha \in (0, \varepsilon)$, we have by the Markov property,

$$\mathbf{P}_{x}\left(\sum_{k=1}^{Z_{np}}\mathbf{1}_{E_{n}^{k}} \le e^{(-\Lambda_{\delta}-\varepsilon)n}, Z_{np}(K) \ge e^{(-\lambda p-\alpha)n}\right)$$

$$= \mathbf{E}_{x}\left[\mathbf{P}_{\mathbf{B}_{np}}\left(\sum_{k=1}^{l}\mathbf{1}_{F_{n}^{k}} \le e^{(-\Lambda_{\delta}-\varepsilon)n}\right)|_{l=Z_{np}}; Z_{np}(K) \ge e^{(-\lambda p-\alpha)n}\right]$$
(5.9)

for

$$F_n^k := \left\{ \begin{aligned} \mathbf{B}_0^{0,k} \in K, \, |\mathbf{B}_{n(1-p)}^{0,k}| > |\mathbf{B}_{n(1-p)}^{0,k} - \mathbf{B}_0^{0,k}| > \delta(n+1) + 1, \\ \sup_{n(1-p) \le s \le n(1-p)+1} |\mathbf{B}_s^{0,k} - \mathbf{B}_{n(1-p)}^{0,k}| < 1 \end{aligned} \right\} \cap D_{0,n(1-p)+1}.$$

Let $\mathbf{x} = (x^1, \dots, x^m)$. Then by the Cauchy-Schwarz inequality,

$$\mathbf{P}_{\mathbf{x}}\left(\sum_{k=1}^{m} \mathbf{1}_{F_{n}^{k}} \leq e^{(-\Lambda_{\delta}-\varepsilon)n}\right) = \mathbf{P}_{\mathbf{x}}\left(\exp\left(-\sum_{k=1}^{m} \mathbf{1}_{F_{n}^{k}}\right) \geq e^{-e^{(-\Lambda_{\delta}-\varepsilon)n}}\right)$$
$$\leq e^{e^{(-\Lambda_{\delta}-\varepsilon)n}} \mathbf{E}_{\mathbf{x}}\left[\exp\left(-\sum_{k=1}^{m} \mathbf{1}_{F_{n}^{k}}\right)\right].$$
(5.10)

Since the events F_n^k $(1 \le k \le m)$ are independent under $\mathbf{P}_{\mathbf{x}}$, we obtain by the inequality $1 - x \le e^{-x}$,

$$\mathbf{E}_{\mathbf{x}}\left[\exp\left(-\sum_{k=1}^{m}\mathbf{1}_{F_{n}^{k}}\right)\right] = \prod_{k=1}^{m}\mathbf{E}_{x^{k}}\left[\exp\left(-\mathbf{1}_{F_{n}^{k}}\right)\right] = \prod_{1\leq k\leq m, x^{k}\in K}\left\{1-(1-e^{-1})\mathbf{P}_{x^{k}}(F_{n}^{k})\right\}$$
$$\leq \prod_{1\leq k\leq m, x^{k}\in K}\exp\left(-(1-e^{-1})\mathbf{P}_{x^{k}}(F_{n}^{k})\right).$$
(5.11)

Let

$$C_n = \left\{ |B_{n(1-p)}| > |B_{n(1-p)} - B_0| > \delta(n+1) + 1 \right\}.$$

Then by the Markov property, we have for any $x \in K$,

$$\mathbf{P}_{x}(F_{n}^{k}) = E_{x} \left[e^{-A_{n(1-p)+1}^{\mu}}; C_{n} \cap \left\{ \sup_{\substack{n(1-p) \leq s \leq n(1-p)+1 \\ n(1-p) \leq s \leq n(1-p)+1 \\ e^{-A_{n(1-p)}^{\mu}} E_{B_{n(1-p)}} \left[e^{-A_{1}^{\mu}}; \sup_{\substack{0 \leq s \leq 1 \\ 0 \leq s \leq 1 \\ e^{-A_{n(1-p)}}} |B_{s} - B_{0}| < 1 \right]; C_{n} \right].$$
(5.12)

Since it follows by the Cauchy-Schwarz inequality that

$$E_y\left[e^{-A_1^{\mu}}; \sup_{0 \le s \le 1} |B_s - B_0| < 1\right] \ge \frac{P_y(\sup_{0 \le s \le 1} |B_s - B_0| < 1)^2}{E_y\left[e^{A_1^{\mu}}\right]} \ge \frac{P_0(\sup_{0 \le s \le 1} |B_s| < 1)^2}{\sup_{z \in \mathbb{R}^d} E_z\left[e^{A_1^{\mu}}\right]}$$

there exists c > 0 by (5.12) such that

$$\mathbf{P}_x(F_n^k) \ge cE_x\left[e^{-A_{n(1-p)}^{\mu}}; C_n\right].$$
(5.13)

By the same calculation as in [35, p.141–142], there exist c' > 0 and c'' > 0 such that for any $x \in \mathbb{R}^d$,

$$P_x(C_n) \ge c' \int_{(\delta(n+1)+1)/\sqrt{n(1-p)}}^{\infty} e^{-r^2/2} r^{d-1} \, \mathrm{d}r \ge c'' n^{(d-2)/2} \exp\left(-\frac{\delta^2 n}{2(1-p)}\right).$$
(5.14)

Then by the Cauchy-Schwarz inequality again, we have for any q > 1,

$$E_{x}\left[e^{-A_{n(1-p)}^{\mu}};C_{n}\right] \geq \frac{P_{x}\left(C_{n}\right)^{q}}{E_{x}\left[e^{A_{n(1-p)}^{\mu}/(q-1)};C_{n}\right]^{q-1}} \geq \frac{\left(c''n^{(d-2)/2}e^{-\delta^{2}n/\{2(1-p)\}}\right)^{q}}{E_{x}\left[e^{A_{n(1-p)}^{\mu}/(q-1)};|B_{n(1-p)}| > \delta(n+1)+1\right]^{q-1}}.$$
(5.15)

Let

$$\beta_q := \lambda\left(\frac{\mu}{q-1}\right) = \inf\left\{\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x - \frac{1}{q-1}\int_{\mathbb{R}^d} u^2 \,\mathrm{d}\mu \mid u \in C_0^\infty(\mathbb{R}^d), \int_{\mathbb{R}^d} u^2 \,\mathrm{d}x = 1\right\}.$$

Then $\sqrt{-2\lambda} \ge \sqrt{-2\beta_q} > 0$ for some q > 1. If we let $p = 1 - \delta/\sqrt{-2\lambda}$, then by Lemma 4.4, there exists $N \ge 1$ for any $\varepsilon' > 0$ such that for all $n \ge N$,

$$\sup_{y \in K} E_y \left[e^{A_{n(1-p)}^{\mu}/(q-1)}; |B_{n(1-p)}| > \delta(n+1) + 1 \right] \le e^{(-\delta^2/\{2(1-p)^2\} + \varepsilon')n(1-p)}.$$

This implies that for any $x \in K$, the last term of (5.15) is greater than

$$\frac{(c_{\delta,p}n^{(d-2)/2}e^{-\delta^2 n/\{2(1-p)\}})^q}{(e^{(-\delta^2/\{2(1-p)^2\}+\varepsilon')n(1-p)})^{q-1}} = c_{\delta,p}^q n^{(d-2)q/2} e^{-\delta^2 n/\{2(1-p)\}} e^{-\varepsilon'(1-p)(q-1)n} =: q_{n,p}.$$

Hence it follows by (5.13) that $\mathbf{P}_x(F_n^k) \ge q_{n,p}$ for all $n \ge N$. Since this yields that

$$\prod_{1 \le k \le m, x^k \in K} \exp\left(-(1 - e^{-1}) \mathbf{P}_{x^k}(F_n^k)\right) \le \exp\left(-(1 - e^{-1})q_{n,p} \sharp\{k \mid x^k \in K\}\right),$$

we have by (5.10) and (5.11),

$$\mathbf{E}_{x}\left[\mathbf{P}_{\mathbf{B}_{np}}\left(\sum_{k=1}^{l}\mathbf{1}_{F_{n}^{k}}\leq e^{(-\Lambda_{\delta}-\varepsilon)n}\right)|_{l=Z_{np}}; Z_{np}(K)\geq e^{(-\lambda p-\alpha)n}\right]$$

$$\leq \mathbf{E}_{x}\left[\exp\left(e^{(-\Lambda_{\delta}-\varepsilon)n}-(1-e^{-1})q_{n,p}Z_{np}(K)\right); Z_{np}(K)\geq e^{(-\lambda p-\alpha)n}\right]$$

$$\leq \exp\left(e^{(-\Lambda_{\delta}-\varepsilon)n}-(1-e^{-1})q_{n,p}e^{(-\lambda p-\alpha)n}\right)$$

$$=\exp\left(-e^{(-\Lambda_{\delta}-\alpha-\varepsilon'(1-p)(q-1))n}\left(cn^{(d-2)q/2}-e^{(\varepsilon'(1-p)(q-1)-(\varepsilon-\alpha))n}\right)\right).$$

If we take $\varepsilon' > 0$ so small that $\varepsilon'(1-p)(q-1) < \varepsilon - \alpha$, then there exists $N'' \ge 1$ such that for all $n \ge N''$, we obtain by (5.9),

$$\mathbf{P}_{x}\left(\sum_{k=1}^{Z_{np}}\mathbf{1}_{E_{n}^{k}} \le e^{(-\Lambda_{\delta}-\varepsilon)n}, Z_{np}(K) \ge e^{(-\lambda p-\alpha)n}\right) \le \exp\left(-e^{(-\Lambda_{\delta}-\alpha-\varepsilon'(1-p)(q-1))n}c'n^{(d-2)q/2}\right).$$

Noting that $\Lambda_{\delta} < 0$ for any $\delta \in (0, \sqrt{-\lambda/2})$, we can take $\varepsilon > 0$ such that $-\Lambda_{\delta} - \varepsilon > 0$. Since this implies that

$$\sum_{n=1}^{\infty} \mathbf{P}_x \left(\sum_{k=1}^{Z_{np}} \mathbf{1}_{E_n^k} \le e^{(-\Lambda_\delta - \varepsilon)n}, Z_{np}(K) \ge e^{(-\lambda p - \alpha)n} \right) < \infty,$$

we see by the Borel-Cantelli lemma that the event

$$\left\{\sum_{k=1}^{Z_{np}} \mathbf{1}_{E_n^k} > e^{(-\Lambda_\delta - \varepsilon)n}\right\} \cup \left\{Z_{np}(K) < e^{(-\lambda p - \alpha)n}\right\}$$

occurs infinitely often. By (2.6), we further obtain

$$\mathbf{P}_x\left(\sum_{k=1}^{Z_{np}} \mathbf{1}_{E_n^k} > e^{(-\Lambda_\delta - \varepsilon)n} \text{ for all sufficiently large } n \mid M_\infty > 0\right) = 1.$$

Hence we have $\mathbf{P}_x(\cdot \mid M_{\infty} > 0)$ -a.s. for all sufficiently large t > 0,

$$Z_{t}^{\delta t} = \sum_{k=1}^{Z_{t}} \mathbf{1}_{\{|\mathbf{B}_{t}^{k}| > \delta t\}} \geq \sum_{k=1}^{Z_{[t]p}} \mathbf{1}_{\{\{\mathbf{B}_{[t]p}^{[t]p,k} \in K, |\mathbf{B}_{s}^{[t]p,k}| > \delta s \text{ for all } s \in [[t], [t]+1]\} \cap D_{[t]p,[t]+1}\}}$$
$$\geq \sum_{k=1}^{Z_{[t]p}} \mathbf{1}_{E_{[t]}^{k}} \geq e^{(-\Lambda_{\delta}-\varepsilon)[t]},$$

which implies that

$$\liminf_{t \to \infty} \frac{1}{t} \log Z_t^{\delta t} \ge -\Lambda_{\delta} - \varepsilon.$$

By letting $\varepsilon' \to +0$, $\alpha \to +0$, and then $\varepsilon \to +0$, we arrive at the desired conclusion. Theorem 3.2 is a consequence of Lemmas 5.1 and 5.2.

6 Proofs of Theorems 3.6 and 3.7

In this section, we prove Theorems 3.6 and 3.7. For the lower bound of (3.2) especially, we take into consideration the effect of $p_0 \neq 0$ as in Lemma 5.2.

6.1 Proof of Theorem 3.6

By the Chebyshev inequality and (2.5),

$$\mathbf{P}_x\left(\frac{L_t}{t} \ge \delta\right) = \mathbf{P}_x(Z_t^{\delta t} \ge 1) \le \mathbf{E}_x\left[Z_t^{\delta t}\right] = E_x\left[e^{A_t^{(Q-1)\mu}}; |B_t| \ge \delta t\right]$$

for any $x \in \mathbb{R}^d$. Theorem 4.1 then implies that for any compact set $K \subset \mathbb{R}^d$,

$$\limsup_{t \to \infty} \frac{1}{t} \log \sup_{x \in K} \mathbf{P}_x \left(\frac{L_t}{t} \ge \delta \right) \le \limsup_{t \to \infty} \frac{1}{t} \log \sup_{x \in K} E_x \left[e^{A_t^{(Q-1)\mu}}; |B_t| \ge \delta t \right] = -\Lambda_{\delta}.$$
(6.1)

We now assume that $\delta \geq \sqrt{-2\lambda}$. Since

$$\mathbf{P}_x\left(\frac{L_t}{t} \ge \delta\right) \ge \mathbf{P}_x\left(\frac{L_t}{t} \ge \delta, t < T\right) = E_x\left[e^{-A_t^{\mu}}; |B_t| \ge \delta t\right],$$

Lemma 4.5 yields that for any compact set $K \subset \mathbb{R}^d$,

$$\liminf_{t \to \infty} \frac{1}{t} \log \inf_{x \in K} \mathbf{P}_x \left(\frac{L_t}{t} \ge \delta \right) \ge \liminf_{t \to \infty} \frac{1}{t} \log \inf_{x \in K} E_x \left[e^{-A_t^{\mu}}; |B_t| \ge \delta t \right] = -\frac{\delta^2}{2}.$$

Combining this with (6.1), we complete the proof of (i).

We next assume that $\sqrt{-\lambda/2} < \delta < \sqrt{-2\lambda}$. Fix $p \in (0,1)$ and $\alpha > 0$. For $x \in \mathbb{R}^d$, we take a compact set $K \subset \mathbb{R}^d$ so that $x \in K$. Let

$$C_t := \left\{ Z_t(K) \ge e^{(-\lambda - \alpha)t} \right\}$$

and

 $D_{s,t} := \{ \text{no branching occurs during the time interval } [s,t] \}.$

Then by the Markov property,

$$\mathbf{P}_{x}(Z_{t}^{\delta t} \geq 1) = \mathbf{P}_{x}\left(\bigcup_{k=1}^{Z_{t}} \left\{ |\mathbf{B}_{t}^{k}| \geq \delta t \right\} \right) \geq \mathbf{P}_{x}\left(\bigcup_{k=1}^{Z_{pt}} \left\{ \mathbf{B}_{pt}^{pt,k} \in K, |\mathbf{B}_{t}^{pt,k}| \geq \delta t \right\}, C_{pt} \cap D_{pt,t} \right)$$
$$= \mathbf{E}_{x}\left[\mathbf{P}_{\mathbf{B}_{pt}}\left(\bigcup_{k=1}^{l} \left\{ \mathbf{B}_{0}^{0,k} \in K, |\mathbf{B}_{(1-p)t}^{0,k}| \geq \delta t \right\} \cap D_{0,(1-p)t} \right) |_{l=Z_{pt}}; C_{pt} \right].$$
(6.2)

For $\mathbf{x} = (x^1, \dots, x^l) \in \mathbf{X}$, since

$$\mathbf{P}_{\mathbf{x}} \left(\bigcup_{k=1}^{l} \left\{ \mathbf{B}_{0}^{0,k} \in K, |\mathbf{B}_{(1-p)t}^{0,k}| \ge \delta t \right\} \cap D_{0,(1-p)t} \right) \\ = 1 - \mathbf{P}_{\mathbf{x}} \left(\bigcap_{k=1}^{l} \left\{ \left\{ \mathbf{B}_{0}^{0,k} \in K, |\mathbf{B}_{(1-p)t}^{0,k}| \ge \delta t \right\} \cap D_{0,(1-p)t} \right\}^{c} \right) \\ = 1 - \prod_{k=1}^{l} \mathbf{P}_{x^{k}} \left(\left\{ \mathbf{B}_{0}^{1} \in K, |\mathbf{B}_{(1-p)t}^{1}| \ge \delta t, T > (1-p)t \right\}^{c} \right) \\ = 1 - \prod_{k=1}^{l} \left(1 - \mathbf{P}_{x^{k}} \left(\mathbf{B}_{0}^{1} \in K, |\mathbf{B}_{(1-p)t}^{1}| \ge \delta t, T > (1-p)t \right) \right),$$

we have by (6.2),

$$\mathbf{P}_{x}(Z_{t}^{\delta t} \ge 1) \ge \mathbf{E}_{x} \left[1 - \prod_{k=1}^{Z_{pt}} \left\{ 1 - \mathbf{P}_{\mathbf{B}_{pt}^{k}} \left(\mathbf{B}_{0}^{1} \in K, |\mathbf{B}_{(1-p)t}^{1}| \ge \delta t, T > (1-p)t \right) \right\}; C_{pt} \right].$$
(6.3)

In what follows, we fix $p \in (0, 1)$ with $\delta/(1-p) > \sqrt{-2\lambda}$. Then by Lemma 4.5, there exists $t_0 > 0$ for any $\varepsilon > 0$ such that for any $y \in K$ and $t \ge t_0$,

$$\mathbf{P}_{y}\left(|\mathbf{B}_{(1-p)t}^{1}| \ge \delta t, T > (1-p)t\right) = E_{y}\left[e^{-A_{(1-p)t}^{\mu}}; |B_{(1-p)t}| \ge \delta t\right]$$
$$\ge \inf_{z \in K} E_{z}\left[e^{-A_{(1-p)t}^{\mu}}; |B_{(1-p)t}| \ge \frac{\delta}{1-p} \cdot (1-p)t\right] \ge \exp\left(\left(-\frac{\delta^{2}}{2(1-p)^{2}} - \varepsilon\right)(1-p)t\right).$$

Hence for any $t \ge t_0$,

$$\mathbf{P}_{x}(Z_{t}^{\delta t} \geq 1) \geq \mathbf{E}_{x} \left[1 - \prod_{1 \leq k \leq Z_{pt}, \mathbf{B}_{pt}^{k} \in K} \left\{ 1 - \exp\left(\left(-\frac{\delta^{2}}{2(1-p)^{2}} - \varepsilon\right)(1-p)t\right)\right\}; C_{pt} \right] \\ = \mathbf{E}_{x} \left[1 - \left\{ 1 - \exp\left(\left(\left(-\frac{\delta^{2}}{2(1-p)^{2}} - \varepsilon\right)(1-p)t\right)\right)\right\}^{Z_{pt}(K)}; C_{pt} \right] \\ \geq \left\{ 1 - \left(1 - \exp\left(\left(\left(-\frac{\delta^{2}}{2(1-p)^{2}} - \varepsilon\right)(1-p)t\right)\right)\right)^{e^{(-\lambda-\alpha)pt}} \right\} \mathbf{P}_{x}(C_{pt}).$$

$$(6.4)$$

By elementary calculation as in the proof of [34, Lemma 3.10],

$$\liminf_{t \to \infty} \frac{1}{t} \log \left\{ 1 - \left(1 - \exp\left(\left(-\frac{\delta^2}{2(1-p)^2} - \varepsilon \right) (1-p)t \right) \right)^{e^{(-\lambda-\alpha)pt}} \right\}$$
$$\geq -\lambda p - \frac{\delta^2}{2(1-p)} - \varepsilon (1-p) - \alpha p \to -\lambda p - \frac{\delta^2}{2(1-p)} \quad (\varepsilon \to +0, \ \alpha \to +0).$$

The right hand side above takes the maximal value $-\lambda - \sqrt{-2\lambda}\delta$ at $p = 1 - \delta/\sqrt{-2\lambda}$. Since it follows by (2.6) that

$$\mathbf{P}_x(C_{pt}) \ge \mathbf{P}_x(C_{pt} \cap \{M_\infty > 0\}) \to \mathbf{P}_x(M_\infty > 0) \quad (t \to \infty),$$

we see by (6.4) that if $\mathbf{P}_x(M_{\infty} > 0) > 0$ and $\delta > \sqrt{-\lambda/2}$, then

$$\liminf_{t \to \infty} \frac{1}{t} \log \mathbf{P}_x \left(\frac{L_t}{t} \ge \delta \right) = \liminf_{t \to \infty} \frac{1}{t} \log \mathbf{P}_x (Z_t^{\delta t} \ge 1) \ge -\lambda - \sqrt{-2\lambda} \delta.$$

Combining this with (6.1), we finish the proof.

6.2 Proof of Theorem 3.7

We first show (i). Assume that $\delta \geq \sqrt{-2\lambda}$. Since it follows by the proof of Theorem 3.6 (i) that

$$E_x\left[e^{-A_t^{\mu}}; |B_t| \ge \delta t\right] \le \mathbf{P}_x\left(\frac{L_t}{t} \ge \delta\right) \le E_x\left[e^{A_t^{(Q-1)\mu}}; |B_t| \ge \delta t\right],\tag{6.5}$$

we get (3.3) by Remarks 4.2 and 4.6.

We next show (ii). Assume that $\delta \in (\sqrt{-\lambda/2}, \sqrt{-2\lambda})$. If $p_0 \equiv 0$ and μ is compactly supported in \mathbb{R}^d , then the upper bound of (3.4) follows by (6.5) and Remark 4.2. For the lower bound of it, we make use of the Feynman-Kac expression of $\mathbf{P}_x(L_t/t \geq \delta)$. Such an approach is similar to that of [11] and due to McKean [28, 29] (see also [19, Section 1.3] and [20, Example 3.4]).

Let us derive the Feynman-Kac expression of $\mathbf{P}_x(L_t/t \ge \delta)$. Let f be a nonnegative Borel measurable function on \mathbb{R}^d such that $0 \le f(x) \le 1$ for any $x \in \mathbb{R}^d$ and let

$$u(t,x) = \mathbf{E}_x \left[\prod_{k=1}^{Z_t} f(\mathbf{B}_t^k) \right]$$

for $t \ge 0$ and $x \in \mathbb{R}^d$. We first give the Feynman-Kac expression of u(t, x) and 1 - u(t, x). Let

$$F_u(t,x) = \sum_{n=1}^{\infty} p_n(x)u(t,x)^n, \quad G_u(t,x) = \sum_{n=1}^{\infty} p_n(x)u(t,x)^{n-1}.$$

If we define

$$H_u(t,x) = \sum_{n=1}^{\infty} p_n(x) \left(\sum_{k=1}^n u(t,x)^{k-1} \right),$$

then

$$(1 - u(t, x))H_u(t, x) = 1 - p_0 - F_u(t, x).$$
(6.6)

Lemma 6.1. Assume that $0 \leq f(x) \leq 1$ for any $x \in \mathbb{R}^d$. Then

$$u(t,x) = E_x \left[\exp\left(\int_0^t (G_u(t-s, B_s) - 1) \, \mathrm{d}A_s^\mu \right) f(B_t) \right]$$
(6.7)

and

$$1 - u(t, x) = E_x \left[\exp\left(\int_0^t (H_u(t - s, B_s) - 1) \, \mathrm{d}A_s^{\mu}\right) (1 - f(B_t)) \right] + E_x \left[\int_0^t \exp\left(\int_0^s (H_u(t - w, B_w) - 1) \, \mathrm{d}A_w^{\mu}\right) p_0(B_s) \, \mathrm{d}A_s^{\mu} \right].$$
(6.8)

Proof. Let $F = F_u$, $G = G_u$ and $H = H_u$. For $s, t \ge 0$ with $t \ge s$, define

$$C_s^{G,t} = \int_0^s G(t-r, B_r) \,\mathrm{d}A_r^\mu.$$

We first prove by induction that for any $n \ge 0$,

$$u(t,x) = \sum_{k=0}^{n} \frac{1}{k!} E_x \left[e^{-A_t^{\mu}} f(B_t) (C_t^{G,t})^k \right] + \frac{1}{n!} E_x \left[\int_0^t e^{-A_s^{\mu}} F(t-s,B_s) (C_s^{G,t})^n \, \mathrm{d}A_s^{\mu} \right].$$
(6.9)

For n = 0, this equality is valid because we have by the strong Markov property,

$$u(t,x) = \mathbf{E}_{x} \left[f(\mathbf{B}_{t}^{1}) : t < T \right] + \mathbf{E}_{x} \left[\prod_{k=1}^{Z_{t}} f(\mathbf{B}_{t}^{k}); T \leq t \right]$$

$$= \mathbf{E}_{x} \left[f(\mathbf{B}_{t}^{1}) : t < T \right] + \mathbf{E}_{x} \left[\mathbf{E}_{\mathbf{B}_{T}} \left[\prod_{k=1}^{Z_{t-s}} f(\mathbf{B}_{t-s}^{k}) \right] |_{s=T}; T \leq t \right]$$

$$= E_{x} \left[e^{-A_{t}^{\mu}} f(B_{t}) \right] + E_{x} \left[\int_{0}^{t} e^{-A_{s}^{\mu}} \sum_{n=1}^{\infty} p_{n}(B_{s}) u(t-s, B_{s})^{n} \, \mathrm{d}A_{s}^{\mu} \right]$$

$$= E_{x} \left[e^{-A_{t}^{\mu}} f(B_{t}) \right] + E_{x} \left[\int_{0}^{t} e^{-A_{s}^{\mu}} F(t-s, B_{s}) \, \mathrm{d}A_{s}^{\mu} \right].$$
(6.10)

Suppose that (6.9) is true for some $n \ge 1$. Then by (6.10),

$$E_{x} \left[\int_{0}^{t} e^{-A_{s}^{\mu}} F(t-s, B_{s}) \left(C_{s}^{G,t} \right)^{n} dA_{s}^{\mu} \right]$$

= $E_{x} \left[\int_{0}^{t} e^{-A_{s}^{\mu}} u(t-s, B_{s}) G(t-s, B_{s}) \left(C_{s}^{G,t} \right)^{n} dA_{s}^{\mu} \right]$
= $E_{x} \left[\int_{0}^{t} e^{-A_{s}^{\mu}} E_{B_{s}} \left[e^{-A_{t-s}^{\mu}} f(B_{t-s}) \right] \left(C_{s}^{G,t} \right)^{n} dC_{s}^{G,t} \right]$
+ $E_{x} \left[\int_{0}^{t} e^{-A_{s}^{\mu}} E_{B_{s}} \left[\int_{0}^{t-s} e^{-A_{w}^{\mu}} F(t-s-w, B_{w}) dA_{w}^{\mu} \right] \left(C_{s}^{G,t} \right)^{n} dC_{s}^{G,t} \right]$
= (I) + (II).

By the Markov property,

$$(\mathbf{I}) = E_x \left[\int_0^t e^{-A_s^{\mu}} E_x \left[e^{-A_{t-s}^{\mu} \circ \theta_s} f(B_{t-s} \circ \theta_s) \mid \mathcal{F}_s \right] (C_s^{G,t})^n \, \mathrm{d}C_s^{G,t} \right] \\ = E_x \left[\int_0^t e^{-A_s^{\mu}} e^{-A_{t-s}^{\mu} \circ \theta_s} f(B_{t-s} \circ \theta_s) (C_s^{G,t})^n \, \mathrm{d}C_s^{G,t} \right] \\ = E_x \left[e^{-A_t^{\mu}} f(B_t) \int_0^t (C_s^{G,t})^n \, \mathrm{d}C_s^{G,t} \right] = \frac{1}{n+1} E_x \left[e^{-A_t^{\mu}} f(B_t) \left(C_t^{G,t}\right)^{n+1} \right]$$

and

$$\begin{aligned} \text{(II)} &= E_x \left[\int_0^t e^{-A_s^{\mu}} E_x \left[\int_0^{t-s} e^{-A_w^{\mu} \circ \theta_s} F(t-s-w, B_w \circ \theta_s) \, \mathrm{d}A_w^{\mu} \circ \theta_s \mid \mathcal{F}_s \right] \left(C_s^{G,t} \right)^n \, \mathrm{d}C_s^{G,t} \right] \\ &= E_x \left[\int_0^t \left(\int_s^t e^{-A_w^{\mu}} F(t-w, B_w) \, \mathrm{d}A_w^{\mu} \right) \left(C_s^{G,t} \right)^n \, \mathrm{d}C_s^{G,t} \right] \\ &= E_x \left[\int_0^t e^{-A_w^{\mu}} F(t-w, B_w) \left(\int_0^w \left(C_s^{G,t} \right)^n \, \mathrm{d}C_s^{G,t} \right) \, \mathrm{d}A_w^{\mu} \right] \\ &= \frac{1}{n+1} E_x \left[\int_0^t e^{-A_w^{\mu}} F(t-w, B_w) \left(C_w^{G,t} \right)^{n+1} \, \mathrm{d}A_w^{\mu} \right]. \end{aligned}$$

Hence the induction is complete by (6.9).

We next show that

$$\lim_{n \to \infty} \frac{1}{n!} E_x \left[\int_0^t e^{-A_s^{\mu}} F(t-s, B_s) \left(C_s^{G,t} \right)^n \, \mathrm{d}A_s^{\mu} \right] = 0.$$
(6.11)

Since $0 \le u(t, x) \le 1$, we have

 $F(t,x) \le G(t,x) \le 1$ for all t > 0 and $x \in \mathbb{R}^d$,

and therefore

$$\int_0^t e^{-A_s^{\mu}} F(t-s, B_s) \left(C_s^{G,t}\right)^n \, \mathrm{d}A_s^{\mu} \le \int_0^t e^{-A_s^{\mu}} (A_s^{\mu})^n \, \mathrm{d}A_s^{\mu}.$$

Since this implies that

$$\sum_{n=0}^{\infty} \frac{1}{n!} E_x \left[\int_0^t e^{-A_s^{\mu}} F(t-s, B_s) \left(C_s^{G,t} \right)^n \, \mathrm{d}A_s^{\mu} \right] \le \sum_{n=0}^{\infty} \frac{1}{n!} E_x \left[\int_0^t e^{-A_s^{\mu}} (A_s^{\mu})^n \, \mathrm{d}A_s^{\mu} \right] = E_x \left[A_t^{\mu} \right] < \infty,$$

we get (6.11). Furthermore, we obtain (6.7) by letting $n \to \infty$ in (6.9).

We let v(t, x) = 1 - u(t, x). Since

$$e^{-A_t^{\mu}} = 1 - \int_0^t e^{-A_s^{\mu}} \,\mathrm{d}A_s^{\mu},$$

we have by (6.6) and (6.10),

$$v(t,x) = E_x \left[e^{-A_t^{\mu}} (1 - f(B_t)) \right] + E_x \left[\int_0^t e^{-A_s^{\mu}} (1 - F(t - s, B_s)) \, \mathrm{d}A_s^{\mu} \right]$$

= $E_x \left[e^{-A_t^{\mu}} (1 - f(B_t)) \right] + E_x \left[\int_0^t e^{-A_s^{\mu}} v(t - s, B_s) H(t - s, B_s) \, \mathrm{d}A_s^{\mu} \right]$ (6.12)
+ $E_x \left[\int_0^t e^{-A_s^{\mu}} p_0(B_s) \, \mathrm{d}A_s^{\mu} \right].$

Then the proof is complete by the induction and calculation similar to those for (6.7). \Box

Let $f_R(x) = \mathbf{1}_{\{|x| < R\}}$ for R > 0. If we define

$$u_R(t,x) = \mathbf{E}_x \left[\prod_{k=1}^{Z_t} f_R(\mathbf{B}_t^k) \right]$$

and $v_R(t,x) = 1 - u_R(t,x)$, then $v_R(t,x) = \mathbf{P}_x(L_t \ge R)$. We also define

$$C_s^{R,t} = \int_0^s (H_{u_R}(t - w, B_w) - 1) \, \mathrm{d}A_w^{\mu}$$

for $s, t \ge 0$ with $t \ge s$. Then

$$C^{R,t}_t = C^{R,t}_s + C^{R,t}_{t-s} \circ \theta_s.$$

For $\delta > 0$, we let $D_s^t = C_s^{\delta t,t}$. Since

$$v_{\delta t}(t,x) = 1 - u_{\delta t}(t,x) = \mathbf{P}_x(L_t/t \ge \delta),$$

we have by (6.8),

$$\mathbf{P}_{x}(L_{t}/t \ge \delta) = E_{x}\left[e^{D_{t}^{t}}; |B_{t}| \ge \delta t\right] + E_{x}\left[\int_{0}^{t} e^{D_{s}^{t}} p_{0}(B_{s}) \,\mathrm{d}A_{s}^{\mu}\right] \ge E_{x}\left[e^{D_{t}^{t}}; |B_{t}| \ge \delta t\right].$$
(6.13)

To derive the decay rate of the right hand side above as $t \to \infty$, we show

Lemma 6.2. Suppose that μ is compactly supported in \mathbb{R}^d and $\sup_{x \in \mathbb{R}^d} \sum_{n=1}^{\infty} n^2 p_n(x) < \infty$. Then for any $p \in (0,1)$ and $\delta > \sqrt{-\lambda/2}$,

$$\lim_{t \to \infty} e^{\lambda p t} E_x \left[e^{D_{pt}^t} \right] = h(x) \int_{\mathbb{R}^d} h(y) \, \mathrm{d}y.$$

Proof. For any v > 0,

$$n - \sum_{k=1}^{n} (1-v)^{k-1} = n - \frac{1 - (1-v)^n}{v} = \frac{(1-v)^n - (1-nv)}{v} \le \frac{n(n-1)}{2}v.$$

This inequality is true also for v = 0. Therefore,

$$Q(x) - H_{1-v_R}(t,x) = \sum_{n=1}^{\infty} p_n(x) \left(n - \sum_{k=1}^{n} (1 - v_R(t,x))^{k-1} \right)$$
$$\leq \frac{1}{2} \sum_{n=1}^{\infty} n(n-1) p_n(x) v_R(t,x).$$

Then by the inequality $1 - e^{-x} \le x$, we have for any $p \in (0, 1)$ and $t \ge 0$,

$$1 - e^{-(A_{pt}^{(Q-1)\mu} - C_{pt}^{R,t})} = 1 - \exp\left(-\int_{0}^{pt} (Q(B_s) - H_{1-v_R}(t-s, B_s)) \, \mathrm{d}A_s^{\mu}\right)$$

$$\leq \int_{0}^{pt} (Q(B_s) - H_{1-v_R}(t-s, B_s)) \, \mathrm{d}A_s^{\mu}$$

$$\leq \int_{0}^{pt} \frac{1}{2} \sum_{n=1}^{\infty} n(n-1)p_n(B_s) v_R(t-s, B_s) \, \mathrm{d}A_s^{\mu} = \int_{0}^{pt} v_R(t-s, B_s) \, \mathrm{d}A_s^{M\mu}$$

for

$$M(x) = \frac{1}{2} \sum_{n=1}^{\infty} n(n-1)p_n(x).$$

Hence

$$0 \le e^{A_{pt}^{(Q-1)\mu}} - e^{C_{pt}^{R,t}} = e^{A_{pt}^{(Q-1)\mu}} \left(1 - e^{-(A_{pt}^{(Q-1)\mu} - C_{pt}^{R,t})}\right) \le e^{A_{pt}^{(Q-1)\mu}} \int_{0}^{pt} v_{R}(t-s, B_{s}) \, \mathrm{d}A_{s}^{M\mu}.$$
(6.14)

If we take $R = \delta t$, then for any $s \in [0, pt]$,

$$v_{\delta t}(t-s,x) = \mathbf{P}_x(L_{t-s} \ge \delta t) \le \mathbf{P}_x(L_{t-s} \ge \delta(t-s)) = \mathbf{P}_x(Z_{t-s}^{\delta(t-s)} \ge 1) \le \mathbf{E}_x\left[Z_{t-s}^{\delta(t-s)}\right].$$

Since $\delta > \sqrt{-\lambda/2}$, Theorem 4.1 yields that for any compact set $K \subset \mathbb{R}^d$ and for any $\varepsilon \in (0, \Lambda_{\delta})$, there exists T > 0 such that for all $t \geq T$ and $s \in [0, pt]$,

$$\sup_{x \in K} v_{\delta t}(t-s,x) \le \sup_{x \in K} \mathbf{E}_x \left[Z_{t-s}^{\delta(t-s)} \right] \le e^{(-\Lambda_{\delta}+\varepsilon)(t-s)} \le e^{(-\Lambda_{\delta}+\varepsilon)(1-p)t}.$$

Taking K as the support of μ , we have

$$\int_0^{pt} v_{\delta t}(t-s, B_s) \mathrm{d}A_s^{M\mu} \le e^{(-\Lambda_{\delta} + \varepsilon)(1-p)t} A_{pt}^{M\mu}.$$

Noting that $D_{pt}^t = C_{pt}^{\delta t,t}$, we get by (6.14),

$$0 \leq E_x \left[e^{A_{pt}^{(Q-1)\mu}} \right] - E_x \left[e^{D_{pt}^t} \right] \leq e^{(-\Lambda_\delta + \varepsilon)(1-p)t} E_x \left[e^{A_{pt}^{(Q-1)\mu}} A_{pt}^{M\mu} \right]$$

$$= e^{-\lambda pt} e^{(-\Lambda_\delta + \varepsilon)(1-p)t} e^{\lambda pt} E_x \left[e^{A_{pt}^{(Q-1)\mu}} A_{pt}^{M\mu} \right].$$
(6.15)

By the same argument as for (4.14), there exist $c_1 > 0$ and $c_2 > 0$ such that for any $\varepsilon_2 > 0$,

$$e^{(\lambda-\varepsilon_2)pt}E_x\left[e^{A_{pt}^{(Q-1)\mu}}A_{pt}^{M\mu}\right] \le E_x\left[\sup_{0\le s\le pt}\left(e^{(\lambda-\varepsilon_2)s}e^{A_s^{(Q-1)\mu}}\right)A_{pt}^{M\mu}\right] \le c(\varepsilon_2)(c_1+c_2t).$$

Since $\Lambda_{\delta} > 0$, there exists $\varepsilon_2 > 0$ for any $\varepsilon \in (0, \Lambda_{\delta})$ such that

$$c := (\Lambda_{\delta} - \varepsilon)(1 - p) - \varepsilon_2 p > 0.$$

Then the last term of (6.15) is less than

$$c(\varepsilon_2)e^{(-\lambda+\varepsilon_2)pt}e^{(-\Lambda_\delta+\varepsilon)(1-p)t}(c_1+c_2t) = c(\varepsilon_2)e^{-\lambda pt}e^{-ct}(c_1+c_2t),$$

that is,

$$0 \le e^{\lambda pt} \left(E_x \left[e^{A_{pt}^{(Q-1)\mu}} \right] - E_x \left[e^{D_{pt}^t} \right] \right) \le c_2(\varepsilon_2) e^{-ct} (c_1 + c_2 t) \to 0 \ (t \to \infty).$$

Hence by (2.3),

$$e^{\lambda pt} E_x \left[e^{D_{pt}^t} \right] = e^{\lambda pt} E_x \left[e^{A_{pt}^{(Q-1)\mu}} \right] + e^{\lambda pt} \left(E_x \left[e^{D_{pt}^t} \right] - E_x \left[e^{A_{pt}^{(Q-1)\mu}} \right] \right)$$
$$\to h(x) \int_{\mathbb{R}^d} h(y) \, \mathrm{d}y \ (t \to \infty).$$

This completes the proof.

We are now in a position to prove the lower bound of (3.4). For any $p \in (0, 1)$, we have by the Markov property,

$$E_{x}\left[e^{D_{t}^{t}};|B_{t}| \geq \delta t\right] = E_{x}\left[e^{D_{pt}^{t}}E_{B_{pt}}\left[e^{D_{(1-p)t}^{t}};|B_{(1-p)t}| \geq \delta t\right]\right].$$

Since $D_{(1-p)t}^t \ge 0$ for any $t \ge 0$, the last term above is greater than

$$E_x\left[e^{D_{pt}^t}P_{B_{pt}}\left(|B_{(1-p)t}| \ge \delta t\right)\right] \ge E_x\left[e^{D_{pt}^t}\right]P_0\left(|B_{(1-p)t}| \ge \delta t\right)$$
(6.16)

by [35, Appendix A]. Then by Lemma 6.2, we have as $t \to \infty$,

$$E_x \left[e^{D_{pt}^t} \right] P_0 \left(|B_{(1-p)t}| \ge \delta t \right)$$

$$\sim \frac{\omega_d}{(2\pi)^{d/2}} \left(\frac{\delta^2 t}{1-p} \right)^{(d-2)/2} \exp\left(-\lambda pt - \frac{\delta^2 t}{2(1-p)} \right) h(x) \int_{\mathbb{R}^d} h(y) \, \mathrm{d}y.$$
(6.17)

If we let $p = 1 - \delta/\sqrt{-2\lambda}$, then the last term of (6.17) becomes

$$\frac{\omega_d(\sqrt{-2\lambda}\delta)^{(d-2)/2}}{(2\pi)^{d/2}}t^{(d-2)/2}e^{(-\lambda-\sqrt{-2\lambda}\delta)t}h(x)\int_{\mathbb{R}^d}h(y)\,\mathrm{d}y.$$

We thus get the lower bound of (3.4) by (6.13).

7 Proof of Theorem 3.9

Our proof of Theorem 3.9 is a refinement of that of Theorem 3.2.

7.1 Proof of (i)

Let $\delta = \sqrt{-\lambda/2}$ and let $\{t_n\}$ be a positive increasing sequence such that $t_n \to \infty$ as $n \to \infty$. Let G(t) be a positive function on $(0, \infty)$. For any $n \ge 1$ and $\varepsilon > 0$, we have by the same way as in (5.1) and (5.2),

$$\mathbf{P}_{x}\left(\max_{t_{n}\leq s\leq t_{n+1}} Z_{s}^{\delta t_{n}}\geq G(t_{n})\right)\leq E_{x}\left[e^{A_{t_{n}}^{(Q-1)\mu}}\mathbf{E}_{B_{t_{n}}}\left[\max_{0\leq s\leq t_{n+1}-t_{n}} Z_{s}^{\delta t_{n}}\right]\right]/G(t_{n}).$$
(7.1)

Let a(t) be a nonnegative function on $(0, \infty)$ such that a(t) = o(t) $(t \to \infty)$ and $R(t) := \delta t - a(t)$. For $s \leq t$, let $\mathbf{B}_s^{(t),k}$ be the position at time s of the kth particle alive at time t. Since

$$\max_{0 \le s \le t_{n+1} - t_n} Z_s^{\delta t_n} \le \sum_{k=1}^{Z_{t_{n+1} - t_n}} \mathbf{1}_{\{\sup_{0 \le s \le t_{n+1} - t_n} | \mathbf{B}_s^{(t_{n+1} - t_n), k} | \ge \delta t_n\}}$$

we have by the same argument as for (5.5),

$$\mathbf{E}_{x} \left[\max_{t_{n} \leq s \leq t_{n+1}} Z_{s}^{\delta t_{n}} \right] \leq E_{x} \left[e^{A_{t_{n}}^{(Q-1)\mu}} E_{B_{t_{n}}} \left[e^{A_{t_{n+1}-t_{n}}^{(Q-1)\mu}}; \sup_{0 \leq s \leq t_{n+1}-t_{n}} |B_{s}| \geq \delta t_{n} \right] \right] \\
= E_{x} \left[e^{A_{t_{n}}^{(Q-1)\mu}} E_{B_{t_{n}}} \left[e^{A_{t_{n+1}-t_{n}}^{(Q-1)\mu}}; \sup_{0 \leq s \leq t_{n+1}-t_{n}} |B_{s}| \geq \delta t_{n} \right]; |B_{t_{n}}| \geq R(t_{n}) \right] \\
+ E_{x} \left[e^{A_{t_{n}}^{(Q-1)\mu}} E_{B_{t_{n}}} \left[e^{A_{t_{n+1}-t_{n}}^{(Q-1)\mu}}; \sup_{0 \leq s \leq t_{n+1}-t_{n}} |B_{s}| \geq \delta t_{n} \right]; |B_{t_{n}}| < R(t_{n}) \right] = (\mathbf{I}) + (\mathbf{II}).$$
(7.2)

In what follows, we suppose that

- $t_{n+1} t_n \to 0$ as $n \to \infty$;
- $a(t_n)^2/(t_{n+1}-t_n) \to \infty$ as $n \to \infty$.

Then by Remark 4.2,

$$(\mathbf{I}) \le E_x \left[e^{A_{t_n}^{(Q-1)\mu}}; |B_{t_n}| \ge R(t_n) \right] \sup_{x \in \mathbb{R}^d} E_x \left[e^{A_{t_{n+1}-t_n}^{(Q-1)\mu}} \right] \asymp e^{\sqrt{-2\lambda}a(t_n)} t_n^{(d-1)/2} \quad (n \to \infty).$$

$$(7.3)$$

By the Cauchy-Schwarz inequality, we have for any $x \in \mathbb{R}^d$ with $|x| \leq R(t_n)$ and for any constants p, q > 1 with 1/p + 1/q = 1,

$$E_{x}\left[e^{A_{t_{n+1}-t_{n}}^{(Q-1)\mu}};\sup_{0\leq s\leq t_{n+1}-t_{n}}|B_{s}|\geq\delta t_{n}\right]\leq E_{x}\left[e^{A_{t_{n+1}-t_{n}}^{(Q-1)\mu}};\sup_{0\leq s\leq t_{n+1}-t_{n}}|B_{s}-x|\geq a(t_{n})\right]$$

$$\leq E_{x}\left[e^{pA_{t_{n+1}-t_{n}}^{(Q-1)\mu}}\right]^{1/p}P_{x}\left(\sup_{0\leq s\leq t_{n+1}-t_{n}}|B_{s}-x|\geq a(t_{n})\right)^{1/q}$$

$$\leq cP_{0}\left(\sup_{0\leq s\leq t_{n+1}-t_{n}}|B_{s}|\geq a(t_{n})\right)^{1/q}.$$
(7.4)

If r(t) is a positive function on $(0, \infty)$ such that $r(t)^2/t \to \infty$ as $t \to +0$, then by [33, Corollary 3.4] and the change of variables,

$$P_0\left(\sup_{0\le s\le t} |B_s| \ge r(t)\right) = P_0\left(\sup_{0\le s\le t/r(t)^2} |B_s| \ge 1\right)$$

$$\approx \int_0^{t/r(t)^2} \frac{e^{-1/(2t)}}{t^{(d+2)/2}} \,\mathrm{d}t = \int_{r(t)^2/t}^\infty e^{-u/2} u^{(d-2)/2} \,\mathrm{d}u \sim 2e^{-r(t)^2/(2t)} \left(\frac{r(t)^2}{t}\right)^{(d-2)/2} \quad (t \to \infty).$$

Hence

$$P_0\left(\sup_{0\le s\le t_{n+1}-t_n} |B_s| \ge a(t_n)\right) \asymp \exp\left(-\frac{a(t_n)^2}{2(t_{n+1}-t_n)}\right) \left(\frac{a(t_n)^2}{t_{n+1}-t_n}\right)^{(d-2)/2} \quad (n\to\infty).$$
(7.5)

For any $x \in \mathbb{R}^d$, since it follows by (2.3) that

$$E_x\left[e^{A_{t_n}^{(Q-1)\mu}}\right] \asymp e^{(-\lambda)t_n} \quad (n \to \infty),$$

we have by (7.4) and (7.5),

$$(II) \leq cE_x \left[e^{A_{t_n}^{(Q-1)\mu}} \right] P_0 \left(\sup_{0 \leq s \leq t_{n+1} - t_n} |B_s| \geq a(t_n) \right)^{1/q}$$

$$\approx e^{(-\lambda)t_n} \exp \left(-\frac{a(t_n)^2}{2q(t_{n+1} - t_n)} \right) \left(\frac{a(t_n)^2}{t_{n+1} - t_n} \right)^{(d-2)/(2q)} \qquad (n \to \infty).$$

$$(7.6)$$

For $c_1 > 0$, $c_2 > 0$ and $\alpha \in (0, 1)$, if we let

$$a(t) \equiv c_1, \quad t_n = c_2 n^{\alpha},$$

then

$$c_2 \alpha (n+1)^{\alpha-1} \le t_{n+1} - t_n \le c_2 \alpha n^{\alpha-1}$$
(7.7)

and therefore,

$$\frac{c_1}{\sqrt{c_2\alpha}} n^{(1-\alpha)/2} \le \frac{a(t_n)}{\sqrt{t_{n+1} - t_n}} \le \frac{c_1}{\sqrt{c_2\alpha}} (n+1)^{(1-\alpha)/2}$$

Since $t_{n+1} - t_n \to 0$ and $a(t_n)^2/(t_{n+1} - t_n) \to \infty$ as $n \to \infty$, we obtain by (7.1), (7.2) (7.3) and (7.6),

$$\mathbf{P}_{x}\left(\max_{t_{n}\leq s\leq t_{n+1}} Z_{s}^{\delta t_{n}} \geq G(t_{n})\right) \\
\leq \frac{c}{G(t_{n})}\left(e^{\sqrt{-2\lambda}a(t_{n})}t_{n}^{(d-1)/2} + e^{(-\lambda)t_{n}}\exp\left(-\frac{a(t_{n})^{2}}{2q(t_{n+1}-t_{n})}\right)\left(\frac{a(t_{n})^{2}}{t_{n+1}-t_{n}}\right)^{(d-2)/(2q)}\right) \\
\leq \frac{c'}{G(t_{n})}\left(e^{\sqrt{-2\lambda}c_{1}}n^{\alpha(d-1)/2} + e^{c_{2}(-\lambda)n^{\alpha}}e^{-c_{1}^{2}n^{1-\alpha}/(2qc_{2}\alpha)}n^{(1-\alpha)(d-2)/(2q)}\right).$$
(7.8)

Here we take $\alpha = 1/2$ and c_1 so large that $c_1 \ge \sqrt{-2q\lambda}c_2$. If we let $G(t) = t^a(\log t)(\log \log t)^{1+\varepsilon}$ for a > 0 and $\varepsilon > 0$, then by (7.8),

$$\mathbf{P}_x\left(\max_{t_n \le s \le t_{n+1}} Z_s^{\delta t_n} \ge G(t_n)\right) \le \frac{cn^{(d-1)/4}}{n^{a/2} (\log n) (\log \log n)^{1+\varepsilon}}.$$

In particular, if we let a = (d+3)/2, then

$$\sum_{n=1}^{\infty} \mathbf{P}_x \left(\max_{t_n \le s \le t_{n+1}} Z_s^{\delta t_n} \ge G(t_n) \right) < \infty.$$

Hence by the Borel-Cantelli lemma, there exists an event of full \mathbf{P}_x -probability and a natural valued random variable $N \geq 1$ such that on this event, we have for all $n \geq N$,

$$\max_{t_n \le s \le t_{n+1}} Z_s^{\delta t_n} \le G(t_n).$$

Moreover, for any $n \geq N$ and $t \in [t_n, t_{n+1}]$,

$$Z_t^{\delta t} \le \max_{t_n \le s \le t_{n+1}} Z_s^{\delta t_n} \le G(t_n) \le G(t),$$

which completes the proof.

7.2 Proof of (ii)

As in the proof of Lemma 5.2, we denote by $\mathbf{B}_s^{t,k}$ the position of a particle at time s starting from \mathbf{B}_t^k at time $t \ (s \ge t)$. Let $\{t_n\}$ be a positive increasing sequence such that $t_n \to \infty$. Fix a constant $p_n \in [0, 1)$ and a compact set $K \subset \mathbb{R}^d$. Then for each index k,

$$\left\{ \mathbf{B}_{p_{n}t_{n}}^{p_{n}t_{n},k} \in K, |\mathbf{B}_{s}^{p_{n}t_{n},k}| > \delta s \text{ for all } s \in [t_{n}, t_{n+1}] \right\}$$

$$\supset \left\{ \mathbf{B}_{p_{n}t_{n}}^{p_{n}t_{n},k} \in K, |\mathbf{B}_{t_{n}}^{p_{n}t_{n},k}| > |\mathbf{B}_{t_{n}}^{p_{n}t_{n},k} - \mathbf{B}_{p_{n}t_{n}}^{p_{n}t_{n},k}| > \delta t_{n+1} + 1, \right\}$$

$$= : E_{n}^{k} \cdot E_{n}^{k}$$

Let G(t) and f(t) be positive functions on $(0,\infty)$ such that $f(t) \to 0$ as $t \to \infty$. Define

$$N_t = \left\{ e^{\lambda t} Z_t(K) \ge f(t) \right\}.$$

Then by the Markov property,

$$\mathbf{P}_{x}\left(\left\{\sum_{k=1}^{Z_{pnt_{n}}}\mathbf{1}_{E_{n}^{k}}\leq G(t_{n})\right\}\cap N_{p_{n}t_{n}}\right)=\mathbf{E}_{x}\left[\mathbf{P}_{\mathbf{B}_{pnt_{n}}}\left(\sum_{k=1}^{l}\mathbf{1}_{F_{n}^{k}}\leq G(t_{n})\right)|_{l=Z_{pnt_{n}}};N_{p_{n}t_{n}}\right]$$
(7.9)

for

$$F_n^k := \left\{ \begin{aligned} \mathbf{B}_0^{0,k} \in K, \ |\mathbf{B}_{(1-p_n)t_n}^{0,k}| > |\mathbf{B}_{(1-p_n)t_n}^{0,k} - \mathbf{B}_0^{0,k}| > \delta t_{n+1} + 1, \\ \sup_{(1-p_n)t_n \le s \le t_{n+1} - p_n t_n} |\mathbf{B}_s^{0,k} - \mathbf{B}_{(1-p_n)t_n}^{0,k}| < 1 \end{aligned} \right\}.$$

Let $\mathbf{x} = (x^1, \dots, x^l)$. Then by the same way as in (5.10) and (5.11),

$$\mathbf{P}_{\mathbf{x}}\left(\sum_{k=1}^{l} \mathbf{1}_{F_{n}^{k}} \le G(t_{n})\right) \le e^{G(t_{n})} \prod_{1 \le k \le m, x^{k} \in K} \exp\left(-(1-e^{-1})\mathbf{P}_{x^{k}}(F_{n}^{k})\right).$$
(7.10)

Let

$$C_n := \left\{ |B_{(1-p_n)t_n}| > |B_{(1-p_n)t_n} - B_0| > \delta t_{n+1} + 1 \right\}.$$

Then for any $x \in K$, we have by the Markov property and the spatial uniformity of the Brownian motion,

$$\mathbf{P}_{x}(F_{n}^{k}) = P_{x} \left(\left\{ \sup_{(1-p_{n})t_{n} \leq s \leq t_{n+1}-p_{n}t_{n}} |B_{s} - B_{(1-p_{n})t_{n}}| < 1 \right\} \cap C_{n} \right)$$
$$= E_{x} \left[P_{B_{(1-p_{n})t_{n}}} \left(\sup_{0 \leq s \leq t_{n+1}-t_{n}} |B_{s} - B_{0}| < 1 \right); C_{n} \right]$$
$$= P_{x}(C_{n})P_{0} \left(\sup_{0 \leq s \leq t_{n+1}-t_{n}} |B_{s} - B_{0}| < 1 \right).$$

In what follows, we suppose that

- $t_{n+1} t_n \to 0$ as $n \to \infty$;
- $t_{n+1}/\sqrt{(1-p_n)t_n} \to \infty$ as $n \to \infty$.

Then by the same way as in (5.14), there exist $c_0 > 0$ and $c_1 > 0$ such that for any $x \in \mathbb{R}^d$,

$$P_x(C_n) \ge c_0 \int_{(\delta t_{n+1}+1)/\sqrt{(1-p_n)t_n}}^{\infty} e^{-r^2/2} r^{d-1} \, \mathrm{d}r \ge c_1 e^{-(\delta t_{n+1})^2/(2(1-p_n)t_n)} \left(\frac{\delta t_{n+1}}{\sqrt{(1-p_n)t_n}}\right)^{d-2},$$

which implies that for any $x \in K$,

$$\mathbf{P}_{x}(F_{n}^{k}) \ge c_{1}e^{-(\delta t_{n+1})^{2}/(2(1-p_{n})t_{n})} \left(\frac{\delta t_{n+1}}{\sqrt{(1-p_{n})t_{n}}}\right)^{d-2} =: c_{1}q_{n}$$

Because of this and (7.10), there exists $c_2 > 0$ such that

$$\mathbf{P}_{\mathbf{x}}\left(\sum_{k=1}^{l} \mathbf{1}_{F_{n}^{k}} \le G(t_{n})\right) \le e^{G(t_{n})} \exp\left(-c_{2}q_{n} \cdot \sharp\left\{k \mid x^{k} \in K\right\}\right).$$

Hence by (7.9),

$$\mathbf{P}_{x}\left(\left\{\sum_{k=1}^{Z_{p_{n}t_{n}}}\mathbf{1}_{E_{n}^{k}}\leq G(t_{n})\right\}\cap N_{p_{n}t_{n}}\right)\leq e^{G(t_{n})}\mathbf{E}_{x}\left[\exp\left(-c_{2}q_{n}Z_{p_{n}t_{n}}(K)\right);N_{p_{n}t_{n}}\right] \qquad (7.11)$$
$$\leq \exp\left(G(t_{n})-c_{2}q_{n}e^{-\lambda p_{n}t_{n}}f(t_{n})\right).$$

Here we note that

$$q_n e^{-\lambda p_n t_n} = e^{g_n(p_n)} \left(\frac{\delta t_{n+1}}{\sqrt{(1-p_n)t_n}}\right)^{d-2}$$

for

$$g_n(p) = -\lambda p t_n - \frac{(\delta t_{n+1})^2}{2(1-p)t_n} \quad (0 \le p < 1).$$

Then the right hand side above takes the maximal value $g_n(p_n^*) = \lambda(t_{n+1} - t_n)$ for $p_n^* = 1 - t_{n+1}/(2t_n)$.

We take $p_n = p_n^*$, $f(t) = (\log \log t)^{-\varepsilon}$ and $G(t) = c_3 t^{(d-2)/2} (\log \log t)^{-\varepsilon}$ for $\varepsilon > 0$ and $c_3 > 0$. Then

$$G(t_n) - c_2 q_n e^{-\lambda p_n t_n} f(t_n) = t_n^{(d-2)/2} (\log \log t_n)^{-\varepsilon} \left\{ c_3 - c_2 e^{\lambda (t_{n+1}-t_n)} \left(-\lambda t_{n+1}/t_n\right)^{(d-2)/2} \right\}.$$
(7.12)

For some $\alpha \in (0, 1]$ and c > 0, if we let

$$t_n = cn^{\alpha},$$

then $t_{n+1}-t_n \to 0$ as $n \to \infty$ and $t_{n+1}/\sqrt{(1-p_n)t_n} = \sqrt{2t_{n+1}} \to \infty$ as $n \to \infty$. Therefore by (7.12), we can take $c_3 > 0$ so small that

$$G(t_n) - c_2 q_n e^{-\lambda p_n t_n} f(t_n) \le -c_4 n^{\alpha(d-2)/2} (\log \log n)^{-\varepsilon}$$

for some $c_4 > 0$. Then by (7.11),

$$\mathbf{P}_x\left(\left\{\sum_{k=1}^{Z_{p_nt_n}}\mathbf{1}_{E_n^k} \le G(t_n)\right\} \cap N_{p_nt_n}\right) \le \exp\left(-c_4 n^{\alpha(d-2)/2} (\log\log n)^{-\varepsilon}\right).$$

In particular, if we assume that $d \geq 3$, then

$$\sum_{n=1}^{\infty} \mathbf{P}_x \left(\left\{ \sum_{k=1}^{Z_{p_n t_n}} \mathbf{1}_{E_n^k} \le G(t_n) \right\} \cap N_{p_n t_n} \right) < \infty.$$

Hence by the Borel-Cantelli lemma, the event

$$\left\{\sum_{k=1}^{Z_{p_n t_n}} \mathbf{1}_{E_n^k} > G(t_n)\right\} \cup (N_{p_n t_n})^c$$

occurs for all sufficiently large n. Since $p_n t_n = t_n - t_{n+1}/2 \to \infty$ as $n \to \infty$, we have by (2.6),

$$e^{\lambda p_n t_n} Z_{p_n t_n}(K) \to M_\infty \int_K h(y) \,\mathrm{d}y, \quad \mathbf{P}_x\text{-a.s.}$$

Since $\int_{K} h(y) \, dy > 0$ and $f(t) \to 0$ as $t \to \infty$, we see that on the event $\{M_{\infty} > 0\}$, the event $(N_{p_n t_n})^c$ occurs only for finite $n \ge 1$, that is, the event $\left\{\sum_{k=1}^{Z_{p_n t_n}} \mathbf{1}_{E_n^k} > G(t_n)\right\}$ occurs for all sufficiently large $n \ge 1$.

For all sufficiently large t > 0, there exists $n = n(t) \in \mathbb{N}$ such that $t_n \leq t < t_{n+1}$ and

$$G(t_n) = c_3 t_n^{(d-2)/2} (\log \log t_n)^{-\varepsilon} \ge c_5 t_{n+1}^{(d-2)/2} (\log \log t)^{-\varepsilon} \ge c_5 t^{(d-2)/2} (\log \log t)^{-\varepsilon}$$

for some $c_5 > 0$. We thus have, $\mathbf{P}_x(\cdot \mid M_\infty > 0)$ -a.s. for all sufficiently large t > 0,

$$Z_t^{\delta t} = \sum_{k=1}^{Z_t} \mathbf{1}_{\{|\mathbf{B}_t^k| > \delta t\}} \ge \sum_{k=1}^{Z_{pnt_n}} \mathbf{1}_{\{\mathbf{B}_{pnt_n}^{pnt_n,k} \in K, |\mathbf{B}_s^{pnt_n,k}| > \delta s \text{ for all } s \in [t_n, t_{n+1}]\}}$$
$$\ge \sum_{k=1}^{Z_{pnt_n}} \mathbf{1}_{E_n^k} \ge G(t_n) \ge c_5 t^{(d-2)/2} (\log \log t)^{-\varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary, the last inequality above is valid by taking $c_5 = 1$.

A Appendix

A.1 Decay rate of the ground state

Let $\mu = \mu^+ - \mu^-$ for some $\mu^+, \mu^- \in \mathcal{K}_{\infty}(1)$. Recall that

$$\lambda(\mu) = \inf\left\{\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x - \int_{\mathbb{R}^d} u^2 \,\mathrm{d}\mu \mid u \in C_0^\infty(\mathbb{R}^d), \int_{\mathbb{R}^d} u^2 \,\mathrm{d}x = 1\right\}.$$

In what follows, we let $\lambda := \lambda(\mu)$ and assume that $\lambda < 0$. As mentioned in Subsection 2.1, λ is the principal eigenvalue of the operator $-\Delta/2 - \mu$ and the corresponding eigenfunction h has a bounded, continuous and strictly positive version.

Lemma A.1. Suppose that $\mu = \mu^+ - \mu^-$ for some $\mu^+, \mu^- \in \mathcal{K}_{\infty}(1)$. Then for any positive constants A_1 and A_2 with $A_2 < \sqrt{-2\lambda} < A_1$, there exist positive constants c_1 , c_2 such that

$$\frac{c_1 e^{-A_1|x|}}{|x|^{(d-1)/2}} \le h(x) \le \frac{c_2 e^{-A_2|x|}}{|x|^{(d-1)/2}}, \quad (|x| \ge 1).$$

Moreover, if μ^- is compactly supported in \mathbb{R}^d , then the inequality above holds with $A_1 = \sqrt{-2\lambda}$. A similar result is valid for μ^+ and A_2 .

Proof. We follow the argument of [10] and [39, Lemma 4.1]. We first discuss the upper bound of h. For r > 0, let

$$\sigma_r := \inf\{t > 0 \mid |B_t| \le r\}.$$

Since $M_t := e^{\lambda t} e^{A_t^{\mu}} h(B_t)$ is a P_x -martingale, we have by the optional stopping theorem and the Hölder inequality,

$$h(x) = E_x \left[e^{\lambda(t \wedge \sigma_r)} e^{A_{t \wedge \sigma_r}^{\mu}} h(B_{t \wedge \sigma_r}) \right] \le \|h\|_{\infty} E_x \left[e^{p(\lambda + \varepsilon)(t \wedge \sigma_r)} \right]^{1/p} E_x \left[e^{-q\varepsilon(t \wedge \sigma_r)} e^{qA_{t \wedge \sigma_r}^{\mu}} \right]^{1/q}$$
(A.1)

for any $\varepsilon \in (0, -\lambda)$ and p, q > 1 with 1/p + 1/q = 1.

Let $\mu_r(\mathrm{d}x) = \mathbf{1}_{|x|>r}(x)\mu(\mathrm{d}x)$ and let \hat{P}_x be the law of the killed process of **M** by the exponential distribution with rate $q\varepsilon$. Then

$$E_x \left[e^{-q\varepsilon(t\wedge\sigma_r)} e^{qA^{\mu}_{t\wedge\sigma_r}} \right] = E_x \left[e^{-q\varepsilon(t\wedge\sigma_r)} e^{qA^{\mu_r}_{t\wedge\sigma_r}} \right] = \hat{E}_x \left[e^{qA^{\mu_r}_{t\wedge\sigma_r}} \right] \le \hat{E}_x \left[e^{qA^{\mu_r}_{\sigma_r}} \right].$$
(A.2)

Since $\mu^+ \in \mathcal{K}_{\infty}(1)$ and

$$\hat{E}_x\left[qA_{\sigma_r}^{\mu_r^+}\right] \le q\hat{E}_x\left[A_{\infty}^{\mu_r^+}\right] = q\int_{|y|\ge r} G_{q\varepsilon}(x,y)\,\mu^+(dy),$$

there exists $R = R(\varepsilon, p) > 0$ such that for any $r \ge R$,

$$\sup_{x \in \mathbb{R}^d} \hat{E}_x \left[q A_{\sigma_r}^{\mu_r^+} \right] \le q \sup_{x \in \mathbb{R}^d} \int_{|y| \ge r} G_{q\varepsilon}(x, y) \, \mu^+(dy) < 1.$$

Then the Khasminskii lemma (see, e.g., [15, Lemma 3.7]) implies that for any $r \ge R$,

$$\sup_{x \in \mathbb{R}^d} \hat{E}_x \left[e^{q A_{\sigma_r}^{\mu_r^+}} \right] < \infty.$$
(A.3)

Hence by (A.1) and (A.2),

$$h(x) \leq \|h\|_{\infty} E_x \left[e^{p(\lambda+\varepsilon)(t\wedge\sigma_r)} \right]^{1/p} \left(\sup_{x\in\mathbb{R}^d} \hat{E}_x \left[e^{qA_{\sigma_r}^{\mu_r^+}} \right] \right)^{1/q}$$

$$\to \|h\|_{\infty} E_x \left[e^{p(\lambda+\varepsilon)\sigma_r} \right]^{1/p} \left(\sup_{x\in\mathbb{R}^d} \hat{E}_x \left[e^{qA_{\sigma_r}^{\mu_r^+}} \right] \right)^{1/q} \quad (A.4)$$

Let ν_r be the equilibrium potential of $B_r := \{x \in \mathbb{R}^d \mid |x| \leq r\}$ (see [18, p.82] for definition). Then

$$E_x\left[e^{p(\lambda+\varepsilon)\sigma_r}\right] = \int_{|y|\le r} G_{-p(\lambda+\varepsilon)}(x,y)\nu_r(\mathrm{d}y) \le \sup_{|y|\le r} G_{-p(\lambda+\varepsilon)}(x,y)\nu_r(B_r).$$
(A.5)

Since we see by (2.1) that for any $x, y \in \mathbb{R}^d$ with $|x| \ge 2r$ and $|y| \le r$,

$$G_{-p(\lambda+\varepsilon)}(x,y) \le c_{\varepsilon,p,r} \frac{e^{-\sqrt{-2p(\lambda+\varepsilon)|x|}}}{|x|^{(d-1)/2}},$$

we have by (A.5),

$$E_x \left[e^{p(\lambda+\varepsilon)\sigma_r} \right] \le c'_{\varepsilon,p,r} \frac{e^{-\sqrt{-2p(\lambda+\varepsilon)}|x|}}{|x|^{(d-1)/2}} \ (|x|\ge 1).$$
(A.6)

Then by (A.4),

$$h(x) \le \|h\|_{\infty} c_{\varepsilon,p,r}'' \left(\frac{e^{-\sqrt{-2p(\lambda+\varepsilon)}|x|}}{|x|^{(d-1)/2}}\right)^{1/p} \ (|x| \ge 1), \tag{A.7}$$

which implies the desired upper bound of h. If we further assume that μ^+ is compactly supported in \mathbb{R}^d , then (A.7) is valid for p = 1 because μ_r^+ vanishes for large r > 0.

We next discuss the lower bound of h. Here we denote by \tilde{P}_x the law of the killed process of \mathbf{M} by the exponential distribution with rate $-\lambda$. Then by the optional stopping theorem again,

$$h(x) = E_x \left[e^{\lambda(t \wedge \sigma_r)} e^{A_{t \wedge \sigma_r}} h(B_{t \wedge \sigma_r}) \right] \ge \inf_{|y| \le r} h(y) \tilde{E}_x \left[e^{-A_{t \wedge \sigma_r}^{\mu^-}} \right] \ge \inf_{|y| \le r} h(y) \tilde{E}_x \left[e^{-A_{\sigma_r}^{\mu^-}}; \sigma_r \le t \right].$$
(A.8)

By the same argument as for (A.3), there exists R = R(p) > 0 for any p > 1 such that for any $r \ge R$,

$$\sup_{x \in \mathbb{R}^d} \tilde{E}_x \left[e^{A_{\sigma_r}^{\mu_r^-}/(p-1)} \right]^{p-1} < \infty.$$

Then by the Hölder inequality, we have for any p > 1 and $r \ge R$,

$$\tilde{E}_{x}\left[e^{-A_{\sigma_{r}}^{\mu^{-}}};\sigma_{r}\leq t\right]\geq\frac{\tilde{P}_{x}(\sigma_{r}\leq t)^{p}}{\tilde{E}_{x}\left[e^{A_{\sigma_{r}}^{\mu^{-}_{r}}/(p-1)}\right]^{p-1}}\geq\frac{\tilde{P}_{x}(\sigma_{r}\leq t)^{p}}{\sup_{y\in\mathbb{R}^{d}}\tilde{E}_{y}\left[e^{A_{\sigma_{r}}^{\mu^{-}_{r}}/(p-1)}\right]^{p-1}}$$
$$\rightarrow\frac{\tilde{P}_{x}(\sigma_{r}<\infty)^{p}}{\sup_{y\in\mathbb{R}^{d}}\tilde{E}_{y}\left[e^{A_{\sigma_{r}}^{\mu^{-}_{r}}/(p-1)}\right]^{p-1}}\quad(t\to\infty).$$
(A.9)

Since

$$\tilde{P}_x(\sigma_r < \infty) = E_x \left[e^{\lambda \sigma_r}; \sigma_r < \infty \right] = \int_{B_r} G_{-\lambda}(x, y) \nu_r(\mathrm{d}y),$$

we have by the same argument as for (A.6),

$$\tilde{P}_x(\sigma_r < \infty) \ge c_r \frac{e^{-\sqrt{-2\lambda}|x|}}{|x|^{(d-1)/2}}.$$

Then by (A.8) and (A.9),

$$h(x) \ge c_{p,r} \left(\frac{e^{-\sqrt{-2\lambda}|x|}}{|x|^{(d-1)/2}}\right)^p \ (|x|\ge 1).$$
 (A.10)

We thus get the desired lower bound of h. If μ^- is compactly supported in \mathbb{R}^d , then (A.10) is valid for p = 1 because μ_r^- vanishes for large r > 0.

By using Lemma A.1 instead of [39, Lemma 4.1], we can follow the argument of [39, Section 4] to get

Theorem A.2. Suppose that $\mu = \mu^+ - \mu^-$ for some $\mu^+, \mu^- \in \mathcal{K}_{\infty}(1)$. Then for any $f \in \mathcal{B}_b(\mathbb{R}^d)$,

$$\lim_{t \to \infty} e^{\lambda t} E_x \left[e^{A_t^{\mu}} f(B_t) \right] = h(x) \int_{\mathbb{R}^d} f(y) h(y) \, \mathrm{d}y \quad (x \in \mathbb{R}^d).$$

A.2 Positivity of M_{∞} and survival

We discuss relations among the positivity of M_{∞} , the finiteness of the total number of branching and the survival property. Note that we already discussed in [34, Proposition 3.6, Theorem 3.7 and Remark 3.14] the relation between the first and third properties for branching symmetric stable processes with absorbing boundary.

Let $\overline{\mathbf{M}} = (\{\mathbf{B}_t\}_{t\geq 0}, \{\mathbf{P}_{\mathbf{x}}\}_{\mathbf{x}\in\mathbf{X}})$ be a branching Brownian motion on \mathbf{X} with branching rate $\mu \in \mathcal{K}_{\infty}$ and branching mechanism $\{p_n(x)\}_{n\geq 0}$. Denote by $G^{\mu}(x, y)$ the Green function associated with the Feynman-Kac semigroup $p_t^{\mu}f(x) = E_x \left[e^{-A_t^{\mu}}f(B_t)\right]$. For a function u on \mathbb{R}^d , define

$$F(u)(x) = \sum_{n=0}^{\infty} p_n(x)u(x)^n$$

if the right hand side makes sense. We first study the solution to the next equation:

$$u(x) = E_x \left[e^{-A_{\infty}^{\mu}} \right] + E_x \left[\int_0^{\infty} e^{-A_t^{\mu}} F(u)(B_t) \, \mathrm{d}A_t^{\mu} \right], \quad 0 \le u(x) \le 1 \quad (x \in \mathbb{R}^d).$$
(A.11)

Lemma A.3. Suppose that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G^{\mu}(x, y) \,\mu(\mathrm{d}y)\mu(\mathrm{d}x) < \infty. \tag{A.12}$$

Let u and v be functions on \mathbb{R}^d such that $0 \leq u(x) \leq v(x) < 1$ on \mathbb{R}^d . If these functions are solutions to the equation (A.11), then $u \equiv v$.

We omit the proof of Lemma A.3 because it is similar to that of [34, Lemma 3.5]. We note that if $\mu(\mathbb{R}^d) < \infty$, then (A.12) is fulfilled because

$$\int_{\mathbb{R}^d} G^{\mu}(x,y)\,\mu(\mathrm{d}y) = E_x\left[\int_0^\infty e^{-A_t^{\mu}}\,\mathrm{d}A_t^{\mu}\right] = 1 - E_x\left[e^{-A_{\infty}^{\mu}}\right] \le 1.$$

We next reveal the relations as we mentioned at the first of this subsection. Let N be the total number of branching for $\overline{\mathbf{M}}$.

Proposition A.4. Suppose that (A.12) holds and $\mathbf{P}_x(M_{\infty} > 0) > 0$. If d = 1, 2, then

$$\{e_0 = \infty\} = \{N = \infty\} = \{M_\infty > 0\}, \quad \mathbf{P}_x \text{-}a.s.$$

On the other hand, if $d \geq 3$, then

$$\{e_0 = \infty\} \supseteq \{N = \infty\} = \{M_\infty > 0\}, \quad \mathbf{P}_x\text{-}a.s.$$

If $d \geq 3$, then the Brownian motion is transient so that the associated particle goes to infinity eventually. Since we assume that the branching rate μ is small at infinity, the number of branching can be small even on the survival event. In fact, branching never occurs with positive probability. On the other hand, if d = 1 or 2, then the Brownian motion is recurrent so that the associated particle can come to the support of μ infinitely often. Therefore, branching occurs infinite times on the survival event. *Proof.* Let $u(x) = \mathbf{P}_x(N < \infty)$ and $v(x) = \mathbf{P}_x(M_\infty = 0)$. Then v(x) < 1 by assumption. Moreover, if $N < \infty$, then Z_t is a finite random constant eventually and thus $M_\infty = 0$. Namely, we obtain $0 \le u(x) \le v(x) < 1$. Since u and v are solutions to the equation (A.11), we obtain $u \equiv v$ by Lemma A.3, whence

$$\{N = \infty\} = \{M_{\infty} > 0\}, \quad \mathbf{P}_x\text{-a.s.}$$

Let $u_e(x) = \mathbf{P}_x(e_0 < \infty)$. Then

$$u_e(x) = E_x \left[\int_0^\infty e^{-A_t^{\mu}} F(u_e)(B_t) \, \mathrm{d}A_t^{\mu} \right].$$

For d = 1, 2, since $P_x(A_{\infty}^{\mu} = \infty) = 1$ by [31, p.426, Proposition 3.11], we have $E_x[e^{-A_{\infty}^{\mu}}] = 0$ so that u_e also satisfies the equation (A.11). Furthermore, since $\{e_0 < \infty\} \subset \{M_{\infty} = 0\}$, we obtain $0 \le u_e(x) \le v(x) < 1$ and thus $u_e(x) = v(x)$ by Lemma A.3. This implies that

$$\{e_0 = \infty\} = \{M_\infty > 0\}, \quad \mathbf{P}_x\text{-a.s.}$$

On the other hand, if $d \geq 3$, then $\sup_{x \in \mathbb{R}^d} E_x[A^{\mu}_{\infty}] < \infty$ by (2.2). Hence by Jensen's inequality,

$$\mathbf{P}_x(T=\infty) = E_x[e^{-A_\infty^{\mu}}] \ge \exp\left(-E_x[A_\infty^{\mu}]\right) \ge \exp\left(-\sup_{x\in\mathbb{R}^d} E_x[A_\infty^{\mu}]\right) > 0.$$

Since

$$\{e_0 < \infty\} \cup \{T = \infty\} \subset \{N < \infty\}, \quad \{e_0 < \infty\} \cap \{T = \infty\} = \emptyset,$$

we have

$$\mathbf{P}_x(N < \infty) \ge \mathbf{P}_x(e_0 < \infty) + \mathbf{P}_x(T = \infty) > \mathbf{P}_x(e_0 < \infty).$$

Then by assumption,

$$\mathbf{P}_x(e_0 = \infty) > \mathbf{P}_x(N = \infty) = \mathbf{P}_x(M_\infty > 0) > 0,$$

which shows that

$$\{e_0 = \infty\} \supseteq \{N = \infty\}, \quad \mathbf{P}_x\text{-a.s.}$$

We thus complete the proof.

$\textbf{A.3} \quad \textbf{Proof of} \ (4.12)$

We evaluate the integral in the right hand side of (4.12). We first recall that a(t) is a function on $(0, \infty)$ such that a(t) = o(t) $(t \to \infty)$ and $R(t) = \delta t + a(t)$ for some $\delta > 0$. We will show that as $t \to \infty$,

$$(R(t) - \varepsilon_1 t)^d \int_0^t e^{(-\lambda + \varepsilon_2)s} \frac{1}{(t-s)^{(d+2)/2}} \exp\left(-\frac{(R(t) - \varepsilon_1 t)^2}{2(t-s)}\right) ds$$

$$\approx \begin{cases} e^{-(\delta - \varepsilon_1)^2 t/2} t^{(d-2)/2} & (\delta > \sqrt{-2\lambda}), \\ e^{(-\lambda + \varepsilon_2)t - \sqrt{2(-\lambda + \varepsilon_2)}(R(t) - \varepsilon_1 t)} t^{(d-1)/2} & (\delta \le \sqrt{-2\lambda}). \end{cases}$$
(A.13)

By the change of variables s = t - v, we get

$$\int_{0}^{t} e^{(-\lambda+\varepsilon_{2})s} \frac{1}{(t-s)^{(d+2)/2}} \exp\left(-\frac{(R(t)-\varepsilon_{1}t)^{2}}{2(t-s)}\right) ds$$

$$= e^{(-\lambda+\varepsilon_{2})t} \int_{0}^{t} e^{-(-\lambda+\varepsilon_{2})v} \frac{e^{-(R(t)-\varepsilon_{1}t)^{2}/(2v)}}{v^{(d+2)/2}} dv$$

$$= e^{(-\lambda+\varepsilon_{2})t-\sqrt{2(-\lambda+\varepsilon_{2})}(R(t)-\varepsilon_{1}t)} \int_{0}^{t} e^{-(\sqrt{(-\lambda+\varepsilon_{2})v}-(R(t)-\varepsilon_{1}t)/\sqrt{2v})^{2}} \frac{1}{v^{(d+2)/2}} dv$$

$$= e^{(-\lambda+\varepsilon_{2})t-\sqrt{2(-\lambda+\varepsilon_{2})}(R(t)-\varepsilon_{1}t)} (\text{III}).$$
(A.14)

If we let

$$w = \sqrt{(-\lambda + \varepsilon_2)v} - \frac{R(t) - \varepsilon_1 t}{\sqrt{2v}},$$

then

(III) =
$$2 \int_{-\infty}^{S(t)} e^{-w^2} \frac{F_t(w)^d}{\sqrt{w^2 + 2\sqrt{2(-\lambda + \varepsilon_2)}(R(t) - \varepsilon_1 t)}} \, \mathrm{d}w$$
 (A.15)

for

$$F_t(w) = \frac{2\sqrt{-\lambda + \varepsilon_2}}{w + \sqrt{w^2 + 2\sqrt{2(-\lambda + \varepsilon_2)}(R(t) - \varepsilon_1 t)}}$$

and

$$S(t) = \sqrt{(-\lambda + \varepsilon_2)t} - \frac{R(t) - \varepsilon_1 t}{\sqrt{2t}}.$$

Assume that $\delta > \sqrt{-2\lambda}$. Fix $\varepsilon_1 \in (0, \delta - \sqrt{-2\lambda})$, and take $\varepsilon_2 > 0$ so that $\delta - \varepsilon_1 > \sqrt{2(-\lambda + \varepsilon_2)}$. Then there exist $c_1 > 0$ and $c_2 > 0$ such that for any $w \leq 0$ and t > 0,

$$c_1 \frac{\sqrt{w^2 + 2\sqrt{2(-\lambda + \varepsilon_2)}(R(t) - \varepsilon_1 t)}}{R(t) - \varepsilon_1 t} \le F_t(w) \le c_2 \frac{\sqrt{w^2 + 2\sqrt{2(-\lambda + \varepsilon_2)}(R(t) - \varepsilon_1 t)}}{R(t) - \varepsilon_1 t}.$$
(A.16)

Since S(t) < 0 for all sufficiently large t > 0, we have as $t \to \infty$,

$$(\text{III}) \approx \frac{1}{(R(t) - \varepsilon_1 t)^d} \int_{-\infty}^{S(t)} e^{-w^2} \left(w^2 + 2\sqrt{2(-\lambda + \varepsilon_2)}(R(t) - \varepsilon_1 t) \right)^{(d-1)/2} \mathrm{d}w$$
$$= \frac{1}{(R(t) - \varepsilon_1 t)^d} \int_{-S(t)}^{\infty} e^{-w^2} \left(w^2 + 2\sqrt{2(-\lambda + \varepsilon_2)}(R(t) - \varepsilon_1 t) \right)^{(d-1)/2} \mathrm{d}w.$$

If $w \ge -S(t)$, then

$$w^{2} \geq \left(\frac{R(t) - \varepsilon_{1}t}{\sqrt{2}t} - \sqrt{(-\lambda + \varepsilon_{2})}\right)^{2} t = \left(\frac{\delta - \varepsilon_{1}}{\sqrt{2}} - \sqrt{(-\lambda + \varepsilon_{2})} + \frac{a(t)}{\sqrt{2}t}\right)^{2} t$$

and hence for all sufficiently large t > 0,

 $c_1(\varepsilon_1)w^2 \le w^2 + 2\sqrt{2(-\lambda + \varepsilon_2)}(R(t) - \varepsilon_1 t) \le c_2(\varepsilon_1)w^2.$

This implies that as $t \to \infty$,

$$(\text{III}) \approx \frac{c_3(\varepsilon_1)}{(R(t) - \varepsilon_1 t)^d} \int_{-S(t)}^{\infty} e^{-w^2} w^{d-1} \, \mathrm{d}w \sim \frac{c_3(\varepsilon_1)}{(R(t) - \varepsilon_1 t)^d} e^{-S(t)^2} (-S(t))^{d-2}$$
$$= \frac{c_4(\varepsilon_1)}{(R(t) - \varepsilon_1 t)^d} \exp\left(-\frac{(R(t) - \varepsilon_1 t)^2}{2t}\right) e^{-((-\lambda + \varepsilon_2)t - \sqrt{2(-\lambda + \varepsilon_2)}(R(t) - \varepsilon_1 t))} t^{(d-2)/2}.$$
(A.17)

We next assume that $\delta \leq \sqrt{-2\lambda}$. Then

$$(\text{III}) = \int_{-\infty}^{0} e^{-w^2} \frac{F_t(w)^d}{\sqrt{w^2 + 2\sqrt{2(-\lambda + \varepsilon_2)}(R(t) - \varepsilon_1 t)}} \, \mathrm{d}w \\ + \int_{0}^{S(t)} e^{-w^2} \frac{F_t(w)^d}{\sqrt{w^2 + 2\sqrt{2(-\lambda + \varepsilon_2)}(R(t) - \varepsilon_1 t)}} \, \mathrm{d}w = (\text{III})_1 + (\text{III})_2.$$

By (A.16) and the change of variables (w = -v), we obtain as $t \to \infty$,

$$(\text{III})_1 \approx \frac{1}{(R(t) - \varepsilon_1 t)^d} \int_0^\infty e^{-v^2} \left(v^2 + 2\sqrt{2(-\lambda + \varepsilon_2)} (R(t) - \varepsilon_1 t) \right)^{(d-1)/2} \, \mathrm{d}v$$
$$\approx \frac{c_4(\varepsilon_1)}{(R(t) - \varepsilon_1 t)^{(d+1)/2}}.$$

Since there exist $c(\varepsilon_1) > 0$ and $c'(\varepsilon_1) > 0$ such that for all sufficiently large t > 0,

$$c(\varepsilon_1)\sqrt{t} \le w + \sqrt{w^2 + 2\sqrt{2(-\lambda + \varepsilon_2)}(R(t) - \varepsilon_1 t)} \le c'(\varepsilon_1)\sqrt{t} \quad (0 \le w \le S(t))$$

and $S(t) \to \infty$ as $t \to \infty$, we also have

$$(\text{III})_2 \asymp \frac{c_5(\varepsilon_1)}{(R(t) - \varepsilon_1 t)^{(d+1)/2}} \quad (t \to \infty),$$

that is,

(III)
$$\approx \frac{c_6(\varepsilon_1)}{(R(t) - \varepsilon_1 t)^{(d+1)/2}} \quad (t \to \infty).$$
 (A.18)

We thus get (A.13) by (A.14), (A.15), (A.17) and (A.18).

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