## KERNELS, TRUTH AND SATISFACTION

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**Prologue.** The well known Kotlarski-Krajewski-Lachlan Theorem [KKL81] says that every model  $\mathcal{M}$  of Peano Arithmetic (PA) has an elementary extension  $\mathcal{N} \succ \mathcal{M}$  having a full satisfaction class (or, equivalently, every resplendent model has a full satisfaction class). Later, Enayat & Visser [EV15] gave another proof. According to [EV15], the proof in [KKL81] used some "rather exotic proof-theoretic technology", while the proof in [EV15] uses "a perspicuous method for the construction of full satisfaction classes". Although not made explicit there, the proof in [EV15], when stripped to its essentials, is seen to ultimately depend on showing that certain digraphs have kernels. This is made explicit here.

There is a lengthy discussion in §4 of [EV15] about the relationship of full satisfaction classes to full truth classes. Satisfaction classes, which are sets of ordered pairs consisting of a formula in the language of arithmetic and an assignment for that formula, are exclusively used in [EV15]. Truth classes are sets of arithmetic sentences that may also have domain constants. By [EV15, Prop. 4.3] (whose "routine but laborious proof is left to the reader"), there is a canonical correspondence between full truth classes and *extensional* full satisfaction classes. The culmination of [EV15, §4] is the construction of extensional full satisfaction classes. In §2 of this paper, we will avoid the intricacies of [EV15, §4] by working exclusively with truth classes to easily obtain the same conclusion.

§1. Digraphs and kernels. A binary relational structure  $\mathcal{A} = (A, E)$  is referred to here as a directed graph, or digraph for short.<sup>1</sup> A subset  $K \subseteq A$  is a kernel of  $\mathcal{A}$  if for every  $a \in A$ ,  $a \in K$  iff whenever aEb, then  $b \notin K$ . According to [BJG09], kernels were introduced by von Neumann [vNM44] and have subsequently found many applications. For  $n < \omega$ , define the binary relation  $E^n$  on A by recursion:  $xE^0y$  iff x = y;  $xE^{n+1}y$  iff xEz and  $zE^ny$  for some  $z \in A$ . A digraph

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<sup>&</sup>lt;sup>1</sup>Henceforth,  $\mathcal{A}$  always denotes a digraph (A, E). If  $B \subseteq A$ , then we often identify B with the the induced subdigraph  $\mathcal{B} = (B, E \cap B^2)$ .

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 $\mathcal{A}$  is a **directed acyclic graph** (**DAG**) if whenever  $n < \omega$  and  $aE^na$ , then n = 0. Some DAGs have kernels while others do not. For example, if < is a linear order of A with no maximum element, then (A, <) is a DAG with no kernel. However, every *finite* DAG has a (unique) kernel, as was first noted in [vNM44].

An element  $b \in A$  for which there is no  $c \in A$  such that bEc is a **sink** of  $\mathcal{A}$ . We say that  $\mathcal{A}$  is **well-founded** if every nonempty subdigraph of  $\mathcal{A}$  has a sink. Every finite DAG is well-founded, and every wellfounded digraph is a DAG having a kernel. The next proposition, for which we need some more definitions, says even more is true. A subset D of a digraph  $\mathcal{A}$  is **closed** if whenever  $d \in D$  and dEa, then  $a \in D$ . If  $X \subseteq A$  and  $k < \omega$ , then define  $\operatorname{Cl}_k^{\mathcal{A}}(X)$  by recursion:  $\operatorname{Cl}_0^{\mathcal{A}}(X) = X$ and  $\operatorname{Cl}_{k+1}^{\mathcal{A}}(X) = X \cup \{a \in A : dEa \text{ for some } d \in \operatorname{Cl}_k^{\mathcal{A}}(X)\}$ . Let  $\operatorname{Cl}^{\mathcal{A}}(X) = \bigcup_{k < \omega} \operatorname{Cl}_k^{\mathcal{A}}(X)$ , which is the smallest closed superset of X.

PROPOSITION 1: Suppose that  $\mathcal{A}$  is a digraph,  $D \subseteq A$  is closed,  $K_0 \subseteq D$  is a kernel of D, and  $A \setminus D$  is well-founded. Then  $\mathcal{A}$  has a (unique) kernel K such that  $K_0 = K \cap D$ .

*Proof.* By recursion on ordinals  $\alpha$ , define  $D_{\alpha}$  so that  $D_0 = D$ ,  $D_{\alpha+1} = D_{\alpha} \cup \{b \in D : b \text{ is a sink of } A \setminus D_{\alpha}\}$ , and  $D_{\alpha} = \bigcup_{\beta < \alpha} D_{\beta}$  if  $\alpha$  is a limit ordinal. Then, there is  $\gamma$  such that  $A = D_{\gamma}$ . For each  $\alpha$ , there is a unique kernel  $K_{\alpha}$  of  $D_{\alpha}$  such that  $K_{\beta} = K_{\alpha} \cap D_{\beta}$  whenever  $\beta < \alpha$ . Let  $K = K_{\gamma}$ .

Let  $\mathcal{A}$  be a digraph. If there is  $k < \omega$  for which there are no  $a, b \in \mathcal{A}$ such that  $aE^{k+1}b$ , then  $\mathcal{A}$  has **finite height**, and we let  $ht(\mathcal{A})$ , the **height** of  $\mathcal{A}$ , be the least such k. If  $\mathcal{A}$  has finite height, then it is wellfounded. We say that  $\mathcal{A}$  has **local finite height** if for every  $m < \omega$ there is  $k < \omega$  such that  $ht(Cl_m^{\mathcal{A}}(F)) \leq k$  for every  $F \subseteq \mathcal{A}$  having cardinality at most m. If  $\mathcal{A}$  has local finite height, then it is a DAG. Having local finite height is a first-order property: if  $\mathcal{B} \equiv \mathcal{A}$  and  $\mathcal{A}$  has local finite height, then so does  $\mathcal{B}$ .

THEOREM 2: Every digraph  $\mathcal{A}$  having local finite height has an elementary extension  $\mathcal{B} \succ \mathcal{A}$  that has a kernel.

*Proof.* This proof is modeled after Theorem 3.2(b)'s in [EV15].

To get  $\mathcal{B}$  with a kernel K, we let  $B_0 = \emptyset$ , and then obtain an elementary chain  $\mathcal{A} = \mathcal{B}_1 \prec \mathcal{B}_2 \prec \mathcal{B}_3 \prec \cdots$  and an increasing sequence  $\emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$  such that for every  $n < \omega, K_n$  is a kernel of  $\operatorname{Cl}^{\mathcal{B}_{n+1}}(B_n)$ . Having these sequences, we let  $\mathcal{B} = \bigcup_{n < \omega} \mathcal{B}_{n+1}$  and

 $K = \bigcup_{n < \omega} K_n$ , so that  $\mathcal{B} \succ \mathcal{A}$  and K is a kernel of  $\mathcal{B}$ . The next lemma allows us to get  $\mathcal{B}_{n+2}$  and  $K_{n+1}$  when we already have  $B_n$ ,  $\mathcal{B}_{n+1}$  and  $K_n$ .

LEMMA 3: Suppose that  $\mathcal{B}_{n+1}$  is a digraph having local finite height, D is a closed subset of  $B_{n+1}$ , and  $K_n$  is a kernel of D. Then there are  $\mathcal{B}_{n+2} \succ \mathcal{B}_{n+1}$  and a kernel  $K_{n+1}$  of  $\operatorname{Cl}^{\mathcal{B}_{n+2}}(B_{n+1})$  such that  $K_n = K_{n+1} \cap D$ .

To prove Lemma 3, let  $\Sigma$  be the union of the following three sets of sentences:

- Th $((\mathcal{B}_{n+1}, a)_{a \in B_{n+1}});$
- { $\sigma_{F,k} : k < \omega$  and  $F \subseteq B_{n+1}$  is finite}, where  $\sigma_{F,k}$  is the sentence  $\forall x \in \operatorname{Cl}_k(F)[U(x) \leftrightarrow \forall y \in \operatorname{Cl}_{k+1}(F)(xEy \to \neg U(y))];$
- $\{U(d): d \in K_n\} \cup \{\neg U(d): d \in D \setminus K_n\}.$

This  $\Sigma$  is a set of  $\mathcal{L}$ -sentences, where  $\mathcal{L} = \{E, U\} \cup B_{n+1}$  and U is a new unary relation symbol.

It suffices to show that  $\Sigma$  is consistent, for then we can let  $(\mathcal{B}_{n+2}, U) \models \Sigma$  and let  $K_{n+1} = U \cap \operatorname{Cl}^{\mathcal{B}_{n+2}}(B_{n+1})$ . To do so, we need only show that every finite subset of  $\Sigma$  is consistent.

Let  $\Sigma_0 \subseteq \Sigma$  be finite. Let  $k_0 < \omega$  and finite  $F_0 \subseteq B_{n+1}$  be such that if  $\sigma_{F,k} \in \Sigma_0$ , then  $k < k_0$  and  $F \subseteq F_0$ . Let  $D = \operatorname{Cl}_{k_0}^{\mathcal{B}_{n+1}}(F_0)$ . Since  $\mathcal{B}_{n+1}$ has local finite height, then D has finite height and, therefore, is wellfounded. By Proposition 1,  $\mathcal{B}_{n+1}$  has a kernel U such that  $K_n = U \cap D$ . Then,  $(\mathcal{B}_{n+1}, U) \models \Sigma_0$ , so  $\Sigma_0$  is consistent, thereby proving Lemma 3 and also Theorem 2.

COROLLARY 4: Every resplendent (or countable, recursively saturated) digraph that has local finite height has a kernel.

*Proof.* This is just a definitional consequence of Theorem 2 and the fact [BS76] that countable, recursively structures are resplendent.  $\Box$ 

§2. Truth Classes. There are various ways that syntax for arithmetic can defined in a model  $\mathcal{M}$  of PA. It usually makes little difference how it is done, so we will choose a way that is very convenient.

We will formalize the language of arithmetic by using just two ternary relation symbols: one for addition and one for multiplication. Suppose that  $\mathcal{M} \models \mathsf{PA}$ . For each  $a \in M$ , we have a constant symbol  $c_a$ . Then let  $\mathcal{L}^M$  consist of the two ternary relations and all the  $c_a$ 's. The only propositional connective we will use is the NOR connective  $\downarrow$ , where  $\sigma_0 \downarrow \sigma_1$  is  $\neg(\sigma_0 \lor \sigma_1)$ . The only quantifier we will use is the "there are none such that" quantifier  $\mathsf{N}$ , where  $\mathsf{N}v\varphi(v)$  is  $\forall v[\neg\varphi(v)]$ . Let  $\mathsf{Sent}^{\mathcal{M}}$  be

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the set of  $\mathcal{L}^{\mathcal{M}}$ -sentences as defined in  $\mathcal{M}$ . A subset  $S \subseteq \mathsf{Sent}^{\mathcal{M}}$  is a **full truth class** for  $\mathcal{M}$  provided the following hold for every  $\sigma \in \mathsf{Sent}^{\mathcal{M}}$ :

- if  $\sigma = \sigma_0 \downarrow \sigma_1$ , then  $\sigma \in S$  iff  $\sigma_0, \sigma_1 \notin S$ ;
- If  $\sigma = \mathsf{M}v\varphi(v)$ , then  $\sigma \in S$  iff there is no  $a \in M$  such that  $\varphi(c_a) \in S$ ;
- If  $\sigma$  is atomic, then  $\sigma \in S$  iff  $\mathcal{M} \models \sigma$ .

Let  $A^{\mathcal{M}} = \{ \sigma \in \mathsf{Sent}^{\mathcal{M}} : \text{ if } \sigma \text{ is atomic, then } \mathcal{M} \models \sigma \}$ . Define the binary relation  $E^{\mathcal{M}}$  on  $A^{\mathcal{M}}$  so that if  $\sigma_1, \sigma_2 \in A^{\mathcal{M}}$ , then  $\sigma_2 E^{\mathcal{M}} \sigma_1$  iff one of the following holds:

- there is  $\sigma_0$  such that  $\sigma_2 = \sigma_0 \downarrow \sigma_1$  or  $\sigma_2 = \sigma_1 \downarrow \sigma_0$ ;
- $\sigma_2 = \mathsf{M} v \varphi(v)$  and  $\sigma_1 = \varphi(c_a)$  for some  $a \in M$ .

Let  $\mathcal{A} = \mathcal{A}^{\mathcal{M}} = (A^{\mathcal{M}}, E^{\mathcal{M}})$ . Obviously,  $\mathcal{A}$  is a DAG. Moreover, it has local finite height: if  $F \subseteq A^{\mathcal{M}}$  is finite and  $m < \omega$ , then  $\operatorname{ht}(\operatorname{Cl}_m^{\mathcal{A}}(F)) \leq (2^{m+1}-1)|F|$ . We easily see that S is a full truth class for  $\mathcal{M}$  iff S is a kernel of  $\mathcal{A}$ .

We can now infer the following version of the KKL Theorem.

COROLLARY 5: Every resplendent (or countable, recursively saturated)  $\mathcal{M} \models \mathsf{PA}$  has a full truth class.

*Proof.* Since  $\mathcal{A}^{\mathcal{M}}$  is definable in  $\mathcal{M}$  and  $\mathcal{M}$  is resplendent, then  $\mathcal{A}^{\mathcal{M}}$  is also resplendent. Thus, by Corollary 4,  $\mathcal{A}^{\mathcal{M}}$  has a kernel, which we have seen is a full truth class for  $\mathcal{M}$ .

Corollary 5 can be improved by replacing PA with any of its subtheories in which enough syntax is definable.

## References

- [BJG09] J. Bang-Jensen and G. Gutin, Digraphs. Theory, algorithms and applications, 2nd edition, Springer Monographs in Mathematics, Springer-Verlag London, Ltd., London, 2009.
- [BS76] Jon Barwise and John Schlipf, An introduction to recursively saturated and resplendent models, J. Symbolic Logic 41 (1976), 531–536.
- [EV15] Ali Enayat and Albert Visser, New constructions of satisfaction classes. Unifying the philosophy of truth, 321–335, Log. Epistemol. Unity Sci., 36, Springer, Dordrecht, 2015.
- [KKL81] H. Kotlarski, S. Krajewski and A. H. Lachlan, Construction of satisfaction classes for nonstandard models, Canad. Math. Bull. 24 (1981), 283–293.
- [vNM44] J. von Neumann and O. Morgenstern, Theory of Games and Economic Behavior Princeton University Press, Princeton, 1944.

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