## Antipodes of monoidal decomposition spaces

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To the memory of Thomas Poguntke

#### Abstract

We introduce a notion of antipode for monoidal (complete) decomposition spaces, inducing a notion of weak antipode for their incidence bialgebras. In the connected case, this recovers the usual notion of antipode in Hopf algebras. In the non-connected case it expresses an inversion principle of more limited scope, but still sufficient to compute the Möbius function as  $\mu = \zeta \circ S$ , just as in Hopf algebras. At the level of decomposition spaces, the weak antipode takes the form of a formal difference of linear endofunctors  $S_{\text{even}} - S_{\text{odd}}$ , and it is a refinement of the general Möbius inversion construction of Gálvez–Kock– Tonks, but exploiting the monoidal structure.

### 1 Introduction

Decomposition spaces were introduced by Gálvez, Kock and Tonks [5, 6, 7] as a very general setting for incidence algebras and Möbius inversion, and independently by Dyckerhoff and Kapranov [2] under the name unital 2-Segal spaces, for use in homological algebra, representation theory and geometry. A decomposition space is a simplicial  $\infty$ -groupoid with a property expressing the ability to decompose objects.

It is the combinatorial perspective that concerns the present contribution. The line of development from the classical theory of incidence coalgebras [9] is summarised by regarding locally finite posets as special instances of Möbius categories [14], which in turn are regarded as simplicial sets via the nerve. The crucial observation from [5] is that the Segal condition (which characterises the ability to compose as in a category) is not needed: the decomposition-space axiom characterises instead the ability to *decompose*, in a sufficiently controlled manner so as to allow the construction of an incidence coalgebra. There are countless examples in combinatorics of coalgebras arising from decomposition spaces but not (directly) from categories or posets. The passage from simplicial sets to simplicial groupoids is motivated by combinatorics to take into account symmetries. The further passage to  $\infty$ -groupoids is harder to justify from combinatorics, but is the natural level of generality from

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a homotopy viewpoint. The decomposition-space approach to incidence algebras is *objective* (in the sense by Lawvere and Menni [13]), meaning that the constructions take place with the combinatorial objects themselves rather than with vector spaces spanned by them. In this way, all proofs are natively 'bijective'. It is an attractive feature of the theory that most arguments boil down to computing pullbacks, by which we always mean *homotopy* pullbacks (i.e. pullbacks in the  $\infty$ -category S of  $\infty$ -groupoids).

Bialgebras and Hopf algebras, rather than just coalgebras, are obtained from *monoidal* decomposition spaces. In examples from combinatorics, the monoidal structure is often disjoint union. It is characteristic for the decomposition-space approach that the bialgebras obtained are (often filtered but) not connected in general. In particular they are not in general Hopf. For example, for the nerve of a category, all identity arrows become group-like elements in the coalgebra.

While the decomposition-space theory was originally modelled on incidence coalgebras and Möbius inversion machinery à la Rota, algebraic combinatorics soon after Rota discovered the more powerful machinery of antipodes, when available. For example, in an incidence Hopf algebra, Möbius inversion amounts to precomposing with the antipode S, exhibiting in particular the Möbius function as  $\mu = \zeta \circ S$ . The work of Schmitt [15, 16] was seminal to the change of emphasis from Möbius inversion to antipodes. The recent work of Aguiar and Ardila [1] represents a striking example of the power of antipodes.

The present note upgrades the Gálvez–Kock–Tonks Möbius-inversion construction [6] to the construction of a kind of antipode in any *monoidal* (complete) decomposition space. Many of the constructions are quite similar; the main innovative idea is that there is a useful weaker notion of antipode for bialgebras even if they are *not* Hopf.

We introduce this notion and establish its main features (and limitations). Briefly, for X a monoidal (complete) decomposition space, the antipode is defined as a formal difference between linear endofunctors of  $S_{/X_1}$ ,

$$S := S_{\text{even}} - S_{\text{odd}},$$

given by multiplying principal edges of nondegenerate simplices (cf. p.7 below). It cannot quite convolution-invert the identity endofunctor, as a true antipode should [17], but it can invert a modification of it, denoted Id':

$$\mathrm{Id}'(f) = \begin{cases} f & \text{if } f \text{ nondegenerate,} \\ \mathrm{id}_u & \text{if } f \text{ degenerate.} \end{cases}$$

Here u is the monoidal unit object, and we write  $id_u$  for  $s_0u$ .

Precisely, our main theorem (3.3) is the inversion formula

$$S_{\text{even}} * \text{Id}' \simeq e + S_{\text{odd}} * \text{Id}'$$

where  $e := \eta \circ \varepsilon$  is the neutral element for convolution. Under the finiteness conditions satisfied by *Möbius* decomposition spaces [6, §8], one can take homotopy cardinality [4, §3] and form the difference  $|S| := |S_{\text{even}}| - |S_{\text{odd}}|$  to arrive at the nicer-looking equation in the Q-vector-space level convolution algebra:

$$|S| * |\mathrm{Id}'| = |e| = |\mathrm{Id}'| * |S|.$$

The three main features justifying the weaker notion of antipode are:

- 1. If the monoidal decomposition space is connected, so that its incidence bialgebra is Hopf, then the homotopy cardinality of S is the usual antipode (cf. Proposition 3.5). (At the objective level of decomposition spaces, the construction of S is new also in the connected case.)
- 2. In any case, S computes the Möbius functor as

$$\mu \simeq \zeta \circ S$$

(cf. Corollary 4.1).

3. More generally, we establish an inversion formula for multiplicative functors (valued in any algebra) that send group-like elements to the unit (Theorem 4.3). The zeta functor is an example of this.

At the algebraic level of  $\mathbb{Q}$ -vector spaces, the weak antipode can be seen as a lift of the true antipode from the connected quotient of the bialgebra. When the bialgebra comes from the nerve of a category, this quotient is obtained by identifying all objects of the category. Recent developments have shown the utility of avoiding this reduction, which destroys useful information. For example, the Faà di Bruno formula for general operads [3], [12] crucially exploits the finer structure of the zeroth graded piece of the incidence bialgebra, and in the bialgebra version [10] of BPHZ renormalisation in perturbative quantum field theory, the zeroth graded piece of the bialgebra of Feynman graphs contains the terms of the Lagrangian (not visible in the quotient Hopf algebra usually employed).

# 2 Preliminaries: monoidal decomposition spaces and their incidence bialgebras

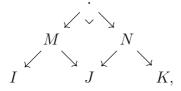
We assume familiarity with the basic theory of decomposition spaces [5, 6, 7], and limit ourselves to a minimal background section, so as at least to establish notation.

Like in [5, 6, 7], we work with simplicial  $\infty$ -groupoids, since it is the natural generality of the theory. However, our results belong to combinatorics, where the examples of interest are merely simplicial groupoids or even simplicial sets (such as the nerve of a poset). The reader may safely substitute 'groupoid' or 'set' for the word ' $\infty$ -groupoid' throughout. This is the viewpoint taken in [8], which may serve as an introduction.

**Linear functors and spans.** Denote by S the  $\infty$ -category of  $\infty$ -groupoids. A functor between slices  $F : S_{/I} \to S_{/J}$  is called *linear* [4, §2] if it is given by a span

$$I \xleftarrow{p} M \xrightarrow{q} J$$

by pullback along p (denoted  $p^*$ ) followed by composition with q (denoted  $q_!$ ), i.e.  $F = q_! \circ p^*$ . The  $\infty$ -groupoid M itself plays the role of an  $(I \times J)$ -indexed matrix. Crucially, composition of linear functors is given by taking pullback like this:



the objective version of matrix multiplication. We denote by **LIN** the  $\infty$ -category of slices of S and linear functors. In suitably finite situations, one can take homotopy cardinality of slices and linear functors to obtain vector spaces and linear maps (see for examples the Appendix of [8]).

**Decomposition spaces.** A simplicial  $\infty$ -groupoid  $X : \Delta^{\text{op}} \to S$  is called a *decomposition space* if it takes active-inert pushouts in  $\Delta$  to pullbacks [5, §3]. The precise meaning is not so important for the present purposes—it suffices here to say that the decomposition-space axiom precisely ensures that the comultiplication given by the linear functor

$$\Delta: \mathfrak{S}_{/X_1} \xrightarrow{(d_2,d_0)_! \circ d_1^*} \mathfrak{S}_{/X_1 \times X_1}$$

defined by the span

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1$$

is up-to-coherent-homotopy coassociative as well as counital (the counit being defined by the span  $X_1 \stackrel{s_0}{\leftarrow} X_0 \to \mathbf{1}$ ) [5, §5]. Here and throughout,  $\mathbf{1}$  denotes any contractible  $\infty$ -groupoid (the terminal  $\infty$ -groupoid), so that  $S_{/1} \simeq S$ . The  $\infty$ -category  $S_{/X_1}$ , with  $\Delta$  and  $\varepsilon$ , is called the *incidence coalgebra* of X. When X is locally finite [6, §7], one can take homotopy cardinality to obtain an ordinary coalgebra in  $\mathbb{Q}$ -vector spaces, namely  $\mathbb{Q}_{\pi_0 X_1}$ , the  $\mathbb{Q}$ -vector space spanned by path components of  $X_1$ . Any Segal space is a decomposition space [5, §3]; an example to keep in mind is the nerve of a small category.

Complete decomposition spaces, and nondegenerate simplices. A decomposition space is *complete* ([6, §2]) when  $s_0 : X_0 \to X_1$  is a monomorphism of  $\infty$ groupoids (i.e. its (homotopy) fibres are either empty or contractible). (For simplicial *sets*, the condition is automatic.) This condition ensures that there is a well-behaved notion of nondegenerate simplices: define the space of nondegenerate 1-simplices  $\vec{X}_1$ as the complement of the essential image of the monomorphism  $s_0 : X_0 \to X_1$ , so that we have

$$X_1 \simeq X_0 + \vec{X}_1,$$

where + denotes the disjoint union. More generally,  $X_n \subset X_n$  is characterised as the complement of the union of the essential images of the degeneracy maps  $s_i: X_{n-1} \to X_n$ , that is

$$\vec{X}_n = X_n \setminus \bigcup_{i=0}^{n-1} \operatorname{Im}(s_i)$$

By definition  $\vec{X}_0 = X_0$ . In a complete decomposition space, an *n*-simplex is nondegenerate if and only if all its *n* principal edges are nondegenerate [6, §2].

Monoidal decomposition spaces. Bialgebras are obtained from decomposition spaces with a *CULF monoidal structure*  $[5, \S 9]$ . This means first of all that there are simplicial maps (unit and multiplication):

$$\mathbf{1} \stackrel{\eta}{\longrightarrow} X \stackrel{\mu}{\longleftarrow} X \times X,$$

but with the important condition imposed that these maps should be CULF, which is a pullback condition required expressly to ensure that there is induced a monoid structure on the incidence coalgebra  $S_{/X_1}$  in the  $\infty$ -category of coalgebras, and hence altogether a bialgebra structure on  $S_{/X_1}$ .

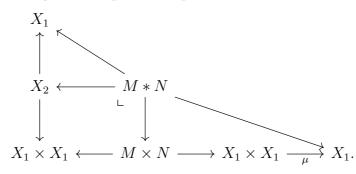
**Connectedness.** A monoidal decomposition space (or its incidence bialgebra) is called *connected* when  $X_0$  is contractible (that is,  $X_0 \simeq \mathbf{1}$ ). Usually, connectedness should refer to a filtration [17]. This filtration does not always exist for a monoidal decomposition space X, but it does exist when X is Möbius (the existence of the length filtration is one characterisation of the Möbius condition [6, §8]). In that case,  $X_0$  spans filtration degree 0, so the condition  $X_0 \simeq \mathbf{1}$  agrees with the usual notion of being connected for filtered coalgebras (or bialgebras). When X is Möbius and connected, its cardinality is a connected filtered bialgebra, and therefore, by standard arguments [17], a Hopf algebra. However, many important incidence bialgebras are not connected.

**Examples.** Let X be the fat nerve of the category of finite sets and surjections. The resulting bialgebra is the Fàa di Bruno bialgebra [8]. The zeroth graded piece is spanned by the invertible surjections (which are all group-like), so is not connected. The monoidal structure is disjoint union and the monoidal unit is the (identity of the) empty set.

More generally, for any reduced operad, the so-called two-sided bar construction X is a monoidal (complete) decomposition space [12]. The groupoid  $X_0$  is the free symmetric monoidal category on the set of objects of the operad. (Note that X is never connected.) The groupoid  $X_1$  is the free symmetric monoidal category on the action groupoid of the symmetric-group actions on the set of operations. The generalisation of the classical Faà di Bruno formula to any operad [3, 12] (the classical case being that of the terminal reduced operad) crucially exploits the typing constraints expressed by the objects in  $X_0$  (which are invisible in the connected quotient Hopf algebra).

# 3 Antipodes for monoidal complete decomposition spaces

**Convolution.** Let X be a monoidal decomposition space. For  $F, G : S_{/X_1} \to S_{/X_1}$ two linear endofunctors, the *convolution product*  $F * G : S_{/X_1} \to S_{/X_1}$  is given by first comultiplying, then composing with the tensor product  $F \otimes G$ , and finally multiplying. If F and G are given by the spans  $X_1 \leftarrow M \to X_1$  and  $X_1 \leftarrow N \to X_1$ , then F \* G is given by the composite of spans



The neutral element for convolution in  $LIN(S_{X_1}, S_{X_1})$  is  $e := \eta \circ \varepsilon$ . By composition of spans, it is given by the span

$$X_1 \xleftarrow{s_0} X_0 \xrightarrow{w} X_1,$$

where w denotes the composite  $X_0 \to \mathbf{1} \xrightarrow{\eta} X_1$ .

**The antipode.** Define the linear endofunctor  $S_n : S_{/X_1} \to S_{/X_1}$  by the span

$$X_1 \stackrel{g}{\longleftarrow} \vec{X_n} \xrightarrow{p} X_1 \times \ldots \times X_1 \xrightarrow{\mu_n} X_1, \tag{1}$$

where g returns the 'long edge' of a simplex, and p returns its n principal edges.

In the case n = 0, we have  $g = s_0$  and  $(X_1)^0 = \mathbf{1}$  and  $\mu_0 = \eta$ , whence  $S_0$  coincides with the neutral element:

 $S_0 = e.$ 

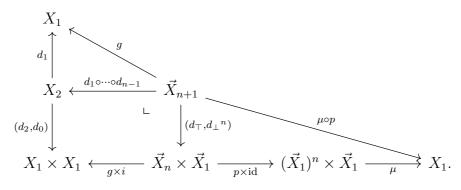
Note also that the functor  $S_1$  is given by the span

$$X_1 \stackrel{i}{\longleftarrow} \vec{X_1} \stackrel{i}{\longrightarrow} X_1. \tag{2}$$

Lemma 3.1. We have

$$S_n \simeq (S_1)^{*n}.$$

*Proof.* The case n = 0 is trivial since  $S_0$  is neutral. In the convolution  $S_n * S_1$ , the main pullback is given by the Lemma 3.5 of [6]:



Commutativity of the upper triangle is precisely the face-map description of g. The lower triangle commutes since  $d_{\perp}^n$  returns itself the last principal edge.

Put

$$S_{\text{even}} := \sum_{n \text{ even}} S_n, \qquad S_{\text{odd}} := \sum_{n \text{ odd}} S_n.$$

Note that the sum of linear functors is given by the sum (disjoint union) of the middle objects of the respesenting spans. Hence  $S_{\text{even}}$  is given by the span

$$X_1 \longleftarrow \sum_{n \text{ even}} \vec{X_n} \longrightarrow X_1,$$

where the left leg returns the long edge of a simplex, and the right leg returns the monoidal product of the principal edges. Similarly of course with  $S_{\text{odd}}$ .

The *antipode* S is defined as the formal difference

$$S := S_{\text{even}} - S_{\text{odd}}.$$

The difference cannot be formed at the objective level where there is no minus sign available, but it does make sense after taking homotopy cardinality to arrive at  $\mathbb{Q}$ -vector spaces. For this to be meaningful, certain finiteness conditions must be imposed: X should be *Möbius*, which means locally finite and of locally finite length, cf. [6, §8]. We shall continue to work with  $S_{\text{even}}$  and  $S_{\text{odd}}$  individually.

The idea of an antipode is that it should be convolution inverse to the identity functor, i.e. S \* Id should be  $\eta \circ \varepsilon$ . This is not in general true for monoidal decomposition spaces. We show instead that S inverts the following modified identity functor.

The linear functor  $\mathrm{Id}': S_{/X_1} \to S_{/X_1}$  is given by the span

$$X_1 \xleftarrow{=} X_0 + \vec{X_1} \xrightarrow{w|i} X_1,$$

where *i* is the inclusion  $\vec{X}_1 \subset X_1$ , and  $w : X_0 \xrightarrow{p} \mathbf{1} \xrightarrow{\eta} X_1$  is the constant map with value  $id_u$ , the identity at the monoidal unit object *u*. In other words,

$$\mathrm{Id}' \simeq S_0 + S_1$$

On elements,

$$\mathrm{Id}'(f) = \begin{cases} f & \text{if } f \text{ nondegenerate,} \\ \mathrm{id}_u, & \text{if } f \text{ degenerate.} \end{cases}$$

**Lemma 3.2.** The linear functors  $S_n$  satisfy

$$S_n * \mathrm{Id}' \simeq S_n + S_{n+1} \simeq \mathrm{Id}' * S_n.$$

*Proof.* Since  $\operatorname{Id}' \simeq S_0 + S_1$ , the result follows from  $S_n * S_1 \simeq S_{n+1} \simeq S_1 * S_n$  (which is a consequence of Lemma 3.1), and  $S_n * S_0 \simeq S_n \simeq S_0 * S_n$  (S<sub>0</sub> is neutral for convolution).

**Theorem 3.3.** Given a monoidal complete decomposition space X, we have explicit equivalences

 $S_{\text{even}} * \text{Id}' \simeq e + S_{\text{odd}} * \text{Id}'$  and  $\text{Id}' * S_{\text{even}} \simeq e + \text{Id}' * S_{\text{odd}}$ 

*Proof.* It follows from Lemma 3.2 that all four functors are equivalent to  $\sum_{n\geq 0} S_n$ .

Finiteness conditions and homotopy cardinality. If the monoidal complete decomposition space X is locally finite (meaning that  $X_1$  is locally finite and  $X_0 \xrightarrow{s_0} X_1 \xleftarrow{d_1} X_2$  are finite maps [6, §8]), then we can take homotopy cardinality [4, §3] to obtain the incidence bialgebra at the Q-vector space level, and obtain also linear endomorphisms

$$|S_n|: \mathbb{Q}_{\pi_0 X_1} \to \mathbb{Q}_{\pi_0 X_1}.$$

If X is furthermore Möbius, the sums involved in the definitions of  $S_{\text{even}}$  and  $S_{\text{odd}}$  are finite, and the difference  $|S| = |S_{\text{even}}| - |S_{\text{odd}}|$  is a well-defined linear endomorphism of  $\mathbb{Q}_{\pi_0 X_1}$ , and we arrive at the following weak antipode formula:

**Proposition 3.4.** If X is a Möbius monoidal decomposition space, then we have

$$|S| * |\mathrm{Id}'| = |e| = |\mathrm{Id}'| * |S|$$

in  $\mathbb{Q}_{\pi_0 X_1}$ , the homotopy cardinality of the incidence bialgebra of X.

**Connectedness and the usual notion of antipode.** We say a monoidal decomposition space is *connected* if  $X_0$  is contractible. In this situation,  $X_0$  contains only the monoidal unit, so that the maps w and  $s_0$  coincide, and hence  $\mathrm{Id}' \simeq \mathrm{Id}$ . (Indeed, note that the identity endofunctor  $\mathrm{Id} : S_{/X_1} \to S_{/X_1}$  is given by the span  $X_1 \stackrel{=}{\leftarrow} X_1 \stackrel{=}{\to} X_1$ , and that  $s_0 | i : X_0 + \vec{X}_1 \to X_1$  is an equivalence.) We then get the following stricter inversion result, yielding the usual notion of antipode in Hopf algebras, after taking homotopy cardinality:

**Proposition 3.5.** If X is a connected monoidal complete decomposition space, then

 $S_{\text{even}} * \text{Id} \simeq e + S_{\text{odd}} * \text{Id}$  and  $\text{Id} * S_{\text{even}} \simeq e + \text{Id} * S_{\text{odd}}$ .

If moreover X is Möbius, we get

$$|S| * |Id| = |e| = |Id| * |S|.$$

Relationship with classical antipode formulae. If X is the nerve of a Möbius category  $\mathcal{C}$ , then the comultiplication formula reads

$$\Delta(f) = \sum_{b \circ a = f} a \otimes b.$$

The decomposition space X becomes monoidal if  $\mathfrak{C}$  is *monoidal extensive* [5, §9], meaning that it has a monoidal structure  $(\mathfrak{C}, \otimes, k)$  with natural equivalences

$$\mathcal{C}/x \times \mathcal{C}/y \xrightarrow{\sim} \mathcal{C}/(x \otimes y), \qquad \mathbf{1} \xrightarrow{\sim} \mathcal{C}/k.$$

In combinatorics, extensive monoidal structures most often arise as disjoint union.

Spelling out the general antipode formula in the case of a monoidal extensive category gives

$$S(f) = \sum_{k \ge 0} (-1)^k \sum_{\substack{a_k \circ \cdots \circ a_1 = f \\ a_i \neq \text{id}}} a_1 \cdots a_k.$$

When  $\mathcal{C}$  is just a locally finite hereditary poset (with intervals regarded as arrows), this is Schmitt's antipode formula for the reduced incidence Hopf algebra of the poset [15].

Schmitt's formula works more generally for hereditary families of poset intervals, meaning classes of poset intervals that are closed under taking subintervals and cartesian products [16]. Our general formula covers that case as well. The intervals of such a family do not necessarily come from a single poset (or even a Möbius category). One can prove that such a family always forms a monoidal decomposition space, the most important case being the family of *all* (finite) poset intervals [7].

Other classical antipode formulae are readily extracted. For example, from the general formula  $S_{n+1} \simeq S_n * S_1$  (see Lemma 3.1), one finds

$$S_{\text{even}} \simeq S_0 + S_{\text{odd}} * S_1, \qquad S_{\text{odd}} \simeq S_{\text{even}} * S_1$$

whence the recursive formula

$$S \simeq S_0 - S * S_1,$$

valid after taking homotopy cardinality. Spelling this out in the case of the nerve of a monoidal extensive Möbius category yields the familiar formula

$$S(f) = S_0(f) - \sum_{\substack{b \circ a = f \\ b \neq \text{id}}} S(a) \cdot b,$$

which also goes back to Schmitt [15], in the poset case.

### 4 Inversion in convolution algebras

**Möbius inversion.** The Möbius inversion formula [6, §3] is recovered easily from Theorem 3.3. Recall that the *zeta functor* is the linear functor  $\zeta : S_{/X_1} \to S$  defined by the span  $X_1 \stackrel{=}{\leftarrow} X_1 \to \mathbf{1}$ .

First we define

$$\Phi_n := \zeta \circ S_n$$

By composition of spans,  $\Phi_n$  is given by

$$X_1 \xleftarrow{g} \vec{X_n} \longrightarrow \mathbf{1}$$

in accordance with [6]. We also get

$$\Phi_{\text{even}} := \zeta \circ S_{\text{even}} = \sum_{n \text{ even}} \Phi_n, \qquad \Phi_{\text{odd}} := \zeta \circ S_{\text{odd}} = \sum_{n \text{ odd}} \Phi_n.$$

The following is now an immediate consequence of Theorem 3.3.

**Corollary 4.1** ([6] Theorem 3.8). For a monoidal complete decomposition space, the Möbius inversion principle holds, expressed by the explicit equivalences

$$\Phi_{\rm even} * \zeta \simeq \varepsilon + \Phi_{\rm odd} * \zeta \quad and \qquad \zeta * \Phi_{\rm even} \simeq \varepsilon + \zeta * \Phi_{\rm odd}$$

This proof is a considerable simplification compared to the proof given in [6], but note that it crucially depends on the monoidal structure. The theorem of [6] is more general in that it works also in the absence of a monoidal structure.

More general inversion. One advantage of the antipode over the Möbius inversion formula is that it gives a uniform inversion principle, rather than just inverting the zeta function. At the Q-vector space level, the result  $|\mu| = |\zeta| \circ |S|$  is readily generalised as follows. Let  $B_X$  denote the homotopy cardinality of the incidence bialgebra of a monoidal Möbius decomposition space X.

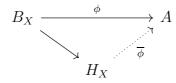
**Lemma 4.2.** For any  $\mathbb{Q}$ -algebra A with unit  $\eta_A$ , consider the convolution algebra  $(\operatorname{Lin}(B_X, A), *, \eta_A \varepsilon)$ . If  $\phi : B_X \to A$  is multiplicative and sends all group-like elements to  $\eta_A$ , then  $\phi$  is convolution invertible with inverse  $\phi \circ S$ .

*Proof.* Indeed, 'multiplicative' ensures that  $\phi \circ (S * \mathrm{Id}') = (\phi \circ S) * (\phi \circ \mathrm{Id}')$ , and the condition on group-like elements ensures that  $\phi \circ \mathrm{Id}' = \phi$  (and that  $\phi \circ \eta_B = \eta_A$ ).  $\Box$ 

The connected quotient  $H_X$  is defined as  $H_X := B_X/J_X$ , where

$$J_X = \langle s_0 x - s_0 u \mid x \in X_0 \rangle$$

which is a Hopf ideal [17] since the elements  $s_0 x$  are group-like. (Here *u* denotes the monoidal unit.) It is clear that  $H_X$  is connected, hence a Hopf algebra. Now the conditions on  $\phi$  in Lemma 4.2 amount precisely to saying that  $\phi$  vanishes on the Hopf ideal  $J_X$ , and hence factors through the quotient Hopf algebra  $H_X$ :



From this perspective, the weak antipode of  $B_X$  does not invert anything that could not have been inverted with classical technology, namely by the true antipode in  $H_X$ . The point of the weak antipode is that it is defined already at the objective level of decomposition spaces, without the need of quotienting. We shall establish the following objective version of Lemma 4.2.

**Theorem 4.3.** Let X be a monoidal complete decomposition space, and let A be a monoidal  $\infty$ -groupoid—this makes  $S_{/A}$  an algebra in **LIN**. Consider the convolution algebra (**LIN**( $S_{/X_1}, S_{/A}$ ),  $*, \eta_A \varepsilon$ ). If a linear functor  $\phi : S_{/X_1} \to S_{/A}$  is multiplicative and contracts degenerate elements, then  $\phi$  is convolution invertible with inverse  $\phi \circ S$ .

The main task is to define the notions involved. Throughout, we let X denote a monoidal complete decomposition space, and A a monoidal  $\infty$ -groupoid. A linear functor  $\phi : S_{/X_1} \to S_{/A}$  given by a span

$$X_1 \stackrel{u}{\leftarrow} F \stackrel{v}{\rightarrow} A$$

is called *multiplicative* if it is a span of monoidal functors with u CULF. This means that we have commutative diagrams

Commutativity of the diagrams expresses of course that the functors u and v are monoidal. CULFness amounts to the pullback conditions indicated, which are required because we need to do pull-push along these squares.

A linear functor  $\phi : S_{/X_1} \to S_{/A}$  given by a span

$$X_1 \stackrel{u}{\leftarrow} F \stackrel{v}{\to} A$$

is said to *contract degenerate elements* if the following condition holds:

Two conditions are expressed by this: the first is that u pulled back along  $s_0$  gives the identity map. (The map  $s_F$  is defined by this pullback.) The second condition says that  $v \circ s_F$  factors through the unit. Altogether, the conditions express the idea of mapping all degenerate elements to the unit object of A.

**Lemma 4.4.** If  $\phi$  contracts degenerate elements (Equation (4)), then it is unital (Equation (3) RHS).

*Proof.* In the diagram

$$1 \xleftarrow{=} 1 \xrightarrow{=} 1 \xrightarrow{=} 1$$

$$\eta_0 \downarrow \qquad \downarrow \qquad \downarrow \eta_0 \qquad \downarrow =$$

$$X_0 \xleftarrow{=} X_0 \xrightarrow{p} 1$$

$$s_0 \downarrow \qquad \downarrow \qquad \downarrow s_F \qquad \downarrow \eta_A$$

$$X_1 \xleftarrow{=} F \xrightarrow{p} A,$$

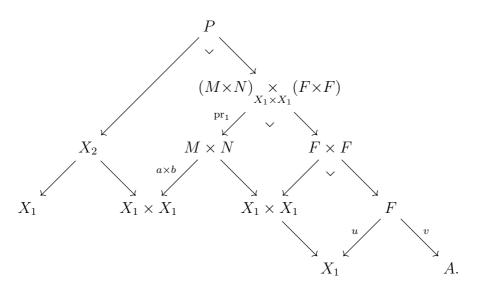
the bottom squares are (4), and the outline diagram is (3) RHS, since the composite vertical arrows are  $\eta_1$ ,  $\eta_F$ , and  $\eta_A$ .

**Lemma 4.5.** If a linear functor  $\phi : S_{/X_1} \to S_{/A}$  is multiplicative, then  $\phi \circ -$  distributes over convolution. Precisely, for any linear endofunctors  $\alpha, \beta : S_{/X_1} \to S_{/X_1}$ , we have

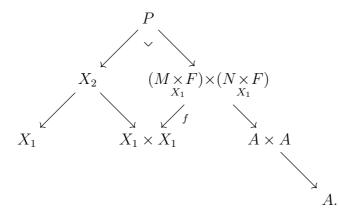
$$\phi \circ (\alpha * \beta) \simeq (\phi \circ \alpha) * (\phi \circ \beta).$$

Note that \* on the left refers to convolution of endofunctors, while \* on the right refer to convolution in  $LIN(S_{/X_1}, S_{/A})$ .

*Proof.* The left-hand side  $\phi \circ (\alpha * \beta)$  is computed by the pullbacks



The right-hand side  $(\phi \circ \alpha) * (\phi \circ \beta)$  is computed by the pullback



Here f is the map  $(a \circ pr_1) \times (b \circ pr_1)$ . These two composed spans agree since clearly

$$(M \times N) \underset{X_1 \times X_1}{\times} (F \times F) \simeq (M \underset{X_1}{\times} F) \times (N \underset{X_1}{\times} F)$$

(and  $f \simeq (a \times b) \circ \operatorname{pr}_1$ ).

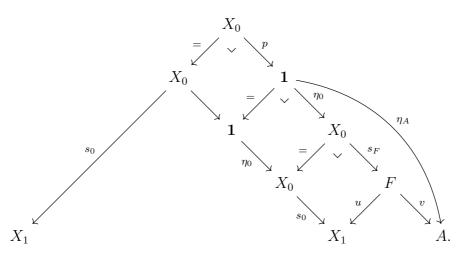
**Lemma 4.6.** If a linear functor  $\phi : S_{/X_1} \to S_{/A}$  contracts degenerate elements, then we have

$$\phi \circ \mathrm{Id}' \simeq \phi.$$

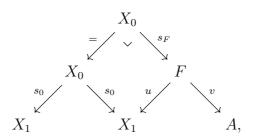
*Proof.* Let  $\omega$  denote the endofunctor defined by the span  $X_1 \xleftarrow{s_0} X_0 \xrightarrow{s_0} X_1$ . Since  $\mathrm{Id}' = S_0 + S_1$  and  $\mathrm{Id} = \omega + S_1$ , it is enough to establish

$$\phi \circ S_0 \simeq \phi \circ \omega.$$

The left-hand side  $\phi \circ S_0$  is computed by the pullbacks



The right-hand side  $\phi \circ \omega$  is computed by the pullback



and the composite  $v \circ s_F$  is again  $\eta_A \circ p$  by hypothesis.

*Proof of Theorem 4.3.* We need to show that  $\phi \circ S$  is convolution inverse to  $\phi$ . With the preparations made, this is now direct:

$$(\phi \circ S) * \phi \stackrel{4.6}{\simeq} (\phi \circ S) * (\phi \circ \mathrm{Id}') \stackrel{4.5}{\simeq} \phi \circ (S * \mathrm{Id}') \stackrel{3.3}{\simeq} \phi \circ \eta_1 \circ \varepsilon \stackrel{4.4}{\simeq} \eta_A \circ \varepsilon.$$

Remark 4.7. The more general Möbius inversion principle of Lemma 4.2 and Theorem 4.3 is of interest for two reasons. Firstly, in the connected case, the general Möbius inversion principle, which we here derived from the antipode, but which can be formulated without reference to S, is actually *equivalent* to the existence of the antipode. Indeed, if one takes A to be  $X_1$  itself (so that at the cardinality level one uses  $B_X$  as the algebra A), and takes  $\phi$  to be the identity map, then the resulting inverse is the antipode.

Secondly, the extra generality serves to highlight the tight analogy between Möbius inversion and abstract Hopf-algebraic renormalisation in perturbative quantum field theory, as explained in [11]. The  $\phi$  are then the (regularised) Feynman rules (which are inherently multiplicative, and can be arranged to send group-like elements to 1). In this generality, the passage from Möbius inversion to renormalisation consists just in adding a Rota–Baxter operator to the formulae (see [11] for details). (The result is then no longer an inverse but rather a counter-term.)

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