CHARACTERIZATIONS OF THE WEAKLY COMPACT IDEAL ON $P_{\kappa}\lambda$

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ABSTRACT. Hellsten [Hel03] gave a characterization of Π_n^1 -indescribable subsets of a Π^1_n -indescribable cardinal in terms of a natural filter base: when κ is a Π^1_n -indescribable cardinal, a set $S \subseteq \kappa$ is Π^1_n -indescribable if and only if $S \cap C \neq \emptyset$ for every *n*-club $C \subseteq \kappa$. We generalize Hellsten's characterization to Π_n^1 -indescribable subsets of $P_{\kappa}\lambda$, which were first defined by Baumgartner. After showing that under reasonable assumptions the Π_0^1 -indescribability ideal on $P_{\kappa}\lambda$ equals the minimal strongly normal ideal NSS_{κ,λ} on $P_{\kappa}\lambda$, and is not equal to $NS_{\kappa,\lambda}$ as may be expected, we formulate a notion of *n*-club subset of $P_{\kappa\lambda}$ and prove that a set $S \subseteq P_{\kappa}\lambda$ is Π^1_n -indescribable if and only if $S \cap C \neq \emptyset$ for every n-club $C \subseteq P_{\kappa}\lambda$. We also prove that elementary embeddings considered by Schanker [Sch13] witnessing near supercompactness lead to the definition of a normal ideal on $P_{\kappa}\lambda$, and indeed, this ideal is equal to Baumgartner's ideal of non- Π_1^1 -indescribable subsets of $P_{\kappa}\lambda$. Additionally, as applications of these results we answer a question of Cox-Lücke [CL17] about \mathcal{F} -layered posets, provide a characterization of Π_n^m -indescribable subsets of $P_{\kappa}\lambda$ in terms of generic elementary embeddings, prove several results involving a two-cardinal weakly compact diamond principle and observe that a result of Pereira [Per17] yields the consistency of the existence of a (κ, κ^+) -semimorasses $\mu \subseteq P_{\kappa}\kappa^+$ which is Π_n^1 -indescribable for all $n < \omega$.

1. INTRODUCTION

Recall that a set $W \subseteq \kappa$ is weakly compact if and only if for every $A \subseteq \kappa$ there is a transitive $M \models \operatorname{ZFC}^-$ with $\kappa, A, W \in M$ and $M^{<\kappa} \subseteq M$, there is a transitive Nand there is an elementary embedding $j: M \to N$ with critical point κ such that $\kappa \in j(W)$. It is well known that κ is weakly compact (as a subset of itself) if and only if the collection NWC_{κ} = { $X \subseteq \kappa : X$ is not weakly compact} is a normal ideal on κ , which we refer to as the weakly compact ideal on κ . Baumgartner [Bau77, Section 2] showed that assuming $\kappa^{<\kappa} = \kappa$, a set $W \subseteq \kappa$ is weakly compact if and only if it is Π_1^1 -indescribable, meaning that for every Π_1^1 -formula φ and every $R \subseteq V_{\kappa}$, if $(V_{\kappa}, \in, R) \models \varphi$ then there exists $\alpha \in W$ such that $(V_{\alpha}, \in, R \cap V_{\alpha}) \models \varphi$. Thus,

NWC_{$$\kappa$$} = $\Pi_1^1(\kappa) =_{def} \{ X \subseteq \kappa : X \text{ is not } \Pi_1^1 \text{-indescribable} \}.$

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Sun [Sun93] proved that the ideal NWC_{κ} can be characterized in terms of a natural filter base as follows. A set $C \subseteq \kappa$ is called 1-*club* if and only if $C \in \mathrm{NS}^+_{\kappa}$ and whenever $\alpha < \kappa$ is inaccessible and $C \cap \alpha \in \mathrm{NS}^+_{\alpha}$ we have $\alpha \in C$. Sun's characterization of weakly compact sets states that, assuming κ is a weakly compact cardinal, a set $W \subseteq \kappa$ is weakly compact if and only if $W \cap C \neq \emptyset$ for every 1-club $C \subseteq \kappa$. Since a set $S \subseteq \kappa$ is Π_0^1 -indescribable if and only if κ is inaccessible and $S \in \mathrm{NS}^+_{\kappa}$, it follows that for κ inaccessible $\mathrm{NS}^+_{\kappa} = \Pi_0^1(\kappa)^+ = \{S \subseteq \kappa : S \text{ is first-order indescribable}\}$ and we can restate Sun's characterization as: for κ weakly compact, a set $W \subseteq \kappa$ is weakly compact (or equivalently Π_1^1 -indescribable) if and only if

$$(\forall C \in \Pi_0^1(\kappa)^+)((\forall \alpha < \kappa)(C \cap \alpha \in \Pi_0^1(\alpha)^+ \implies \alpha \in C) \implies C \cap W \neq \emptyset).$$
(1.1)

In this article we prove similar results for the weakly compact ideal and the Π_n^1 indescribability ideals on $P_{\kappa}\lambda$, which apparently¹ were first defined by Baumgartner in [Bau], and have since been studied in [Car85], [Joh90], [Abe98], [MU12], [Usu13] and [MU15]. In [Bau], Baumgartner defined a notion of Π_n^m -indescribability for subsets of $P_{\kappa}\lambda$ using a natural $P_{\kappa}\lambda$ -version of the cumulative hierarchy (see Section 4 below), which gives rise to the Π_n^m -indescribability ideal on $P_{\kappa}\lambda$

$$\Pi_n^m(\kappa,\lambda) = \{ X \subseteq P_\kappa \lambda : X \text{ is not } \Pi_n^m \text{-indescribable} \}.$$

Abe [Abe98] showed that when $P_{\kappa}\lambda$ is Π_n^m -indescribable the ideal $\Pi_n^m(\kappa, \lambda)$ is normal. In light of the version of Sun's characterization of weakly compact subsets of κ in (1.1), it seems natural to attempt to give a similar characterization for Baumgartner's notion of Π_1^1 -indescribability for subsets of $P_{\kappa}\lambda$. We will show that when κ is inaccessible, the ideal $\Pi_0^1(\kappa, \lambda)$ of non- Π_0^1 -indescribable (i.e. non-first-order indescribable) subsets of $P_{\kappa}\lambda$ is equal to the minimal strongly normal ideal NSS_{κ,λ} of non strongly stationary subsets of $P_{\kappa}\lambda$ (see Section 2 below) and is not equal to NS_{κ,λ} as may be expected. The fact that $\Pi_0^1(\kappa, \lambda) = \text{NSS}_{\kappa,\lambda}$ and the fact that Sun's characterization (1.1) holds, suggests that the correct notion of "1-club subset of $P_{\kappa}\lambda$ " needed to generalize Sun's characterization to $P_{\kappa}\lambda$ should be stated using NSS⁺_{κ,λ} instead of NS⁺_{κ,λ}. Recall that for $x \in P_{\kappa}\lambda$ we define $\kappa_x =_{\text{def}} |x \cap \kappa|$.

Definition 1.1. We say that $C \subseteq P_{\kappa}\lambda$ is 1-*club* if and only if

- (1) $C \in \text{NSS}^+_{\kappa,\lambda}$ and
- (2) C is 1-closed, that is, for every $x \in P_{\kappa}\lambda$, if κ_x is an inaccessible cardinal and $C \cap P_{\kappa_x} x \in \text{NSS}^+_{\kappa_x,x}$ then $x \in C$.

In Section 5, we generalize Sun's characterization of Π_1^1 -indescribable subsets of κ by showing that the notion of 1-club subset of $P_{\kappa}\lambda$ in Definition 1.1 can indeed be used to characterize the Π_1^1 -indescribable subsets of $P_{\kappa}\lambda$. In fact, in Section 5, we develop a notion of *n*-club subset of $P_{\kappa}\lambda$ where $n < \omega$ and prove the following.

Theorem 1.2. Suppose $\kappa \leq \lambda$ are cardinals with $\lambda^{<\kappa} = \lambda$, $n < \omega$ and $P_{\kappa}\lambda$ is Π^1_n -indescribable. Then $S \subseteq P_{\kappa}\lambda$ is Π^1_n -indescribable if and only if for all n-clubs $C \subseteq P_{\kappa}\lambda$ we have $S \cap C \neq \emptyset$.

In Section 5, we also prove that Baumgartner's Π_1^1 -indescribable subsets of $P_{\kappa}\lambda$ can be characterized using elementary embeddings which resemble the usual elementary embeddings witnessing the weak compactness of subsets of κ . Recall that,

¹Baumgartner's handwritten notes seem to be unavailable.

for cardinals $\kappa < \lambda$, κ is λ -supercompact if and only if there is an elementary embedding $j: V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and $j'' \lambda \in M$. Such embeddings can be assumed to be ultrapowers by normal fine κ -complete ultrafilters on $P_{\kappa}\lambda$, in which case $M^{\lambda} \cap V \subseteq M$. Schanker [Sch13] fused the notions of weak compactness and λ -supercompactness as follows: κ is said to be *nearly* λ supercompact if for every $A \subseteq \lambda$ there is a transitive $M \models \text{ZFC}^-$ with $\lambda, A \in M$ and $M^{<\kappa} \cap V \subseteq M$, there is a transitive N and an elementary embedding $j: M \to N$ with critical point κ such that $j(\kappa) > \lambda$ and $j'' \lambda \in N$. As observed by Schanker, it is clear that if κ is λ -supercompact then κ is nearly λ -supercompact and the converse is not true in general; for example, the least cardinal κ which is nearly κ^+ -supercopmact is not κ^+ -supercompact. Even though κ is supercompact if and only if κ is nearly λ -supercompact for every $\lambda \geq \kappa$, Schanker proved that for any fixed $\lambda > \kappa$. κ being nearly λ -supercompact need not imply that κ is measurable. For example, Schanker proved that if κ is nearly κ^+ -supercompact and GCH holds, then there is a cofinality-preserving forcing extension in which κ remains nearly κ^+ -supercompact and GCH fails first at κ . Furthermore, if κ is λ -supercompact and GCH holds then there is a cofinality-preserving forcing extension [CGHS15] in which κ is the least weakly compact cardinal and κ is nearly λ -supercompact.²

We prove that the elementary embeddings considered by Schanker in [Sch13] lead to a normal ideal on $P_{\kappa}\lambda$ as follows, and indeed this ideal is equal to Baumgartner's ideal $\Pi_1^1(\kappa, \lambda)$.

Definition 1.3. We say that a set $W \subseteq P_{\kappa}\lambda$ is weakly compact³ if and only if for every $A \subseteq \lambda$ there is a transitive $M \models \text{ZFC}^-$ with $\lambda, A, W \in M$, a transitive Nand an elementary embedding $j: M \to N$ with critical point κ such that $j(\kappa) > \lambda$ and $j"\lambda \in j(W)$. The weakly compact ideal on $P_{\kappa}\lambda$ is defined to be

 $NWC_{\kappa,\lambda} = \{ X \subseteq P_{\kappa}\lambda : X \text{ is not weakly compact} \}.$

Theorem 1.4. Suppose $\kappa \leq \lambda$ are cardinals such that κ is inaccessible and $\lambda^{<\kappa} = \lambda$. Then a set $W \subseteq P_{\kappa}\lambda$ is Π_1^1 -indescribable if and only if it is weakly compact.

In Section 6, we provide several applications. In Section 6.1, we answer a question of Cox and Lücke [CL17]. Before stating the question, let us review some terminology from [CL17]. For partial orders $\mathbb{Q} \subseteq \mathbb{P}$, we say that \mathbb{Q} is a regular suborder of \mathbb{P} if the inclusion map preserves incompatibility and maximal antichains in \mathbb{Q} are also maximal in \mathbb{P} . Given a partial order \mathbb{P} , we let $\operatorname{Reg}_{\kappa}(\mathbb{P})$ denote the collection of all regular suborders of \mathbb{P} of cardinality less than κ . In [CL17], the authors consider various properties of partial orders that imply $\operatorname{Reg}_{\kappa}(\mathbb{P})$ is *large* in a certain sense. For example, suppose κ is a regular uncountable cardinal, a partial order \mathbb{P} is called κ -stationarily layered if $\operatorname{Reg}_{\kappa}(\mathbb{P})$ is stationary in $P_{\kappa}\mathbb{P}$. Among other things, Cox-Lücke showed [CL17, Theorem 1.8] that such properties can be used to characterize weakly compact cardinals: κ is a weakly compact cardinal if and only if every partial order \mathbb{P} satisfying the κ -chain condition is κ -stationarily layered. Cox and Lücke also consider another notion of *largeness* of $\operatorname{Reg}_{\kappa}(\mathbb{P})$: a partial order \mathbb{P} holds for every surjection $s : \lambda \to \mathbb{P}$. Question 7.4 of [CL17] states, assuming

²GCH must fail at κ is such an extension.

³We prefer this terminology to saying that "W is nearly λ -supercompact" because we will prove that $W \subseteq P_{\kappa}\lambda$ is weakly compact if and only if W is Π_1^1 -indescribable, and thus this terminology conforms to more of the existing literature.

 $(\kappa^+)^{<\kappa} = \kappa^+$, "Let κ be an inaccessible cardinal such that there is a normal filter \mathcal{F} on $P_{\kappa}\kappa^+$ with the property that every partial order of cardinality κ^+ that satisfies the κ -chain condition is \mathcal{F} -layered. Must κ be a measurable cardinal?" By generalizing the work of Schanker [Sch13], we show that the answer is no by proving the following.

Theorem 1.5. Suppose $P_{\kappa}\lambda$ is weakly compact, GCH holds and $\lambda^{<\lambda} = \lambda$. There is a cofinality-preserving forcing extension V[G] in which

- (1) $(P_{\kappa}\lambda)^{V[G]}$ is weakly compact and hence the filter $\mathcal{F} = (\text{NWC}^*_{\kappa,\lambda})^{V[G]}$ is normal and nontrivial,
- (2) every partial order of cardinality λ that satisfies the κ -c.c. is \mathcal{F} -layered,
- (3) κ is not measurable and
- (4) $\lambda^{<\kappa} = \lambda$.

In Section 6.2, we consider properties of generic ultrapowers by the Π_n^m -indescribability ideals on $P_{\kappa}\lambda$. Indeed, we give a characterization of Π_n^m -indescribable subsets of $P_{\kappa}\lambda$ in terms of generic elementary embeddings.

In Section 6.3, generalizing similar principles considered by Hellsten [Hel03], we use the weakly compact ideal NWC_{κ,λ} to formulate a two-cardinal weakly compact diamond principle as follows.

Definition 1.6. Suppose $W \in \text{NWC}^+_{\kappa,\lambda}$. We say that weakly compact diamond holds on W and write $\diamondsuit_{\kappa,\lambda}^{\text{wc}}(W)$ if and only if there is a sequence $\langle a_z \subseteq z : z \in P_{\kappa}\lambda \rangle$ such that for every $A \subseteq \lambda$ we have $\{z \in W : a_z = A \cap z\} \in \text{NWC}^+_{\kappa,\lambda}$. When $W = P_{\kappa}\lambda$ we write simply $\diamondsuit_{\kappa,\lambda}^{\text{wc}}$ instead of $\diamondsuit_{\kappa,\lambda}^{\text{wc}}(P_{\kappa}\lambda)$.

As an application of the 1-club characterization of weakly compact subsets of $P_{\kappa\lambda}$ obtained by combining Theorem 1.2 and Theorem 1.4, we prove that for any $W \in \text{NWC}_{\kappa,\lambda}^+$, if κ is λ -supercompact then $\diamondsuit_{\kappa,\lambda}^{\text{wc}}(W)$ holds. We also show that, assuming GCH, there is a natural way to force $\diamondsuit_{\kappa,\lambda}^{\text{wc}}(W)$ without collapsing cofinalities from the assumption that $P_{\kappa\lambda}$ is weakly compact and $\lambda^{<\lambda} = \lambda$.

In Section 6.4, we use a result of Pereira [Per17] to show that if κ is κ^+ -supercompact and GCH holds then there is a cofinality-preserving forcing extension in which there is a (κ, κ^+) -semimorass $\mu \subseteq (\kappa, \kappa^+)$ which is Π^1_n -indescribable for all $n < \omega$.

We close the paper with a discussion of several open questions concerning reflection properties of weakly compact sets $W \subseteq P_{\kappa}\lambda$ and generalizations of club shooting forcing to the context of the weakly compact ideal on $P_{\kappa}\lambda$.

2. Preliminaries on strongly normal ideals and strong stationarity

Throughout this section we assume $\kappa \leq \lambda$ are cardinals, κ is a regular cardinal and X is a set of ordinals. Recall that an ideal I on $P_{\kappa}X$ is normal if for every $S \in I^+$ and every function $f: P_{\kappa}X \to X$ with $\{x \in S: f(x) \in x\} \in I^+$ there is a $T \in P(S) \cap I^+$ such that $f \upharpoonright T$ is constant. Equivalently, an ideal I on $P_{\kappa}X$ is normal if and only if for every $\{Z_x: x \in X\} \subseteq I$ the set $\bigtriangledown_{x \in X}Z_x =_{def}$ $\{y \in P_{\kappa}X: y \in Z_x \text{ for some } x \in y\}$ is in I (see [For10, Proposition 2.19]). An ideal I on $P_{\kappa}\lambda$ is fine if and only if $\{\alpha\} =_{def} \{x \in P_{\kappa}\lambda : \alpha \in x\} \in I^*$ for every $\alpha < \lambda$. Jech [Jec73] generalized the notion of closed unbounded and stationary subsets of cardinals to subsets $P_{\kappa}\lambda$. Recall that a set $C \subseteq P_{\kappa}\lambda$ is club in $P_{\kappa}\lambda$ if (1) for every $x \in P_{\kappa}\lambda$ there is a $y \in C$ with $x \subseteq y$ and (2) whenever $X \subseteq C$ is

directed under the ordering \subsetneq and $|X| < \kappa$ we have $\bigcup X \in C$. A set $S \subseteq P_{\kappa}\lambda$ is stationary if $S \cap C \neq \emptyset$ for all clubs $C \subseteq P_{\kappa}\lambda$. Jech proved that the collection $NS_{\kappa,\lambda} = \{X \subseteq P_{\kappa}\lambda : X \text{ is not stationary in } P_{\kappa}\lambda\}$ is a normal fine κ -complete ideal on $P_{\kappa}\lambda$. Carr [Car82] proved that, when κ is a regular cardinal, the nonstationary ideal $NS_{\kappa,\lambda}$ is the minimal normal fine κ -complete ideal on $P_{\kappa}\lambda$.

When considering ideals on $P_{\kappa}\lambda$ or $P_{\kappa}X$ for κ inaccessible, it is quite fruitful to work with a different notion of closed unboundedness obtained by replacing the structure $(P_{\kappa}\lambda, \subseteq)$ with a different one. For $x \in P_{\kappa}X$ we define $\kappa_x = |x \cap \kappa|$ and we define an ordering $(P_{\kappa}X, \sqsubset)$ by letting

 $x \sqsubset y$ if and only if $x \in P_{\kappa_u} y$.

Given a function $f: P_{\kappa}X \to P_{\kappa}X$ we let

$$C_f =_{\text{def}} \{ x \in P_{\kappa} X : x \cap \kappa \neq \emptyset \land f[P_{\kappa_x} x] \subseteq P_{\kappa_x} x \}.$$

We say that a set $C \subseteq P_{\kappa}X$ is weakly closed unbounded if there is an f such that $C = C_f$. Note that it is straightforward to see that when κ is inaccessible, every club $C \subseteq P_{\kappa}X$ contains a weak club (see Lemma 2.4 below). However, in general, it is not the case that every weak club contains a club (this follows from Corollary 2.3 below). A set $S \subseteq P_{\kappa}X$ is called strongly stationary if for every f we have $C_f \cap S \neq \emptyset$. An ideal I on $P_{\kappa}X$ is strongly normal if for any $S \in I^+$ and function $f : P_{\kappa}X \to P_{\kappa}X$ such that $f(x) \sqsubset x$ for all $x \in X$ there is $Y \in P(X) \cap I^+$ such that $f \upharpoonright Y$ is constant. It follows easily that an ideal I on $P_{\kappa}X$ is strongly normal if κ is λ -supercompact then the prime ideal dual to a normal fine ultrafilter on $P_{\kappa}\lambda$ is strongly normal. Matet [Mat88] showed that if κ is Mahlo then the collection of non-strongly stationary sets

$$\mathrm{NSS}_{\kappa,X} =_{def} \{ X \subseteq P_{\kappa}X : \exists f : P_{\kappa}X \to P_{\kappa}X \text{ such that } X \cap C_f = \emptyset \}$$

is the minimal strongly normal ideal on $P_{\kappa}X$. Improving this, Carr, Levinski and Pelletier obtained the following.

Theorem 2.1 (Carr-Levinski-Pelletier [CLP90]). Suppose κ is a regular cardinal and X is a set of ordinals with $\kappa \leq |X|$. There is a strongly normal ideal on $P_{\kappa}X$ if and only if κ is Mahlo or $\kappa = \mu^+$ where $\mu^{<\mu} = \mu$; moreover, in this case $\text{NSS}_{\kappa,X}$ is the minimal such ideal.

In these cases, since every strongly normal ideal on $P_{\kappa}\lambda$ is normal, we have

$$NS_{\kappa,\lambda} \subseteq NSS_{\kappa,\lambda}.$$

The following lemma, due to Zwicker (see the discussion on page 61 of [CLP90]), shows that if κ is weakly inaccessible the previous containment is strict. We include a proof for the reader's convenience.

Lemma 2.2 (Zwicker). If κ is weakly inaccessible then $NS_{\kappa,\lambda}$ is not strongly normal.

Proof. Let $A = \{x \in P_{\kappa}\lambda : x \cap \kappa \text{ is an uncountable cardinal with } cf(x \cap \kappa) = \omega\}$. First we show that A is a stationary subset of $P_{\kappa}\lambda$. Let $C \subseteq P_{\kappa}\lambda$ be a club and recall that the set $C^* = \{x \in P_{\kappa}\lambda : x \cap \kappa \in \kappa\}$ is a club subset of $P_{\kappa}\lambda$. We inductively define a sequence $\langle x_i : i < \omega \rangle$ with $x_i \in C \cap C^*$ as follows. Let $\kappa_0 = \omega_1$ and choose $x_0 \in C \cap C^*$ with $\kappa_0 \subseteq x_0$. Let $\kappa_{i+1} > x_i \cap \kappa$ and choose $x_{i+1} \in C \cap C^*$ with $\kappa_{i+1} \subsetneq x_{i+1}$. Now let $x_\omega = \bigcup_{i < \omega} x_i$ and notice that $x_\omega \in C \cap C^* \cap A$.

Now define $F: A \to P_{\kappa}\lambda$ by letting F(x) be some countable and cofinal subset of $x \cap \kappa$. Then $F(x) \sqsubset x$ for all $x \in A$ and F is not constant on any stationary subset of A.

Corollary 2.3. If κ is Mahlo then $NSS_{\kappa,\lambda}$ is nontrivial and $NS_{\kappa,\lambda} \subseteq NSS_{\kappa,\lambda}$.

Lemma 2.4. Suppose κ is an inaccessible cardinal and X is a set of ordinals with $\kappa \leq |X|$. If C is a club subset of $P_{\kappa}X$ and $f : P_{\kappa}X \to P_{\kappa}X$ is such that $z \subsetneq f(z) \in C$ for every $z \in P_{\kappa}X$, then

$$C_f = \{ x \in P_{\kappa}X : x \cap \kappa \neq \emptyset \land f[P_{\kappa_x}x] \subseteq P_{\kappa_x}x \}$$

is a subset of C.

Proof. Suppose C is a club subset of $P_{\kappa}X$ and f is as in the statement of the lemma. Suppose $x \in P_{\kappa}X$ and $f[P_{\kappa_x}x] \subseteq P_{\kappa_x}x$. It follows that $C \cap P_{\kappa_x}x$ is directed since if $y, z \in C \cap P_{\kappa_x}x$ then $y \cup z \in P_{\kappa_x}x$ and hence $y \cup z \subsetneq f(y \cup z) \in C \cap P_{\kappa_x}x$. Since $C \cap P_{\kappa_x}x$ is a directed subset of C with size at most $|x|^{<\kappa_x} < \kappa$, it follows that $x = \bigcup (C \cap P_{\kappa_x}x) \in C$. Thus $C_f \subseteq C$.

The next lemma shows that ${\rm NSS}_{\kappa,\lambda}$ can be obtained by restricting ${\rm NS}_{\kappa,\lambda}$ to a particular stationary sets

Lemma 2.5 ([CLP90], Corollary 3.3). If $\lambda^{<\kappa} = \lambda$ and $\text{NSS}_{\kappa,\lambda}$ is nontrivial, then for any bijection $c: P_{\kappa}\lambda \to \lambda$,

$$\mathrm{NSS}_{\kappa,\lambda} = \mathrm{NS}_{\kappa,\lambda} \upharpoonright S_c$$

where $S_c = \{x \in P_{\kappa}\lambda : c[P_{\kappa_x}x] \subseteq x\}.$

3. Elementary embeddings and the weakly compact ideal on $P_\kappa\lambda$

There are many ways to characterize the weakly compact subsets of $P_{\kappa}\lambda$ from Definition 1.3 using elementary embeddings. Indeed, all six characterizations of the near λ -supercompactness of a cardinal κ given in [Sch13] can be generalized to provide characterizations of weakly compact subsets of $P_{\kappa}\lambda$. Here we summarize the pertinent characterizations without proof; note that the proof is very similar to that of [Sch13, Theorem 1.4].

Lemma 3.1 (Schanker). For cardinals $\kappa \leq \lambda$ with $\lambda^{<\kappa} = \lambda$ and $W \subseteq P_{\kappa}\lambda$, the following are equivalent.

- W is a weakly compact subset of P_κλ; in other words, for every A ⊆ λ there is a transitive M ⊨ ZFC⁻ with λ, A, W ∈ M and M^{<κ} ∩ V ⊆ M, a transitive N and an elementary embedding j : M → N with critical point κ such that j(κ) > λ and j"λ ∈ j(W).
- (2) For all $\delta \geq \kappa$ and every transitive $M \models \text{ZFC}^-$ of size λ with $\lambda, W \in M$ and $M^{<\delta} \cap V \subseteq M$, there is a transitive N of size λ with $N^{<\delta} \cap V \subseteq N$ and $P(\lambda)^M \subseteq N$ and an elementary embedding $j : M \to N$ with critical point κ such that $j(\kappa) > \lambda$, $j"\lambda \in j(W)$ and $N = \{j(f)(j"\lambda) : f \in M \text{ is a function with domain } P_{\kappa}\lambda\}.$
- (3) For every collection \mathcal{A} of at most λ subsets of $P_{\kappa}\lambda$ with $W \in \mathcal{A}$ and every collection \mathcal{F} of at most λ functions from $P_{\kappa}\lambda$ to λ , there exists a κ -complete filter F on $P_{\kappa}\lambda$ such that $W \in F$; $(\forall \alpha < \lambda)(\{\alpha\} =_{def} \{x \in P_{\kappa}\lambda : \alpha \in x\} \in F)$; F measures all sets in \mathcal{A} , meaning that for all $X \in \mathcal{A}$ either $X \in F$ or $P_{\kappa}\lambda \setminus X \in F$, and finally, F is \mathcal{F} -normal, in the sense that for every

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 $f \in \mathcal{F}$ which is regressive on some set in F, there is $\alpha_f < \lambda$ such that $\{x \in P_{\kappa}\lambda : f(x) = \alpha_f\} \in F.$

By assuming a little bit more about cardinal arithmetic we obtain another characterization which will be useful for forcing arguments.

Lemma 3.2. If $\lambda^{<\lambda} = \lambda$ and $W \subseteq P_{\kappa}\lambda$ then W is a weakly compact subset of $P_{\kappa}\lambda$ if and only if for every $A \in H(\lambda^+)$ there is a transitive $M \models \text{ZFC}^-$ of size λ with $\lambda, A, W \in M$ and $M^{<\lambda} \cap V \subseteq M$, there is a transitive N of size λ with $N^{<\lambda} \cap V \subseteq N$ and an elementary embedding $j: M \to N$ with critical point κ such that

- (1) $j(\kappa) > \lambda$, (2) $j''\lambda \in j(W)$, (3) $N = \{j(f)(j''\lambda) : f \in M \text{ is a function with domain } P_{\kappa}\lambda\}$ and
- (4) if $X \in M$ with $|X|^M \leq \lambda$ and $X \in N$ then $j \upharpoonright X \in N$.

Proof. The reverse direction is easy. For the forward direction, assume W is a weakly compact subset of $P_{\kappa\lambda}$ and $A \in H(\lambda^+)$. Since $\lambda^{<\lambda} \subseteq \lambda$ we can use an iterative Skolem-hull argument to build a transitive $M \prec H(\lambda^+)$ of size λ with $\lambda, A, W \in M$ and $M^{<\lambda} \cap V \subseteq M$. Now, applying Lemma 3.1 (3), there is a κ -complete fine M-normal M-ultrafilter F with $W \in F$. Let $j: M \to \text{Ult}(M, F) = ({}^{\kappa}M \cap M)/F$ be the corresponding ultrapower embedding, which is well-founded since F is κ -complete. Thus we may identify Ult(M, F) with its transitive collapse N and obtain $j: M \to N$. To see that (1) - (3) hold one may apply standard arguments. For (4), suppose $X \in M$ with $|X|^M \leq \lambda$ and $X \in N$. Let $b: \lambda \to X$ be a bijection in M. By elementarity $j(b): j(\lambda) \to j(X)$ is a bijection in N and $j(b)[j"\lambda] = j"X$. Furthermore, working in N, we may define a function f with domain X such that $f(x) = j(b)(j(b^{-1}(x))) = j(b)(j(b^{-1})(j(x))) = j \upharpoonright X(x)$.

The next definition will make negating the definition of weakly compact set easier.

Definition 3.3. Suppose $\kappa \leq \lambda$ are cardinals and $Z \in \text{NWC}_{\kappa,\lambda}$. We say that $A \subseteq \lambda$ witnesses that Z is not weakly compact or witnesses $Z \in \text{NWC}_{\kappa,\lambda}$ if and only if whenever $M \models \text{ZFC}^-$ is transitive with $\lambda, A, Z \in M$ and whenever N is transitive and $j: M \to N$ is an elementary embedding with critical point κ such that $j(\kappa) > \lambda$, we must have $j''\lambda \notin j(Z)$; in other words, A being in M guarantees that $j''\lambda \notin j(Z)$.

Proposition 3.4. If $P_{\kappa}\lambda$ is weakly compact then the non-weakly compact ideal NWC_{κ,λ} is a strongly normal proper ideal.

Proof. Let us show that NWC_{κ,λ} is strongly normal; the rest is routine. Suppose $Z_a \in \text{NWC}_{\kappa,\lambda}$ for all $a \in P_{\kappa}\lambda$ and let $Z = \bigtriangledown_{\Box} \{Z_a : a \in P_{\kappa}\lambda\} =_{\text{def}} \{x \in P_{\kappa}\lambda : x \in Z_a \text{ for some } a \in P_{\kappa_x}x\}$. For each $a \in P_{\kappa}\lambda$ there is some $A_a \subseteq \lambda$ witnessing that $Z_a \in \text{NWC}_{\kappa,\lambda}$. Since $|P_{\kappa}\lambda| = |\lambda|^{<\kappa} = |\lambda|$ there is a single set $A \subseteq \lambda$ coding all of the A_a 's as well as the sequence $\vec{Z} = \langle Z_a : a \in P_{\kappa}\lambda \rangle$ in the sense that whenever M is transitive with $A \in M$ then $A_a \in M$ for all $a \in P_{\kappa}\lambda$ and $\vec{Z} \in M$. Clearly we have that for every $a \in P_{\kappa}\lambda$ the set A witnesses that $Z_a \in \text{NWC}_{\kappa,\lambda}$. Let us argue that A witnesses that $Z \in \text{NWC}_{\kappa,\lambda}$. Suppose $M \models \text{ZFC}^-$ is transitive of size λ with $\lambda, A, Z \in M, N$ is transitive and $j : M \to N$ is an elementary embedding with critical point κ such that $j(\kappa) > \lambda$. We must argue that $j'' \lambda \notin j(Z)$. Since $A \in M$

we have $\vec{Z} \in M$ and we let $j(\vec{Z}) = \langle \bar{Z}_b : b \in j(P_\kappa \lambda) \rangle$. Notice that $\bar{Z}_{j(a)} = j(Z_a)$ for all $a \in P_\kappa \lambda$ by elementarity of j. By definition of Z,

$$j(Z) = \{x \in j(P_{\kappa}\lambda) : x \in \overline{Z}_b \text{ for some } b \in P_{j(\kappa)_x}x\}$$

where $j(\kappa)_{j^{n}\lambda} = |j^{n}\lambda \cap j(\kappa)|^{N} = \kappa$. For the sake of contradiction, suppose $j^{n}\lambda \in j(Z)$. Then $j^{n}\lambda \in \overline{Z}_{b}$ for some $b \in P_{j(\kappa)_{j^{n}\lambda}}j^{n}\lambda = P_{\kappa}j^{n}\lambda$. Since the critical point of j is κ we see that b = j(a) for some $a \in P_{\kappa}\lambda$ and hence $j^{n}\lambda \in \overline{Z}_{j(a)} = j(Z_{a})$ for some $a \in P_{\kappa}\lambda$. This contradicts the fact that A witnesses $Z_{a} \in NWC_{\kappa,\lambda}$.

Since $NSS_{\kappa,\lambda}$ is the minimal strongly normal ideal on $P_{\kappa\lambda}$ we obtain.

Corollary 3.5. If $P_{\kappa}\lambda$ is weakly compact then $NSS_{\kappa,\lambda} \subseteq NWC_{\kappa,\lambda}$.

To see that $\text{NSS}_{\kappa,\lambda} \subseteq \text{NWC}_{\kappa,\lambda}$ when $P_{\kappa}\lambda$ is weakly compact, let us consider the set $S = \{x \in P_{\kappa}\lambda : |x \cap \kappa| = |x|\}$, which played an important role in various results on almost disjoint partitions of elements of $\text{NS}^+_{\kappa,\lambda}$ (see the discussion around Proposition 25.5 in [Kan03]).

Proposition 3.6 (Proposition 25.5, [Kan03]). Suppose that $\kappa \leq \lambda$ and $S = \{x \in P_{\kappa}\lambda : |x \cap \kappa| = |x|\}$. Then:

- (a) $S \in \mathrm{NS}^+_{\kappa,\lambda}$
- (b) If κ is a successor cardinal then $S \in NS^*_{\kappa,\lambda}$.
- (c) If $\kappa < \lambda$ and κ is λ -supercompact then $S \notin NS^*_{\kappa,\lambda}$.
- (d) (Baumgartner) If $X \subseteq S$ is stationary then X can be partitioned into λ disjoint stationary sets.

Straight forward arguments show that if $NSS_{\kappa,\lambda}$ is nontrivial, Proposition 3.6 can be improved by replacing $NS_{\kappa,\lambda}$ with $NSS_{\kappa,\lambda}$ in (a) and (b) and by weakening the hypothesis and strengthening the conclusion of (c).

Proposition 3.7. Suppose $\kappa \leq \lambda$ and $S = \{x \in P_{\kappa}\lambda : |x \cap \kappa| = |x|\}$. Then the following hold.

- (a) If $\text{NSS}_{\kappa,\lambda}$ is nontrivial then $S \in \text{NSS}^+_{\kappa,\lambda}$.
- (b) If κ is a successor cardinal then $S \in NSS^*_{\kappa,\lambda}$.
- (c) If $P_{\kappa}\lambda$ is weakly compact then $S \in NWC_{\kappa,\lambda}$.

Proof. For (a), first notice that if κ is a successor then by Proposition 3.6 (a), $S \in \mathrm{NS}^*_{\kappa,\lambda} \subseteq \mathrm{NSS}^+_{\kappa,\lambda}$. On the other hand if κ is a limit, then by Theorem 2.1, κ is Mahlo since $\mathrm{NSS}_{\kappa,\lambda}$ is nontrivial. Fix $f: P_{\kappa}\lambda \to P_{\kappa}\lambda$ and recursively define x(n)and y(n) as follows. Let $y(n+1) = \bigcup f[P_{\kappa_{x(n)}}x(n)] \in P_{\kappa}\lambda$ and define x(n+1) = $y(n) \cup |y(n)|$. Notice that $x(n+1) \in S$. Now it follows that $x(\omega) =_{\mathrm{def}} \bigcup_{n < \omega} x(n) \in S$ and $f[P_{\kappa_{x(\omega)}}x(\omega)] \subseteq P_{\kappa_{x(\omega)}}x(\omega)$.

For (b), if κ is a successor, say $\kappa = \mu^+$ then $\{x \in P_{\kappa}\lambda : |x \cap \kappa| = \mu\}$ is in $\text{NSS}^*_{\kappa,\lambda}$ and is a subset of S.

For (c), fix $A \subseteq \lambda$ and let M be a (κ, λ) -model with $A, S \in M$. Since $P_{\kappa}\lambda$ is weakly compact there is a $j : M \to N$ with critical point κ such that $j(\kappa) > \lambda$ and $j"\lambda \in N$. In N we have $|\kappa| < |j"\lambda|$, and hence $j"\lambda \in j((P_{\kappa}\lambda) \setminus S)$. Thus $S \in \text{NWC}_{\kappa,\lambda}$.

Corollary 3.8. If $P_{\kappa\lambda}$ is weakly compact then $NSS_{\kappa,\lambda} \subsetneq NWC_{\kappa,\lambda}$.

Proof. If $P_{\kappa}\lambda$ is weakly compact then $\text{NSS}_{\kappa,\lambda} \subseteq \text{NWC}_{\kappa,\lambda}$ by Corollary 3.5 and $S = \{x \in P_{\kappa}\lambda : |x \cap \kappa| = |x|\} \in \text{NWC}_{\kappa,\lambda} \setminus \text{NSS}_{\kappa,\lambda}$.

4. Indescribability of subsets of $P_{\kappa}\lambda$

According to [Abe98] and [Car85], in a set of handwritten notes, Baumgartner [Bau] defined a notion of indescribability for subsets of $P_{\kappa}\lambda$ as follows. Give a regular cardinal κ and a set of ordinals A with $\kappa \leq |A|$, consider the hierarchy:

$$V_{0}(\kappa, A) = A$$

$$V_{\alpha+1}(\kappa, A) = P_{\kappa}(V_{\alpha}(\kappa, A)) \cup V_{\alpha}(\kappa, A)$$

$$V_{\alpha}(\kappa, A) = \bigcup_{\beta < \alpha} V_{\beta}(\kappa, A) \text{ for } \alpha \text{ a limit}$$

Clearly $V_{\kappa} \subseteq V_{\kappa}(\kappa, A)$ and if A is transitive then so is $V_{\alpha}(\kappa, A)$ for all $\alpha \leq \kappa$. See [Car85, Section 4] for a discussion of the restricted axioms of ZFC satisfied by $V_{\kappa}(\kappa, \lambda)$ when κ is inaccessible.

Definition 4.1 (Baumgartner [Bau]). Suppose κ is a regular cardinal and A is a set of ordinals with $\kappa \leq |A|$. Let $S \subseteq P_{\kappa}A$. We say that S is $\prod_{n=1}^{1}$ -indescribable in $P_{\kappa}A$ if whenever $(V_{\kappa}(\kappa, A), \in, R_1, \ldots, R_k) \models \varphi$ where $k < \omega, R_1, \ldots, R_k \subseteq V_{\kappa}(\kappa, A)$ and φ is a $\prod_{n=1}^{1}$ sentence, there is an $x \in S$ such that

$$x \cap \kappa = \kappa_x$$
 and $(V_{\kappa_x}(\kappa_x, x), \in, R \cap V_{\kappa_x}(\kappa_x, x)) \models \varphi$.

Abe [Abe
98, Lemma 4.1] showed that if $P_\kappa A$ is
 $\Pi^1_n\mbox{-indescribable}$ then

$$\Pi^1_n(\kappa, A) = \{ X \subseteq P_{\kappa}A : X \text{ is not } \Pi^1_n \text{-indescribable} \}$$

is a strongly normal proper ideal on $P_{\kappa}A$.

Lemma 4.2. [Abe98, page 270] Assuming $|A| \ge \kappa$, there is a Π_1^1 -sentence σ such that $(V_{\kappa}(\kappa, A), \in) \models \sigma$ if and only if κ is inaccessible.

Lemma 4.3. [Abe98, Lemma 1.3] Suppose $\kappa \leq \lambda$ are cardinals and κ is regular.

- (1) $V_{\kappa}(\kappa,\lambda) = \bigcup_{x \in P_{\kappa}\lambda} V_{\kappa_x}(\kappa_x,x).$
- (2) If $y \sqsubset x$ then $V_{\kappa_y}(\kappa_y, y) \in V_{\kappa_x}(\kappa_x, x)$.
- (3) If $\kappa_x = x \cap \kappa$ is inaccessible then $V_{\kappa_x}(\kappa_x, x) = \bigcup_{y \vdash x} V_{\kappa_y}(\kappa_y, y)$.
- (4) For any bijection $h: V_{\kappa}(\kappa, \lambda) \to P_{\kappa}\lambda, \{x \in P_{\kappa}\lambda : h[V_{\kappa_x}(\kappa_x, x)] = P_{\kappa_x}x\} \in NSS^*_{\kappa,\lambda}.$
- (5) If κ is inaccessible then $\{x \in P_{\kappa}\lambda : V_{\kappa_x}(\kappa_x, x) \prec V_{\kappa}(\kappa, \lambda)\} \in \mathrm{NSS}^*_{\kappa,\lambda}$.

Abe states Lemma 4.3 without proof; we now present a restatement and proof of Lemma 4.3 (5) (see Lemma 4.7 below) since it is vital to our proof of Theorem 4.10 and seems to be somewhat nontrivial.

Lemma 4.4. Suppose $\kappa \leq \lambda$ are cardinals and κ is regular. If $x, y \in P_{\kappa}\lambda$ and $x \subseteq y$ then $V_{\beta}(\kappa_x, x) \subseteq V_{\beta}(\kappa_y, y)$ for all $\beta \leq \kappa_x$ and $V_{\kappa_x}(\kappa_x, x) \subseteq V_{\kappa_y}(\kappa_y, y)$.

Proof. Notice that $\kappa_x \leq \kappa_y$. We proceed by induction. Clearly $V_0(\kappa_x, x) = x \subseteq y = V_0(\kappa_y, y)$. Suppose $V_\alpha(\kappa_x, x) \subseteq V_\alpha(\kappa_y, y)$, then

$$V_{\alpha+1}(\kappa_x, x) = P_{\kappa_x}(V_{\alpha}(\kappa_x, x)) \cup V_{\alpha}(\kappa_x, x) \subseteq P_{\kappa_y}(V_{\alpha}(\kappa_y, y)) \cup V_{\alpha}(\kappa_y, y) = V_{\alpha+1}(\kappa_y, y).$$

Assume that $V_{\alpha}(\kappa_x, x) \subseteq V_{\alpha}(\kappa_y, y)$ for all $\alpha < \gamma \leq \kappa_x$ where γ is a limit ordinal. Then $V_{\gamma}(\kappa_x, x) \subseteq V_{\gamma}(\kappa_y, y)$ follows easily by definition. \Box **Lemma 4.5.** Let κ be an inaccessible cardinal and X a set of ordinals with $\kappa \leq |X|$. Suppose $f : P_{\kappa}X \to P_{\kappa}X$ is a function such that for every $z \in P_{\kappa}X$ we have $\sup(z \cap \kappa)^+ \subseteq f(z)$. For $x \in P_{\kappa}X$, if $x \cap \kappa \in \kappa$ and $f[P_{\kappa_x}x] \subseteq P_{\kappa_x}x$ then $\kappa_x = x \cap \kappa$ is a weakly inaccessible cardinal.

Proof. Suppose $x \cap \kappa$ were singular. Then some $z \in P_{\kappa_x} x$ is cofinal in $x \cap \kappa$. But then since $f[P_{\kappa_x} x] \subseteq P_{\kappa_x} x$ we have $\sup(z \cap \kappa)^+ \subseteq f(z) \in P_{\kappa_x} x$, which is impossible since $\sup(z \cap \kappa)^+ > \kappa_x$. This implies that $x \cap \kappa$ is a regular ordinal and thus a cardinal.

Suppose $x \cap \kappa$ were a successor cardinal, say $x \cap \kappa = \mu^+$. Since $\kappa_x = x \cap \kappa = \mu^+$ it follows that $\mu \in P_{\kappa_x} x$ and since $f[P_{\kappa_x} x] \subseteq P_{\kappa_x} x$ we conclude that $\sup(\mu)^+ = \mu^+ = \kappa_x \subseteq f(\mu) \in P_{\kappa_x} x$, which is impossible.

Lemma 4.6. Suppose κ is an inaccessible cardinal and A is a set of ordinals with $\kappa \leq |A|$. If $X \subseteq V_{\kappa}(\kappa, A)$ and $|X| < \kappa$, then there exists $x \in P_{\kappa}A$ such that $X \subseteq V_{\kappa_x}(\kappa_x, x)$.

Proof. We will prove by induction that for all $\alpha < \kappa$, if $X \subseteq V_{\alpha}(\kappa, A)$ and $|X| < \kappa$ then there is an $x \in P_{\kappa}A$ such that $X \subseteq V_{\kappa_x}(\kappa_x, x)$. Suppose $X \subseteq V_0(\kappa, A) = A$ and $|X| < \kappa$, then $X \in P_{\kappa}A$. By the inaccessibility of κ we may let $x \in P_{\kappa}A$ be such that $X \subseteq x$ and $|X| < \kappa_x$. Then $X \in P_{\kappa_x} x \subseteq V_{\kappa_x}(\kappa_x, x)$. Suppose $X \subseteq V_{\alpha}(\kappa, A)$ for some limit $\alpha < \kappa$. Let $\langle \beta(i) : i < \operatorname{cf}(\alpha) \rangle$ be cofinal in α . For each $i, X \cap V_{\beta(i)}(\kappa, A)$ is a subset of $V_{\beta(i)}(\kappa, A)$ of size $< \kappa$. Thus, by induction, for each i there is some $x(i) \in P_{\kappa}A$ such that $X \cap V_{\beta(i)}(\kappa, A) \subseteq V_{\kappa_{x(i)}}(\kappa_{x(i)}, x(i))$. Let $x = \bigcup_{i < \operatorname{cf}(\alpha)} x(i)$. Since $x(i) \subseteq x$, by Lemma 4.4, we have $V_{\kappa_{x(i)}}(\kappa_{x(i)}, x(i)) \subseteq V_{\kappa_x}(\kappa_x, x)$ for each $i < \operatorname{cf}(\alpha)$. Thus

$$X = \bigcup_{i < \mathrm{cf}(\alpha)} (X \cap V_{\beta(i)}(\kappa, A)) \subseteq \bigcup_{i < \mathrm{cf}(\alpha)} V_{\kappa_{x(i)}}(\kappa_{x(i)}, x(i)) \subseteq V_{\kappa_{x}}(\kappa_{x}, x).$$

Now suppose $X \subseteq V_{\alpha+1}(\kappa, A) = P_{\kappa}(V_{\alpha}(\kappa, A)) \cup V_{\alpha}(\kappa, A)$. Then we may write $X = Y \cup Z$ for some $Y \subseteq P_{\kappa}(V_{\alpha}(\kappa, A))$ and $Z \subseteq V_{\alpha}(\kappa, A)$ with $Y \cap Z = \emptyset$. Let $X' = (\bigcup Y) \cup Z$, then we have $|X'| < \kappa$ and $X' \subseteq V_{\alpha}(\kappa, A)$. By the inductive hypothesis there is some $y \in P_{\kappa}A$ such that $X' \subseteq V_{\kappa_y}(\kappa_y, y)$. Now choose $x \in P_{\kappa}A$ with $\kappa_y \sqsubset \kappa_x$. Then $X' \subseteq V_{\kappa_y}(\kappa_y, y) \subseteq V_{\kappa_y}(\kappa_x, x)$ implies $X \subseteq X' \cup P_{\kappa}X' \subseteq V_{\kappa_y+1}(\kappa_x, x) \subseteq P_{\kappa_x}(\kappa_x, x)$ and thus $X \subseteq V_{\kappa_x}(\kappa_x, x)$.

Here we present, with proof, a slight modification of Lemma 4.3(5).

Lemma 4.7. Suppose κ is inaccessible and A is a set of ordinals with $\kappa \leq |A|$. If $R \subseteq V_{\kappa}(\kappa, A)$ then

$$C = \{ x \in P_{\kappa}A : (V_{\kappa_x}(\kappa_x, x), \in, R \cap V_{\kappa_x}(\kappa_x, x)) \prec (V_{\kappa}(\kappa, A), \in, R) \}$$

is in NSS^{*}_{$\kappa,A}; in other words, there is a function <math>f: P_{\kappa}A \to P_{\kappa}A$ such that $C_f =_{def} \{x \in P_{\kappa}A : f[P_{\kappa_x}x] \subseteq P_{\kappa_x}x\} \subseteq C.$ </sub>

Proof. Let \triangleleft be a wellordering of $P_{\kappa}A$. Define $f: P_{\kappa}A \rightarrow P_{\kappa}A$ by letting f(z) be the \triangleleft -least $y \in P_{\kappa}A$ such that

- (1) $z \subsetneq y$,
- (2) $\sup(z \cap \kappa)^+ \subseteq y$,
- (2) $\operatorname{Sd}_{V_{\kappa}(\kappa,A)}^{V_{\kappa}(\kappa,A)} = 3$, (3) $\operatorname{Sk}^{V_{\kappa}(\kappa,A)}(V_{\kappa_{z}}(\kappa_{z},z),\in) \subseteq V_{\kappa_{y}}(\kappa_{y},y)$ and
- (4) $|y| = y \cap \kappa = \kappa_y$.

That such a y can be found follows from Lemma 4.6 and the fact that $\{z \in P_{\kappa}A : |z| = z \cap \kappa = \kappa_z\}$ is unbounded. Suppose $f[P_{\kappa_x}x] \subseteq P_{\kappa_x}x$. Since $x \cap \kappa \in \kappa$, it follows by Lemma 4.5 that $\kappa_x = x \cap \kappa$ is a weakly inaccessible cardinal.

Next we will argue that conditions (1) and (4) imply that $|x| = \kappa_x$. Notice that the set $D =_{\text{def}} \{z \in P_{\kappa}A : |z| = z \cap \kappa = \kappa_z\}$ is a Jech club, meaning that $D \in NS^*_{\kappa,A}$. Since $z \subsetneq f(z) \in D$ for every $z \in P_{\kappa}A$ we may apply Lemma 2.4 to see that $C_f \subseteq D$ and hence $x \in D$, which implies $|x| = x \cap \kappa = \kappa_x$.

Since $|x| = x \cap \kappa = \kappa_x$ is a weakly inaccessible cardinal, we may fix a bijection b from the set of successor ordinals less than κ_x to x. We recursively define a \subsetneq -increasing sequence $\langle x(i) : i < \kappa_x \rangle$ in $P_{\kappa_x} x$ and an elementary chain $\langle M_i : i < \kappa_x \rangle$ of substructures of $(V_{\kappa}(\kappa, A), \in)$ as follows. Choose $x(0) \in P_{\kappa_x} x$ and let $M_0 = \operatorname{Sk}^{V_{\kappa}(\kappa,A)}(V_{\kappa_{x(0)}}(\kappa_{x(0)}, x(0)), \in)$. By (3) and the fact that $f[P_{\kappa_x} x] \subseteq P_{\kappa_x} x$, it follows that $M_0 \subseteq V_{\kappa_y}(\kappa_y, y)$ for some $y \in P_{\kappa_x} x$ with $x(0) \subseteq y$. Given x(i) and M_i , let x(i+1) be the \lhd -least element of $P_{\kappa_x} x$ such that $x(i) \cup \{b(i+1)\} \subseteq x(i+1)$ and $M_i \subseteq V_{\kappa_x(i+1)}(\kappa_{x(i+1)}, x(i+1))$. Define $M_{i+1} = \operatorname{Sk}^{V_{\kappa}(\kappa,A)}(V_{\kappa_{x(i+1)}}(\kappa_{x(i+1)}, x(i+1)), \epsilon)$. If $i < \kappa_x$ is a limit let $x(i) = \bigcup_{j < i} x(j)$ and $M_i = \bigcup_{j < i} M_j$. It follows by induction, applying the elementary chain lemma at limit stages, that for every $j < \kappa_x$ we have i < j implies $M_i \prec M_j$. Thus $\langle M_i : i < \kappa_x \rangle$ is indeed an elementary chain of substructures of $(V_{\kappa}(\kappa, A), \epsilon)$. Since $b(i+1) \in x(i+1)$ for all $i < \kappa_x$, and b is a bijection from the successor ordinals less than κ_x to x, it follows that

$$x = \bigcup_{i < \kappa_x} x(i). \tag{4.1}$$

Thus $\kappa_x = \bigcup_{i < \kappa_x} \kappa_{x(i)}$. Furthermore we have

$$\bigcup_{i < \kappa_x} M_i = \bigcup_{i < \kappa_x} V_{\kappa_x(i)}(\kappa_{x(i)}, x(i))$$
 (by construction)
$$= \bigcup_{y \prec x} V_{\kappa_y}(\kappa_y, y)$$
 (use (4.1) and κ_x inaccessible)
$$= V_{\kappa_x}(\kappa_x, x)$$
 (Lemma 4.3)

Since $\langle M_i : i < \kappa_x \rangle$ is an elementary chain, each M_i is an elementary substructure of $(V_{\kappa_x}(\kappa_x, x), \in)$ and hence $(V_{\kappa_x}(\kappa_x, x), \in) \prec V_{\kappa}(\kappa, A)$ by the Tarski-Vaught test.

Lemma 4.8. If δ is inaccessible and A is a set of ordinals with $\delta \leq |A|$, then $V_{\delta}(\delta, A) \cap V_{\delta} = V_{\delta}$. In particular, if $x \in P_{\kappa}\lambda$ and κ_x is inaccessible then $V_{\kappa_x}(\kappa_x, x) \cap V_{\kappa_x} = V_{\kappa_x}$.

Proof. It suffices to show that for every $\alpha < \delta$ we have $V_{\alpha}(\delta, A) \cap V_{\alpha} = V_{\alpha}$, which can be done using an easy induction argument.

Lemma 4.9. For $x \in P_{\kappa}\lambda$ with $1 \leq x \cap \kappa = \kappa_x \in \kappa$ we have $P_{\kappa}\lambda \cap V_{\kappa_x}(\kappa_x, x) = P_{\kappa_x}x$.

Proof. Notice that $V_1(\kappa_x, x) = (P_{\kappa_x}x) \cup x \subseteq V_{\kappa_x}(\kappa_x, x)$. Hence $P_{\kappa_x}x \subseteq P_{\kappa\lambda} \cap V_{\kappa_x}(\kappa_x, x)$. For the converse, it suffices to prove by induction that for every $\alpha < \kappa_x$ we have $P_{\kappa\lambda} \cap V_{\alpha}(\kappa_x, x) \subseteq P_{\kappa_x}x$. For $\alpha = 0$ notice that $P_{\kappa\lambda} \cap V_0(\kappa_x, x) = (P_{\kappa\lambda}) \cap x = (x \cap \kappa) = \kappa_x \subseteq P_{\kappa_x}x$. Assuming that $P_{\kappa\lambda} \cap V_{\alpha}(\kappa_x, x) \subseteq P_{\kappa_x}x$, let

us consider $P_{\kappa}\lambda \cap V_{\alpha+1}(\kappa_x, x)$. Since $P_{\kappa}\lambda \cap P_{\kappa_x}(V_{\alpha}(\kappa_x, x)) \subseteq P_{\kappa_x}x$ it follows that $P_{\kappa}\lambda \cap V_{\alpha+1}(\kappa_x, x) = P_{\kappa}\lambda \cap P_{\kappa_x}(V_{\alpha}(\kappa_x, x)) \subseteq P_{\kappa_x}x$. The limit case is trivial. \Box

As mentioned in Section 1, the next theorem suggests that one should use strong stationarity instead of stationarity when generalizing the notion of 1-club subset of κ to that of $P_{\kappa}\lambda$.

Theorem 4.10. If κ is Mahlo and A is a set of ordinals with $\kappa \leq |A|$, then $S \subseteq P_{\kappa}A$ is in $\text{NSS}^+_{\kappa,A}$ if and only if S is Π^1_0 -indescribable in $P_{\kappa}A$ (i.e. first-order indescribable); in other words,

$$\Pi_0^1(\kappa, A) = \text{NSS}_{\kappa, A}.$$

Proof. Suppose S is in $\text{NSS}_{\kappa,A}^+$, $R \subseteq V_{\kappa}(\kappa, A)$ and let φ be a first order sentence with $(V_{\kappa}(\kappa, A), \in, R) \models \varphi$. By Lemma 4.7, there is a weak club C_f such that $x \in C_f$ implies $(V_{\kappa_x}(\kappa_x, x), \in, R \cap V_{\kappa_x}(\kappa_x, x)) \prec (V_{\kappa}(\kappa, A), \in, R)$. If we choose $x \in C_f \cap S \neq \emptyset$ then $(V_{\kappa_x}(\kappa_x, x), \in, R \cap V_{\kappa_x}(\kappa_x, x)) \models \varphi$.

Conversely, suppose S is Π_0^1 -indescribable, i.e. $S \in \Pi_0^1(\kappa, A)^+$, and let $C_f \subseteq P_{\kappa}A$ be in NSS^{*}_{κ,A} where $f: P_{\kappa}A \to P_{\kappa}A$. Since $V_1(\kappa, A) = (P_{\kappa}A) \cup A$, it follows that $f \subseteq V_3(\kappa, A) \subseteq V_{\kappa}(\kappa, A)$. We have

$$(V_{\kappa}(\kappa, A), \in, f, P_{\kappa}A) \models (\forall y \in P_{\kappa}A)(\exists z \in P_{\kappa}A)(f(y) = z).$$

$$(4.2)$$

Since S is Π_0^1 -indescribable we may fix an $x \in S$ with $x \cap \kappa = \kappa_x$ to which the formula in (4.2) reflects. Since $x \cap \kappa = \kappa_x$, we may apply Lemma 4.9 to obtain $P_{\kappa}A \cap V_{\kappa_x}(\kappa_x, x) = P_{\kappa_x}x$. Since $f \cap V_{\kappa_x}(\kappa_x, x)$, it follows that

$$(V_{\kappa_x}(\kappa_x, x), \in, f \upharpoonright P_{\kappa_x}x, P_{\kappa_x}x) \models (\forall y \in P_{\kappa_x}x)(\exists z \in P_{\kappa_x}x)((f \upharpoonright P_{\kappa_x}x)(y) = z).$$

Therefore $x \in S \cap C_f$.

5. Indescribability, n-clubs and weak compactness

First, we show that under reasonable assumptions, a set $S \subseteq P_{\kappa}\lambda$ is Π_n^1 -indescribable if and only if $S \cap C \neq \emptyset$ for all *n*-clubs $C \subseteq P_{\kappa}\lambda$. Let us define the notion of *n*-club subset of $P_{\kappa}\lambda$. Recall that Definition 1.1 states $C \subseteq P_{\kappa}\lambda$ is 1-club if and only if

- (1) $C \in \text{NSS}^+_{\kappa \lambda}$ and
- (2) C is 1-closed, that is, for every $x \in P_{\kappa}\lambda$, if κ_x is an inaccessible⁴ cardinal and $C \cap P_{\kappa_x} x \in \text{NSS}^+_{\kappa_x,x}$ then $x \in C$.

We generalize Hellsten's [Hel03, Section 2.4] notion of n-club subset of a cardinal to the two-cardinal context as follows.

Definition 5.1. Suppose κ is an inaccessible cardinal and A is a set of ordinals with $\kappa \leq |A|$. A set $C \subseteq P_{\kappa}A$ is 0-club in $P_{\kappa}A$ if and only if it is a weak club in $P_{\kappa}A$. For $n < \omega$, we say that $C \subseteq P_{\kappa}A$ is (n + 1)-club in $P_{\kappa}A$ if and only if $C \in \Pi_n^1(\kappa, A)^+$ and whenever $x \in P_{\kappa}A$ is such that $C \cap P_{\kappa_x}x \in \Pi_n^1(\kappa_x, x)^+$ and κ_x is inaccessible, then we have $x \in C$.

Lemma 5.2. Suppose $\kappa \leq \lambda$ are cardinals where κ is Mahlo and $n < \omega$. If $P_{\kappa}\lambda$ is Π_n^1 -indescribable so that $\Pi_n^1(\kappa, \lambda)$ is a proper ideal, then every n-club subset of $P_{\kappa}\lambda$ is (n+1)-club.

⁴Notice that we could replace "inaccessible" with "Mahlo" here to obtain an equivalent definition because $\text{NSS}^+_{\kappa_x,x} \neq \emptyset$ and κ_x inaccessible implies κ_x is Mahlo, by Theorem 2.1.

Proof. For n = 0 we must show that if $C \subseteq P_{\kappa}\lambda$ is a weak club, then it is a 1-club. Suppose $f: P_{\kappa}\lambda \to P_{\kappa}\lambda$ is a function with $C = C_f = \{x \in P_{\kappa}\lambda : f[P_{\kappa_x}x] \subseteq P_{\kappa_x}x\}$. Clearly $C_f \in \text{NSS}^+_{\kappa,\lambda}$. Suppose $x \in P_{\kappa}\lambda$ is such that $C_f \cap P_{\kappa_x}x \in \text{NSS}^+_{\kappa_x,x}$ and κ_x is inaccessible. Suppose $x \notin C_f$. Then there is some $y \in P_{\kappa_x}x$ such that $f(y) \notin P_{\kappa_x}x$. Since $\hat{y} =_{\text{def}} \{z \in P_{\kappa_x}x : y \in P_{\kappa_z}z\} \in \text{NS}^*_{\kappa_x,x}$, it follows from Lemma 2.4, that there is a $g: P_{\kappa_x}x \to P_{\kappa_x}x$ such that $C_g \subseteq \hat{y}$. Since $C_f \cap P_{\kappa_x}x$ has nontrivial intersection with every weak club subset of $P_{\kappa_x}x$, we conclude that there is some $w \in (C_f \cap P_{\kappa_x}x) \cap C_g$. Since $w \in C_g \subseteq \hat{y}$ we have $y \in P_{\kappa_w}w$, but since $f(y) \notin P_{\kappa_x}x$ we also have $f(y) \notin P_{\kappa_w}w$ and thus $w \notin C_f$.

Suppose n > 0 and $C \subseteq P_{\kappa}\lambda$ is an *n*-club. Since $\Pi^1_n(\kappa, \lambda)$ is a proper ideal, we see that $C \in \Pi^1_n(\kappa, \lambda)^+$. To show that *C* is an (n+1)-club, suppose $x \in P_{\kappa}\lambda$ is such that $C \cap P_{\kappa_x} x \in \Pi^1_n(\kappa_x, x)^+$ and κ_x is inaccessible. Then $C \cap P_{\kappa_x} x \in \Pi^1_{n-1}(\kappa_x, x)^+$, and hence $x \in C$ since *C* is *n*-club.

Let us consider the following generalization of a standard fact (see [Kan03, Corollary 6.9]).

Lemma 5.3. For every $n < \omega$ there is a Π_{n+1}^1 sentence φ_n such that for any inaccessible cardinal κ and any set of ordinals A with $\kappa \leq |A|$ we have S is Π_n^1 -indescribable in $P_{\kappa}A$ if and only if $(V_{\kappa}(\kappa, A), \in, S, P_{\kappa}A) \models \varphi_n$.

Proof. For n = 0, since the Π_0^1 -indescribability of S is equivalent to its strong stationarity (by Theorem 4.10), we let φ_0 be the natural Π_1^1 description of the strong stationarity of S; that is, φ_0 is the Π_1^1 statement "for all functions $f : P_{\kappa}A \to P_{\kappa}A$ there is an $x \in S$ such that $f[P_{\kappa_x}x] \subseteq P_{\kappa_x}x$ ".

Suppose n > 0. As noted in [Abe98, Section 4], there is a *universal* Π_n^1 formula $\psi_{1,n}(X,Y)$ where X a second-order variable and Y a first-order variable, in the sense that for any Π_n^1 formula $\varphi(X)$ there is a $k < \omega$ such that whenever δ is an inaccessible cardinal, A is a set of ordinals with $|A| \ge \kappa$ and $R \subseteq V_{\delta}(\delta, A)$ we have

 $(V_{\delta}(\delta, A), \in) \models \varphi[R]$ if and only if $(V_{\delta}(\delta, A), \in) \models \psi_{1,n}[R, k]$.

Since $\psi_{1,n}$ is Π_n^1 , it follows that the statement

$$\forall X \forall Y(\psi_{1,n}[X,Y] \to \exists x \in S((V_{\kappa_x}(\kappa_x,x),\in) \models \psi_{1,n}[X \cap V_{\kappa_x}(\kappa_x,x),Y])),$$

which we denote by φ_n , is Π_{n+1}^1 . It is straightforward to see that φ_n satisfies the conclusion of the lemma.

Generalizing [Hel03, Theorem 2.4.2] we obtain the following result which was mentioned in Section 1.

Theorem 1.2. Suppose $\kappa \leq \lambda$ are cardinals with $\lambda^{<\kappa} = \lambda$, $n < \omega$ and $P_{\kappa}\lambda$ is Π^1_n -indescribable. Then $S \subseteq P_{\kappa}\lambda$ is Π^1_n -indescribable if and only if for all n-clubs $C \subseteq P_{\kappa}\lambda$ we have $S \cap C \neq \emptyset$.

Proof. The cases n = 0 and n = 1 follow directly from Theorem 4.10.

Suppose $1 \leq n < \omega$. For the forward direction, suppose $S \subseteq P_{\kappa}\lambda$ is Π_{n-1}^1 indescribable and suppose $C \subseteq P_{\kappa}\lambda$ is an *n*-club. Since $C \in \Pi_{n-1}^1(\kappa, \lambda)^+$, it follows that $(V_{\kappa}(\kappa, \lambda), \in, C, P_{\kappa}\lambda) \models \varphi_{n-1} \land \sigma$ where φ_{n-1} is the Π_n^1 sentence from Lemma 5.3 and σ is the Π_1^1 sentence from Lemma 4.2 asserting that κ is inaccessible. Since $S \in \Pi_n^1(\kappa, \lambda)^+$ there is an $x \in S$ such that $x \cap \kappa = \kappa_x$ and

$$(V_{\kappa_x}(\kappa_x, x), \in, C \cap P_{\kappa_x}x, P_{\kappa_x}x) \models \varphi_{n-1} \wedge \sigma.$$

This implies $C \cap P_{\kappa_x} x$ is in $\Pi_{n-1}^1(\kappa_x, x)^+$ and κ_x is inaccessible. Thus $x \in C$ since C is *n*-club.

Conversely, suppose S intersects every n-club subset of $P_{\kappa}\lambda$. Let $R \subseteq V_{\kappa}(\kappa,\lambda)$ and let $\varphi = \forall X\psi(X)$ be a Π_n^1 sentence where $\psi(X)$ is a Σ_{n-1}^1 formula such that $(V_{\kappa}(\kappa,\lambda), \in, R) \models \varphi$. It suffices to show that

$$D = \{ x \in P_{\kappa}\lambda : (V_{\kappa_x}(\kappa_x, x), \in, R \cap V_{\kappa_x}(\kappa_x, x)) \models \varphi \}$$

is n-club.

First let us show that $D \in \Pi_{n-1}^1(\kappa, \lambda)^+$. Suppose not, then

$$E = P_{\kappa}\lambda \setminus D = \{x \in P_{\kappa}\lambda : (V_{\kappa_x}(\kappa_x, x), \in, R \cap V_{\kappa_x}(\kappa_x, x)) \models \exists X \neg \psi(X)\}$$

is in $\Pi_{n-1}^1(\kappa,\lambda)^*$. By our inductive hypothesis, this implies that E contains an (n-1)-club subset of $P_{\kappa}\lambda$, and since (n-1)-clubs are *n*-clubs we see that E contains an *n*-club. Since $\Pi_n^1(\kappa,\lambda)$ is a proper ideal, E is therefore Π_n^1 -indescribable. Thus, from our assumption that $(V_{\kappa}(\kappa,\lambda), \in, R) \models \forall X\psi(X)$, we may conclude that there is an $x \in E$ such that $(V_{\kappa_x}(\kappa_x, x), \in, R \cap V_{\kappa_x}(\kappa_x, x)) \models \forall X\psi(X)$, a contradiction.

Next, we show that for every $x \in P_{\kappa}\lambda$, if $D \cap P_{\kappa_x}x \in \Pi^1_{n-1}(\kappa_x, x)^+$ then $x \in D$. Suppose $D \cap P_{\kappa_x}x \in \Pi^1_{n-1}(\kappa_x, x)^+$ but $x \notin D$. Then $(V_{\kappa_x}(\kappa_x, x), \in, R \cap V_{\kappa_x}(\kappa_x, x)) \models \neg \psi[A]$ for some $A \subseteq V_{\kappa_x}(\kappa_x, x)$. Since $\neg \psi[A]$ is a Π^1_{n-1} sentence, it follows that there is some $y \in D \cap P_{\kappa_x}x$ such that

$$(V_{\kappa_y}(\kappa_y, y), \in, R \cap V_{\kappa_y}(\kappa_y, y), A \cap V_{\kappa_y}(\kappa_y, y)) \models \neg \psi[A],$$

a contradiction.

From Theorem 1.2 we can easily show that n-clubs are measure one with respect to any supercompactness ultrafilter.

Corollary 5.4. Suppose $\kappa \leq \lambda$ are cardinals with $\lambda^{<\kappa} = \lambda$ and κ is λ -supercompact. If U is a normal fine κ -complete nonprincipal ultrafilter on $P_{\kappa}\lambda$ then for all $n < \omega$ if $C \subseteq P_{\kappa}\lambda$ is n-club then $C \in U$.

Proof. Let $j : V \to N$ be the ultrapower by U. So the critical point of j is κ , $j(\kappa) > \lambda$ and $j"\lambda \in N$. It is easy to see that every 0-club is in U.

For n > 0, suppose C is n-club in $P_{\kappa}\lambda$. Then by elementarity, in N, j(C) is nclub in $j(P_{\kappa}\lambda)$. It will suffice to show that the set $j(C) \cap P_{j(\kappa)_{j'\lambda}} j''\lambda = j(C) \cap P_{\kappa}j''\lambda$ is a Π_{n-1}^1 -indescribable subset of $P_{\kappa}j''\lambda$ in N, because this implies $j''\lambda \in j(C)$. By Theorem 1.2, it will suffice to show that, in N, $j(C) \cap P_{\kappa}j''\lambda$ intersects every (n-1)club subset of $P_{\kappa}j''\lambda$. In N, fix an (n-1)-club $D \subseteq P_{\kappa}j''\lambda$. Since $j \upharpoonright P_{\kappa}\lambda \to P_{\kappa}j''\lambda$ is a bijection and j is a supercompactness ultrapower, it follows that $j^{-1}[D]$ is an (n-1)-club subset of $P_{\kappa}\lambda$. Since C is an n-club subset of $P_{\kappa}\lambda$ in V, it follows that C is Π_{n-1}^1 -indescribable, and thus there is some $x \in C \cap j^{-1}[D]$. Then we have $j(x) = j''x \in j(C) \cap D \cap P_{\kappa}j''\lambda$. Thus, in N, $j(C) \cap P_{\kappa}j''\lambda$ is Π_{n-1}^1 -indescribable and so $j''\lambda \in j(C)$.

Corollary 5.5. Suppose that κ is λ -supercompact where $\kappa \leq \lambda$ and $\lambda^{<\kappa} = \lambda$. If U is a normal fine κ -complete nonprincipal ultrafilter on $P_{\kappa}\lambda$ then for all $n < \omega$ we have

$$\Pi_n^1(\kappa,\lambda)^* \subseteq U \subseteq \Pi_n^1(\kappa,\lambda)^+.$$

Next, let us establish Theorem 1.4, which we restate here for the reader's convenience.

Theorem 1.4. Suppose $\kappa \leq \lambda$ are cardinals such that κ is inaccessible and $\lambda^{<\kappa} = \lambda$. Then a set $W \subseteq P_{\kappa}\lambda$ is Π_1^1 -indescribable if and only if it is weakly compact.

To do this we will use the filter characterization of weakly compact subsets of $P_{\kappa}\lambda$ due to Schanker [Sch13] given above in Lemma 3.1(3), which strongly resembles a filter characterization of Π_1^1 -indescribable sets due to Carr [Car85].

Definition 5.6 ([Car85]). The normal ultrafilter property for $X \in NS^+_{\kappa,\lambda}$, written $NUP_{\kappa,\lambda,X}$ states that for any κ -complete field B of subsets of $P_{\kappa}\lambda$ such that $|B| = \lambda$, $X \in B$ and $(\forall \alpha < \lambda)(\widetilde{\{\alpha\}} =_{def} \{x \in P_{\kappa}\lambda : \alpha \in x\} \in B)$, and for any collection $G = \{g_{\alpha} : \alpha < \lambda\}$ of regressive functions on $P_{\kappa}\lambda$ such that $(\forall \alpha < \lambda)(\forall \beta < \lambda)(g^{-1}_{\alpha}(\{\beta\}) \in B)$, there is a κ -complete ultrafilter U in B such that $X \in U$, $(\forall \alpha < \lambda)(\widetilde{\alpha} \in U)$ and every function in G is constant on a set in U.

For the reader's convenience, let us recall that Carr showed that, under reasonable assumptions, $\text{NUP}_{\kappa,\lambda,X}$ is equivalent to X being Π_1^1 -indescribable by using the following generalization of a characterization of weakly compact cardinals due to Shelah [She79].

Definition 5.7 ([Car85]). We say that $X \subseteq P_{\kappa}\lambda$ has the λ -Shelah property if and only if for every sequence of functions $\langle f_x : x \in X \rangle \in \prod \{x_2 : x \in X\}$

$$(\exists f: \lambda \to \lambda) (\forall x \in P_{\kappa}\lambda) (\exists y \in X \cap \widetilde{x}) (f_y \upharpoonright x = f \upharpoonright x)$$

where $\widetilde{x} = \{z \in P_{\kappa}\lambda : x \subseteq y\}.$

Theorem 5.8 (Theorem 3.5 and Theorem 4.7 in [Car85]). Suppose $\kappa \leq \lambda$ are cardinals, κ is inaccessible and $\lambda^{<\kappa} = \lambda$. For every set $X \subseteq P_{\kappa}\lambda$ we have

X is Π_1^1 -indescribable \iff X is λ -Shelah \iff NUP_{κ,λ,X}.

Proof of Theorem 1.4. By Theorem 5.8 and Theorem 3.1, it suffices to show that Carr's filter property $\operatorname{NUP}_{\kappa,\lambda,W}$ is equivalent to Schanker's filter property Theorem 3.1(3), but it is easy to see that Schanker's filter property is a slight reformulation of Carr's filter property. Notice that if \mathcal{A} is a collection of subsets of $P_{\kappa}\lambda$ is in Theorem 3.1(3), then by the inaccessibility of κ , there is a κ -complete collection B of subsets of $P_{\kappa}\lambda$. Also notice that in Carr's statement of $\operatorname{NUP}_{\kappa,\lambda,W}$, the assertion that U is a κ -complete ultrafilter in U means the same thing as Schanker's statement that U is a κ -complete filter measuring all sets in B.

Corollary 5.9. Suppose $\kappa \leq \lambda$ are cardinals and $\lambda^{<\kappa} = \lambda$. If $P_{\kappa}\lambda$ is weakly compact then the following hold.

$$\begin{aligned} \operatorname{NWC}_{\kappa,\lambda} &= \Pi_1^1(\kappa,\lambda) = \{ Z \subseteq P_\kappa \lambda : Z \cap C = \varnothing \text{ for some } 1\text{-club } C \subseteq P_\kappa \lambda \} \\ \operatorname{NWC}_{\kappa,\lambda}^+ &= \Pi_1^1(\kappa,\lambda)^+ = \{ W \subseteq P_\kappa \lambda : W \cap C \neq \varnothing \text{ for every } 1\text{-club } C \subseteq P_\kappa \lambda \} \\ \operatorname{NWC}_{\kappa,\lambda}^* &= \Pi_1^1(\kappa,\lambda)^* = \{ C \subseteq P_\kappa \lambda : C \text{ contains } a \text{ } 1\text{-club} \} \end{aligned}$$

6. Applications

6.1. On a question of Cox-Lücke. The following question was posed in Cox-Lücke. See Section 1 for relevant background and definitions.

Question 6.1. [CL17, Question 7.4] Assume $(\kappa^+)^{<\kappa} = \kappa^+$. Let κ be an inaccessible cardinal such that there is a normal filter \mathcal{F} on $P_{\kappa}\kappa^+$ with the property that every partial order of cardinality κ^+ that satisfies the κ -chain condition is \mathcal{F} -layered. Must κ be measurable?

The answer is no. The proof of the following lemma is very similar to that of [CL17, Lemma 4.3].

Lemma 6.2. Suppose $P_{\kappa}\lambda$ is weakly compact. Then every partial order of cardinality at most λ that satisfies the κ -chain condition is NWC^{*}_{κ,λ}-layered.

Proof. Fix a surjection $s : \lambda \to \mathbb{P}$. We must show that $X = \{x \in P_{\kappa}\lambda : s[x] \in \operatorname{Reg}_{\kappa}(\mathbb{P})\} \in \operatorname{NWC}_{\kappa,\lambda}^{*}$. Let $M \models \operatorname{ZFC}^{-}$ be transitive of size $\lambda^{<\kappa}$ with $\lambda, X, \mathbb{P}, s, \operatorname{Reg}_{\kappa}(\mathbb{P}), \ldots \in M$ and let $j : M \to N$ be an elementary embedding with critical point κ such that $j(\kappa) > \lambda$ and $j'' \lambda \in N$.

Just as in Cox-Lücke, $j[\mathbb{P}]$ is a suborder of $j(\mathbb{P})$ and $j \upharpoonright \mathbb{P} : \mathbb{P} \to j[\mathbb{P}]$ is an isomorphism of partial orders. If A is a maximal antichain of $j[\mathbb{P}]$ then $j^{-1}[A]$ is a maximal antichain of \mathbb{P} and hence $|A| < \kappa$. Since $\operatorname{crit}(j) = \kappa$, it follows by elementarity that $A = j[j^{-1}[A]] = j(j^{-1}[A])$ is a maximal antichain of $j(\mathbb{P})$. Hence, in $N, j[\mathbb{P}]$ is a regular suborder of $j(\mathbb{P})$. Since $j(s)[j^*\lambda] = j[s[\lambda]] = j[\mathbb{P}] \in \operatorname{Reg}_{j(\kappa)}(j(\mathbb{P}))^N = j(\operatorname{Reg}_{\kappa}(\mathbb{P}))$ we conclude that $j^*\lambda \in j(X)$. Thus $X \in \operatorname{NWC}^*_{\kappa,\lambda}$.

Recall that, as a matter of terminology, $P_{\kappa}\lambda$ is weakly compact if and only if κ is nearly λ -supercompact. Schanker proved that if the near κ^+ -supercompactness of κ is indestructible by the forcing to add κ^+ Cohen subsets of κ , then the near κ^+ -supercompactness of κ is indestructible by the forcing to add any number of Cohen subsets of κ . This allowed Schanker to then show [Sch13, Theorem 4.10 (2)] that if κ is nearly κ^+ -supercompact and $2^{\kappa} = \kappa^+$ then there is a forcing extension V[G] in which κ is nearly κ^+ -supercompact and the GCH fails first at κ , hence κ is not measurable. Translating Schanker's results into our terminology we obtain the following.

Proposition 6.3. Suppose $P_{\kappa}\kappa^+$ is weakly compact and GCH holds. There is a cofinality-preserving forcing \mathbb{P} such that if $G \subseteq \mathbb{P}$ is generic then in V[G] the following hold.

- (1) GCH fails first at κ , hence $(\kappa^+)^{<\kappa} = \kappa^+$ and κ is not measurable.
- (2) $(P_{\kappa}\kappa^{+})^{V[G]}$ is weakly compact, hence $\mathcal{F} =_{def} (NWC^{*}_{\kappa,\kappa^{+}})^{V[G]}$ is a nontrivial normal ideal and by Lemma 6.2, every partial order of cardinality at most κ^{+} that satisfies the κ -chain condition is \mathcal{F} -layered.

This answers Question 6.1 and establishes Theorem 1.5 in the case that $\lambda = \kappa^+$. The case in which $\lambda \geq \kappa^{++}$ requires more work: the usual reflection arguments [Sch13, Theorem 4.10 (3)] show if κ is nearly κ^{++} -supercompact then GCH cannot fail first at κ . However, by carrying out a delicate argument using the *lottery* preparation [Ham00] and the fact that in forcing extensions $V \subseteq V[G]$ satisfying the δ -approximation and cover properties for some $\delta < \kappa$, definable elementary embeddings $h: V[G] \to N$ with critical point κ must lift ground model embeddings [Ham03, Corollary 8], Schanker proved the following.

Theorem 6.4 (Schanker [Sch13]). If κ is nearly λ -supercompact for some $\lambda \geq 2^{\kappa}$ such that $\lambda^{<\lambda} = \lambda$, then there exists a forcing extension preserving all cardinals and cofinalities above κ where κ is nearly λ -supercompact but not measurable. Furthermore, in this extension $2^{\kappa} = \lambda^+$, and if the SCH hold below κ in the ground model, then no cardinals or cofinalities were collapsed.

Again, by translating the previous theorem of Schanker's to our terminology, and applying Lemma 6.2 we obtain Theorem 1.5.

6.2. **Two-cardinal indescribability and generic embeddings.** Let us recall the following standard fact concerning generic ultrapowers (see [For10] for more information).

Lemma 6.5 (Folklore). Suppose κ is regular, $\kappa \leq \lambda$ with $\lambda^{<\kappa} = \lambda$, and I is a κ -complete normal fine ideal on $P_{\kappa}\lambda$. If $G \subseteq P(P_{\kappa}\lambda)/I$ is generic and $j: V \to M = V^{P_{\kappa}\lambda}/G \subseteq V[G]$ is the corresponding generic ultrapower then the following conditions hold.

- (1) G extends the filter I^* dual to I.
- (2) $\operatorname{crit}(j) = \kappa \text{ and } j(\kappa) > \lambda.$
- (3) $[\mathrm{id}]_G = j^* \lambda \in M$ and thus for all $X \in P(P_{\kappa}\lambda)^V$ we have $X \in G$ if and only if $j^* \lambda \in M$ j(X).
- (4) For every function $f: P_{\kappa}\lambda \to V$ in V we have $j(f)(j"\lambda) = [f]_G$.
- (5) M is wellfounded up to $(\lambda^+)^V$.

The previous lemma easily yields the following standard result.

Proposition 6.6 (Folklore). Suppose κ is regular, $\kappa \leq \lambda$ and $\lambda^{<\kappa} = \lambda$. A set $S \subseteq P_{\kappa}\lambda$ is stationary if and only if there is a generic elementary embedding $j : V \to M \subseteq V[G]$ with critical point κ such that $j(\kappa) > \lambda$ and $j" \lambda \in j(S) \cap M$.

Proof. Suppose S is stationary and let $G \subseteq P(\kappa)/(\mathrm{NS}_{\kappa,\lambda} \upharpoonright S)$ be generic. Since $\mathrm{NS}_{\kappa,\lambda} \upharpoonright S$ is a κ -complete normal ideal on $P_{\kappa}\lambda$ we have $\operatorname{crit}(j) = \kappa$ and $[\operatorname{id}]_G = j^{"}\lambda$. Hence $j^{"}\lambda \in M$. Since $\lambda = j(f)(j^{"}\lambda) = [f]_G$ where $f(x) = \operatorname{ot}(x)$ and $j(\kappa) = [c_{\kappa}]_G$ we have $\lambda < j(\kappa)$. Clearly $S \in (\mathrm{NS}_{\kappa,\lambda} \upharpoonright S)^* \subseteq G$ and hence $\kappa \in j(S)$.

Conversely, suppose that $j: V \to M \subseteq V[G]$ is a generic elementary embedding with critical point κ such that $j(\kappa) > \lambda$ and $j"\lambda \in j(S) \cap M$, where j is obtained by forcing with some poset \mathbb{P} . Fix a club $C \subseteq P_{\kappa}\lambda$ in V. Then

$$I = \{ X \in P(P_{\kappa}\lambda) : \Vdash_{\mathbb{P}} j"\lambda \notin j(X) \}$$

is a normal ideal on $P_{\kappa}\lambda$ in V. This implies that $C \in I^*$, in other words, $\Vdash_{\mathbb{P}} j^{"}\lambda \in j(C)$. Thus $M \models j(S) \cap j(C) \neq \emptyset$ and by elementarity $S \cap C \neq \emptyset$. \Box

The following lemma is a straightforward generalization of standard fact about Π_n^m -indescribable filter on a cardinal κ (taking n = m = 1 in the next lemma yields a result proven above; see the proof of Theorem 1.2 in Section 5).

Lemma 6.7. Suppose $\kappa \leq \lambda$ are cardinals and $P_{\kappa}\lambda$ is Π_n^m -indescribable. Further suppose that $R_1, \ldots, R_k \subseteq V_{\kappa}(\kappa, \lambda)$ where $k < \omega$ and φ is a Π_n^m sentence. If $(V_{\kappa}(\kappa, \lambda), \in, R_1, \ldots, R_k) \models \varphi$ then the set

$$\{x \in P_{\kappa}\lambda : (V_{\kappa_x}(\kappa_x, x), \in, R_1 \cap V_{\kappa_x}(\kappa_x, x), \dots, R_k \cap V_{\kappa_x}(\kappa_x, x)) \models \varphi\}$$

is in the filter $\Pi_n^m(\kappa,\lambda)^*$.

Next, we will provide a characterization of Π_n^m -indescribable subsets of $P_{\kappa}\lambda$ in terms of generic embeddings.

Proposition 6.8. Suppose $n, m < \omega$, κ is regular, $\lambda \ge \kappa$ is a cardinal with $\lambda^{<\kappa} = \lambda$ and $S \subseteq P_{\kappa}\lambda$. The following are equivalent.

- (1) S is Π_n^m -indescribable in $P_{\kappa}\lambda$
- (2) There is a generic embedding $j: V \to M \subseteq V[G]$ with critical point κ such that $\operatorname{crit}(j) = \kappa, \ j(\kappa) > \lambda, \ j"\lambda \in j(S) \cap M$ and for every Π_n^m -sentence φ

over $(V_{\kappa}(\kappa,\lambda), \in, R_1, \ldots, R_k)$ where $R_1, \ldots, R_k \in P(V_{\kappa}(\kappa,\lambda))^V$ it follows that

$$((V_{\kappa}(\kappa,\lambda),\in,R_1,\ldots,R_k)\models\varphi)^V$$

implies

$$((V_{\kappa}(\kappa, j^{*}\lambda), \in, j(R_1) \cap V_{\kappa}(\kappa, j^{*}\lambda), \dots, j(R_k) \cap V_{\kappa}(\kappa, j^{*}\lambda)) \models \varphi)^M.$$

Proof. To see that (1) implies (2), suppose $S \subseteq P_{\kappa}\lambda$ is Π_n^m -indescribable. Let G be generic for the poset $P(P_{\kappa}\lambda)/(\Pi_n^m(\kappa,\lambda) \upharpoonright S) - \{[\varnothing]\}$. Since G extends the filter $(\Pi_n^m(\kappa,\lambda) \upharpoonright S)^*$ and $S \in (\Pi_n^m(\kappa,\lambda) \upharpoonright S)^*$, it follows that $S \in G$. Thus, by Lemma 6.5, if we let $j: V \to M = V^{P_{\kappa}\lambda}/G \subseteq V[G]$ be the generic ultrapower obtained from G, then $j^n \lambda \in j(S) \cap M$. Furthermore, if φ and R_1, \ldots, R_k are as in the statement of the proposition and

$$((V_{\kappa}(\kappa,\lambda),\in,R_1,\ldots,R_k)\models\varphi)^V,$$

then it follows by Lemma 6.7 and Lemma 6.5 that

$$((V_{\kappa}(\kappa, j"\lambda), \in, j(R_1) \cap V_{\kappa}(\kappa, j"\lambda), \dots, j(R_k) \cap V_{\kappa}(\kappa, j"\lambda)) \models \varphi)^M.$$

Conversely, if (2) holds then it follows by elementarity that (1) holds.

6.3. Two-cardinal weakly compact diamond. First, as an application of the 1-club characterization of weak compactness in Corollary 5.9, we will show that if κ is large enough then for every $\lambda \geq \kappa$ with $\lambda^{<\kappa} = \lambda$ and every weakly compact $W \subseteq P_{\kappa}\lambda, \diamondsuit_{\kappa,\lambda}^{\mathrm{wc}}(W)$ holds (see Definition 1.6).

Remark 6.9. In what follows we will identify subsets $X \subseteq \lambda$ with functions X: $ot(X) \to \lambda$ enumerating the elements of X in increasing order; in other words, $X(\alpha)$ denotes the α -th element of X where $\alpha < ot(X)$.

Proposition 6.10. Suppose κ is supercompact and $\lambda \geq \kappa$ is a cardinal with $\lambda^{<\kappa} = \lambda$. If $W \subseteq P_{\kappa}\lambda$ weakly compact then $\diamondsuit_{\kappa,\lambda}^{wc}(W)$ holds.

Proof. Suppose κ is supercompact and let $\ell : \kappa \to V_{\kappa}$ be a Laver function [Lav78], that is, for any λ and any $x \in H_{\lambda^+}$, there is a λ -supercompactness embedding $j: V \to M$ with critical point κ , $j(\kappa) > \lambda$, $j"\lambda \in M$ and $j(\ell)(\kappa) = x$.

Fix $\lambda \geq \kappa$ with $\lambda^{<\kappa} = \lambda$ and fix a weakly compact set $W \subseteq P_{\kappa}\lambda$. For each $z \in W$ with $z \cap \kappa \in \kappa$ and $\ell(z \cap \kappa) \subseteq$ ORD define $a_z = \{z(\beta) : \beta < \operatorname{ot}(z) \land \ell(z \cap \kappa)(\beta) = 1\}$ where $\ell(z \cap \kappa)(\beta)$ is the β -th element of $\ell(z \cap \kappa)$. Otherwise let a_z be arbitrary. Let us argue that the set $E_X = \{z \in P_{\kappa}\lambda : X \cap z = a_z\}$ is weakly compact.

Fix a 1-club $C \subseteq P_{\kappa}\lambda$. By Corollary 5.9, it will suffice to show that $j^n\lambda \in j(C) \cap j(E_X)$ where j is a λ -supercompactness embedding.⁵ Take $j: V \to M$ to be a λ -supercompactness embedding with $j(\ell)(\kappa) = E_X$. We certainly have $j^n\lambda \in j(C)$. Let $j(\langle a_z: z \in P_{\kappa}\lambda \rangle) = \langle \bar{a}_z: z \in j(P_{\kappa}\lambda) \rangle$. Since $\bar{a}_{j^n\lambda} = \{j^n\lambda(\beta): \beta < \lambda \land j(\ell)(\kappa)(\beta) = 1\} = j^n X = j(X) \cap j^n\lambda$, it follows that $j^n\lambda \in j(E_X)$.

When arguing that ultrapower embeddings $j: V \to M$ by normal fine κ -complete measures on $P_{\kappa}\lambda$ can be extended to forcing extensions by an Easton-support iteration \mathbb{P} , one often uses a function $f: \kappa \to \kappa$ satisfying $j(f)(\kappa) > \lambda$ to ensure that the tail of $j(\mathbb{P})$ will be sufficiently closed. The same is true when lifting elementary embeddings witnessing the weak compactness of subsets of $P_{\kappa}\lambda$.

⁵Let us emphasize that, in this context $j"\lambda \in j(E_X)$ does not directly imply E_X is weakly compact because j is a supercompactness embedding, but the fact that E_X is weakly compact follows from Corollary 5.9 or Corollary 5.5.

Definition 6.11. Suppose $P_{\kappa}\lambda$ is weakly compact. We say that a function $f: \kappa \to \kappa$ has the *Menas property for weakly compact subsets of* $P_{\kappa}\lambda$ if and only if for every weakly compact $W \subseteq P_{\kappa}\lambda$ and every $A \subseteq \lambda$ there is a transitive $M \models \text{ZFC}^-$ of size $\lambda^{<\kappa}$ with $\lambda, A, W, f \in M$, a transitive N and an elementary embedding $j: M \to N$ with critical point κ such that $j(\kappa) > \lambda$, $j^*\lambda \in j(W)$ and $j(f)(\kappa) > \lambda$.

The proof of the following lemma is essentially the same as that of [Sch13, Lemma 3.3]

Lemma 6.12. Suppose $P_{\kappa}\lambda$ is weakly compact. Then there is a function $f: \kappa \to \kappa$ with the Menas property for weakly compact subsets of $P_{\kappa}\lambda$.

Theorem 6.13. Suppose $W \subseteq P_{\kappa}\lambda$ is weakly compact and GCH holds. There is a cofinality-preserving forcing extension V[G] in which W is a weakly compact subset of $(P_{\kappa}\lambda)^{V[G]}$ and $\diamondsuit_{\kappa,\lambda}^{wc}(W)$ holds.

Proof. Let $f: \kappa \to \kappa$ be a function with the Menas property for weakly compact subsets of $P_{\kappa}\lambda$. Let $\mathbb{P}_{\kappa+1} = \langle \mathbb{P}_{\alpha}, \dot{\mathbb{Q}}_{\beta} : \alpha \leq \kappa + 1, \beta \leq \kappa \rangle$ be the Easton-support iteration such that if $\gamma \leq \kappa$ is inaccessible and $f"\gamma \subseteq \gamma$ then $\dot{\mathbb{Q}}_{\gamma}$ is a \mathbb{P}_{γ} -name for the forcing to add a single Cohen subset to γ , and otherwise $\dot{\mathbb{Q}}_{\gamma}$ is a \mathbb{P}_{γ} -name for trivial forcing. Let $G_{\kappa+1} \cong G_{\kappa} * H_{\kappa} \subseteq \mathbb{P}_{\kappa} * \dot{\mathbb{Q}}_{\kappa}$ be generic over V and let $h_{\kappa} = \bigcup H_{\kappa} : \kappa \to 2$.

We will identify each $z \in P_{\kappa}\lambda$ with a function $z : \operatorname{ot}(z) \to \lambda$ enumerating its elements in increasing order; in other words, $z(\alpha)$ denotes the α -th element of zwhere $\alpha < \operatorname{ot}(z)$. For each $z \in P_{\kappa}\lambda$ with $z \cap \kappa \in \kappa$ we define $a_z = \{z(\beta) : \beta < \operatorname{ot}(z) \land h_{\kappa}(z \cap \kappa + \beta) = 1\}$ and let $\vec{a} = \langle a_z : z \in P_{\kappa}\lambda \rangle$. Standard arguments show that $\mathbb{P}_{\kappa+1}$ preserves cofinalities under GCH, so it remains to show that, in $V[G_{\kappa+1}]$, W remains weakly compact and that $\langle a_z : z \in P_{\kappa}\lambda \rangle$ is a weakly compact diamond sequence on W.

Fix $X \in P(\lambda)^{V[G_{\kappa+1}]}$. It will suffice to show that $E_X(W) = \{z \in W : X \cap z = a_z\} \in \operatorname{NWC}_{\kappa,\lambda}^{V[G_{\kappa+1}]}$. Fix $A \in P(\lambda)^{V[G_{\kappa+1}]}$ and let $\dot{A}, \dot{X}, \dot{E}_X(W), \dot{h}_{\kappa}, \dot{\vec{a}} \in H_{\lambda^+}^V$ be $\mathbb{P}_{\kappa+1}$ -names for the appropriate sets. We assume that \dot{A} and \dot{X} are nice names for subsets of λ . Working in V, let $M \models \operatorname{ZFC}^-$ be transitive of size λ with $\lambda, \dot{A}, \dot{X}, \dot{E}_X(W), \dot{h}_{\kappa}, \dot{\vec{a}}, \mathbb{P}_{\kappa+1}, \ldots \in M$. Since W is weakly compact in V there is a $j: M \to N$ such that $\operatorname{crit}(j) = \kappa, j(\kappa) > \lambda, j(f)(\kappa) > \lambda$ and $j"\lambda \in j(W)$, as in Lemma 3.2.

We now show that standard arguments allow us to lift j to have domain $M[G_{\kappa}]$. Since $N^{<\lambda} \cap V \subseteq N$ we have $j(\mathbb{P}_{\kappa}) \cong \mathbb{P}_{\kappa} * \dot{\mathbb{Q}}_{\kappa} * \dot{\mathbb{P}}'_{\kappa,j(\kappa)}$ where $\dot{\mathbb{P}}'_{\kappa,j(\kappa)}$ is a $\mathbb{P}_{\kappa} * \dot{\mathbb{Q}}_{\kappa}$ name for the tail of the iteration $j(\mathbb{P}_{\kappa})$ as defined in N. Since $j(f)(\kappa) > \lambda$, it follows that the next stage of nontrivial forcing in $j(\mathbb{P}_{\kappa})$ after κ occurs beyond λ . Thus, it follows that in $N[G_{\kappa} * H_{\kappa}]$, the forcing $\mathbb{P}'_{\kappa,j(\kappa)} =_{\text{def}} (\dot{\mathbb{P}}'_{\kappa,j(\kappa)})_{G_{\kappa} * H_{\kappa}}$ is $<\lambda$ closed. Since $|N[G_{\kappa} * H_{\kappa}]|^{V[G_{\kappa} * H_{\kappa}]} \leq \lambda$, the poset $\mathbb{P}_{\kappa,j(\kappa)}$ has at most λ -dense sets in $N[G_{\kappa} * H_{\kappa}]$. The model $N[G_{\kappa} * H_{\kappa}]$ is closed under $<\lambda$ -sequences in $V[G_{\kappa} * H_{\kappa}]$ and thus, working in $V[G_{\kappa} * H_{\kappa}]$ we may build a filter $G'_{\kappa,j(\kappa)} \in V[G_{\kappa} * H_{\kappa}]$ which is generic for $\mathbb{P}'_{\kappa,j(\kappa)}$ over $N[G_{\kappa} * H_{\kappa}]$. Since the critical point of j is κ it follows that $j[G_{\kappa}] \subseteq G_{\kappa} * H_{\kappa} * G'_{\kappa,j(\kappa)}$ and thus j lifts to $j : M[G_{\kappa}] \to N[\hat{G}_{j(\kappa)}]$ where $\hat{G}_{j(\kappa)} = G_{\kappa} * H_{\kappa} * G'_{j(\kappa)}$.

We define $m : \lambda \to 2$ by letting $m \upharpoonright \kappa = h_{\kappa}$ and $m(\kappa + \alpha) = X(\alpha)$ for $\kappa + \alpha < \lambda$, where we are identifying X with it's characteristic function; that is, $X(\xi) = 1$ if and only if $\xi \in X$. Since h_{κ} is clearly in $N[\hat{G}_{j(\kappa)}]$, to check that $m \in j(\mathbb{Q}_{\kappa})$ it will suffice to show that $X \in N[\hat{G}_{j(\kappa)}]$. Since \dot{X} is a nice $\mathbb{P}_{\kappa+1}$ -name for a subset of λ , it follows that $\dot{X} = \bigcup_{\alpha < \lambda} \{\alpha\} \times A_{\alpha}$ where A_{α} is an antichain of $P_{\kappa+1}$ for each $\alpha < \lambda$. Since $j"\lambda \in N$ we have $j"\dot{X} = \bigcup_{\alpha < \lambda} \{j(\alpha)\} \times j"A_{\alpha} \in$, and since $j \upharpoonright \mathbb{P}_{\kappa+1} \in M$, it follows that $\dot{X} \in M$. Hence we have $X = \dot{X}_{G_{\kappa+1}} \in N[\hat{G}_{j(\kappa)}]$.

Since *m* is a condition in $j(\mathbb{Q}_{\kappa})$ we may build a filter $\hat{H}_{j(\kappa)} \subseteq j(\mathbb{Q}_{\kappa})$ with $m \in \hat{H}_{j(\kappa)}$ which is generic over $N[\hat{G}_{j(\kappa)}]$. Since $j^{"}H_{\kappa} \subseteq \hat{H}_{j(\kappa)}$, we may lift *j* to *j* : $M[G_{\kappa} * H_{\kappa}] \to N[\hat{G}_{j(\kappa)} * \hat{H}_{j(\kappa)}]$.

Since $\vec{a} \in M[G_{\kappa} * H_{\kappa}]$ we may let $j(\vec{a}) = \langle \bar{a}_z : z \in j(P_{\kappa}\lambda) \rangle$. Since $m \in \hat{H}_{j(\kappa)}$, it follows that $j(h_{\kappa})(\kappa + \beta) = X(\beta)$ for all $\beta < \lambda$. By definition $a_z = \{z(\beta) : \beta < ot(z) \land h_{\kappa}(z \cap \kappa + \beta) = 1\}$, thus by elementarity

$$j(\vec{a})(j^{"}\lambda) = \bar{a}_{j^{"}\lambda} = \{(j^{"}\lambda)(\beta) : \beta < \operatorname{ot}(j^{"}\lambda) \land j(h_{\kappa})(j^{"}\lambda \cap j(\kappa) + \beta) = 1\}$$
$$= \{j(\beta) : \beta < \lambda \land j(h_{\kappa})(\kappa + \beta) = 1\}$$
$$= \{j(\beta) : \beta < \lambda \land X(\beta) = 1\}$$
$$= j(X) \cap j^{"}\lambda.$$

Thus $j'' \lambda \in j(E_X(W))$.

Standard arguments can be used to prove the following.

Proposition 6.14. $\diamondsuit_{\kappa,\lambda}^{wc}(W)$ implies that NWC_{κ,λ} $\upharpoonright W$ is not λ -saturated.

6.4. Indescribable semimorasses. For cardinals $\kappa \leq \lambda$, a (κ, λ) -semimorass is a subset $\mu \subseteq P_{\kappa}\lambda$ which is well-founded with respect to \subsetneq and satisfies certain homogeneity properties (see Definition 6.15 below). The reader may consult [Kos95] or [Kos17] for some additional information and applications of semimorasses. If $\mu \subseteq P_{\kappa}\lambda$ and $x \in P_{\kappa}\lambda$, we define

$$\mu \upharpoonright x = \{ y \in \mu : y \subsetneq x \}.$$

Definition 6.15. Let κ and λ be cardinals with $\kappa \leq \lambda$. A (κ, λ) -semimorass is a family $\mu \subseteq P_{\kappa}\lambda$ which satisfies the following properties.

- (1) μ is well-founded with respect to \subsetneq .
- (2) μ is *locally small*, that is, for all $x \in \mu$, $|\mu \upharpoonright x| < \kappa$.
- (3) μ is homogeneous, that is, if $x, y \in \mu$ and $\operatorname{rk}(x) = \operatorname{rk}(y)$ then x and y have the same order type and $\mu \upharpoonright y = \{f_{x,y}[z] : z \in \mu \upharpoonright x\}$, where $f_{x,y} : x \to y$ is the unique order-preserving isomorphism from x to y.
- (4) μ is directed with respect to \subseteq , that is, for all $x, y \in \mu$ there is $z \in \mu$ such that $x, y \subseteq z$.
- (5) μ is *locally semi-directed*, that is, for all $x \in \mu$ either
 - (a) $\mu \upharpoonright x$ is directed, or
 - (b) there are $x_1, x_2 \in \mu$ such that $\operatorname{rk}(x_1) = \operatorname{rk}(x_2)$ and $x = x_1 * x_2$, that is, x is the amalgamation of x_1 and x_2 with respect to μ .
- (6) μ covers λ , that is, $\bigcup \mu = \lambda$.
- (7) μ has height κ .

Koszmider [Kos95] proved that if $\kappa^{<\kappa} = \kappa$ then there is a $<\kappa$ -directed closed κ^+ -c.c. poset $\mathbb{K}_{\kappa,\kappa^+}$ such that if $g \subseteq \mathbb{K}_{\kappa,\kappa^+}$ is a generic filter over V then $\mu = \bigcup g$ is a stationary (κ, κ^+) -semimorass. Pereira [Per17] used an Easton-support iteration of Koszmider forcings to prove that if κ is κ^+ -supercompact and GCH holds then

there is a cofinality-preserving forcing extension V[G] in which there is a normal fine κ -complete nonprincipal ultrafilter $U \in V[G]$ on $(P_{\kappa}\kappa^+)^{V[G]}$ which contains a (κ, κ^+) -semimorass $\mu \in U$. It is an easy consequence of Corollary 5.5 and Pereira's result that if κ is κ^+ -supercompact and GCH holds then there is a cofinalitypreserving forcing extension in which there is a (κ, κ^+) -semimorass $\mu \subseteq P_{\kappa}\kappa^+$ which is \prod_n^1 -indescribable for all $n < \omega$.

Corollary 6.16. If κ is κ^+ -supercompact and GCH holds then there is a cofinalitypreserving forcing extension V[G] in which there is a (κ, κ^+) -semimorass $\mu \subseteq P_{\kappa}\lambda$ which is \prod_n^1 -indescribable for all $n < \omega$.

7. QUESTIONS

7.1. Shooting 1-clubs. In many ways 1-club shooting forcings, and more generally n-club shooting forcings, are more well-behaved than club shooting. For example, Hellsten proved [Hel03] that if $W \subseteq \kappa$ is any weakly compact set and GCH holds then there is a cofinality-preserving forcing extension in which W contains a 1-club and all weakly compact subsets of W are preserved, whereas the forcing to shoot a club through a given stationary set $S \subseteq \kappa$ may collapse cardinals unless S contains the singular cardinals less than κ . Similarly, in [CGLH] it is shown that if $W \subseteq \kappa$ is any Π_n^1 -indescribable set and GCH holds then there is a cofinality-preserving forcing extension in which W contains an n-club and all Π_n^1 -indescribable subsets of W are preserved. Can these results be generalized to the two-cardinal context?

Question 7.1. Suppose $W \subseteq P_{\kappa}\lambda$ is weakly compact and GCH holds. Is there a cofinality-preserving forcing extension in which W contains a 1-club and all weakly compact subsets of W remains weakly compact?

The work of Gitik [Git85] seems to be relevant to answering Question 7.1, however this remains open. When attempting to answer Question 7.1, the author considered various Easton-support iterations, and in an attempt to build master conditions for such forcings, the author was led to the following related questions (Question 7.2 and Question 7.3 below).

Question 7.2. Is it consistent that there is a weakly compact set $W \subseteq P_{\kappa}\kappa^+$ which does not contain a 1-club and

for all $x \in W$ if $y \subseteq x$ and $|y| \ge \kappa_x$ then $y \in W$?

Koszmider (see [Kos17, Proposition 4.3] or [Kos95, Proposition 10]) has shown that if μ is a (κ, λ) -semimorass (see Definition 6.15 above) then it satisfies the following non-reflection property:

for every proper subset $X \subsetneq \lambda$ with $|X| \ge \kappa$ we have $\mu \cap P_{\kappa}X \in NS_{\kappa,X}$. (K)

Combined with Corollary 6.16, this shows the consistency of the existence of a set $W \subseteq P_{\kappa}\kappa^+$ which is Π_n^1 -indescribable for all $n < \omega$ and which satisfies Koszmider's non-reflection property (K). When attempting to build master conditions for various forcings, the author arrived at the following question concerning additional non-reflection properties of semimorasses.

Question 7.3. Is it consistent that there is a weakly compact set $W \subseteq P_{\kappa}\kappa^+$ such that W is a *special* (κ, κ^+) -semimorass, meaning that it is a (κ, κ^+) -semimorass and it satisfies the non-reflection property: for every proper subset $X \subsetneq \kappa^+$ with $|X| \ge \kappa$ we have $\mu \cap P_{\kappa_X} X \in \mathrm{NS}_{\kappa_X, X}$?

7.2. The $P_{\kappa}\lambda$ -weakly compact reflection principle. Schanker proved that if κ is κ^+ -supercompact cardinal and GCH holds then there is a forcing extension in which κ remains nearly κ^+ -supercompact and the GCH fails first at κ , hence κ is not κ^+ -supercompact or even measurable. Can we obtain a similar forcing result while preserving GCH?

Question 7.4. If κ is κ^+ -supercompact and GCH holds, is there a cofinalitypreserving forcing extension in which κ remains nearly κ^+ -supercompact, GCH holds and κ is not κ^+ -supercompact? Phrased in our preferred terminology: if $P_{\kappa}\kappa^+$ is weakly compact and GCH holds, is there a cofinality-preserving forcing extension V[G] in which $(P_{\kappa}\kappa^+)^{V[G]}$ is weakly compact, GCH holds and κ is not κ^+ -supercompact?

In order to address this question, let us use the following definition.

Definition 7.5. Suppose κ is inaccessible and X is a set of ordinals with $\kappa \leq |X|$ and $|X|^{<\kappa} = |X|$. We say that a set $W \subseteq P_{\kappa}X$ is *weakly compact* if and only if it is Π_1^1 -indescribable as a subset of $P_{\kappa}X$ via Definition 4.1.

Under the assumptions of the previous definition, the collection NWC_{κ,X} = $\Pi^1_1(\kappa, X)$ is a strongly normal ideal on $P_{\kappa}X$ (see Section 4). If κ is κ^+ -supercompact, then it follows that for every weakly compact $W \subseteq P_{\kappa}\kappa^+$ there is an $x \in P_{\kappa}\kappa^+$ such that $W \cap P_{\kappa_x}x \in \text{NWC}^+_{\kappa_x,x}$.

Definition 7.6. For cardinals $\kappa \leq \lambda$, we say that $W \subseteq P_{\kappa}\lambda$ is a non-reflecting weakly compact set if and only if W is weakly compact and for all $x \in P_{\kappa}\lambda$ the set $W \cap P_{\kappa_x}x$ is not a weakly compact subset of $P_{\kappa_x}x$. We say that the $P_{\kappa}\lambda$ -weakly compact reflection principle holds and write $\operatorname{Refl}_{WC}(\kappa, \lambda)$ if and only if every weakly compact $W \subseteq P_{\kappa}\lambda$ reflects at some $x \in P_{\kappa}\lambda$.

Hence, one could answer Question 7.4 in the affirmative by showing that if κ is κ^+ -supercompact then there is a forcing extension V[G] in which there is a non-reflecting weakly compact subset of $(P_{\kappa}\kappa^+)^{V[G]}$ and $(P_{\kappa}\kappa^+)^{V[G]}$ is weakly compact. However, it seems that subtle issues involved with building master conditions prevent one from using the usual forcing techniques.

Question 7.7. Suppose $P_{\kappa}\kappa^+$ is weakly compact and GCH holds. Is there a forcing extension V[G] in which $(P_{\kappa}\kappa^+)^{V[G]}$ is weakly compact, cofinalities are preserved and there is a weakly compact set $W \subseteq (P_{\kappa}\kappa^+)^{V[G]}$ such that for all $x \in (P_{\kappa}\kappa^+)^{V[G]}$ we have $(W \cap P_{\kappa_x} x \notin \text{NWC}^+_{\kappa_x} x)^{V[G]}$?

The proof of the following is a straightforward application of the fact that Π_1^1 -indescribability can be expressed by a Π_2^1 -sentence (see Lemma 5.3).

Proposition 7.8. Suppose $\kappa \leq \lambda$ are cardinals with $\lambda^{<\kappa} = \lambda$ and $P_{\kappa}\lambda$ is Π_2^1 -indescribable. Then Refl_{WC}(κ, λ) holds.

Question 7.9. Is it consistent that $P_{\kappa}\lambda$ is weakly compact where $\kappa \leq \lambda$ and $\lambda^{<\kappa} = \lambda$, Refl_{WC}(κ, λ) holds and $P_{\kappa}\lambda$ is not Π_2^1 -indescribable?

7.3. Alternative 1-clubs.

Question 7.10. One can formulate a notion of 1-club subset of $P_{\kappa\lambda}$ using Jech's $NS_{\kappa,\lambda}$ instead of $NSS_{\kappa,\lambda}$. In other words, we define $C \subseteq P_{\kappa\lambda}$ to be Jech-1-club if and only if $C \in NS^+_{\kappa\lambda}$ and whenever $x \in P_{\kappa\lambda}$ is such that κ_x is inaccessible and

 $C \cap P_{\kappa_x} x \in \mathrm{NS}^+_{\kappa_x,x}$ we have $x \in C$. What is the relationship between 1-club and 1'-club subsets of $P_{\kappa}\lambda$ when κ is Mahlo?

References

- [Abe98] Yoshihiro Abe. Combinatorial characterization of Π_1^1 -indescribability in $P_{\kappa}\lambda$. Arch. Math. Logic, 37(4):261–272, 1998.
- [Bau] J. E. Baumgartner. Generalizing weak compactness to the (κ, λ) -context. Handwritten Notes.
- [Bau77] James E. Baumgartner. Ineffability properties of cardinals. II. In Logic, foundations of mathematics and computability theory (Proc. Fifth Internat. Congr. Logic, Methodology and Philos. of Sci., Univ. Western Ontario, London, Ont., 1975), Part I, pages 87–106. Univ. Western Ontario Ser. Philos. Sci., Vol. 9. Reidel, Dordrecht, 1977.
- [Car82] Donna M. Carr. The minimal normal filter on $P_{\kappa}\lambda$. Proc. Amer. Math. Soc., 86(2):316–320, 1982.
- [Car85] Donna M. Carr. $P_{\kappa}\lambda$ -generalizations of weak compactness. Z. Math. Logik Grundlag. Math., 31(5):393–401, 1985.
- $\label{eq:GHS15} \begin{array}{l} \mbox{Brent Cody, Moti Gitik, Joel David Hamkins, and Jason A. Schanker. The least weakly compact cardinal can be unfoldable, weakly measurable and nearly $$\theta$-supercompact. Arch. Math. Logic, 54(5-6):491-510, 2015. \end{array}$
- [CGLH] Brent Cody, Victoria Gitman, and Chris Lambie-Hanson. A $\Box(\kappa)$ like principle consistent with weak compactness. (*preprint*, available at https://arxiv.org/abs/1902.04146).
- [CL17] Sean Cox and Philipp Lücke. Characterizing large cardinals in terms of layered posets. Ann. Pure Appl. Logic, 168(5):1112–1131, 2017.
- [CLP90] Donna M. Carr, Jean-Pierre Levinski, and Donald H. Pelletier. On the existence of strongly normal ideals over $P_{\kappa}\lambda$. Arch. Math. Logic, 30(1):59–72, 1990.
- [For10] Matthew Foreman. Ideals and generic elementary embeddings. In Handbook of set theory. Vols. 1, 2, 3, pages 885–1147. Springer, Dordrecht, 2010.
- [Git85] Moti Gitik. Nonsplitting subset of $P_{\kappa}(\kappa^+)$. J. Symbolic Logic, 50(4):881–894 (1986), 1985.
- [Ham00] Joel David Hamkins. The lottery preparation. Ann. Pure Appl. Logic, 101(2-3):103– 146, 2000.
- [Ham03] Joel David Hamkins. Extensions with the approximation and cover properties have no new large cardinals. Fund. Math., 180(3):257–277, 2003.
- [Hel03] Alex Hellsten. Diamonds on large cardinals. Ann. Acad. Sci. Fenn. Math. Diss., (134):48, 2003. Dissertation, University of Helsinki, Helsinki, 2003.
- [Jec73] Thomas J. Jech. Some combinatorial problems concerning uncountable cardinals. Ann. Math. Logic, 5:165–198, 1972/73.
- [Joh90] C. A. Johnson. Some partition relations for ideals on $P_{\kappa}\lambda$. Acta Math. Hungar., 56(3-4):269–282, 1990.
- [Kan03] Akihiro Kanamori. The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings. Springer, second edition, 2003.
- [Kos95] Piotr Koszmider. Semimorasses and nonreflection at singular cardinals. Ann. Pure Appl. Logic, 72(1):1–23, 1995.
- [Kos17] Piotr Koszmider. On constructions with 2-cardinals. Arch. Math. Logic, 56(7-8):849–876, 2017.
- [Lav78] Richard Laver. Making the supercompactness of κ indestructible under κ -directed closed forcing. Israel J. Math., 29(4):385–388, 1978.
- [Mat88] Pierre Matet. Un principe combinatoire en relation avec l'ultranormalité des idéaux. C. R. Acad. Sci. Paris Sér. I Math., 307(2):61–62, 1988.
- [MU12] Pierre Matet and Toshimichi Usuba. Two-cardinal versions of weak compactness: partitions of pairs. Ann. Pure Appl. Logic, 163(1):1–22, 2012.
- [MU15] Pierre Matet and Toshimichi Usuba. Two-cardinal versions of weak compactness: partitions of triples. J. Math. Soc. Japan, 67(1):207–230, 2015.
- [Per17] L. Pereira. Morasses, semimorasses and supercompact ultrafilters. Acta Math. Hungar., 152(1):257–268, 2017.

- [Sch13] Jason Aaron Schanker. Partial near supercompactness. Ann. Pure Appl. Logic, 164(2):67–85, 2013.
- [She79] S. Shelah. Weakly compact cardinals: a combinatorial proof. J. Symbolic Logic, 44(4):559–562, 1979.
- [Sun93] Wen Zhi Sun. Stationary cardinals. Arch. Math. Logic, 32(6):429–442, 1993.
- [Usu13] Toshimichi Usuba. Hierarchies of ineffabilities. *MLQ Math. Log. Q.*, 59(3):230–237, 2013.

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