ON THE MOTIVIC CLASS OF AN ALGEBRAIC GROUP

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ABSTRACT. Let F be a field of characteristic zero admitting a biquadratic field extension. We give an example of a torus G over F whose classifying stack BG is stably rational and such that $\{BG\} \neq \{G\}^{-1}$ in the Grothendieck ring of algebraic stacks over F. We also give an example of a finite étale group scheme A over F such that BA is stably rational and $\{BA\} \neq 1$.

1. INTRODUCTION

Let F be a field. The Grothendieck ring of algebraic stacks $K_0(\operatorname{Stacks}_F)$ was introduced by Ekedahl in [8], following up on earlier works [1], [14], [20]. It is a variant of the Grothendieck ring of varieties $K_0(\operatorname{Var}_F)$. By definition, $K_0(\operatorname{Stacks}_F)$ is generated as an abelian group by the equivalence classes $\{X\}$ of all algebraic stacks X of finite type over F with affine stabilizers. These classes are subject to the scissor relations $\{X\} = \{Y\} + \{X \setminus Y\}$ for every closed substack $Y \subseteq X$, and the relations $\{E\} = \{\mathbb{A}^n \times X\}$ for every vector bundle E of rank n over X. The product is defined by $\{X\} \cdot \{Y\} := \{X \times Y\}$, and extended by linearity.

Given a group scheme G over F, we may consider the class $\{BG\}$ of its classifying stack in $K_0(\operatorname{Stacks}_F)$. The problem of computing $\{BG\}$ appears to be related to the problem of the stable rationality of BG, although no direct implications are known. Recall that BG is stably rational if for one (equivalently, every) generically free representation V of G, the rational quotient V/G is stably rational. An equivalent terminology is that the Noether problem for stable rationality has a positive solution for G; see [12, §3]. The case of a finite (constant) group G was considered in [7]: it frequently happens that $\{BG\} = 1$ (notably for the symmetric groups, see [7, Theorem 4.3]), although there are examples of finite groups G for which $\{BG\} \neq 1$; see [7, Corollary 5.2, Corollary 5.8]. Further work on the triviality of $\{BG\}$ for finite groups G has been done in [16] and [17]. So far, all the known examples of finite group schemes G for which $\{BG\} \neq 1$ are such that BG is not stably rational. This suggests the following question.

Question 1.1. (cf. $[7, \S6]$) Is it true that, for a finite group scheme G, the following two conditions are equivalent?

- BG is stably rational;
- $\{BG\} = 1$ in $K_0(\operatorname{Stacks}_F)$.

We will answer Question 1.1 in the negative in Theorem 1.6.

Now let G be a connected linear algebraic group. Recall that G is special if every G-torsor is Zariski-locally trivial. For example, $\operatorname{GL}_n, \operatorname{SL}_n$ and Sp_n are special; see [5]. It was shown by Ekedahl that if $P \to S$ is a torsor under the special group G, then $\{P\} = \{G\}\{S\}$. This is immediate if S is a scheme, but less obvious

when S is a stack; see [3, Corollary 2.4]. Applying this to the universal G-torsor Spec $F \to BG$, one obtains $\{BG\}\{G\} = 1$.

The equality $\{BG\} = \{G\}^{-1}$ appears to be the analogue for connected groups of the relation $\{BG\} = 1$ for finite group schemes. In [3], these equalities are referred to as *expected class formulas*, and there is a sense in which they are "almost" true. In [8, §2] Ekedahl defines a generalized Euler characteristic

 $\chi_{\rm c}: K_0({\rm Stacks}_F) \to K_0({\rm Coh}_F)$

taking values in a Grothendieck ring $K_0(\operatorname{Coh}_F)$ of Galois representations over F. If G is a finite group scheme, the equality $\chi_c(\{BG\}) = 1$ always holds [7, Proposition 3.1]. On the other hand, if G is connected, then $\chi_c(\{BG\}\{G\}) = 1$; see [3, §2.2]. Since $\{BG\} \neq 1$ for some finite groups G, the following question naturally arises.

Question 1.2. Let F be a field. Is it true that

$$\{BG\} = \{G\}^{-1}$$

in $K_0(\operatorname{Stacks}_F)$ for every connected group G?

In Theorem 1.5, we show that the answer to Question 1.2 is also negative. Computations for non-special G have been carried out for PGL₂ and PGL₃ in [3], for SO_n and n odd in [6], for SO_n and n even and O_n for any n in [19], and for Spin₇, Spin₈ and G₂ in [18]. In each of these cases, (1.3) was found to be true. The expectation was that, for a connected linear algebraic group G over a field F of characteristic 0, Question 1.4 below should have an affirmative answer. If F is an algebraically closed field, then there are no examples of connected G where BG is known not to be stably rational. If F is not assumed to be algebraically closed, then such examples exist. The following variant of Question 1.1 seems natural in this context.

Question 1.4. (cf. [19, §1] and [18, Remark 4.1]) Is it true that, for a connected linear algebraic group G, the following two conditions are equivalent?

- BG is stably rational;
- $\{BG\} = \{G\}^{-1}$ in $K_0(Stacks_F)$.

Our first result gives a negative answer to Question 1.2 and Question 1.4.

Theorem 1.5. Let F be a field of characteristic zero which admits a biquadratic field extension K, let E_1 and E_2 be two distinct quadratic subextensions of K/F, and set $G := R^{(1)}_{E_1 \times E_2/F}(\mathbb{G}_m)$. Then

- (a) BG is stably rational, and
- (b) $\{BG\} \neq \{G\}^{-1}$ in $K_0(\text{Stacks}_F)$.

The torus G is an example of a norm-one torus; see Section 2 for the definition. It follows from Theorem 1.5 that counterexamples H to (1.3) exist in any dimension dim $H \ge 3$: consider for example $H := G \times \mathbb{G}_m^r$ for $r \ge 0$.

The key ingredient in the proof of Theorem 1.5 is the *refined Euler characteristic* of Ekedahl, introduced in [8, §6, 3]; see Section 4.

Our second result gives a negative answer to Question 1.1.

Theorem 1.6. Let F be a field of characteristic zero which admits a biquadratic field extension K, and let E_1 and E_2 be two distinct quadratic subextensions of K/F. Define $G := R_{E_1 \times E_2/F}^{(1)}(\mathbb{G}_m)$, and let A := G[2] be the 2-torsion subgroup of G. Then

- (a) BA is stably rational, and
- (b) $\{BA\} \neq 1$ in $K_0(\operatorname{Stacks}_F)$.

Questions 1.1, 1.2 and 1.4 remain open in the case, where the base field F is assumed to be algebraically closed. Our arguments do not shed any new light in this setting.

The remainder of this paper is structured as follows. In Section 2 we review well known computations of motivic classes for non-split tori. In Section 3 we obtain explicit formulas for the motivic classes of G and BG, and in Section 4 we give the required background on the refined Euler characteristic. In Section 5 we prove Theorem 1.5, and in Section 6 we prove Theorem 1.6.

2. Preliminaries

Let F be a field. We will write \mathbb{L} for the class $\{\mathbb{A}^1\}$ in $K_0(\operatorname{Var}_F)$ or $K_0(\operatorname{Stacks}_F)$. If E is an étale algebra over F, we will denote by $\{E\}$ the class $\{\operatorname{Spec} E\}$ in $K_0(\operatorname{Var}_F)$ or $K_0(\operatorname{Stacks}_F)$. If X is a quasi-projective scheme over E, we will denote by $R_{E/F}(X)$ the Weil restriction of X to F. By definition, for every F-scheme S one has $R_{E/F}(X)(S) = X(S_E)$. We refer the reader to [21, §3.12] for an account of the main properties of the Weil restriction.

Let G be a linear algebraic group over F, and $\alpha \in H^1(F,G)$ be represented by a G-torsor $P \to \operatorname{Spec} F$. For every quasi-projective F-scheme Z, we denote by ${}^{\alpha}Z$ the *twist* of Z by P, that is,

$$^{\alpha}Z := (Y \times P)/G,$$

where G acts diagonally. We refer the reader to [10, Section 2] for the definition and the basic properties of the twisting operation.

We will write C_2 for the cyclic group of two elements, and S_n for the symmetric group on n symbols.

The following observations will be repeatedly used in the sequel.

Lemma 2.1. Let X be a scheme over F, E an étale algebra of degree n over F, $\alpha \in H^1(F, S_n)$ the class corresponding to E/F.

(a) Let S_n act on the disjoint union $\coprod_{i=1}^n X$ by permuting the *n* copies of X. Then

$$^{\alpha}(\coprod_{i=1}^{n}X)\cong X_{E}.$$

(b) Let S_n act on X^n by permuting the n factors. Then

$$^{\alpha}(X^n) \cong R_{E/F}(X).$$

Proof. (a) Let $Y := \coprod_{i=1}^{n} X$, and let S_n act on Y by permuting the copies of X. By definition,

$$^{\alpha}Y = (Y \times \operatorname{Spec} E)/S_n \cong (Y \times_X X_E)/S_n,$$

where S_n acts diagonally. This shows that ${}^{\alpha}Y$ is the twist of X_E by the trivial S_n -torsor $Y \to X$ in the category of X-schemes, which implies ${}^{\alpha}Y \cong X_E$.

(b) See the bottom of page 5 in [11].

Lemma 2.2. Let

$$1 \to N \to G \to H \to 1$$

be an exact sequence of group schemes over F, and assume that G is special. Then

$$\{BN\} = \{H\}/\{G\}$$

Proof. See [3, Proposition 2.9].

Let F_s be a separable closure of F. Recall that a group scheme T over F is called a *torus* if $T_{F_s} \cong \mathbb{G}^n_{\mathrm{m},F_s}$ for some $n \geq 0$. The *character lattice* of T is the finitely generated \mathbb{Z} -free $\mathrm{Gal}(F)$ -module $\mathrm{Hom}_{F_s}(T_{F_s}, \mathbb{G}_{\mathrm{m},F_s})$. The character lattice induces an anti-equivalence between the category of F-tori and the category of $\mathrm{Gal}(F)$ lattices, i.e., \mathbb{Z} -free continuous $\mathrm{Gal}(F)$ -modules; see [9, §2]. Similarly, for every separable finite extension L/F, we have an anti-equivalence between $\mathrm{Gal}(L/F)$ lattices and F-tori T split by L, i.e., such that $T_L \cong \mathbb{G}^n_{\mathrm{m},L}$ for some $n \geq 0$. The *dual torus* of T is the torus T' whose character lattice is dual to that of T.

Let *E* be an étale algebra over *F*. If *G* is a group scheme over *E*, then $R_{E/F}(G)$ is a group scheme over *F*. The group $R_{E/F}(\mathbb{G}_m) := R_{E/F}(\mathbb{G}_{m,E})$ is an *F*-torus. Tori of this kind are called *quasi-split*. They are special groups, and they correspond to permutation Gal(*F*)-lattices, that is, lattices admitting a \mathbb{Z} -basis that is permuted by Gal(*F*); see [21, §3.12, Example 19].

Lemma 2.3. Let T be an algebraic torus over F, and let T' be its dual. Assume that T is stably rational. Then

- (a) BT' is stably rational;
- (b) $\{BT'\}\{T\} = 1$ in $K_0(\text{Stacks}_F)$.

Proof. Since T is stably rational, by [21, §4.7, Theorem 2] there is a short exact sequence

$$(2.4) 1 \to T_1 \to T_2 \to T \to 1$$

where T_1 and T_2 are quasi-split. Since quasi-split tori are isomorphic to their dual, the sequence dual to (2.4),

$$(2.5) 1 \to T' \to T_2 \to T_1 \to 1,$$

shows that T' embeds in T_2 . We may view T_2 as a maximal torus inside GL_n , where $n = \operatorname{rank} T_2$. This gives a faithful representation of T' with quotient birational to T_1 . Since quasi-split tori are rational, it follows that BT' is stably rational.

Quasi-split tori are special, so we may apply Lemma 2.2 to (2.4) and (2.5). We obtain $\{T\} = \{T_2\}/\{T_1\}$ and $\{BT'\} = \{T_1\}/\{T_2\}$, so $\{BT'\}\{T\} = 1$.

Let E/F be an étale algebra, and let $R_{E/F}(\mathbb{G}_m)$ be the associated quasi-split torus. The kernel of the norm homomorphism $R_{E/F}(\mathbb{G}_m) \to \mathbb{G}_m$ is called a *norm-one torus*, and is denoted by $R_{E/F}^{(1)}(\mathbb{G}_m)$. Its dual torus is isomorphic to $R_{E/F}(\mathbb{G}_m)/\mathbb{G}_m$.

Lemma 2.6. Assume that char $F \neq 2$. Let $E := F(\sqrt{m})$ be a separable quadratic field extension, and let α denote the class of E/F in $H^1(F, C_2)$. Then:

- (a) $R_{E/F}^{(1)}(\mathbb{G}_{\mathrm{m}}) \cong R_{E/F}(\mathbb{G}_{\mathrm{m}})/\mathbb{G}_{\mathrm{m}}.$
- (b) Let $\operatorname{Gal}(E/F)$ act on \mathbb{P}^1 via $z \mapsto z^{-1}$. Then ${}^{\alpha}\mathbb{P}^1 \cong \mathbb{P}^1$.
- (c) $R_{E/F}(\mathbb{G}_m)/\mathbb{G}_m$ is rational and

$$\{R_{E/F}(\mathbb{G}_{\mathrm{m}})/\mathbb{G}_{\mathrm{m}}\} = \{B(R_{E/F}(\mathbb{G}_{\mathrm{m}})/\mathbb{G}_{\mathrm{m}})\}^{-1} = \mathbb{L} - \{E\} + 1.$$

- (d) $\{R_{E/F}(\mathbb{G}_{m})\} = \{BR_{E/F}(\mathbb{G}_{m})\}^{-1} = (\mathbb{L}-1)(\mathbb{L}-\{E\}+1).$
- (e) $\{R_{E/F}(\mathbb{P}^1)\} = \mathbb{L}^2 + \{E\}\mathbb{L} + 1.$

Proof. (a) Both tori correspond to the unique non-trivial $\operatorname{Gal}(E/F)$ -lattice of rank 1. Here $\operatorname{Gal}(E/F) \cong C_2$.

(b) The C_2 -action on \mathbb{P}^1 has a fixed point z = 1, hence ${}^{\alpha}\mathbb{P}^1$ has an F-point. By Châtelet's Theorem [13, Theorem 5.1.3], a form of \mathbb{P}^n which admits an F-point is trivial (the case n = 1 is particularly simple, see [13, Remark 1.3.5]). We conclude that ${}^{\alpha}\mathbb{P}^1 \cong \mathbb{P}^1$.

(c) Let $T := R_{E/F}^{(1)}(\mathbb{G}_m) \cong R_{E/F}(\mathbb{G}_m)/\mathbb{G}_m$. The open embedding $\mathbb{G}_m \hookrightarrow \mathbb{P}^1$, as the complement of $Z := \{0, \infty\}$, is equivariant under the C_2 -action on \mathbb{G}_m and \mathbb{P}^1 given by $z \mapsto z^{-1}$. Twisting by α , we obtain by (b) an open embedding of T in \mathbb{P}^1 as the complement of ${}^{\alpha}Z$. In particular, T is rational. By Lemma 2.1(a), ${}^{\alpha}Z \cong \operatorname{Spec} E$, so

$$\{T\} = \{\mathbb{P}^1\} - \{^{\alpha}Z\} = \mathbb{L} + 1 - \{E\}.$$

Now (c) follows from Lemma 2.3(b).

(d) The first equality holds because $R_{E/F}(\mathbb{G}_m)$ is special. Consider the short exact sequence

$$1 \to \mathbb{G}_{\mathrm{m}} \to R_{E/F}(\mathbb{G}_{\mathrm{m}}) \to T \to 1.$$

Since $R_{E/F}(\mathbb{G}_m)$ is special, Lemma 2.2 yields

$$\{R_{E/F}(\mathbb{G}_{\mathrm{m}})\} = (\mathbb{L} - 1)\{BT\}^{-1},\$$

thus (d) follows from (c).

(e) Write $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$, and consider the C_2 -equivariant decomposition

 $(\mathbb{P}^1)^2 = (\mathbb{A}^1)^2 \amalg (\mathbb{A}^1 \times \{\infty\} \cup \{\infty\} \times \mathbb{A}^1) \amalg \{(\infty, \infty)\}.$

By Hilbert's Theorem 90 and Lemma 2.1(a), twisting by α gives

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$$R_{E/F}(\mathbb{P}^1) = \mathbb{A}^2 \amalg \mathbb{A}^1_E \amalg \operatorname{Spec} F,$$

thus $\{R_{E/F}(\mathbb{P}^1)\} = \mathbb{L}^2 + \{E\}\mathbb{L} + 1.$

3. The classes of G and BG

Let F be a field of characteristic not 2, and assume that there exists a biquadratic extension

$$K := F(\sqrt{m_1}, \sqrt{m_2})$$

of F. Let

$$E_1 := F(\sqrt{m_1}), \qquad E_2 := F(\sqrt{m_2}), \qquad E_{12} := F(\sqrt{m_1m_2}), \qquad E := E_1 \times E_2,$$

and let $\Gamma := \operatorname{Gal}(K/F) \cong C_2^2$ be the Galois group of K/F . We define the torus

$$G := R^{(1)}_{E/F}(\mathbb{G}_{\mathrm{m}})$$

and let

$$G' := R_{E/F}(\mathbb{G}_{\mathrm{m}}) / \mathbb{G}_{\mathrm{m}}$$

be the dual torus of G. By definition, we have a short exact sequence

(3.1)
$$1 \to G \to R_{E/F}(\mathbb{G}_{\mathrm{m}}) \xrightarrow{N} \mathbb{G}_{\mathrm{m}} \to 1,$$

where N is the norm homomorphism.

The purpose of this section is the proof of Proposition 3.7, which expresses $\{BG\}$ and $\{G\}$ as rational functions in \mathbb{L} , with coefficients classes of étale algebras.

Let σ_1 and σ_2 be generators for Γ such that $E_1 = K^{\sigma_1}$ and $E_2 = K^{\sigma_2}$. Consider the Γ -action on \mathbb{G}^2_m , where $\sigma_1(u, v) = (v^{-1}, u^{-1})$ and $\sigma_2(u, v) = (v, u)$, and set

$$(3.2) T := {}^{\alpha}(\mathbb{G}_{\mathrm{m}}^2),$$

where $\alpha \in H^1(F, \Gamma)$ corresponds to the extension K/F.

Lemma 3.3. We have

$$\{T\} = \mathbb{L}^2 + (\{E_{12}\} - \{K\})\mathbb{L} + \{K\} - \{E_1\} - \{E_2\} + 1.$$

Proof. The embedding of \mathbb{G}_m in \mathbb{P}^1 as the complement of $Z := \{0, \infty\}$ gives an open embedding $\mathbb{G}_m^2 \hookrightarrow (\mathbb{P}^1)^2$ such that the Γ -action on \mathbb{G}_m^2 extends to $(\mathbb{P}^1)^2$. By definition

$$^{\alpha}(\mathbb{P}^1)^2 = ((\mathbb{P}^1)^2 \times \operatorname{Spec} K) / \Gamma,$$

where $\Gamma = \langle \sigma_1, \sigma_2 \rangle$ acts diagonally. We first take the quotient by the subgroup $\langle \sigma_1 \sigma_2 \rangle$. Since $\sigma_1 \sigma_2(u, v) = (u^{-1}, v^{-1})$ and $E_{12} = K^{\sigma_1 \sigma_2}$, by Lemma 2.6(b)

$$^{\alpha}(\mathbb{P}^1)^2 = ((\mathbb{P}^1)^2 \times \operatorname{Spec} E_{12})/C_2,$$

where C_2 acts on $(\mathbb{P}^1)^2$ by switching the two factors. Here we are using the fact that every automorphism of $(\mathbb{P}^1)^2$ must respect the ruling (because it respects the intersection form), and so $\operatorname{Aut}((\mathbb{P}^1)^2) = (\operatorname{Aut}(\mathbb{P}^1))^2 \rtimes C_2$, where C_2 switches the two factors. By Lemma 2.1(b) we deduce that ${}^{\alpha}(\mathbb{P}^1)^2 \cong R_{E_{12}/F}(\mathbb{P}^1)$, so by Lemma 2.6(e)

(3.4)
$$\{^{\alpha}(\mathbb{P}^1)^2\} = \mathbb{L}^2 + \{E_{12}\}\mathbb{L} + 1.$$

We may partition $(\mathbb{P}^1)^2 \setminus \mathbb{G}_m^2$ in two strata

$$Z_1 := Z \times Z, \qquad Z_2 := (Z \times \mathbb{G}_m) \amalg (\mathbb{G}_m \times Z).$$

The Γ -action on Z_1 has two orbits, and Γ acts on Z_2 by transitively permuting the components as the Klein subgroup of S_4 . By Lemma 2.1(a), ${}^{\alpha}Z_1 = \operatorname{Spec} E_1 \amalg$ Spec E_2 and ${}^{\alpha}Z_2 = \mathbb{G}_{\mathrm{m}} \times \operatorname{Spec} K$. By (3.4)

$$\{T\} = \{^{\alpha} (\mathbb{P}^{1})^{2}\} - \{^{\alpha} Z_{1}\} - \{^{\alpha} Z_{2}\}$$

= $\mathbb{L}^{2} + \{E_{12}\}\mathbb{L} + 1 - \{E_{1}\} - \{E_{2}\} - \{K\}(\mathbb{L} - 1)$
= $\mathbb{L}^{2} + (\{E_{12}\} - \{K\})\mathbb{L} + \{K\} - \{E_{1}\} - \{E_{2}\} + 1.$

Proposition 3.5. There is a short exact sequence of tori

$$1 \to \mathbb{G}_{\mathrm{m}} \to G \to T \to 1$$

where T is the torus of (3.2).

Proof. Let *P*, *M* and Z be the character lattices of $R_{E/F}(\mathbb{G}_m)$, *G* and \mathbb{G}_m , respectively. We may view *P* as the Γ-lattice with a basis e_1, e_2, e_3, e_4 , such that σ_1 acts by switching e_1 with e_2 and fixing e_3 and e_4 , and σ_2 switches e_3 with e_4 and fixes e_1 and e_2 . The sequence of Γ-lattices dual to (3.1) identifies *M* with the cokernel of the Γ-homomorphism $\mathbb{Z} \to P$ given by $1 \mapsto e_1 + e_2 + e_3 + e_4$; denote by $\overline{e_i} \in M$ the projection of e_i . Following Kunyavskiĭ [15, §3, Proposition 1(b)], we consider an exact sequence of Γ-lattices

$$(3.6) 0 \to N \to M \xrightarrow{\pi} \mathbb{Z} \to 0.$$

The map π is defined by $\pi(\sum a_i \overline{e}_i) = a_1 + a_2 - a_3 - a_4$, and $N := \text{Ker } \pi$. A basis for N is given by $v_1 := \overline{e}_1 + \overline{e}_3$ and $v_2 := \overline{e}_1 + \overline{e}_4$. With respect to the basis (v_1, v_2) ,

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the Γ -action on N is given by $\sigma_1(a,b) = (-b,-a)$ and $\sigma_2(a,b) = (b,a)$. It is now clear that N is the character lattice of the torus T of (3.2), hence the proof is complete.

Proposition 3.7. (a) BG is stably rational. (b) $\{BG\}\{G'\} = 1$ in $K_0(\operatorname{Stacks}_F)$.

Proof. Consider the sequence

$$(3.8) 1 \to \mathbb{G}_{\mathrm{m}} \to G' \to (R_{E_1/F}(\mathbb{G}_{\mathrm{m}})/\mathbb{G}_{\mathrm{m}}) \times (R_{E_2/F}(\mathbb{G}_{\mathrm{m}})/\mathbb{G}_{\mathrm{m}}) \to 1,$$

which exhibits G' as a \mathbb{G}_m -torsor over a rational variety, by Lemma 2.6(c). We deduce that G' is rational, and now (a) and (b) follow from Lemma 2.3.

Proposition 3.9. We have

(3.10)
$$\{G\} = (\mathbb{L} - 1)(\mathbb{L}^2 + (\{E_{12}\} - \{K\})\mathbb{L} + \{K\} - \{E_1\} - \{E_2\} + 1)$$

and

(3.11)
$$\{BG\}^{-1} = (\mathbb{L} - 1)(\mathbb{L} - \{E_1\} + 1)(\mathbb{L} - \{E_2\} + 1)$$

in $K_0(\operatorname{Stacks}_F)$.

Proof. By Proposition 3.5, G is a \mathbb{G}_m -torsor over T. Since \mathbb{G}_m is special, $\{G\} = (\mathbb{L} - 1)\{T\}$. The class of T was determined in Lemma 3.3.

By Proposition 3.7(b), $\{BG\}^{-1} = \{G'\}$. Since \mathbb{G}_{m} is special, by (3.8), $\{G'\} = (\mathbb{L}-1)\{R_{E_1/F}^{(1)}(\mathbb{G}_{\mathrm{m}})\}\{R_{E_2/F}^{(1)}(\mathbb{G}_{\mathrm{m}})\}$. Now (3.11) follows from Lemma 2.6(c).

4. The refined Euler characteristic

Let F be a field of characteristic zero. Using the computations of the previous section, we will reduce Theorem 1.5(b) to the assertion that a certain polynomial in \mathbb{L} with coefficients motivic classes of étale algebras is a non-zero element of $K_0(\operatorname{Var}_F)$. To prove the assertion, we will use a simplified version of the refined Euler characteristic, introduced by Ekedahl in [8].

Fix a prime number p, and let \mathcal{G} be a profinite group. The representation ring $a_p(\mathcal{G})$ of \mathcal{G} is the Grothendieck ring of continuous \mathcal{G} -representations [M] of finite dimension over \mathbb{F}_p , subject to the relations $[M \oplus N] = [M] + [N]$. Note that no relations for non-split short exact sequences are imposed. The product structure on $a_p(\mathcal{G})$ is given by tensor product of representations. The next observation is well known when \mathcal{G} is assumed to be finite; see [2, §5.1].

Lemma 4.1. As an abelian group, $a_p(\mathcal{G})$ is freely generated by the set of isomorphism classes of indecomposable representations.

Proof. It is clear that $a_p(\mathcal{G})$ is generated by isomorphism classes of indecomposable representations. Assume that $\sum a_i[M_i] - \sum b_j[N_j] = 0$ in $a_p(\mathcal{G})$, for some positive integers a_i, b_j and some pairwise non-isomorphic indecomposable \mathcal{G} -representations M_i and N_j .

As a group, $a_p(\mathcal{G})$ is the quotient group F/I, where F is the free abelian group with one generator $\langle P \rangle$ for every isomorphism class of \mathcal{G} -representations P, and I is the subgroup generated by all elements of the form $\langle P \oplus Q \rangle - \langle P \rangle - \langle Q \rangle$. It follows that we may find a \mathcal{G} -representation X such that

$$(\oplus_i M_i^{\oplus a_i}) \oplus X \cong (\oplus_j N_i^{\oplus b_j}) \oplus X.$$

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Let \mathcal{G}_0 be a finite quotient of \mathcal{G} such that \mathcal{G} acts on M_i , N_j and X through \mathcal{G}_0 . Then $M \oplus X \cong N \oplus X$ as \mathcal{G}_0 -representations. By the Krull-Schmidt Theorem applied to the group algebra $\mathbb{F}_p[\mathcal{G}_0]$, this implies $M \cong N$ as \mathcal{G}_0 -modules, hence as \mathcal{G} -modules. This is impossible, because the indecomposable representations M_i and N_j are pairwise non-isomorphic. \Box

Proposition 4.2. Let F be a field of characteristic zero, let Gal(F) be the absolute Galois group of F, and let $R_p := a_p(Gal(F))$. There is a ring homomorphism

$$\mu: K_0(\operatorname{Var}_F) \to R_p[t]$$

such that for every smooth complete variety X we have $\mu(X) = \sum_{i} [H^{i}(\overline{X}_{\acute{e}t}, \mathbb{F}_{p})]t^{i}$.

Proof. See the proof of [8, Proposition 3.2(i)]. To show that μ is well-defined, one needs to assume that char F = 0 in order to invoke Bittner's presentation of $K_0(\operatorname{Var}_F)$; see [4, Theorem 3.1].

5. Proof of Theorem 1.5

Theorem 1.5(a) was proved in Proposition 3.7(b), so we will focus on Theorem 1.5(b). We maintain the notation given at the beginning of Section 3.

Proof of Theorem 1.5(b). Assume by contradiction that $G = R_{E/F}^{(1)}(\mathbb{G}_m)$ satisfies (1.3). Then by Proposition 3.9 we have

$$(\mathbb{L} - 1)(\mathbb{L} - \{E_1\} + 1)(\mathbb{L} - \{E_2\} + 1) =$$
$$= (\mathbb{L} - 1)(\mathbb{L}^2 + (\{E_{12}\} - \{K\})\mathbb{L} + \{K\} - \{E_1\} - \{E_2\} + 1)$$

in $K_0(\text{Stacks}_F)$. Since $\mathbb{L} - 1$ is invertible in $K_0(\text{Stacks}_F)$, we may divide by $\mathbb{L} - 1$ on both sides. Subtracting \mathbb{L}^2 on the left and on the right, we arrive to

$$(2-\{E_1\}-\{E_2\})\mathbb{L}+(1-\{E_1\})(1-\{E_2\}) = (\{E_{12}\}-\{K\})\mathbb{L}+\{K\}-\{E_1\}-\{E_2\}+1,$$

that is

that is

$$(\{K\} - \{E_1\} - \{E_2\} - \{E_{12}\} + 2)\mathbb{L} = 0$$

in $K_0(\operatorname{Stacks}_F)$.

Recall that $K_0(\operatorname{Stacks}_F)$ is the localization of $K_0(\operatorname{Var}_F)$ at \mathbb{L} and the cyclotomic polynomials in \mathbb{L} ; see [8, Theorem 1.2]. It follows that

(5.1)
$$(\{K\} - \{E_1\} - \{E_2\} - \{E_{12}\} + 2)f(\mathbb{L}) = 0$$

in $K_0(\operatorname{Var}_F)$, where $f(x) \in \mathbb{Z}[x]$ is a monic polynomial of some degree n.

In order to obtain a contradiction, we now want to apply the homomorphism μ of (4.2), with respect to the prime p = 2. If L/F is an étale algebra of degree n, $\mu(\{L\})$ consists of the permutation representation of Gal(F) associated to L, concentrated in degree 0. Since we have chosen p = 2, $\mu(\{\mathbb{P}^1\})$ consists of one copy of the trivial representation in degree 0 and 2 (in the case p > 2 one would need a Tate twist in degree 2). Since $\mathbb{L} = \{\mathbb{P}^1\} - 1$, we deduce that $\mu(\mathbb{L}) = t^2$, and hence $\mu(f(\mathbb{L})) = f(t^2)$.

If X is a finite $\operatorname{Gal}(F)$ -set, we denote by $\mathbb{F}_2[X]$ the permutation representation over \mathbb{F}_2 associated to X. Recall from Section 3 that we denote $\operatorname{Gal}(K/F)$ by $\Gamma = \langle \sigma_1, \sigma_2 \rangle$. Applying μ to (5.1) and looking at degree 2n, we obtain

$$[\mathbb{F}_{2}[\Gamma]] - [\mathbb{F}_{2}[\Gamma/\langle\sigma_{1}\rangle]] - [\mathbb{F}_{2}[\Gamma/\langle\sigma_{2}\rangle]] - [\mathbb{F}_{2}[\Gamma/\langle\sigma_{12}\rangle]] + 2[\mathbb{F}_{2}] = 0$$

in R_2 . This is a non-trivial relation of linear dependence in R_2 among classes of indecomposable representations. This is in contradiction with Lemma 4.1, hence $\{BG\} \neq \{G\}^{-1}$, as desired.

Remark 5.2. By [21, §4.9, Example 7] every torus of rank 2 is rational, so by Proposition 3.5 the torus G is rational. By Lemma 2.3, BG' is stably rational and $\{BG'\} = \{G\}^{-1}$. By Proposition 3.7(b) we have $\{BG\} = \{G'\}^{-1}$, so $\{BG'\}\{G'\} = \{BG\}^{-1}\{G\}^{-1}$. Since $\{BG\}\{G\} \neq 1$, the conclusions of Theorem 1.5(a) and (b) hold for G' as well.

6. Proof of Theorem 1.6

We maintain the notation of Section 3.

Proof of Theorem 1.6. Let $\Gamma := \operatorname{Gal}(K/F)$, let M be the character lattice of G, so that M/2M is the character module of A, and let P be the character lattice of $R_{E/F}(\mathbb{G}_m)$. As in the proof of Proposition 3.5, we view P as the lattice freely generated by e_1, e_2, e_3, e_4 , such that σ_1 acts by switching e_1 with e_2 , and σ_2 by switching e_3 with e_4 . Using (3.1), we may construct a commutative diagram of Γ -modules

with exact rows. Here \mathbb{Z} denotes the trivial one-dimensional Γ -lattice, $\iota(1) := e_1 + e_2 + e_3 + e_4$, and N is the kernel of φ , that is,

$$N = \{ \sum_{i=1}^{4} a_i e_i : a_1 \equiv a_2 \equiv a_3 \equiv a_4 \pmod{2} \}.$$

Applying the snake lemma to (6.1), we obtain a short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\iota} N \to M \to 0.$$

Define $\pi : N \to \mathbb{Z}$ by sending $\sum a_i e_i$ to $(a_1 + a_2)/2$. Then π is a Γ -homomorphism and ι is a section of π . Therefore, we have an isomorphism $N \cong \mathbb{Z} \oplus M$.

Let S be an F-torus with character lattice N. Since $N \cong \mathbb{Z} \oplus M$, we have $S \cong \mathbb{G}_m \times G$. The bottom row of (6.1) corresponds to the short exact sequence of group schemes

$$1 \to A \to R_{E/F}(\mathbb{G}_{\mathrm{m}}) \to \mathbb{G}_{\mathrm{m}} \times G \to 1.$$

By Lemma 2.2, we have $\{BA\} = \{\mathbb{G}_m\}\{G\}/\{R_{E/F}(\mathbb{G}_m)\}$. Applying Lemma 2.2 to (3.1), we see that $\{BG\} = \{\mathbb{G}_m\}/\{R_{E/F}(\mathbb{G}_m)\}$. Therefore, $\{BA\} = \{BG\}\{G\}$. By Theorem 1.6 we have $\{BG\} \neq \{G\}^{-1}$, hence $\{BA\} \neq 1$, as desired. \Box

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References

- Kai Behrend and Ajneet Dhillon. On the motivic class of the stack of bundles. Advances in Mathematics, 212(2):617–644, 2007.
- [2] D. J. Benson. Representations and cohomology. II, volume 31 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, second edition, 1998. Cohomology of groups and modules.
- [3] Daniel Bergh. Motivic classes of some classifying stacks. Journal of the London Mathematical Society, 93(1):219-243, 2015.
- [4] Franziska Bittner. The universal Euler characteristic for varieties of characteristic zero. Compositio Mathematica, 140(4):1011–1032, 2004.
- [5] Séminaire Claude Chevalley and JP Serre. Espaces fibrés algébriques. Séminaire Claude Chevalley, 3:1–37, 1958.
- [6] Ajneet Dhillon, Matthew B Young, et al. The motive of the classifying stack of the orthogonal group. The Michigan Mathematical Journal, 65(1):189–197, 2016.
- [7] Torsten Ekedahl. A geometric invariant of a finite group. arXiv preprint arXiv:0903.3148, 2009.
- [8] Torsten Ekedahl. The Grothendieck group of algebraic stacks. arXiv preprint arXiv:0903.3143, 2009.
- [9] Giordano Favi and Mathieu Florence. Tori and essential dimension. J. Algebra, 319(9):3885– 3900, 2008.
- [10] Mathieu Florence. On the essential dimension of cyclic p-groups. Invent. Math., 171(1):175– 189, 2008.
- [11] Mathieu Florence and Zinovy Reichstein. On the rationality problem for forms of moduli spaces of stable marked curves of positive genus. *arXiv preprint arXiv:1709.05696*, 2017.
- [12] Mathieu Florence and Zinovy Reichstein. The rationality problem for forms of $M_{0,n}$. Bulletin of the London Mathematical Society, 50(1):148–158, 2018.
- [13] Philippe Gille and Tamás Szamuely. Central Simple Algebras and Galois Cohomology. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2006.
- [14] Dominic Joyce. Motivic invariants of Artin stacks and stack functions. The Quarterly Journal of Mathematics, 58(3):345–392, 2007.
- [15] B. È. Kunyavskiĭ. Three-dimensional algebraic tori. In *Investigations in number theory (Russian)*, pages 90–111. Saratov. Gos. Univ., Saratov, 1987. Translated in Selecta Math. Soviet. 9 (1990), no. 1, 1–21.
- [16] Ivan Martino. The Ekedahl invariants for finite groups. Journal of Pure and Applied Algebra, 220(4):1294–1309, 2016.
- [17] Ivan Martino. Introduction to the Ekedahl Invariants. Mathematica scandinavica, 120(2):211– 224, 2017.
- [18] Roberto Pirisi and Mattia Talpo. On the motivic class of the classifying stack of G_2 and the spin groups. To appear in International Mathematics Research Notices. arXiv preprint arXiv:0903.3143.
- [19] Mattia Talpo and Angelo Vistoli. The motivic class of the classifying stack of the special orthogonal group. Bulletin of the London Mathematical Society, 49(5):818–823, 2017.
- [20] Bertrand Toën. Grothendieck rings of Artin n-stacks. arXiv preprint math/0509098, 2005.
- [21] V. E. Voskresenskii. Algebraic groups and their birational invariants, volume 179 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, 1998. Translated from the Russian manuscript by Boris Kunyavski [Boris È. Kunyavskiĭ].