ANNIHILATING-IDEAL GRAPH OF C(X)

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ABSTRACT. In this article we study the annihilating-ideal graph of the ring C(X). We have tried to associate the graph properties of $\mathbb{AG}(X)$, the ring properties of C(X) and the topological properties of X. We have shown that X has an isolated point if and only if \mathbb{R} is a direct summand of C(X) if and only if $\mathbb{AG}(X)$ is not triangulated. Radius, girth, dominating number and clique number of the $\mathbb{AG}(X)$ are investigated. We have proved that $c(X) \leq \operatorname{dt}(\mathbb{AG}(X)) \leq w(X)$ and clique $\mathbb{AG}(X) = \chi \mathbb{AG}(X) = c(X)$.

1. Introduction

Let $G = \langle V(G), E(G) \rangle$ be an undirected graph. A vertex which adjacent to just one vertex is called a *leaf vertex*. The degree of a vertex of G is the number of edges incident to the vertex. If G has a vertex which adjacent to all another vertices, then G is called a star graph. For each vertices u and v in V(G), the length of the shortest path between u and v, denoted by d(u,v), is called the distance between u and v. The diameter of G is defined $\sup\{d(u,v):u,v\in V(G)\},\$ the diameter of G is denoted by diam(G). The eccentricity of a vertex u of G is denoted by ecc(u) and is defined to be maximum of $\{d(u,v): u \in G\}$. The minimum of $\{ecc(u): u \in G\}$, denoted by Rad(G), is called the radius of G. For every $u, v \in V(G)$, we denote the length of the shortest cycle containing u and v by gi(u,v) and the minimum length of cycles in G, is denoted by girth(G) and is called the girth of graph, so $girth(G) = min\{gi(u,v) : u,v \in V(G)\}$. We say G is triangulated (hypertriangulated) if each vertex (edge) of G is vertex (edge) of some triangle. A subset D of V(G) is called dominating set if for each $u \in V(G) \setminus D$, there is some $v \in D$, such that v is adjacent to u. The dominating number of G, denoted by dt(G), is the smallest cardinal number of dominating sets of G. We say two vertices u and v are orthogonal and denote by $u \perp v$, if u and v are adjacent and there are no any vertex which adjacent to both vertices u and v. If for every $u \in V(G)$, there is some $v \in V(G)$ such that $u \perp v$, then G is called complemented. A complete subgraph of G is called a clique of G. The supremum of the cardinality of cliques of G, denoted by clique(G), is said the *clique number* of G. The chromatic number of G, denoted by $\chi(G)$, is the minimum cardinal number of colors needed to color vertices of G so that no two vertices have that same color. Clearly, clique(G) $\leq \chi(G)$.

Throughout the paper R is denoted the commutative ring with unity. For each ideal I of R and each element a of R, we denote the ideal $\{x \in R : ax \in I\}$ by (I : a). When $I = \langle 0 \rangle$ we write Ann(a) instead of $(\langle 0 \rangle : a)$ and call this the *annihilator* of

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a. If for each subset S of R there is some $a \in R$ such that Ann(S) = Ann(a), then we say R is satisfy infinite annihilating condition (R is an i.a.c ring). We denote the family of all non-zero ideal with non-zero annihilating by $A(R)^*$. AG(R) is a graph with vertices $A(R)^*$ and two distinct vertices I and J are adjacent, if $IJ = \{0\}$.

In this paper C(X) is denote the set of all real-valued continuous function on a Tychonoff space X. The weight of X, denoted by w(X), is the infimum of the cardinalities of bases of X. The cellularity of X, denoted by c(X), is defined

 $\sup\{|\mathcal{U}|:\mathcal{U} \text{ is a family of mutually disjoint nonempty open subsets of } X\}$

For any $f \in C(X)$, we denote $f^{-1}\{0\}$ and $X \setminus f^{-1}\{0\}$ by Z(f) and $\operatorname{Coz}(f)$, respectively. Every set of the form $Z(f)(\operatorname{Coz}(f))$ is called zeroset (cozeroset). An ideal I of C(X) is called fixed (free) if $\bigcap_{f \in I} Z(f) \neq \emptyset$ ($\bigcap_{f \in I} Z(f) = \emptyset$). Suppose that $A \subseteq X$, We denote $\{f \in C(X) : A \subseteq Z(f)\}$ and $\{f \in C(X) : A \subseteq Z(f)^\circ\}$ by M_A and O_A , respectively. When $A = \{p\}$ we write M_p and O_p instead of $M_{\{p\}}$ and $O_{\{p\}}$, respectively, it is clear that $M_A = \bigcap_{p \in A} M_p$ and $O_A = \bigcap_{p \in A} O_p$. By [11, Theroem 7.3(Gelfand-Kolmogoroff)], $\{M^p : p \in \beta X\}$ is the family of all maximal ideal of C(X). An ideal I of C(X) is called z-ideal, if Z(f) = Z(g) and $f \in I$, then $g \in I$. Clearly, for every ideal I of C(X), $Z^{-1}(Z(I))$ is a the smallest z-ideal containing I. For more details we refer the reader to [9] [11], [4] and [16].

The studding of some graphs on C(X) is an interesting. In these investigations were tried to associate the ring properties of C(X), the graph properties of graphs on C(X) and the topological properties of X. In [5, 3, 6] the zero-divisor graph, the comaximal ideal graph of C(X) and comaximal graph of C(X) were studied. In [7, 8], the studying annihilating-ideal graph of commutative rings were started. On later, this investigation was continued in many papers, for instance see [1, 13, 14, 10, 2, 12, 15].

In this article we study the annihilating-ideal graph of C(X). We abbreviate $\mathbb{A}(C(X))^*$ and $\mathbb{AG}(C(X))$ by $\mathbb{A}(X)^*$ and $\mathbb{AG}(X)$, respectively. If |X| = 1, then $\mathbb{A}(X)^* = \emptyset$, so we assume |X| > 1, throughout the paper.

In the reminder of this section we give some propositions which were concluded immediately from the native algebraic properties of C(X) and [7, 8, 5]. In Section 2, we define maps \mathbf{O} from the family of all subsets of C(X) onto the family of all open subsets of X and \mathbf{I} from the family of all subset of X into the family of all ideals of C(X). We study these maps and use of these notions to study the graph. We show that I is adjacent of I if and only if I if I if I is adjacent of I if and only if I if I is adjacent of I if and only if I if I is adjacent of I if and only if I if I is adjacent of I if and only if I if I is adjacent of I if and only if I if I is adjacent of I if and only if I if I if I if I if I if I is adjacent of I if and only if I if I

I is zero divisor if and only if $\overline{\mathbf{O}(I)} \neq X$, $I(U) \in \mathbb{A}(X)^*$ if and only if $\overline{\mathbf{O}(I)} \neq \emptyset$. In Section 3, we investigate the radius of the graph and we show that $\mathbb{AG}(X)$ is star if and only if |X| = 2. Section 4, is devoted to the girth of the graph. In this section we show that if |X| > 2, then $\mathrm{girth}\mathbb{AG}(X) = 3$. Also, we show that an ideal in $\mathbb{A}(X)^*$ is a leaf vertex if and only if $X \setminus \overline{\mathbf{O}(I)} = X$. The studying the dominating number of the graph is the subject of Section 5. In this section we show that the clique number and chromatic number of $\mathbb{AG}(X)$, and cellularity of X are equal.

Proposition 1.1. The following statements are equivalent.

- (a) |X| = 2.
- (b) $\operatorname{diam}(\mathbb{AG}(X)) = 1$.
- (c) clique AG(X) = 2.
- (d) $\mathbb{AG}(X)$ is a bipartite graph by two nonempty parts.
- (e) $\mathbb{AG}(X)$ is a complete bipartite graph by two nonempty parts.

Proof. It is concludes from [8, Theorem 1.4] and [14, Corollary 2.1].

Proposition 1.2. The following statements hold.

- (a) X has at least 3 points if and only if $diam(\mathbb{AG}(X)) = 3$.
- (b) $\chi(\mathbb{AG}(X)) = \text{clique}(\mathbb{AG}(X)).$

Proof. (a). It implies from [8, Proposition 1.1], [5, Corollary 1.3] and the previous proposition.

(b). It is evident, by [8, Corollary 2.11].

The following proposition is an immediate consequence of [7, Theorem 1.4], [8, Corollaries 2.11 and 2.12] and this fact that X is finite if and only if C(X) has just finitely many ideals.

Proposition 1.3. The following statements are equivalent.

- (a) $\mathbb{AG}(X)$ is a finite graph.
- (b) C(X) has only finitely many ideals.
- (c) Every vertex of $\mathbb{AG}(X)$ has a finite degree.
- (d) X is finite.
- (e) $\chi(\mathbb{AG}(X))$ is finite.
- (f) clique($\mathbb{AG}(X)$) is finite.
- (g) $\mathbb{AG}(X)$ does not have an infinite clique.
- (h) $\chi(\Gamma(C(X)))$ is finite.

2.
$$\mathbf{I}(U)$$
 and $\mathbf{O}(I)$

For each subset S of C(X), we denote $\bigcup_{f \in S} \operatorname{Coz}(f)$ by $\mathbf{O}(S)$, and for each subset U of X, we denote $\{f \in C(X) : U \subseteq \operatorname{Z}(f)\} = M_U = \bigcap_{a \in U} M_a$ by $\mathbf{I}(U)$. It is clear that $\mathbf{O}(S) = X \setminus \left(\bigcap_{f \in S} \operatorname{Z}(f)\right)$ and if G is an open set in X, then $\mathbf{I}(G) = O_G$. First, in this section we study the properties of these maps, then by these maps the edges and vertices of $\mathbb{AG}(X)$ are investigated.

Lemma 2.1. Let S and T be two subsets of C(X), f be an element of C(X) and U and V be two subsets of X. The following hold.

- (a) If $S \subseteq T$, then $\mathbf{O}(S) \subseteq \mathbf{O}(T)$.
- (b) If $U \subseteq V$, then $\mathbf{I}(V) \subseteq \mathbf{I}(U)$.
- (c) $\mathbf{O}(S) = \emptyset$ if and only if $S = \{0\}$.
- (d) $\mathbf{O}(S) = X$ if and only if $\langle S \rangle$ is a free ideal.
- (e) $\mathbf{I}(U) = \{0\}$ if and only if U is dense in X.
- (f) $\mathbf{I}(U) = C(X)$ if and only if $U = \emptyset$.
- (g) $\mathbf{O}(\langle f \rangle) = \operatorname{Coz}(f)$.
- (h) $\mathbf{I}(U) = \mathbf{I}(\overline{U})$.

Proof. It is straightforward.

Proposition 2.2. Let S be a subset of C(X). If $I = \langle S \rangle$, then $\mathbf{O}(I) = \mathbf{O}(S)$.

Proof. By Lemma 2.1, $\mathbf{O}(S) \subseteq \mathbf{O}(I)$. Suppose that $g \in I$, so a finite family $\{f_i\}_{i=1}^n$ of elements of S and a finite family $\{g_i\}_{i=1}^n$ of elements of C(X) exist such that $g = \sum_{i=1}^n h_i f_i$, then

$$\operatorname{Coz}(g) = \operatorname{Coz}\left(\sum_{i=1}^{n} h_i f_i\right) \subseteq \bigcup_{i=1}^{n} \operatorname{Coz}(h_i f_i) \subseteq \bigcup_{i=1}^{n} \operatorname{Coz}(f_i) \subseteq \mathbf{O}(S)$$

Hence
$$\mathbf{O}(I) = \bigcup_{g \in I} \operatorname{Coz}(g) \subseteq \mathbf{O}(S)$$
 and consequently $\mathbf{O}(I) = \mathbf{O}(S)$.

Proposition 2.3. Let $\{I_{\alpha}\}_{{\alpha}\in A}$ be a family of ideals of C(X), I and J be ideals of C(X), $\{U_{\alpha}\}_{{\alpha}\in A}$ be a family of subsets of X and U and V be subsets of X. Then the following hold.

- (a) $\mathbf{O}\left(\sum_{\alpha\in A}I_{\alpha}\right)=\bigcup_{\alpha\in A}\mathbf{O}(I_{\alpha}).$
- (b) $\mathbf{O}\left(\bigcap_{\alpha\in A}I_{\alpha}\right)\subseteq\bigcap_{\alpha\in A}\mathbf{O}(I_{\alpha}).$ (c) $\mathbf{I}\left(\bigcup_{\alpha\in A}U_{\alpha}\right)=\bigcap_{\alpha\in A}\mathbf{I}(U_{\alpha}).$ (d) $\mathbf{O}(I\cap J)=\mathbf{O}(I)\cap\mathbf{O}(J).$

- (e) $\mathbf{I}(U \cap V) \supseteq \mathbf{I}(U) + \mathbf{I}(V)$.

Proof. (a). By Proposition 2.2,

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$$\mathbf{O}\left(\sum_{\alpha\in A}I_{\alpha}\right) = \mathbf{O}\left(\bigcup_{\alpha\in A}I_{\alpha}\right) = \bigcup_{f\in\bigcup_{\alpha\in A}I_{\alpha}}\operatorname{Coz}(f) = \bigcup_{\alpha\in A}\bigcup_{f\in I_{\alpha}}\operatorname{Coz}(f) = \bigcup_{\alpha\in A}\mathbf{O}(I_{\alpha}).$$

- (b). It follows immediately from Proposition 2.1.
- (c). Lemma 2.1, concludes that $\mathbf{I}\left(\bigcup_{\alpha\in A}U_{\alpha}\right)\subseteq\bigcap_{\alpha\in A}\mathbf{I}(U_{\alpha})$.

$$f \in \bigcap_{\alpha \in A} \mathbf{I}(U_{\alpha}) \quad \Rightarrow \quad f \in \mathbf{I}(U_{\alpha}) \quad \forall \alpha \in A$$

$$\Rightarrow \quad U_{\alpha} \subseteq \mathbf{Z}(f) \quad \forall \alpha \in A$$

$$\Rightarrow \quad \bigcup_{\alpha \in A} U_{\alpha} \subseteq \mathbf{Z}(f)$$

$$\Rightarrow \quad f \in \mathbf{I}\left(\bigcup_{\alpha \in A} U_{\alpha}\right)$$

Thus $\bigcap_{\alpha \in A} \mathbf{I}(U_{\alpha}) \subseteq \mathbf{I}\left(\bigcup_{\alpha \in A} U_{\alpha}\right)$ and consequently $\mathbf{I}\left(\bigcup_{\alpha \in A} U_{\alpha}\right) = \bigcap_{\alpha \in A} \mathbf{I}(U_{\alpha})$. (d). By Lemma 2.1, $\mathbf{O}(I \cap J) \subseteq \mathbf{O}(I) \cap \mathbf{O}(J)$.

$$\mathbf{O}(I) \cap \mathbf{O}(J) = \left(\bigcup_{f \in I} \operatorname{Coz}(f)\right) \cap \left(\bigcup_{g \in I} \operatorname{Coz}(g)\right)$$

$$= \bigcup_{\substack{f \in I \\ g \in J}} \left(\operatorname{Coz}(f) \cap \operatorname{Coz}(g)\right) = \bigcup_{\substack{f \in I \\ g \in J}} \operatorname{Coz}(fg)$$

$$\subseteq \bigcup_{h \in I \cap J} \operatorname{Coz}(h) = \mathbf{O}(I \cap J)$$

Thus the equality holds.

In the following examples we show that the equality in parts (b) and (e) of the above proposition need not establish.

Example 2.4. Consider $C(\mathbb{R})$. For each $r \in \mathbb{Q}$, we have $\mathbf{O}(M_r) = \mathbb{R} \setminus \{r\}$, thus $\bigcap_{r\in Q} \mathbf{O}(M_r) = \mathbb{R} \setminus \mathbb{Q}. \ Also \bigcap_{r\in \mathbb{Q}} M_r = M_{\mathbb{Q}} = \{0\}, \ so \ \mathbf{O}\left(\bigcap_{r\in \mathbb{Q}} M_r\right) = \mathbf{O}(\{0\}) = \mathbb{Q}(\{0\})$

Example 2.5. Consider $C(\mathbb{R})$. Easily we can see that, $\mathbf{I}(\mathbb{Q}) = \{0\} = \mathbf{I}(\mathbb{R} \setminus \mathbb{Q})$, and thus $\mathbf{I}(\mathbb{Q} \cap \mathbb{R} \setminus \mathbb{Q}) = \mathbf{I}(\emptyset) = C(\mathbb{R}) \neq \{0\} = \mathbf{I}(\mathbb{Q}) + \mathbf{I}(\mathbb{R} \setminus \mathbb{Q}).$

Corollary 2.6. Let U and V be subsets of X. The following are equivalent.

- (a) $U \cup V$ is dense in X.
- (b) $\mathbf{I}(U) \cap \mathbf{I}(V) = \{0\}.$
- (c) $I(U)I(V) = \{0\}.$

Proof. It follows from Lemma 2.1 and Proposition 2.3.

Proposition 2.7. Let I be an ideal of C(X) and U be a subset of X. The following hold.

- (a) $\mathbf{O}(\mathbf{I}(U)) = (X \setminus U)^{\circ}$.
- (b) I(O(I)) = Ann(I).
- (c) $(IO)^3(I) = (IO)(I)$.
- (d) $\mathbf{O}(\operatorname{Ann}(I)) = (X \setminus \mathbf{O}(I))^{\circ}$.

$$\begin{array}{lll} \textit{Proof.} \ \, (a). & \mathbf{O}(\mathbf{I}(U)) \, = \, \bigcup_{f \in \mathbf{I}(U)} \mathrm{Coz}(f) \, = \, \bigcup_{\mathbf{Z}(f) \supseteq U} \mathrm{Coz}(f) \, = \, \bigcup_{\mathbf{Coz}(f) \subseteq X \backslash U} \mathrm{Coz}(f) \, = \\ (X \backslash U)^{\circ}. & \\ (b). & f \in \mathrm{Ann}(I) \quad \Leftrightarrow & \forall g \in I \quad fg = 0 \\ & \Leftrightarrow & \forall g \in I \quad \mathbf{Z}(f) \cup \mathbf{Z}(g) = X \\ & \Leftrightarrow & \forall g \in I \quad \mathrm{Coz}(g) \subseteq \mathbf{Z}(f) \\ & \Leftrightarrow & \bigcup_{g \in I} \mathrm{Coz}(g) \subseteq \mathbf{Z}(f) \\ & \Leftrightarrow & \mathbf{O}(I) \subseteq \mathbf{Z}(f) \\ & \Leftrightarrow & f \in \mathbf{I}(\mathbf{O}(I)) \end{array}$$

Thus $Ann(I) = \mathbf{I}(\mathbf{O}(I))$.

- (c). Since $Ann^3(I) = Ann(I)$, it follows from (c), immediately.
- (d). By (b) and (a), $\mathbf{O}(\mathrm{Ann}(I)) = \mathbf{O}(\mathbf{I}(\mathbf{O}(I))) = (X \setminus \mathbf{O}(I))^{\circ}$.

Corollary 2.8. If I is a zero divisor ideal of C(X), then I is a fixed ideal.

Proof. Since I is a zero divisor, $\operatorname{Ann}(I) \neq \{0\}$, so Proposition 2.7, deduces $\mathbf{I}(\mathbf{O}(I)) \neq \{0\}$, hence $\mathbf{O}(I)$ is not dense, by Lemma 2.1. Thus $\mathbf{O}(I) \neq X$ and therefore $\bigcap_{f \in I} Z(f) = X \setminus \mathbf{O}(I) \neq \emptyset$, so I is a fixed ideal.

The converse of the above corollary need not be true because for instance $M_0 \subseteq C(\mathbb{R})$ is a fixed ideal which is not a zero divisor ideal.

Corollary 2.9. Let P be a prime ideal of C(X). P is a zero divisor if and only if there is some isolated point p in X such that $P = M_p = O_p$.

Proof. (⇒). By Corollary 2.8, P is fixed, so there is some $p \in X$ such that $O_p \subseteq P \subseteq M_p$. Thus $\mathbf{O}(P) = X \setminus \left(\bigcap_{f \in P} \mathbf{Z}(f)\right) = X \setminus \{p\}$. Since P is a zero divisor, Ann $(P) \neq \{0\}$ and therefore $\mathbf{I}(\mathbf{O}(I)) \neq \{0\}$, by Proposition 2.7. Now Lemma 2.1, deduces $X \setminus \{p\}$ is not dense and thus p is an isolated point. Consequently, $P = M_p = O_p$.

(\Leftarrow). Since $P = M_p$, $\mathbf{O}(P) = X \setminus \left(\bigcap_{f \in P} Z(f)\right) = X \setminus \{p\}$. Since p is an isolated point, it follows that $\mathbf{O}(P)$ is not dense in X, thus $\mathbf{I}(\mathbf{O}(P)) \neq \{0\}$, by Lemma 2.1. Now Proposition 2.7, follows that $\mathrm{Ann}(P) \neq \{0\}$ and therefore P is a zero divisor.

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Lemma 2.10. If G is an open subset of X, then an ideal I exists such that O(I) = G. i.e., O maps the family of all ideals of C(X) onto the family of all open subsets of X.

Proof. Put $I = \langle \{ f \in C(X) : \text{Coz}(f) \subseteq G \} \rangle$. Then by Proposition 2.2,

$$\mathbf{O}(I) = \mathbf{O}(\langle \{f : \operatorname{Coz}(f) \subseteq G\} \rangle) = \bigcup_{\operatorname{Coz}(f) \subseteq G} \operatorname{Coz}(f) = G \qquad \Box$$

Lemma 2.11. For each ideal I of C(X), we have $\mathbf{O}(I_z) = \mathbf{O}(I)$.

Proof. Since $Z(I) = Z(I_z)$, so $\{Coz(f) : f \in I\} = \{Coz(f) : f \in I_z\}$ and therefore $O(I_z) = O(I)$.

Theorem 2.12. O is a map from the family of all z-ideals of C(X) onto the family of all open sets of X.

Proof. It is clear by Lemmas 2.10 and 2.11.

Theorem 2.13. Let I and J be two ideals of C(X). The following statements hold

- (a) $IJ = \{0\}$ if and only if $\mathbf{O}(I) \cap \mathbf{O}(J) = \emptyset$.
- (b) $IAnn(J) = \{0\}$ if and only if $\mathbf{O}(I) \subseteq \overline{\mathbf{O}(J)}$.
- (c) $\operatorname{Ann}(I)\operatorname{Ann}(J) = \{0\}$ if and only if $\overline{\mathbf{O}(I) \cup \mathbf{O}(J)} = \emptyset$.
- (d) $\overline{\mathbf{O}(I)} = \overline{\mathbf{O}(J)}$ if and only if $\mathrm{Ann}(I) = \mathrm{Ann}(J)$.
- (e) $\mathbf{I}(U)I = \{0\}$ if and only if $\mathbf{O}(I) \subseteq \overline{U}$.

Proof. (a \Rightarrow). Since $IJ = \{0\}$, $I \subseteq \text{Ann}(J)$, thus $I \subseteq \mathbf{I}(\mathbf{O}(J))$, by Proposition 2.7. Now suppose that $f \in I$, then $f \in \mathbf{I}(\mathbf{O}(J))$, hence $\mathbf{Z}(f) \supseteq \mathbf{O}(J)$, so $\mathbf{Coz}(f) \subseteq X \setminus \mathbf{O}(I)$. This follows that $\mathbf{O}(I) = \bigcup_{f \in I} \mathbf{Coz}(f) \subseteq X \setminus \mathbf{O}(I)$ and therefore $\mathbf{O}(I) \cap \mathbf{O}(J) = \emptyset$.

- (a \Leftarrow). Suppose that $f \in I$ and $g \in J$, then $Coz(f) \subseteq \mathbf{O}(I)$ and $Coz(g) \subseteq \mathbf{O}(J)$, thus $Coz(f) \cap Coz(g) \subseteq \mathbf{O}(I) \cap \mathbf{O}(J) = \emptyset$, so fg = 0 and therefore $IJ = \{0\}$.
- (b). By part (a), $I \text{Ann}(J) = \{0\}$ if and only if $\mathbf{O}(I) \cap \mathbf{O}(\text{Ann}(J)) = \emptyset$. By Proposition 2.7, it is equivalent to $\mathbf{O}(I) \cap (X \setminus \mathbf{O}(I))^{\circ} = \emptyset$. One can see easily that, it is equivalent to say that $\mathbf{O}(I) \subseteq \overline{\mathbf{O}(I)}$.
- (c). By part (a), $\operatorname{Ann}(I)\operatorname{Ann}(J) = \{0\}$ if and only if $\mathbf{O}(\operatorname{Ann}(I)) \cap \mathbf{O}(\operatorname{Ann}(J)) = \emptyset$; if and only if $(X \setminus \mathbf{O}(J))^{\circ} \cap (X \setminus \mathbf{O}(I))^{\circ} = \emptyset$, By Proposition 2.7. It easy to see that, it is equivalent to say that $\overline{\mathbf{O}(I)} \cup \overline{\mathbf{O}(I)} = X$.
 - (d). By part (b),

$$\overline{\mathbf{O}(I)} = \overline{\mathbf{O}(J)} \quad \Leftrightarrow \quad \mathbf{O}(I) \subseteq \overline{\mathbf{O}(J)} \text{ and } \mathbf{O}(J) \subseteq \overline{\mathbf{O}(I)}$$

$$\Leftrightarrow \quad I\mathrm{Ann}(J) = \{0\} \text{ and } J\mathrm{Ann}(I) = \{0\}$$

$$\Leftrightarrow \quad \mathrm{Ann}(J) \subseteq \mathrm{Ann}(I) \text{ and } \mathrm{Ann}(I) \subseteq \mathrm{Ann}(J)$$

$$\Leftrightarrow \quad \mathrm{Ann}(I) = \mathrm{Ann}(J) .$$

(e). By part (a) and Proposition 2.7,

$$\begin{split} I\mathbf{I}(U) &= \{0\} &\Leftrightarrow & \mathbf{O}(I) \cap \mathbf{O}(\mathbf{I}(U)) = \emptyset \\ &\Leftrightarrow & \mathbf{O}(I) \cap (X \setminus U)^\circ = \emptyset \\ &\Leftrightarrow & \mathbf{O}(I) \cap X \setminus \overline{U} = \emptyset \\ &\Leftrightarrow & \mathbf{O}(I) \subseteq \overline{U} \;. \end{split}$$

By the above theorem two ideals I and J are adjacent if and only if each maximal ideal of C(X) contains either I or J. Now we can conclude the following corollary from Lemmas 2.1 and 2.10 and Theorem 2.13.

Corollary 2.14. Suppose that I is a non-zero ideal of C(X) and $U \subseteq X$.

- (a) $I \in \mathbb{A}(X)^*$ if and only if $\overline{\mathbf{O}(I)} \neq X$.
- (b) $\mathbf{I}(U) \in \mathbb{A}(X)^*$ if and only if $\overline{U}^{\circ} \neq \emptyset$.

Corollary 2.15. Suppose that $I, J \in \mathbb{A}(X)^*$. I and J are orthogonal if and only if $\mathbf{O}(I) \cap \mathbf{O}(J) = \emptyset$ and $\overline{\mathbf{O}(I) \cup \mathbf{O}(J)} = X$.

Proof. It is easy to verify, by Theorem 2.13 and Corollary 2.14. \Box

Proposition 2.16. If the image of the mapping O is the family of all cozero sets of X, then C(X) is an i.a.c ring.

Proof. Suppose that S is a subset of X and set $I = \langle S \rangle$. For some $f \in C(X)$, we have $\mathbf{O}(I) = \mathbf{O}(S) = \mathbf{Coz}(f)$, by the assumption and Proposition 2.2. Thus by Lemma 2.1, Proposition 2.2 and Theorem 2.13,

$$g \in \text{Ann}(I) \Leftrightarrow gI = \{0\} \Leftrightarrow \mathbf{O}(\langle g \rangle) \cap \mathbf{O}(I) = \emptyset$$

 $\Leftrightarrow \text{Coz}(g) \cap \text{Coz}(f) = \emptyset \Leftrightarrow gf = 0 \Leftrightarrow g \in \text{Ann}(f)$

Hence
$$Ann(S) = Ann(I) = Ann(f)$$
, i.e. $C(X)$ is an i.a.c. ring.

3. Radius of the graph

In this section, some topological properties of X are linked to distance and eccentricity of vertices of $\mathbb{AG}(X)$, then by these facts we study the radius of the graph.

Lemma 3.1. For any ideals I and J in $\mathbb{A}(X)^*$,

- (a) d(I, J) = 1 if and only if $\mathbf{O}(I) \cap \mathbf{O}(J) = \emptyset$.
- (b) d(I,J) = 2 if and only if $\mathbf{O}(I) \cap \mathbf{O}(J) \neq \emptyset$ and $\overline{\mathbf{O}(I) \cup \mathbf{O}}(J) \neq X$.
- (c) d(I,J) = 3 if and only if $\mathbf{O}(I) \cap \mathbf{O}(J) \neq \emptyset$ and $\overline{\mathbf{O}(I) \cup \mathbf{O}(J)} = X$.

Proof. (a). It is clear, by Theorem 2.13.

- (b \Leftarrow). Since I is not adjacent to J, $\mathbf{O}(I) \cap \mathbf{O}(J) \neq \emptyset$, by Theorem 2.13. By the assumption there is an ideal K in $\mathbb{A}(X)^*$ such that K is adjacent to both ideals I and J, so Lemma 2.1, concludes that $\mathbf{O}(K) \neq \emptyset$ and also Theorem 2.13, deduces that $\mathbf{O}(I) \cap \mathbf{O}(K) = \emptyset$ and $\mathbf{O}(J) \cap \mathbf{O}(K) = \emptyset$, thus $\mathbf{O}(K) \cap (\mathbf{O}(I) \cup \mathbf{O}(J)) = \emptyset$, hence $\mathbf{O}(I) \cup \mathbf{O}(J) \neq X$.
- (b \Leftarrow). Theorem 2.13, follows that I is not adjacent to J. Set $H = \mathbf{O}(I) \cup \mathbf{O}(J)$ and $K = \mathbf{I}(H)$. Since $\emptyset \neq H \subseteq \overline{H}^{\circ} \neq \emptyset$, by Lemma 2.10 and Corollary 2.14, $\mathbf{I}(H) \in \mathbb{A}(X)^*$. Since $\mathbf{O}(I), \mathbf{O}(J) \subseteq H \subseteq \overline{H}$, $IK = JK = \{0\}$, by Theorem 2.13. Hence K is adjacent to both ideals I and J, thus d(I, J) = 3.
 - (c). It follows from (a), (b) and [7, Theorem 2.1].

Lemma 3.2. Let $f \in C(X)$ and I be an ideal of C(X) and $p \in \mathbf{O}(I)$. If $Coz(f) \subseteq \{p\}$ and p is an isolated point of X, then $f \in I$.

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Proof. Since $p \in \mathbf{O}(I)$, there is some $g \in I$ such that $p \in \mathrm{Coz}(g)$. Set

$$h(x) = \begin{cases} \frac{f(p)}{g(p)} & x = p\\ 0 & x \neq p \end{cases}$$

Since p is an isolated point, $h \in C(X)$. Now we have f = gh and therefore $f \in I$.

Proposition 3.3. Suppose that I is a non-zero annihilating ideal of C(X). The following statements hold.

- (a) ecc(I) = 3 if and only if O(I) is not singleton.
- (b) ecc(I) = 2 if and only if O(I) is singleton and |X| > 2.
- (c) ecc(I) = 1 if and only if O(I) is singleton and |X| = 2.

Proof. (a \Rightarrow). There is some $J \in \underline{\mathbb{A}(X)^*}$ such that d(I,J) = 3. Lemma 3.1, concludes that $\mathbf{O}(I) \cap \mathbf{O}(J) \neq \emptyset$ and $\overline{\mathbf{O}(I) \cup \mathbf{O}(J)} = X$. If $\mathbf{O}(I)$ is singleton, then $\mathbf{O}(I) \subseteq \mathbf{O}(J)$ and therefore $\overline{\mathbf{O}(J)} = \overline{\mathbf{O}(I) \cup \mathbf{O}(J)} = X$, so $J \notin \mathbb{A}(X)^*$, by Corollary 2.14, which is a contradiction.

(a \Leftarrow). There are distinct points p and q in $\mathbf{O}(I)$, so there are disjoint open sets $H, K \subseteq \mathbf{O}(I)$ such that $p \in H$ and $q \in K$. By Lemma 2.10, there is some ideal J such that $\mathbf{O}(J) = H \cup X \setminus \overline{\mathbf{O}(I)}$. Since $q \notin \overline{\mathbf{O}(J)}$ and $p \in \mathbf{O}(J)$, Lemma 2.1 and Corollary 2.14, concludes that $J \in \mathbb{A}(X)^*$. Then

$$H\subseteq \mathbf{O}(I)\cap \mathbf{O}(J)\quad \Rightarrow \quad \mathbf{O}(I)\cap \mathbf{O}(J)\neq \emptyset$$

$$\overline{\mathbf{O}(I) \cup \mathbf{O}(J)} \supseteq \overline{\mathbf{O}(I)} \cup \left(X \setminus \overline{\mathbf{O}(I)}\right) = X \quad \Rightarrow \quad \overline{\mathbf{O}(I) \cup \mathbf{O}(J)} = X$$

Hence d(I, J) = 3, by Lemma 3.1. Consequently, ecc(I) = 3.

(c \Rightarrow). There is some ideal $I \in \mathbb{A}(X)^*$ which is adjacent to any element of $\mathbb{A}(X)^*$. By (a), $\mathbf{O}(I)$ is singleton, thus there is some isolated point $p \in X$ such that $\mathbf{O}(I) = \{p\}$. Since $\emptyset \neq X \setminus \{p\}$ is open, by Lemma 2.10, there is some ideal J, such that $\mathbf{O}(J) = X \setminus \{p\}$. Since $\mathbf{O}(J) \neq \emptyset$ and $\overline{\mathbf{O}(J)} = X \setminus \{p\} \neq \emptyset$, $J \in \mathbb{A}(X)^*$, by Lemma 2.1 and Corollary 2.14. Since $\mathrm{ecc}(I) = 1$, $\mathrm{ecc}(J) \leqslant 2$, so $\mathbf{O}(J)$ is singleton, by part (a), and therefore |X| = 2.

(c \Leftarrow). $C(X) \cong \mathbb{R} \oplus \mathbb{R}$, so $\mathbb{AG}(X)$ is a star graph, by [7, Corollary 2.3], since $\mathbb{A}(X)^*$ has just two element, it follows that $\operatorname{ecc}(I) = 1$.

The following corollary is an immediate consequences of the above theorem.

Corollary 3.4. |X| = 2 if and only if $\mathbb{AG}(X)$ is star.

Now we can determine the radius of the graph.

Theorem 3.5. For any topological space X,

$$\operatorname{Rad}(\mathbb{AG}(X)) = \begin{cases} 1 & \text{if } |X| = 2 \\ 2 & \text{if } |X| > 2 \text{ and } X \text{ has an islated point.} \\ 3 & \text{if } |X| > 2 \text{ and } X \text{ has no any isloated point.} \end{cases}$$

Proof. It is straight consequence of Lemma 2.10 and Proposition 3.3.

4. Girth of the graph

In this section, first we correspond an equivalent topological property to leaf vertices, then we show that if $\mathbb{AG}(X)$ has a cycle then girth $\mathbb{AG}(X) = 3$. Finally we try to associate the graph properties of $\mathbb{AG}(X)$, the ring properties of C(X) and the topological properties of X.

Lemma 4.1. Suppose that Y is a clopen subset of X. For each ideal I of C(X), there are ideals I_1 and I_2 of C(X) such that $I = I_1 \oplus I_2$ and I_1 and I_2 are ideals of $M_Y \cong C(X \setminus Y)$ and $M_{X \setminus Y} \cong C(Y)$, respectively.

Proof. By this fact that since Y is clopen, $C(X) \cong C(Y) \oplus C(X \setminus Y)$, it is straightforward.

Proposition 4.2. Let $I \in \mathbb{A}(X)^*$. $X \setminus \overline{\mathbf{O}(I)}$ is singleton if and only if I is a leaf vertex.

Proof. ⇒). Suppose that $X \setminus \overline{\mathbf{O}(I)} = \{p\}$. Since $\{p\}$ is open, by Lemma 2.10, there is an ideal J such that $\mathbf{O}(J) = \{p\}$, then $\overline{\mathbf{O}(J)} = \{p\}$, and therefore $J \in \mathbb{A}(X)^*$, by Lemma 2.1 and Corollary 2.14. Also $\mathbf{O}(I) \cap \mathbf{O}(J) = \emptyset$, so I is adjacent to J, by Theorem 2.13. Suppose that K is adjacent to I and $Y = \overline{\mathbf{O}(I)}$. Then $\mathbf{O}(K) \cap \mathbf{O}(I) = \emptyset$, by Theorem 2.13, thus $\mathbf{O}(K) \subseteq X \setminus \overline{\mathbf{O}(I)} = \{p\}$. By Lemma 2.1, $\mathbf{O}(K) \neq \emptyset$, so $\mathbf{O}(K) = \{p\}$. Since $\{p\}$ is clopen, by Lemma 4.1, it follows that there are ideals K_1 and K_2 of $M_p \cong C(Y)$ and $M_Y \cong C(\{p\}) \cong \mathbb{R}$, respectively, such that $K = K_1 \oplus K_2$. If $K_1 \neq \{0\}$, then $0 \neq f \in K_1 \subseteq K$ exists, so there is some $q \in Y$ such that $f(q) \neq 0$, thus $p \neq q \in \mathbf{O}(K)$, which is a contradiction. Hence $K_1 = \{0\}$, since $K \neq \{0\}$, it follows that $K_2 = M_Y$, thus $K = M_Y$, and this completes the proof.

 \Leftarrow). Suppose that $X \setminus \overline{\mathbf{O}(I)}$ is not singleton, so distinct points p,q in $X \setminus \overline{\mathbf{O}(I)}$ exist. Using Lemma 2.1 and Corollary 2.14, it is easily to verify that there are disjoint open sets H_1 and H_2 containing p and q, respectively, which $H_1 \cap \mathbf{O}(I) = H_2 \cap \mathbf{O}(I) = \emptyset$. Now Lemma 2.10, implies that there are ideals J_1 and J_2 such that $\mathbf{O}(J_1) = H_1$ and $\mathbf{O}(J_2) = H_2$, clearly $J_1, J_2 \in \mathbb{A}(X)^*$. Then $\mathbf{O}(I) \cap \mathbf{O}(J_1) = \mathbf{O}(I) \cap \mathbf{O}(J_2) = \emptyset$. So, by Theorem 2.13, I is adjacent to both ideals J_1 and J_2 . \square

Lemma 4.3. Suppose that $I, J \in \mathbb{A}(X)^*$ and they are not leaf vertices. The following statements hold.

- (a) $\mathbf{O}(I) \cap \mathbf{O}(J) = \emptyset$ and $\mathbf{O}(I) \cup \mathbf{O}(J) \neq X$ if and only if gi(I, J) = 3.
- (b) If $\mathbf{O}(I) \cap \mathbf{O}(J) = \emptyset$ and $\overline{\mathbf{O}(I) \cup \mathbf{O}(J)} = X$, then gi(I, J) = 4.
- (c) If $\mathbf{O}(I) \cap \mathbf{O}(J) \neq \emptyset$ and $\overline{\mathbf{O}(I)} = \overline{\mathbf{O}(J)}$, then gi(I, J) = 4.
- (d) Suppose that $\mathbf{O}(I) \cap \mathbf{O}(J) \neq \emptyset$ and $\mathbf{O}(I) \neq \mathbf{O}(J)$. Then $X \setminus \mathbf{O}(I) \cup \mathbf{O}(J)$ is not singleton if and only if gi(I, J) = 4.
- (e) $\mathbf{O}(I) \cap \mathbf{O}(J) \neq \emptyset$, $\overline{\mathbf{O}(I)} \neq \overline{\mathbf{O}(J)}$ and $X \setminus \overline{\mathbf{O}(I) \cup \mathbf{O}(J)}$ is singleton if and only if gi(I, J) = 5.

Proof. (a \Rightarrow). Set $H = \mathbf{O}(I) \cup \mathbf{O}(J)$ and $K = \mathbf{I}(H)$. Since $\overline{H} \neq X$ and $\overline{H}^{\circ} \neq \emptyset$, $K \in \mathbb{A}(X)^*$, by Lemma 2.1 and Corollary 2.14. Since $\mathbf{O}(I), \mathbf{O}(J) \subseteq H \subseteq \overline{H}$, by Theorem 2.13, K is adjacent to both ideals I and J. By the assumption and Theorem 2.13, I adjacent to I, hence $\mathbf{gi}(I,J) = 3$.

(a \Leftarrow). By the assumption, I is adjacent to J and some $K \in \mathbb{A}(X)^*$ exists such that K is adjacent to both ideals I and J, so $\mathbf{O}(I) \cap \mathbf{O}(J) = \emptyset$, $\mathbf{O}(I) \cap \mathbf{O}(K) = \emptyset$

and $\mathbf{O}(J) \cap \mathbf{O}(K) = \emptyset$, by Theorem 2.13. Hence $(\mathbf{O}(I) \cup \mathbf{O}(J)) \cap \mathbf{O}(K) = \emptyset$. Since $K \neq \{0\}$, $\mathbf{O}(K) \neq \emptyset$, by Lemma 2.1, and therefore $\mathbf{O}(I) \cup \mathbf{O}(J) \neq X$.

- (b). The assumption and part (a) imply that $gi(I, J) \ge 4$ and Theorem 2.13, concludes that I is adjacent to J and Ann(I) is adjacent to Ann(J). Since I and J is adjacent to Ann(I) and Ann(J), respectively, the proof is complete.
- (c). We can conclude from the assumption and part (a), that $gi(I, J) \ge 4$. Since $\mathbf{O}(I) \subseteq \overline{\mathbf{O}(J)}$ and $\mathbf{O}(J) \subseteq \overline{\mathbf{O}(J)}$, by Theorem 2.13, I and J are adjacent to $\mathrm{Ann}(J)$ and $\mathrm{Ann}(I)$, respectively. Since I and J are adjacent to $\mathrm{Ann}(I)$ and $\mathrm{Ann}(J)$, respectively, the proof is complete.
- $(d \Rightarrow)$. It easy to see that there are two distinct open sets H_1 and H_2 such that $H_1 \cap \mathbf{O}(I) = H_1 \cap \mathbf{O}(J) = H_2 \cap \mathbf{O}(I) = H_2 \cap \mathbf{O}(J) = \emptyset$. Then, by Lemma 2.10, there are two ideals K_1 and K_2 such that $\mathbf{O}(K_1) = H_1$ and $\mathbf{O}(K_2) = H_2$, it is clear that $K_1, K_2 \in \mathbb{A}(X)^*$, by Lemma 2.1 and Corollary 2.14. Now Theorem 2.13, concludes that both vertices I and J are adjacent to both vertices K_1 and K_2 , thus gi(I, J) = 4, by part (a).
- (d \Leftarrow). By Theorem 2.13, I is adjacent to J, since gi(I, J) = 4, it follows that there are distinct vertices K_1 and K_2 which are adjacent to both vertices I and J, so I + J is adjacent to both vertices K_1 and K_2 . Now Propositions 2.3 and 4.2, conclude that $X \setminus \overline{\mathbf{O}(I) \cup \mathbf{O}(J)} = X \setminus \overline{\mathbf{O}(I + J)}$ is not singleton.
- (e \Rightarrow). By parts (a) and (d), $gi(I,J) \geqslant 5$. If $\mathbf{O}(I) \subseteq \overline{\mathbf{O}(J)}$, then $\overline{\mathbf{O}(I)} \subseteq \overline{\mathbf{O}(J)}$, so $X \setminus \overline{\mathbf{O}(I) \cup \mathbf{O}(J)} = X \setminus \overline{\mathbf{O}(J)}$ and therefore $X \setminus \overline{\mathbf{O}(J)}$ is singleton, by the assumption. Now Proposition 4.2, concludes that J is a leaf vertex, which contradicts the assumption, so $\mathbf{O}(I) \not\subseteq \overline{\mathbf{O}(J)}$, one can show similarly that $\mathbf{O}(J) \not\subseteq \overline{\mathbf{O}(I)}$, so $H_1 = \mathbf{O}(I) \setminus \overline{\mathbf{O}(J)}$ and $H_2 = \mathbf{O}(J) \setminus \overline{\mathbf{O}(I)}$ are nonempty open sets, thus, Lemma 2.10, implies that there are ideals K_1 and K_2 such that $\mathbf{O}(K_1) = H_1$ and $\mathbf{O}(K_2) = H_2$, it is evident that $K_1, K_2 \in \mathbb{A}(X)^*$, by Lemma 2.1 and Corollary 2.14. Since $X \setminus \overline{\mathbf{O}(I) \cup \mathbf{O}(J)}$ is nonempty open set, there is an ideal K_3 such that $\mathbf{O}(K_3) = X \setminus \overline{\mathbf{O}(I) \cup \mathbf{O}(J)}$, it is clear that $K_3 \in \mathbb{A}(X)^*$, by Lemma 2.1 and Corollary 2.14. Then

$$\mathbf{O}(I) \cap \mathbf{O}(K_2) = \mathbf{O}(K_2) \cap \mathbf{O}(K_1) = \mathbf{O}(K_1) \cap \mathbf{O}(J)$$
$$= \mathbf{O}(J) \cap \mathbf{O}(K_3) = \mathbf{O}(K_3) \cap \mathbf{O}(I) = \emptyset$$

so gi(I, J) = 5.

$$(e \Leftarrow)$$
. It is clear, by Parts (a)-(d).

It is clear that if |X| = 2, then AG(X) has no any cycle. In the following theorem we show that if AG(X) has an cycle then the girth of the graph is 3.

Theorem 4.4. If |X| > 2, then girth $A\mathbb{G}(X) = 3$.

Proof. It easy to see that there are mutually disjoint nonempty open sets G_1 , G_2 and G_3 . By Lemma 2.10, there are ideals I_1 , I_2 and I_3 , such that $\mathbf{O}(I_1) = G_1$, $\mathbf{O}(I_2) = G_2$ and $\mathbf{O}(I_3) = G_3$, evidently, $I_1, I_2, I_3 \in \mathbb{A}(X)^*$, by Lemma 2.1 and Corollary 2.14. By Theorem 2.13, I_1 is adjacent to I_2 , I_2 is adjacent to I_3 and I_3 is adjacent to I_1 , hence girth $\mathbb{AG}(X) = 3$.

Theorem 4.5. The following statements are equivalent.

- (a) X has an isolated point.
- (b) \mathbb{R} is a direct summand of C(X).
- (c) $\mathbb{AG}(X)$ has a leaf vertex.

(d) $\mathbb{AG}(X)$ is not triangulated.

Proof. (a \Leftrightarrow b) and (c \Rightarrow d) are clear and (a \Leftrightarrow c) follows from Proposition 4.2. (d \Rightarrow a) Suppose that X has no any isolated point and $I \in \mathbb{A}(X)^*$. Then $X \setminus \overline{\mathbf{O}(I)}$ is not singleton, so it has two distinct points p and q, so there are disjoint open sets G_1 and G_2 , such that $G_1 \cap \mathbf{O}(I) = G_2 \cap \mathbf{O}(I) = \emptyset$. By Lemma 2.10, there are $J, K \in \mathbb{A}(X)^*$, such that $\mathbf{O}(J) = G_1$ and $\mathbf{O}(K) = G_2$. Thus I is adjacent to J, J is adjacent to K and K is adjacent I. Consequently, $\mathbb{AG}(X)$ is triangulated. \square

5. Dominating number

In the last section, we give an upper bound and a lower bound for dominating number of the graph by topological notions, then the chromatic number and the clique number of the graph are studied. Finally, we introduce the disjoint open set graph on a topology and give the radius, dominating number, diameter and the girth of this graph.

Theorem 5.1. $c(X) \leq dt(\mathbb{AG}(X)) \leq w(X)$, for each topological space X.

Proof. Suppose that \mathcal{U} is a family of mutually disjoint nonempty open sets. If $\overline{\bigcup \mathcal{U}} \neq X$, then $\mathcal{V} = \mathcal{U} \cup \left\{ X \setminus \overline{\bigcup \mathcal{U}} \right\}$ is a family of mutually open sets which $\overline{\bigcup \mathcal{V}} = X$, so without loss of generality we can assume that $\overline{\bigcup \mathcal{U}} = X$. For each $U \in \mathcal{U}$, there is some $I_U \in \mathbb{A}(X)^*$ such that $\mathbf{O}(I_U) = U$, by Lemma 2.10. Since $U \neq \emptyset$ and $\overline{U} \neq X$, it follows that $I_U \in \mathbb{A}(X)^*$, by Lemma 2.1 and Corollary 2.14. Now suppose that D is a dominating set, then for each $U \in \mathcal{U}$, there is some ideal J_U in D such that J_U is adjacent to $\sum_{U \neq V \in \mathcal{U}} I_V$, now Theorem 2.13, implies that $\mathbf{O}(J_U) \cap \mathbf{O}\left(\mathbf{O}\left(\sum_{U \neq V \in \mathcal{U}} I_V\right)\right) = \emptyset$, thus $\mathbf{O}(J_U) \cap \left(\bigcup_{U \neq V \in \mathcal{U}} U\right) = \emptyset$. Suppose that $J_U = J_{U'}$, then $\mathbf{O}(J_U) = \mathbf{O}(J_{U'})$. If $U \neq U'$, then

$$\begin{aligned} \mathbf{O}(J_{\scriptscriptstyle U}) \cap \bigcap \mathcal{U} &= \mathbf{O}(J_{\scriptscriptstyle U}) \cap \left[\left(\bigcup_{U \neq V \in \mathcal{U}} V \right) \cup \left(\bigcup_{U' \neq V \in \mathcal{U}} V \right) \right] \\ &= \left[\mathbf{O}(J_{\scriptscriptstyle U}) \cap \left(\bigcup_{U \neq V \in \mathcal{U}} V \right) \right] \cup \left[\mathbf{O}(J_{\scriptscriptstyle U}) \cap \left(\bigcup_{U' \neq V \in \mathcal{U}} V \right) \right] = \emptyset. \end{aligned}$$

Thus $\overline{\bigcup \mathcal{U}} \neq X$, which contradicts our assumption. Hence U = U', so $|\mathcal{U}| \leq |D|$, and consequently $c(X) \leq \operatorname{dt}(\mathbb{AG}(X))$.

Now suppose that \mathcal{B} is a base for X, without loss of generality we can assume that every element of \mathcal{B} is not empty. For each $B \in \mathcal{B}$, there is some $0 \neq f_B \in C(X)$ such that $\emptyset \neq \operatorname{Coz}(f_B) \subseteq B$. Clearly, we can choose f such that $\overline{\operatorname{Coz}(f_B)} \neq X$. Lemma 2.1, concludes that $\mathbf{O}(\langle f_B \rangle) = \operatorname{Coz}(f_B)$, so $\mathbf{O}(\langle f_B \rangle) \neq \emptyset$ and $\overline{\mathbf{O}(\langle f_B \rangle)} \neq X$, for each $B \in \mathcal{B}$, thus $\langle f_B \rangle \in \mathbb{A}(X)^*$, by Lemma 2.1 and Corollary 2.14. For each $J \in \mathbb{A}(X)^*$, $\overline{\mathbf{O}(I)} \neq X$, by Corollary 2.14, so $(X \setminus \mathbf{O}(I))^\circ \neq \emptyset$, thus $B \in \mathcal{B}$ exists such that $B \subseteq (X \setminus \mathbf{O}(I))^\circ$, hence $\mathbf{O}(\langle f_B \rangle) \subseteq X \setminus \mathbf{O}(I)$, consequently, $\mathbf{O}(\langle f_B \rangle) \cap \mathbf{O}(I) = \emptyset$, therefore Theorem 2.13, implies that $\langle f_B \rangle$ is adjacent to I. Hence $\{\langle f_B \rangle : B \in \mathcal{B}\}$ is a dominating set, since $|\{\langle f_B \rangle : B \in \mathcal{B}\}| \leqslant |\mathcal{B}|$, it follows that $\operatorname{dt}(\mathbb{AG}(X)) \leqslant w(X)$.

Corollary 5.2. If X is discrete, then $dt(\mathbb{AG}(X)) = |X|$.

Proof. For each $p \in X$, since $\{p\}$ is open, there is some $I_p \in \mathbb{A}(X)^*$ such that $\mathbf{O}(I_p) = \{p\}$, by Lemma 2.10. For each $I \in \mathbb{A}(X)^*$, $\mathbf{O}(I) \neq X$, by Corollary 2.14, thus $p \in X \setminus \mathbf{O}(I)$ exists, hence $\mathbf{O}(I) \cap \mathbf{O}(I_p) = \emptyset$, so I is adjacent to I_p . This shows that $\{I_p : p \in X\}$ is a dominating set, this implies that $\mathrm{dt}(\mathbb{AG}(X)) \leqslant |X|$. It is clear that |X| = c(X), so $|X| \leqslant \mathrm{dt}(\mathbb{AG}(X))$, by Theorem 5.1. Consequently, $\mathrm{dt}(\mathbb{AG}(X)) = |X|$.

Theorem 5.3. $dt(\mathbb{AG}(X))$ is finite if and only if |X| is finite. In this case, $dt(\mathbb{AG}(X)) = |X|$.

Proof. \Rightarrow). Suppose that |X| is infinite. Clearly c(X) is infinite, so $dt(\mathbb{AG}(X))$ is infinite, by Theorem 5.1.

 \Leftarrow). If |X| is finite, then X is discrete, so $dt(\mathbb{AG}(X)) = |X|$ is finite, by Corollary 5.2.

Theorem 5.4. $\chi \mathbb{AG}(X) = \text{clique}\mathbb{AG}(X) = c(X)$, for each topological space X.

Proof. It is an immediate consequence of Proposition 1.2, Lemma 2.10 and Theorem 2.13. \Box

Definition 5.5. Suppose that (X, τ) is a topological space, set $\tau^* = \{G \in \tau : G \neq \emptyset \text{ and } (X \setminus G)^\circ \neq \emptyset\}$. We define a graph with vertices of elements of τ^* , where two distinct vertices G and H are adjacent if and only if $G \cap H = \emptyset$. We call this graph disjoint open set graph and denote by $\mathbb{DG}(X)$.

Lemma 5.6. Suppose that G and G' are two graphs. If φ is a map from the vertices of G onto vertices G' such that $\{u,v\} \in E(G)$ if and only if $\{\varphi(u), \varphi(v)\} \in E(G')$ and $\{u,v\} \in E(G)$ implies that $\varphi(u) \neq \varphi(v)$, then

- (a) diam(G') = diam(G).
- (b) Rad(G') = Rad(G).
- (c) $girth(G') \leq girth(G)$.
- (d) $dt(G') \leq dt(G)$.
- (e) $\operatorname{clique}(G') = \operatorname{clique}(G)$.
- (f) $\chi(G') = \chi(G)$.
- (g) G is complemented if and only if G' is complemented.

Proof. Suppose that $u, v \in V(G)$. $u = u_0 - u_1 - u_2 - \cdots - u_n = v$ is a path if and only if $\varphi(u) = \varphi(u_0) - \varphi(u_1) - \cdots - \varphi(u_n) = \varphi(v)$ is a path, so $d(\varphi(u), \varphi(v)) = d(u, v)$. By this fact the proof of the statements (a),(b) and (c) are clear.

- (d). Readily, we can see that if $A \subseteq V(G)$ is a dominating set, then $\varphi(A) \subseteq V(G')$ is a dominating set, so $dt(G') \leq dt(G)$.
- (e). Similar to the proof part (d) we can show that $\operatorname{clique}(G') \leq \operatorname{clique}(G)$. Now suppose that $S \subseteq V(G')$ is a clique set. Put $u_v \in \varphi^{-1}(u)$, $\operatorname{clearly} \{u_v\}_{v \in S}$ is a clique set and $|\{u_v\}_{v \in S}| = |S|$, hence $\operatorname{clique}(G) \leq \operatorname{clique}(G')$ and consequently the equality holds.
- (f). Suppose that for some cardinal number x, the map $f:G'\to x$ is a coloring, (i.e., if $\{u,v\}\in E(G)$, then $f(u)\neq f(v)$). Then, clearly, $f\circ\varphi:G\to x$ is a coloring, so $\chi(G')\leqslant \chi(G)$. Now suppose that $g:G\to y$ is a coloring, for some cardinal number y. For each $v\in V(G)$, choose $u_v\in V(G)$ such that $\varphi(u_v)=v$. Let $\bar{g}:G'\to y$ be given by $\bar{g}(v)=g(u_v)$. Evidently, \bar{g} is a coloring, thus $\chi(G')\leqslant \chi(G)$, and consequently $\chi(G)=\chi(G')$.

(g). It is easy. \Box

The following Theorem is an immediate consequence of the above Lemma, Proposition 1.2, Theorems 3.5, 4.4, and 5.4 and this fact that the annihilating-ideal graph of a ring is complemented.

Theorem 5.7. Let X be a topological space. The following statements hold.

(a)

$$\operatorname{diam}(\mathbb{DG}(X)) = \begin{cases} 1 & \text{if } |X| = 2\\ 3 & \text{if } |X| > 2 \end{cases}$$

(b) |X| = 2 if and only if $\mathbb{DG}(X)$ is a star graph.

(c)

$$\operatorname{Rad}(\mathbb{DG}(X)) = \begin{cases} 1 & \text{if } |X| = 2 \\ 2 & \text{if } |X| > 2 \text{ and } X \text{ has an islated point.} \\ 3 & \text{if } |X| > 2 \text{ and } X \text{ has no any isloated point.} \end{cases}$$

- (d) If |X| > 2, then $girth(\mathbb{DG}(X)) = 3$.
- (e) $\chi(\mathbb{DG}(X)) = \text{clique}\mathbb{DG}(X) = c(X)$.
- (g) $\mathbb{DG}(X)$ is complemented.

Finally, we note that it is clear that $\chi(\mathbb{DG}(X)) = c(X)$.

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