Decomposition formulas associated with the multivariable confluent hypergeometric functions

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The main object of this work is to show how some rather elementary techniques based upon certain inverse pairs of symbolic operators would lead us easily to several decomposition formulas associated with confluent hypergeometric functions of two and more variables. Many operator identities involving these pairs of symbolic operators are first constructed for this purpose. By means of these operator identities several decomposition formulas are found, which express the aforementioned hypergeometric functions in terms of such simpler functions as the products of the Gauss hypergeometric functions.

Keywords: decomposition formulas; multiple confluent hypergeometric functions; inverse pairs of symbolic operators; Gauss hypergeometric function; multiple Lauricella functions; Bessel function of many variables

1 Introduction

A great interest in the theory of multiple hypergeometric functions is motivated essentially by the fact that the solutions of many applied problems involving, for example, partial differential equations are obtainable with the help of such hypergeometric functions (see, for details, [25], p.47 et seq., Section 1.7; see also the works [18, 19] and the references cited therein). For instance, the energy absorbed by some nonferromagnetic conductor sphere included in an internal magnetic field can be calculated with the help of such functions [15]. Hypergeometric functions in several variables are used in physical and quantum chemical applications as well (cf. [17, 23]). Especially, many problems in gas dynamics lead to solutions of degenerate second-order partial differential equations which are then solvable in terms of multiple hypergeometric functions.

We note that Riemann's functions and the fundamental solutions of the degenerate second-order partial differential equations are expressible by means of hypergeometric functions in several variables [16, 21, 22, 24]. In investigation of the boundary-value problems for these partial differential equations, we need decompositions for hypergeometric functions in several variables in terms of simpler hypergeometric functions of the Gauss and Appell types.

In addition to the Gaussian functions, which have received the greatest attention in the literature, confluent functions have been considered. For example, twenty confluent hypergeometric functions of two variables exist; seven were introduced by Humbert [13], and the remaining ones by Horn [12] and by Borngässer [2]. Certain confluent functions in three variables were considered by Jain [14] and by Exton [9], but the entire set has not been given, i.e. confluent functions in all directions of research have been little studied with respect to other hypergeometric functions. While a brief account of such functions is presented in [25], we shall include the definition of the important special class of confluent functions and find the decomposition formulas for these functions.

Burchnall and Chaundy introduced the symbolic operators ∇ and Δ(see [3, 4]) by means of which they presented a number of expansion and decomposition formulas for some double hypergeometric functions (only seven of which are confluent functions) in terms of the classical Gauss hypergeometric function of one variable. Recently Hasanov and Srivastava [10, 11] generalized the Burchnall-Chaundy's operators and by making use of some technique based upon certain inverse pairs of symbolic operators, the authors investigate several decomposition formulas associated with Lauricella's (but no confluent) hypergeometric functions of many variables when a number of variables exceeds two.

In this paper we introduce other multivariable analog of Burchnall-Chaundy's operators and find the decomposition formulas for some confluent hypergeometric functions of two and more variables.

2 Symbolic operators

Burchnall and Chaundy [3, 4], and Chaundy [5], give a number of expansions of double hypergeometric functions in series of simpler hypergeometric functions. Their method is based upon the inverse pair of symbolic operators

$$\nabla (h) := \frac{\Gamma (h) \Gamma (\delta_1 + \delta_2 + h)}{\Gamma (\delta_1 + h) \Gamma (\delta_2 + h)}, \quad \Delta (h) := \frac{\Gamma (\delta_1 + h) \Gamma (\delta_2 + h)}{\Gamma (h) \Gamma (\delta_1 + \delta_2 + h)}, \tag{2.1}$$

where

$$\delta_1 := x_1 \frac{\partial}{\partial x_1}, \delta_2 := x_2 \frac{\partial}{\partial x_2}. \tag{2.2}$$

The symbolic operators defined by (2.1) and (2.2) are limited only to functions of two variables, therefore recently Hasanov and Srivastava [10, 11] generalized these operators in the forms

$$\tilde{\nabla}_{x_1:x_2,...,x_m} (h) := \frac{\Gamma (h) \Gamma (\delta_1 + ... + \delta_m + h)}{\Gamma (\delta_1 + h) \Gamma (\delta_2 + ... + \delta_m + h)} \tag{2.3}$$

and

$$\tilde{\Delta}_{x_1:x_2,...,x_m} (h) := \frac{\Gamma (\delta_1 + h) \Gamma (\delta_2 + ... + \delta_m + h)}{\Gamma (h) \Gamma (\delta_1 + ... + \delta_m + h)}, \tag{2.4}$$

where

$$\delta_i := x_i \frac{\partial}{\partial x_i} \quad (i = 1, ..., m).$$

With the help of symbolic operators defined by (2.3) and (2.4), decomposition formulas for many multiple hypergeometric functions have been found. For example [10, 11],

$$\begin{aligned} &F_{\mathbf{A}}^{(m)} (a, b_1, ..., b_m; c_1, ..., c_m; x_1, ..., x_m) \\ &= \tilde{\nabla}_{x_1:x_2,...,x_m} (a) F (a, b_1; c_1; x_1) F_{\mathbf{A}}^{(m-1)} (a, b_2, ..., b_m; c_2, ..., c_m; x_2, ..., x_m), \\ &F_{\mathbf{B}}^{(m)} (c_1, ..., c_m, b_1, ..., b_m; a; x_1, ..., x_m) \\ &= \tilde{\Delta}_{x_1:x_2,...,x_m} (a) F (c_1, b_1; a; x_1) F_{\mathbf{B}}^{(m-1)} (c_2, ..., c_m, b_2, ..., b_m; a; x_2, ..., x_m), \end{aligned} \tag{2.5}$$

where

$$F (a, b, c; x) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!}, |x| < 1, \tag{2.6}$$

$$\begin{aligned} &F_{\mathbf{A}}^{(m)} (a, b_1, ..., b_m; c_1, ..., c_m; x_1, ..., x_m) \\ &= \sum_{i_1,...,i_m=0}^{\infty} \frac{(a)_{i_1+...+i_m} (b_1)_{i_1} \cdots (b_m)_{i_m}}{(c_1)_{i_1} \cdots (c_m)_{i_m}} \frac{x_1^{i_1} \ldots x_m^{i_m}}{i_1! \ldots i_m!}, \ |x_1| + |x_2| + ... + |x_m| < 1; \end{aligned} \tag{2.7}$$

$$\begin{aligned} &F_{\mathbf{B}}^{(m)} (c_1, ..., c_m, b_1, ..., b_m; a; x_1, ..., x_m) = \\ &\sum_{i_1,...,i_m=0}^{\infty} \frac{(c_1)_{i_1} \cdots (c_m)_{i_m} (b_1)_{i_1} \cdots (b_m)_{i_m}}{(a)_{i_1+...+i_m}} \frac{x_1^{i_1} \ldots x_m^{i_m}}{i_1! \ldots i_m!}, \max (|x_1|, |x_2|, ..., |x_m|) < 1. \end{aligned}$$

Here $(\mu)_k := \Gamma (\mu + k) / \Gamma (\mu)$ is a Pochhammer symbol; F is Gauss hypergeometric function of one variable [7, Chapter 2]; $F_{\mathbf{A}}^{(m)}$ and $F_{\mathbf{B}}^{(m)}$ are multiple Lauricella hypergeometric functions [1, p.115].

However, the recurrence of this formula did not allow further advancement in the direction of increasing the number of variables.

Further study of the properties of the Lauricella function defined by (2.7) showed that the formula (2.5) can be reduced to a more convenient form.

Lemma 1[8]. *The following formula holds true at* $m \in N$

$$\begin{aligned} &F_{A}^{(m)}(a, b_1, ..., b_m; c_1, ..., c_m; x_1, ..., x_m) = \sum_{\substack{n_{i,j}=0 \\ (2 \le i \le j \le m)}}^{\infty} \frac{(a)_{N_2(m,m)}}{\underset{(2 \le i \le j \le m)}{n_{2,2}! n_{2,3}! \cdots n_{i,j}! \cdots n_{m,m}!}} \\ &\cdot \prod_{k=1}^{m} \frac{(b_k)_{M_2(k,m)}}{(c_k)_{M_2(k,m)}} x_k^{M_2(k,m)} F \left[a + N_2(k,m), b_k + M_2(k,m); c_k + M_2(k,m); x_k\right], \end{aligned} \tag{2.8}$$

where

$$M_l(k,m) = \sum_{i=l}^{k} n_{i,k} + \sum_{i=k+1}^{m} n_{k+1,i}, \quad N_l(k,m) = \sum_{i=l}^{k+1} \sum_{j=i}^{m} n_{i,j}, \ l \in N.$$

It should be noted that the symbolic operators δ_1 and δ_2 defined by (2.2) in the one-dimensional case take the form $\delta := xd/dx$ and such an operator is used in solving problems of the operational calculus [20, p.26].

We now introduce here the other multivariable analogues of the Burchnall-Chaundy symbolic operators $\nabla(h)$ and $\Delta(h)$ defined by (2.1):

$$\tilde{\nabla}_{x;y}^{m,n}(h) := \frac{\Gamma(h)\Gamma(h+\delta_1+...+\delta_m-\sigma_1-...-\sigma_n)}{\Gamma(h+\delta_1+...+\delta_m)\Gamma(h-\sigma_1-...-\sigma_n)} \tag{2.9}$$

$$= \sum_{k=0}^{\infty} \frac{(-\delta_1-...-\delta_m)_k(\sigma_1+...+\sigma_n)_k}{(h)_k k!}, \tag{2.10}$$

$$\tilde{\Delta}_{x;y}^{m,n}(h) := \frac{\Gamma(h+\delta_1+...+\delta_m)\Gamma(h-\sigma_1-...-\sigma_n)}{\Gamma(h)\Gamma(h+\delta_1+...+\delta_m-\sigma_1-...-\sigma_n)} \tag{2.11}$$

$$= \sum_{k=0}^{\infty} \frac{(\delta_1+...+\delta_m)_k(-\sigma_1-...-\sigma_n)_k}{(1-h)_k k!}, \tag{2.12}$$

where

$$x := (x_1,...,x_m), y := (y_1,...,y_n), \tag{2.13}$$

$$\delta_i := x_i\frac{\partial}{\partial x_i}, \sigma_j := y_j\frac{\partial}{\partial y_j}, \quad i=1,...,m,\ j=1,...,n;\ m,n\in \mathrm{N}.$$

In addition, we consider operators which are equal to the Hasanov-Srivastava's symbolic operators $\tilde{\nabla}(h)$ and $\tilde{\Delta}(h)$ defined by (2.3) and (2.4):

$$\tilde{\nabla}_{x;-}^{m,0}(h) := \tilde{\nabla}_{x_1:x_2,...,x_m}(h), \quad \tilde{\Delta}_{x;-}^{m,0}(h) := \tilde{\Delta}_{x_1:x_2,...,x_m}(h), m\in \mathrm{N};$$

$$\tilde{\nabla}_{-;y}^{0,n}(h) := \tilde{\nabla}_{-y_1:-y_2,...,-y_n}(h), \quad \tilde{\Delta}_{-;y}^{0,n}(h) := \tilde{\Delta}_{-y_1:-y_2,...,-y_n}(h), n\in \mathrm{N}.$$

It is obvious that

$$\tilde{\nabla}_{x;-}^{1,0}(h) = \tilde{\Delta}_{x;-}^{1,0}(h) = \tilde{\nabla}_{-;y}^{0,1}(h) = \tilde{\Delta}_{-;y}^{0,1}(h) = 1.$$

Lemma 2. *Let be $f := f(x,y)$ function with variables x and y in (2.13). Then following equalities hold true for any m, $n \in \mathrm{N}$:*

$$\left(-\sum_{i=1}^{m} x_i\frac{\partial}{\partial x_i}\right)_k f = (-1)^k k! \sum_{K}\prod_{s=1}^{m}\frac{x_s^{i_s}}{i_s!}\cdot\frac{\partial^k f}{\partial x_1^{i_1}...\partial x_m^{i_m}},\ \ k\in \mathrm{N}\cup\{0\}; \tag{2.14}$$

$$\left(\sum_{j=1}^{n} y_j\frac{\partial}{\partial y_j}\right)_k f = \begin{cases} f, & k=0, \\ k!\sum\limits_{l=1}^{k}\begin{pmatrix} l-1 \\ k-1\end{pmatrix}\sum\limits_{L}\prod\limits_{s=1}^{n}\frac{y_s^{j_s}}{j_s!}\cdot\frac{\partial^l f}{\partial y_1^{j_1}...\partial y_n^{j_n}}, & k\in \mathrm{N}, \end{cases} \tag{2.15}$$

where

$$K := \{(i_1,...,i_m) : i_1\geq 0,...,i_m\geq 0, i_1+...+i_m = k\},$$

$$L := \{(j_1,...,j_n) : j_1\geq 0,...,j_n\geq 0, j_1+...+j_n = l\}.$$

The lemma 2 is proved by method of mathematical induction.

3 Decomposition formulas in the two-variable case

In this section we shall give the decomposition formulas for the following hypergeometric functions of two variables [7, pp.225-226]:

$$H_2(a,b,c,e,d;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(b)_m(c)_n(e)_n}{(d)_m m!n!}x^m y^n,$$

$$\mathrm{H}_2(a,b,c;d;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(b)_m(c)_n}{(d)_m m!n!}x^m y^n,\ |x|<1, \tag{3.1}$$

$$\mathrm{H}_3(a,b;d;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(b)_m}{(d)_m m!n!}x^m y^n,\ |x|<1, \tag{3.2}$$

$$\mathrm{H}_4(a,b;d;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(b)_n}{(d)_m m!n!}x^m y^n, \tag{3.3}$$

$$\mathrm{H}_5(a;d;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}}{(d)_m m!n!}x^m y^n, \tag{3.4}$$

$$\mathrm{H}_{11}(a,b,c;d;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(b)_n(c)_n}{(d)_m m!n!}x^m y^n,\ |y|<1, \tag{3.5}$$

where a,b,c,d,e are complex numbers, $d\neq 0,-1,-2,....$ We note that hypergeometric functions defined by (3.1)-(3.5) are confluent functions (In the literature it is customary to denote the confluent functions through the capital letters of the Greek alphabet).

In the special case when $m=n=1$, the symbolic operators (2.9)-(2.12) and equalities (2.14)-(2.15) take a simpler forms:

$$\tilde{\nabla}_{x,y}^{1,1}(h):=\frac{\Gamma(h)\Gamma(h+\delta-\sigma)}{\Gamma(h+\delta)\Gamma(h-\sigma)}=\sum_{k=0}^{\infty}\frac{(-\delta)_k(\sigma)_k}{(h)_k k!}, \tag{3.6}$$

$$\tilde{\Delta}_{x,y}^{1,1}(h):=\frac{\Gamma(h+\delta)\Gamma(h-\sigma)}{\Gamma(h)\Gamma(h+\delta-\sigma)}=\sum_{k=0}^{\infty}\frac{(\delta)_k(-\sigma)_k}{(1-h)_k k!}, \tag{3.7}$$

$$\left(-x\frac{\partial}{\partial x}\right)_k f=(-1)^k x^k\frac{\partial^k f}{\partial x^k},\ \ k\in\mathrm{N}\cup\{0\}, \tag{3.8}$$

$$\left(y\frac{\partial}{\partial y}\right)_k f=\begin{cases} f, & k=0,\\ k!\sum\limits_{l=1}^{k}\begin{pmatrix} l-1\\ k-1\end{pmatrix}\frac{y^l}{l!}\frac{\partial^l f}{\partial y^l}, & k\in\mathrm{N},\end{cases} \tag{3.9}$$

where

$$\delta:=x\frac{\partial}{\partial x},\quad \sigma:=y\frac{\partial}{\partial y}.$$

By applying the pair of symbolic operators (3.6) and (3.7), we find the following set of operator identities:

$$H_2(a,b,c,d;e;x,y)=\tilde{\nabla}_{x,y}^{1,1}(a)F(a,b;e;x)F(c,d;1-a;-y), \tag{3.10}$$

$$F(a,b;e;x)F(c,d;1-a;y)=\tilde{\Delta}_{x,y}^{1,1}(a)H_2(a,b,c,d;e;x,-y), \tag{3.11}$$

$$\mathrm{H}_2(a,b,c;d;x,y)=\tilde{\nabla}_{x,y}^{1,1}(a)F(a,b;d;x)\,{}_1F_1(c;1-a;-y), \tag{3.12}$$

$$F(a,b;d;x)\,{}_1F_1(c;1-a;-y)=\tilde{\Delta}_{x,y}^{1,1}(a)\mathrm{H}_2(a,b,c;d;x,y), \tag{3.13}$$

$$\mathrm{H}_3(a,b;d;x,y)=\tilde{\nabla}^{1,1}_{x,y}(a)\,F(a,b;d;x)\,{}_0F_1(1-a;-y)\,, \tag{3.14}$$

$$F(a,b;d;x)\,{}_0F_1(1-a;y)=\tilde{\Delta}^{1,1}_{x,y}(a)\,\mathrm{H}_3(a,b;d;x,-y)\,, \tag{3.15}$$

$$\mathrm{H}_4(a,b;d;x,y)=\tilde{\nabla}^{1,1}_{x,y}(a)\,{}_1F_1(a;d;x)\,{}_1F_1(b;1-a;-y)\,, \tag{3.16}$$

$${}_1F_1(a;d;x)\,{}_1F_1(b;1-a;y)=\tilde{\Delta}^{1,1}_{x,y}(a)\,\mathrm{H}_4(a,b;d;x,-y)\,, \tag{3.17}$$

$$\mathrm{H}_5(a;d;x,y)=\tilde{\nabla}^{1,1}_{x,y}(a)\,{}_1F_1(a;d;x)\,{}_0F_1(1-a;-y)\,, \tag{3.18}$$

$${}_1F_1(a;d;x)\,{}_0F_1(1-a;y)=\tilde{\Delta}^{1,1}_{x,y}(a)\,\mathrm{H}_5(a;d;x,-y)\,, \tag{3.19}$$

$$\mathrm{H}_{11}(a,b,c;d;x,y)=\tilde{\nabla}^{1,1}_{x,y}(a)\,{}_1F_1(a;d;x)\,F(b,c;1-a;-y)\,, \tag{3.20}$$

$${}_1F_1(a;d;x)\,F(b,c;1-a;-y)=\tilde{\Delta}^{1,1}_{x,y}(a)\,\mathrm{H}_{11}(a,b,c;d;x,y)\,, \tag{3.21}$$

where

$${}_pF_q(a_1,...,a_p;b_1,...,b_q;x):=\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n}\frac{x^n}{n!},\,|x|<1,$$

is a generalized Gauss hypergeometric function, F is the famous Gauss function defined by (2.6).

By using equalities (3.8) and (3.9) from the operator identities (3.10) to (3.21) we can derive the following decomposition formulas for double hypergeometric functions H_2 , $\mathrm{H}_2-\mathrm{H}_5$ and H_{11} :

$$\begin{aligned}H_2(a,b,c,d;e;x,y)&=F(a,b;e;x)\,F(c,d;1-a;-y)\,+\sum_{k=1}^{\infty}\sum_{l=1}^{k}\frac{(-1)^{k+l}(k-1)!}{(l-1)!l!(k-l)!}\\&\cdot\frac{(b)_k(c)_l(d)_l}{(1-a)_l(e)_k}x^ky^lF(a+k,b+k;e+k;x)\,F(c+l,d+l;1-a+l;-y)\,,\end{aligned} \tag{3.22}$$

$$\begin{aligned}F(a,b;e;x)\,F(c,d;1-a;y)&=H_2(a,b,c,d;e;x,-y)+\sum_{k=1}^{\infty}\sum_{l=1}^{k}\frac{(k-1)!}{(l-1)!l!(k-l)!}\\&\cdot\frac{(-1)^{k-l}(b)_l(c)_k(d)_k}{(1-a)_k(1-a)_{k-l}(e)_l}x^ly^kH_2(a-k+l,b+l,c+k,d+k;e+l;x,-y)\,,\end{aligned} \tag{3.23}$$

$$\begin{aligned}\mathrm{H}_2(a,b,c;d;x,y)&=F(a,b;d;x)\,{}_1F_1(c;1-a;-y)+\sum_{k=1}^{\infty}\sum_{l=1}^{k}\frac{(-1)^{k+l}(k-1)!}{(l-1)!l!(k-l)!}\\&\cdot\frac{(b)_k(c)_l}{(1-a)_l(d)_k}x^ky^lF(a+k,b+k;d+k;x)\,{}_1F_1(c+l;1-a+l;-y)\,,\end{aligned} \tag{3.24}$$

$$\begin{aligned}F(a,b;d;x)\,{}_1F_1(c;1-a;y)&=\mathrm{H}_2(a,b,c;d;x,-y)+\sum_{k=1}^{\infty}\sum_{l=1}^{k}\frac{(-1)^{k-l}(k-1)!}{(l-1)!l!(k-l)!}\\&\cdot\frac{(b)_l(c)_k}{(1-a)_k(1-a)_{k-l}(d)_l}x^ly^k\mathrm{H}_2(a-k+l,b+l,c+k;d+l;x,-y)\,,\end{aligned} \tag{3.25}$$

$$\begin{aligned}\mathrm{H}_3(a,b;d;x,y)&=F(a,b;d;x)\,{}_0F_1(1-a;-y)+\sum_{k=1}^{\infty}\sum_{l=1}^{k}\frac{(-1)^{k+l}(k-1)!}{(l-1)!l!(k-l)!}\\&\cdot\frac{(b)_k}{(1-a)_l(d)_k}x^ky^lF(a+k,b+k;d+k;x)\,{}_0F_1(1-a+l;-y)\,,\end{aligned} \tag{3.26}$$

$$\begin{aligned}F(a,b;d;x)\,{}_0F_1(1-a;y)&=\mathrm{H}_3(a,b;d;x,-y)+\sum_{k=1}^{\infty}\sum_{l=1}^{k}\frac{(-1)^{k-l}(k-1)!}{(l-1)!l!(k-l)!}\\&\cdot\frac{(b)_l}{(1-a)_k(1-a)_{k-l}(d)_l}x^ly^k\mathrm{H}_3(a-k+l,b+l;d+l;x,-y)\,,\end{aligned} \tag{3.27}$$

$$\begin{aligned}\mathrm{H}_4(a,b;d;x,y)&={}_1F_1(a;d;x)\,{}_1F_1(b;1-a;-y)+\sum_{k=1}^{\infty}\sum_{l=1}^{k}\frac{(-1)^{k+l}(k-1)!}{(l-1)!l!(k-l)!}\\&\cdot\frac{(b)_l}{(1-a)_l(d)_k}x^ky^l\,{}_1F_1(a+k;d+k;x)\,{}_1F_1(b+l;1-a+l;-y)\,,\end{aligned} \tag{3.28}$$

$$
{}_1F_1\left(a;d;x\right){}_1F_1\left(b;1-a;y\right)=\mathrm{H}_4\left(a,b;d;x,-y\right)+\sum_{k=1}^{\infty}\sum_{l=1}^{k}\frac{(-1)^{k-l}(k-1)!}{(l-1)!l!(k-l)!}
\cdot\frac{(b)_k}{(1-a)_k(1-a)_{k-l}(d)_l}x^ly^k\mathrm{H}_4\left(a-k+l,b+k;d+l;x,-y\right), \tag{3.29}
$$

$$
\mathrm{H}_5\left(a;d;x,y\right)={}_1F_1\left(a;d;x\right){}_0F_1\left(1-a;-y\right)+\sum_{k=1}^{\infty}\sum_{l=1}^{k}\frac{(-1)^{k+l}(k-1)!}{(l-1)!l!(k-l)!}
\cdot\frac{1}{(1-a)_l(d)_k}x^ky^l{}_1F_1\left(a+k;d+k;x\right){}_0F_1\left(1-a+l;-y\right), \tag{3.30}
$$

$$
{}_1F_1\left(a;d;x\right){}_0F_1\left(1-a;y\right)=\mathrm{H}_5\left(a,b;d;x,-y\right)+\sum_{k=1}^{\infty}\sum_{l=1}^{k}\frac{(-1)^{k-l}(k-1)!}{(l-1)!l!(k-l)!}
\cdot\frac{1}{(1-a)_k(1-a)_{k-l}(d)_l}x^ly^k\mathrm{H}_5\left(a-k+l;d+l;x,-y\right), \tag{3.31}
$$

$$
\mathrm{H}_{11}\left(a,b,c;d;x,y\right)={}_1F_1\left(a;d;x\right)F\left(b,c;1-a;-y\right)+\sum_{k=1}^{\infty}\sum_{l=1}^{k}\frac{(-1)^{k+l}(k-1)!}{(l-1)!l!(k-l)!}
\cdot\frac{(b)_l(c)_l}{(1-a)_l(d)_k}x^ky^l{}_1F_1\left(a+k;d+k;x\right)F\left(b+l,c+l;1-a+l;-y\right), \tag{3.32}
$$

$$
{}_1F_1\left(a;d;x\right)F\left(b,c;1-a;y\right)=\mathrm{H}_{11}\left(a,b,c;d;x,-y\right)+\sum_{k=1}^{\infty}\sum_{l=1}^{k}\frac{(-1)^{k-l}(k-1)!}{(l-1)!l!(k-l)!}
\cdot\frac{(b)_k(c)_k}{(1-a)_k(1-a)_{k-l}(d)_l}x^ly^k\mathrm{H}_{11}\left(a-k+l,b+k,c+k;d+l;x,-y\right). \tag{3.33}
$$

The expansions (3.22)-(3.33) can be proved without symbolic methods by comparing coefficients of equal powers of x and y on both sides.

4 Decomposition formulas for the multivariable confluent hypergeometric function

An interesting unification (and generalization) of multiple Lauricella's functions $F_{\mathbf{A}}^{(m)}$ and $F_{\mathbf{B}}^{(m)}$ and Horn's functions of two variables H_2 was considered by Erdélyi [6](see also [25, p.74]), who defined his general function in the form:

$$
H_{m+n,m}\left(a,b_1,...,b_m,d_1,...,d_n,e_1,...,e_n;c_1,...,c_m;x,y\right)
=\sum\frac{(a)_{i_1+...+i_m-j_1-...-j_n}(b_1)_{i_1}...(b_m)_{i_m}(d_1)_{j_1}...(d_n)_{j_n}(e_1)_{j_1}...(e_n)_{j_n}}{(c_1)_{i_1}...(c_m)_{i_m}}\frac{x_1^{i_1}...x_m^{i_m}}{i_1!...i_m!}\frac{y_1^{j_1}...y_n^{j_n}}{j_1!...j_n!}, \tag{4.1}
$$

where $m,n\in\mathrm{N}\cup\{0\}$, x and y are variables defined in (2.13). In the series defined by (4.1) $i_1,...,i_m$ and $j_1,...,j_n$ run from 0 to ∞.

Evidently, we have

$$
H_{m,m}=F_{\mathbf{A}}^{(m)},H_{n,0}=F_{\mathbf{B}}^{(n)},H_{2,1}=H_2.
$$

From the hypergeometric function (4.1) we shall define the following confluent hypergeometric function

$$
\mathrm{H}_A^{(m,n)}\left(a,b_1,...,b_m;c_1,...,c_m;x,y\right)
=\lim_{\varepsilon\to0}H_{m+n,m}\left(a,b_1,...,b_m,\tfrac{1}{\varepsilon},...,\tfrac{1}{\varepsilon};c_1,...,c_m;x,\varepsilon^2y\right).
$$

At the determination of the confluent hypergeometric function $\mathrm{H}_A^{(m,n)}$ the equality [1, p.124] $\lim\limits_{\varepsilon\to0}(1/\varepsilon)_k\cdot\varepsilon^k=1$ (k is a natural number) has been used. The found confluent hypergeometric function has the following form

$$
H_A^{(m,n)}\left(a,b_1,...,b_m;c_1,...,c_m;x,y\right)
=\sum\frac{(a)_{i_1+...+i_m-j_1-...-j_n}(b_1)_{i_1}...(b_m)_{i_m}}{(c_1)_{i_1}...(c_m)_{i_m}}\frac{x_1^{i_1}...x_m^{i_m}}{i_1!...i_m!}\frac{y_1^{j_1}...y_n^{j_n}}{j_1!...j_n!},\left(|x_1|+...+|x_m|<1\right), \tag{4.2}
$$

Now we apply the symbolic operators $\tilde{\nabla}_{x,y}^{m,n}\left(h\right)$ and $\tilde{\Delta}_{x,y}^{m,n}\left(h\right)$ to the studying of properties of confluent hypergeometric function $\mathrm{H}_A^{(m,n)}$ defined by (4.2).

Using the formulas (2.9) and (2.11), we obtain

$$\mathrm{H}_A^{(m,n)}\left(a,b_1,...,b_m;c_1,...,c_m;x,y\right)=\tilde{\nabla}_{x,y}^{m,n}\left(a\right)F_A^{(m)}\left(a,b_1,...,b_m;c_1,...,c_m;x\right)J_{-a}^{(n)}\left(y\right), \tag{4.3}$$

$$F_A^{(m)}\left(a,b_1,...,b_m;c_1,...,c_m;x\right)J_{-a}^{(n)}\left(y\right)=\tilde{\Delta}_{x,y}^{m,n}\left(a\right)\mathrm{H}_A^{(m,n)}\left(a,b_1,...,b_m;c_1,...,c_m;x,y\right), \tag{4.4}$$

where $J_a^{(n)}$ is the Bessel function in n variables:

$$J_a^{(n)}\left(y\right):=\sum_{j_1,...,j_n=0}^{\infty}\frac{(-1)^{j_1+...j_n}}{(1+a)_{j_1+...j_n}}\frac{y_1^{j_1}...y_n^{j_n}}{j_1!...j_n!}. \tag{4.5}$$

Now by virtue of formulas (2.14) and (2.15) from the formulas (4.3) and (4.4) we have

$$\begin{array}{l}\mathrm{H}_A^{(m,n)}\left(a,b_1,...,b_m;c_1,...,c_m;x,y\right)=F_A^{(m)}\left(a,b_1,...,b_m;c_1,...,c_m;x\right)J_{-a}^{(n)}\left(y\right)\\+\sum\limits_{k=1}^{\infty}\sum\limits_{l=1}^{k}\sum\limits_{M}\frac{(-1)^{k+l}k!(k-1)!(b)_i}{(l-1)!(k-l)!(1-a)_l(c)_i}\frac{x_1^{i_1}...x_m^{i_m}}{i_1!...i_m!}\frac{y_1^{j_1}...y_n^{j_n}}{j_1!...j_n!}\\\cdot F_A^{(m)}\left(a+k,b_1+i_1,...,b_m+i_m;c_1+i_1,...,c_m+i_m;x\right)J_{-a+l}^{(n)}\left(y\right),\end{array} \tag{4.6}$$

$$\begin{array}{l}F_A^{(m)}\left(a,b_1,...,b_m,;c_1,...,c_m;x\right)J_{-a}^{(n)}\left(y\right)=\mathrm{H}_A^{(m,n)}\left(a,b_1,...,b_m;c_1,...,c_m;x,y\right)\\+\sum\limits_{k=1}^{\infty}\sum\limits_{l=1}^{k}\sum\limits_{N}\frac{k!(k-1)!(a)_{l-k}(b)_i}{(l-1)!(k-l)!(1-a)_k(c)_i}\frac{x_1^{i_1}...x_m^{i_m}}{i_1!...i_m!}\frac{y_1^{j_1}...y_n^{j_n}}{j_1!...j_n!}\\\cdot\mathrm{H}_A^{(m,n)}\left(a+l-k,b_1,...,b_m;c_1,...,c_m;x\right)J_{-a+k}^{(n)}\left(y\right),\end{array} \tag{4.7}$$

where

$$M:=\left\{(i_1,...,i_m;j_1,...,j_n):i_1\geq 0,...,i_m\geq 0,j_1\geq 0,...,j_n\geq 0,i_1+...+i_m=k,j_1+...+j_n=l\right\},$$

$$N:=\left\{(j_1,...,j_n):i_1\geq 0,...,i_m\geq 0,j_1\geq 0,j_1\geq 0,...,j_n\geq 0,i_1+...+i_m=l,j_1+...+j_n=k\right\}.$$

Thus, we have obtained the decomposition formulas for the multiple confluent function defined by (4.2). We recall that the multiple Lauricella function $F_A^{(m)}$ has an expansion formula (2.8).

It is easy to see that in the case when $m=n=1$ the decomposition formulas (4.6) and (4.7)coincide with the formulas defined by (3.26) and (3.27), respectively.

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