

AXIOMS FOR THE REAL NUMBERS: A CONSTRUCTIVE APPROACH

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ABSTRACT. We present axioms for the real numbers by omitting the field axioms and then derive the field properties of the real numbers. We prove all our theorems constructively.

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Keywords: constructive mathematics; real numbers; axiomatization; ordered algebraic structures

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1. INTRODUCTION

We axiomatically define the real numbers as the completion of the rational numbers. We keep the usual strict order and field structure on the rational numbers, and we embed the rational numbers in a certain way into a complete set; we will later say what that certain way is and what we mean by ‘complete’. That complete set with the rational numbers is what we call the completion of the rational numbers. We do not need the field axioms to say what the real numbers are because the ring properties of the rational numbers can be extended to the completion (Theorem 18) and the invertibility of the nonzero real numbers can be proved (Theorem 20).

Much of the theory of the completion of an ordered set has been developed in [4]; we will use here only what we need from [4].

We prove our theorems constructively, meaning we abstain from using the law of excluded middle. More about constructive mathematics can be found in [1, 2, 3] or on the Web. Some of the theorems below have been proved in [4], so their proofs are omitted.

2. COMPLETION OF AN ORDERED SET

In [4], an *ordered set* is defined as a set X with a binary relation $<$ such that, for all $x, y, z \in X$,

- $x < y$ implies $y < x$ is false; (Asymmetry)
- $x < y$ implies $x < z$ or $z < y$; (Cotransitivity)
- $x < y$ is false and $y < x$ is false imply $x = y$; (Negative Antisymmetry).

We write $x \leq y$ for $y < x$ is false. Also, in [4], a subset S of an ordered set X is *almost dense* in X if $x < y$ in X implies $x \leq s < s' \leq y$ for some $s, s' \in S$, and S is *bicofinal* in X if, for each $x \in X$, $s \leq x \leq s'$ for some $s, s' \in S$. We say S is *finitely enumerable* if S is empty or there is a positive integer n and a function from $\{1, \dots, n\}$ onto S . A subset S of X is *upper order located* if, $x < y$ in X implies either $x < s$ for some $s \in S$ or $u < y$ for some upper bound u of S . An ordered set X is *complete* if each nonempty, bounded above, and upper order located subset of X has a supremum in X . We call a nonempty, bounded above, and upper order located subset of X a *supable* subset of X .

In [4], a *completion* of an ordered set X is an ordered set Y together with an embedding f of X into Y such that

- Y is complete;
- $f(X)$ is almost dense in Y , and
- $f(X)$ is bicofinal in Y .

An *embedding* f is a function such that $x < x'$ if and only if $f(x) < f(x')$. In [4], we also construct a completion of an arbitrary ordered set and show that a completion of an ordered set is unique, up to isomorphism. As mentioned above, we keep the usual strict order and the field structure on the rational numbers \mathbb{Q} , and we define the real numbers \mathbb{R} as the completion of \mathbb{Q} .

3. FIELD STRUCTURE ON THE REAL NUMBERS

In what follows, we will show under what conditions we can define a unique addition, a unique additive inversion, and a unique multiplication on \mathbb{R} and then prove that \mathbb{R} is a field.

3.1. Archimedean ordered abelian groups. In [4], we prove that the completion of an Archimedean ordered abelian group is an Archimedean ordered abelian group. Let us give some of the details.

An ordered abelian group is *Archimedean* if $x, y > 0$ implies there is a positive integer n such that $x \leq ny$. An addition on an ordered set is *compatible with the order* if $x + z < y + z$ if and only if $x < y$, and $z + x < z + y$ if and only if $x < y$. If X is an ordered set containing an Archimedean ordered abelian group A as an almost dense, bicofinal subset, we say an addition on X is *admissible* if it extends the addition on A and is compatible with the order on X . The following is proved in [4] (Theorem 58):

Theorem 1. *Let A be an Archimedean ordered abelian group, and let X be an ordered set containing A as an almost dense, bicofinal subset. Then any two admissible additions on X are equal.*

To define an admissible addition on an ordered set X containing an Archimedean ordered abelian group A as an almost dense, bicofinal subset, we need this lemma, which is proved in [4] (Lemmas 59 & 60):

Lemma 2. *Let A be an Archimedean ordered abelian group.*

- (1) *A subset S of A is supable if and only if, for each $\epsilon > 0$ in A and for each positive integer m , there exist $a \in S$ and an upper bound u of S in A such that $m(u - a) < \epsilon$.*
- (2) *The sum of two supable subsets of A is supable.*

For an Archimedean ordered abelian group A , an admissible addition on the completion \overline{A} is defined as follows: for $x, y \in \overline{A}$, consider the supable subsets $L_x = \{a \in A : a \leq x\}$ and $L_y = \{a \in A : a \leq y\}$. Since $L_x + L_y$ is supable by Lemma 2(2), we define $x + y$ to be $\sup(L_x + L_y)$.

For the additive inversion, we define $-x$ to be $\inf -L_x$, for each $x \in \overline{A}$.

Lemma 3. *Let A be an Archimedean ordered abelian group, and let X be an ordered set containing A as an almost dense, bicofinal subset. Let $x, y \in X$. If, for each $\epsilon > 0$ in A , there are $u, v \in A$ such that $x, y \in [u, v]$ and $v - u < \epsilon$, then $x = y$. ([4], Lemma 56)*

Theorem 4. *Let A be an Archimedean ordered abelian group. Then \overline{A} is an Archimedean ordered abelian group. ([4], Theorem 62).*

3.2. Commutative ordered monoids. We call a multiplication on an ordered set X with a distinguished element 0 *preadmissible* if, for each x, y, z in X , the following conditions hold:

- (1) $0x = 0 = x0$;
- (2) if $x < y$ and $z > 0$, then $xz < yz$;
- (3) if $x < y$ and $z < 0$, then $xz > yz$.

Lemma 5. *Let X be an ordered set and $x, y, x', y' \in X$:*

- (1) *If $x < y$ implies $x' < y'$, and $x = y$ implies $x' = y'$, then $x \leq y$ implies $x' \leq y'$.*
- (2) *If $x < y$ implies $x' > y'$, and $x = y$ implies $x' = y'$, then $x \leq y$ implies $x' \geq y'$.*

Proof. (1) Suppose $x \leq y$ and $y' < x'$. If $x < y$, then $x' < y'$, which is false by asymmetry, so $y \leq x$. By negative antisymmetry, $x = y$, so $x' = y'$, implying $x' < x'$, which is false by asymmetry.

(2) The proof goes as in (1). □

Theorem 6. *Let X be an ordered set with a distinguished element 0 and a preadmissible multiplication. For each x, y, z in X ,*

- (1) *if $x \leq y$ and $z > 0$, then $xz \leq yz$;*
- (2) *if $x \leq y$ and $z < 0$, then $xz \geq yz$.*

Proof. (1) Suppose $z > 0$. If $x < y$, then $xz < yz$, because of condition 2 for a preadmissible multiplication. Also, if $x = y$, then $xz = yz$ because multiplication, as a function on $X \times X$, is well defined. Hence, $x \leq y$ implies $xz \leq yz$ by Lemma 5(1).

(2) Suppose $z < 0$. If $x < y$, then $xz > yz$ because of condition 3 for a preadmissible multiplication. Also, if $x = y$, then $xz = yz$. Hence, $x \leq y$ implies $xz \geq yz$, by Lemma 5(2). □

Remark. In the proof of Theorem 6(1), we have shown

$$z > 0 \Rightarrow [((x < y \Rightarrow xz < yz) \text{ and } (x = y \Rightarrow xz = yz)) \Rightarrow (x \leq y \Rightarrow xz \leq yz)],$$

which is equivalent to condition 2 for a preadmissible multiplication implying Theorem 6(1). That move is a special form of the general reasoning law:

$$C \Rightarrow [(A \Rightarrow B) \Rightarrow (A' \Rightarrow B')] \text{ is equivalent to } ((C \text{ and } A) \Rightarrow B) \Rightarrow ((C \text{ and } A') \Rightarrow B').$$

Theorem 7. *Let X be an ordered set with a distinguished element 0 and a preadmissible multiplication. Let a, b, x, y be in X with $a \leq x \leq b$. Then*

- (1) *if l is a lower bound of $\{ay, by\}$ and u is an upper bound of $\{ay, by\}$, then $l \leq xy \leq u$;*
- (2) *if l' is a lower bound of $\{ya, yb\}$ and u' is an upper bound of $\{ya, yb\}$, then $l' \leq yx \leq u'$.*

Proof. (1) If $xy < l$, then $xy < ay$ and $xy < by$. If $y > 0$, then $ay \leq xy \leq by$ by Theorem 6(1), which is false since $xy < ay$, so $y \leq 0$. If $y < 0$, then $ay \geq xy \geq by$ by Theorem 6(2), which is false since $xy < by$, so $y \geq 0$. Hence, $y = 0$, by negative antisymmetry. But if $y = 0$, then $x0 < a0$, giving $0 < 0$, which is false by asymmetry. Therefore, $l \leq xy$. Now, if $u < xy$, then $ay < xy$ and $by < xy$. If $y > 0$, then $ay \leq xy \leq by$, which is false, so $y \leq 0$. If $y < 0$, then $ay \geq xy \geq by$, which is false, so $y \geq 0$. Hence, $y = 0$. But if $y = 0$, then $a0 < x0$, giving $0 < 0$, which is false. Therefore, $xy \leq u$.

(2) The proof goes as in (1). □

Theorem 8. *Let $a \leq x \leq b$ and $a' \leq y \leq b'$ in X . If l is a lower bound of $\{aa', ab', ba', bb'\}$ and u is an upper bound of $\{aa', ab', ba', bb'\}$, then $l \leq xy \leq u$.*

Proof. Since u is an upper bound of $\{aa', ab', ba', bb'\}$, u is also an upper bound of $\{aa', ab'\}$ and an upper bound of $\{ba', bb'\}$. By Theorem 7(2), $ay \leq u$ and $by \leq u$, so u is an upper bound of $\{ay, by\}$. Therefore, $xy \leq u$, by Theorem 7(1). Similarly, $l \leq xy$. □

By a *commutative ordered monoid* M with a distinguished element 0 , we mean a monoid such that $xy = yx$, and $0 < x$ and $0 < y$ imply $0 < xy$. We call M *locally bounded* if every finitely enumerable subset of M has a minimum element and a maximum element.

For an ordered set X containing a locally bounded commutative ordered monoid M with a distinguished element 0 , as an almost dense, bicofinal subset, and satisfying $0 \leq 1$, a multiplication on X is *admissible* if

- (1) the multiplication on X is preadmissible;
- (2) the multiplication on X extends the multiplication on M ;
- (3) for $c, d \in M$, $c \leq xy \leq d$ implies there are $a, b, a', b' \in M$ such that $a \leq x \leq b$, $a' \leq y \leq b'$, and $c \leq \min \{aa', ab', ba', bb'\} \leq xy \leq \max \{aa', ab', ba', bb'\} \leq d$.

In what remains, we will assume that $0 \leq 1$ in M .

Lemma 9. *Let X be an ordered set containing a locally bounded commutative ordered monoid M with a distinguished element 0 , as an almost dense, bicofinal subset. Let X have a preadmissible multiplication. Let x, y be in X and let a, b, a', b' be in M . If $a \leq x \leq b$ and $a' \leq y \leq b'$, then $\min \{aa', ab', ba', bb'\} \leq xy \leq \max \{aa', ab', ba', bb'\}$.*

Proof. Since $\min \{aa', ab', ba', bb'\}$ is a lower bound of $\{aa', ab', ba', bb'\}$ and $\max \{aa', ab', ba', bb'\}$ is an upper bound of $\{aa', ab', ba', bb'\}$,

$$\min \{aa', ab', ba', bb'\} \leq xy \leq \max \{aa', ab', ba', bb'\},$$

by Theorem 8. □

Lemma 10. *Let X be an ordered set containing an almost dense, bicofinal subset M . Let $x, y \in X$. If, for all $a, b \in M$, $a \leq x \leq b$ if and only if $a \leq y \leq b$, then $x = y$.*

Proof. If $x < y$, then $a \leq x \leq c < d \leq y \leq b$ for some $a, c, d, b \in M$, since M is almost dense and bicofinal in X , so $c < x$ and $y < d$, which is impossible, by asymmetry. Therefore, $y \leq x$. Symmetrically, $x \leq y$. Hence, $x = y$, by negative antisymmetry. □

Theorem 11. *Let X be an ordered set containing a locally bounded commutative ordered monoid M with a distinguished element 0 , as an almost dense, bicofinal subset. Any two admissible multiplications on X are equal.*

Proof. For two admissible multiplications $\hat{\cdot}$ and $\tilde{\cdot}$ on X , suppose for all $c, d \in M$, $c \leq x \hat{\cdot} y \leq d$. Then there are $a, b, a', b' \in M$ such that $a \leq x \leq b$, $a' \leq y \leq b'$, and $c \leq \min \{aa', ab', ba', bb'\} \leq x \hat{\cdot} y \leq \max \{aa', ab', ba', bb'\} \leq d$. Therefore, $c \leq \min \{aa', ab', ba', bb'\} \leq x \tilde{\cdot} y \leq \max \{aa', ab', ba', bb'\} \leq d$ by Lemma 9, so $c \leq x \tilde{\cdot} y \leq d$. Similarly, $c \leq x \tilde{\cdot} y \leq d$ implies $c \leq x \hat{\cdot} y \leq d$. Hence, $x \hat{\cdot} y = x \tilde{\cdot} y$, by Lemma 10. □

Lemma 12. *Let X be an ordered set containing a locally bounded commutative ordered monoid M with a distinguished element 0 , as an almost dense, bicofinal subset. Let $x, y \in X$. Suppose also that the multiplication on M be preadmissible. For fixed $r_0, s_0, r'_0, s'_0 \in M$ and for all $a, b, a', b' \in M$, if $r_0, a \leq x \leq s_0, b$ and $r'_0, a' \leq y \leq s'_0, b'$, then $\min \{aa', ab', ba', bb'\} \leq \max \{r_0 r'_0, r_0 s'_0, s_0 r'_0, s_0 s'_0\}$.*

Proof. Since M is locally bounded and almost dense in X , there are $m, m' \in M$ such that $r_0, a \leq m \leq s_0, b$ and $r'_0, a' \leq m' \leq s'_0, b'$. Therefore,

$$\min \{r_0 r'_0, r_0 s'_0, s_0 r'_0, s_0 s'_0\}, \min \{aa', ab', ba', bb'\} \leq mm' \leq \max \{r_0 r'_0, r_0 s'_0, s_0 r'_0, s_0 s'_0\}, \max \{aa', ab', ba', bb'\}$$

by Lemma 9. □

Theorem 13. *Let X be an ordered set containing a locally bounded commutative ordered monoid M with a distinguished element 0 , as an almost dense, bicofinal subset. In addition, assume that the multiplication on M be preadmissible and satisfy the following (*): for $x, y \in X$, if $c < d$ in M , then there are $a, b, a', b' \in M$ such that $a \leq x \leq b$; $a' \leq y \leq b'$; and either $c < \min \{aa', ab', ba', bb'\}$ or $\max \{aa', ab', ba', bb'\} < d$. Let $x, y \in X$; then the subset $P_{x,y} = \{\min \{aa', ab', ba', bb'\} : a, b, a', b' \in M, a \leq x \leq b \text{ and } a' \leq y \leq b'\}$ is supable.*

Proof. The subset $P_{x,y}$ is nonempty because M is bicofinal in X and M is locally bounded, and $P_{x,y}$ is bounded above by Lemma 12. Upper order locatedness of $P_{x,y}$ follows from almost density of M in X and from (*). □

Let M be a locally bounded commutative ordered monoid with a distinguished element 0 , and let the multiplication on M be continuous¹, be preadmissible, and satisfy the condition (*) in Theorem 13. For $x, y \in \overline{M}$, let the multiplication on \overline{M} be $xy = \sup P_{x,y}$, with $P_{x,y}$ being defined in Theorem 13. Under these hypotheses, we will show the following theorems about this multiplication on \overline{M} .

¹More precisely, the multiplication on M is continuous in the product topology. In [4] (p. 18), the order topology on an ordered set is introduced.

Theorem 14. *For each $x \in \overline{M}$, $x0 = 0 = 0x$.*

Proof. If $0 < x0$, then $0 < \min \{aa', ab', ba', bb'\}$ for some $a, b, a', b' \in M$ with $a \leq x \leq b$ and $a' \leq 0 \leq b'$ because $x0 = \sup P_{x,0}$, by definition. Then

$$\begin{aligned} 0 &< aa', \\ 0 &< ab', \\ 0 &< ba', \\ 0 &< bb'. \end{aligned}$$

Since $0 < aa'$, it is false that $a = 0$ and it is false that $a > 0$. Since $0 < ab'$, it is false that $a = 0$ and it is false that $a < 0$. Hence, $a = 0$ by negative antisymmetry, which is impossible since the multiplication on M is admissible, $0 < aa'$, and $0 < ab'$. Therefore, $x0 \leq 0$.

If $x0 < 0$, then $x0 \leq r < s \leq 0$ for some $r, s \in M$, since M is almost dense in \overline{M} . By condition (*) in Theorem 13, there are $a, b, a', b' \in M$ such that $a \leq x \leq b$; $a' \leq 0 \leq b'$; and either $r < \min \{aa', ab', ba', bb'\}$ or $\max \{aa', ab', ba', bb'\} < s$. If $r < \min \{aa', ab', ba', bb'\}$, then $\min \{aa', ab', ba', bb'\} \leq x0 < r < \min \{aa', ab', ba', bb'\}$, which is impossible by asymmetry. If $\max \{aa', ab', ba', bb'\} < s$, then

$$\begin{aligned} aa' &< 0, \\ ab' &< 0, \\ ba' &< 0, \\ bb' &< 0. \end{aligned}$$

Since $aa' < 0$, it is false that $a = 0$ and it is false that $a < 0$. Since $ab' < 0$, it is false that $a = 0$ and it is false that $a > 0$. Hence, $a = 0$ by negative antisymmetry, which is impossible. Therefore, $x0 \geq 0$.

Since $x0 \leq 0$ and $x0 \geq 0$, it follows $x0 = 0$, by negative antisymmetry. Similarly, $0 = 0x$. \square

Theorem 15. *Let $x, y \in \overline{M}$. For $c, d \in M$, $c \leq xy \leq d$ implies there are $a, b, a', b' \in M$ such that $a \leq x \leq b$, $a' \leq y \leq b'$, and $c \leq \min \{aa', ab', ba', bb'\} \leq xy \leq \max \{aa', ab', ba', bb'\} \leq d$.*

Proof. In [4] (Theorem 32), we proved that a nonempty almost dense subset of an ordered set is topologically dense in that ordered set, so M is topologically dense in \overline{M} . Since elements of M can be arbitrarily close to x, y and to themselves, elements of $M \times M$ can be arbitrarily close to (x, y) and to themselves in the product topology. Hence, take $a, b, a', b' \in M$ close enough to x, y such that $a \leq x \leq b$ and $a' \leq y \leq b'$ and such that aa', ab', ba', bb' are very close to each other by continuity of multiplication on M . Therefore, $c \leq \min \{aa', ab', ba', bb'\} \leq xy \leq \max \{aa', ab', ba', bb'\} \leq d$; note that $\min \{aa', ab', ba', bb'\} \leq xy$ by the definition of the multiplication on \overline{M} and that $xy \leq \max \{aa', ab', ba', bb'\}$ because $\max \{aa', ab', ba', bb'\}$ is an upper bound of $P_{x,y}$ by Lemma 12 and xy is the least upper bound of $P_{x,y}$ in Theorem 13. \square

Theorem 16. *The multiplication on \overline{M} extends the multiplication on M .*

Proof. Let $m, m' \in M$. Since $m \leq m \leq m$ and $m' \leq m' \leq m'$ and since the multiplication on M is preadmissible, $m \cdot_M m' \leq m \cdot_{\overline{M}} m' \leq m \cdot_M m'$, by Lemma 9, so $m \cdot_M m' = m \cdot_{\overline{M}} m'$, by negative antisymmetry. \square

Theorem 17. *Let M be a locally bounded commutative ordered monoid with a distinguished element 0, and let the multiplication on M be continuous, be preadmissible, and satisfy the condition (*) in Theorem 13. Then \overline{M} is a commutative ordered monoid.*

Proof. Let $x, y, z \in \overline{M}$. For all $a, b \in M$, suppose $a \leq x \leq b$. Since $1 \in M$ and $0 \leq 1$, $a \leq x1 \leq b$. Now suppose $a \leq x1 \leq b$. Then there are $c, d, a', b' \in M$ such that $c \leq x \leq d$, $a' \leq 1 \leq b'$, and $a \leq \min\{ca', cb', da', db'\} \leq x1 \leq \max\{ca', cb', da', db'\} \leq b$, by Theorem 15. If $x < a$, then $e \leq x \leq m < m' \leq a$ for some $e, m, m' \in M$, so $e \leq x1 \leq m$ by the definition of the multiplication on \overline{M} . Hence, $x1 < x1$, which is impossible, so $a \leq x$. Similarly, $x \leq b$. Therefore, $x1 = 1$, by Lemma 10.

Associativity and commutativity of multiplication on \overline{M} follows from associativity and commutativity on M , Theorem 15, and Lemma 10.

Now suppose $0 < x, y$. Then $0 < a \leq x \leq b$ and $0 < a' \leq y \leq b'$ for some $a, b, a', b' \in M$, since M is almost dense and bicofinal in \overline{M} . Since M is an ordered monoid, $0 < aa', ab', ba', bb'$, so $0 < \min\{aa', ab', ba', bb'\} \leq xy$. Hence, $0 < xy$. \square

By a *(locally bounded) commutative ordered ring* X , we mean an ordered set with an addition $+$ and multiplication \cdot such that $(X, +)$ is an ordered abelian group, (X, \cdot) is a (locally bounded) commutative ordered monoid, and the distributive law holds in X . We say X is Archimedean if $(X, +)$ is Archimedean. Note that the multiplication on a commutative ordered ring is preadmissible.

Theorem 18. *The real numbers $\mathbb{R} = \overline{\mathbb{Q}}$ are an Archimedean, commutative ordered ring.*

Proof. By Theorem 4, $(\mathbb{R}, +)$ is an Archimedean ordered abelian group. By Theorem 17, (\mathbb{R}, \cdot) is a commutative ordered monoid. To prove the distributive law $x(y + z) = xy + xz$, note that addition and multiplication on \mathbb{R} are continuous functions, so the functions on \mathbb{R}^3 defined by $L(x, y, z) = x(y + z)$ and by $R(x, y, z) = xy + xz$ are continuous. Since \mathbb{Q}^3 is topologically dense in \mathbb{R}^3 (because \mathbb{Q} is topologically dense in \mathbb{R}) and since $L = R$ when restricted to \mathbb{Q}^3 , it follows $L = R$ on \mathbb{R} . Hence, the distributive law holds in \mathbb{R} . \square

3.3. Invertibility of nonzero elements. Any ordered set admits a tight apartness defined by $x \neq y$ if $x < y$ or $y < x$. A tight apartness is a positive notion for the negative notion of difference, which is two elements are “different” if they are “not equal”. A tight apartness is discussed in [5]. In an ordered set with a distinguished element 0, we say an element x is *nonzero* if $x \neq 0$. An element x in a monoid M is *invertible* if $xy = 1$ for some $y \in M$.

Lemma 19. *Let x be a nonzero real number and $D_x = \{1/b \in \mathbb{Q} : 0 < x \leq b \text{ or } x \leq b < 0\}$. Then D_x is supable.*

Proof. If $0 < x$, let ϵ be a positive rational number and r a rational number such that $0 < r < x$. Then there are rational numbers a and b such that $r < a \leq x \leq b$ and $b - a < r^2\epsilon$. Also $r^2 < ab$, $1/a$ is an upper bound of D_x , and

$$(1/b) - (1/a) = (a - b)/ba \leq (b - a)/ba < \epsilon.$$

If $x < 0$, let r be a rational number such that $x < r < 0$. Then there are rational numbers a and b such that $a \leq x \leq b < r$ and $b - a < r^2\epsilon$. Then

$$(1/b) - (1/a) = (a - b)/ba < \epsilon.$$

Therefore, D_x is supable by Lemma 2(1). \square

Theorem 20. *A real number is nonzero if and only if it is invertible.*

Proof. Let x be a nonzero real number. If $0 < x$, let ϵ be a positive rational number, and let r and s be rational numbers such that $0 < r < x < s$. Then there are rational numbers a and b such that $r < a \leq x \leq b < s$ and $b - a < r^2\epsilon/2s$, so

$$1/b \leq \sup D_x \leq 1/a,$$

by the definition of D_x . Since $\min \{a/b, 1, b/a\} = a/b$ and $\max \{a/b, 1, b/a\} = b/a$,

$$a/b \leq x \sup D_x \leq b/a$$

by Lemma 12 and the definition of multiplication. Also $a/b \leq 1 \leq b/a$ because $0 < a \leq b$, and

$$(b/a) - (a/b) = (b^2 - a^2) / ab = ((b + a) / ab) (b - a) < \epsilon.$$

Hence, $x \sup D_x = 1$, by Lemma 3. If $x < 0$, let r' and s' be rational numbers such that $r' < x < s' < 0$. Then

$$r' < a \leq x \leq b < s' < 0$$

and

$$b - a < (s'^2 / -2r')\epsilon$$

for some rational numbers a and b , so $1/b \leq \sup D_x \leq 1/a$. Thus,

$$b/a \leq x \sup D_x \leq a/b$$

with $b/a \leq 1 \leq a/b$ and $(a/b) - (b/a) < \epsilon$. Therefore,

$$x \sup D_x = 1.$$

Conversely, let x be an invertible real number, so $xy = 1$ for some $y \in \mathbb{R}$. Then there is a positive integer N such that $-N < y < N$. Also, by cotransitivity, either $-1/N < x$ or $x < 0$, and either $0 < x$ or $x < 1/N$. It is impossible that $x < 0$ and $0 < x$, by irreflexivity, nor is it possible that $-1/N < x < 1/N$; if it were, $-1 < xy < 1$. Therefore, $x < 0$ or $0 < x$, so $x \neq 0$. \square

A *Heyting field* is a field with a tight apartness. Heyting fields are discussed in [5].

Corollary 21. *The real numbers are a Heyting field.*

Proof. By Theorems 18 and 20. \square

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