

Identity Types in Algebraic Model Structures and Cubical Sets

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Abstract

We give a general technique for constructing a functorial choice of very good paths objects, which can be used to implement identity types in models of type theories in direct manner with little reliance on general coherence results. We give a simple proof that applies in algebraic model structures that possess a notion of structured weak equivalence, in a sense that we define here. We then give a more direct proof that applies both to the original BCH cubical set model and more recent variants. We give an explanation how this construction relates to the one used in the CCHM cubical set model of type theory.

1 Introduction

In the original Bezem-Coquand-Huber cubical set model of type theory in [2], Bezem, Coquand and Huber only showed the J -computation rule for identity types held only up to propositional equality rather than the more usual definitional equality. In [20] the author gave both an explanation for why this was the case and a solution. The explanation was a Brouwerian counterexample based on the nerve of a complete metric space, demonstrating that there is no constructive proof that the necessary strict equality in the J -computation rule holds. The solution was to consider a second more elaborate definition of identity type.

The motivation for the work here is to give a more conceptually clear proof of this construction, by viewing it in terms of the cofibration-trivial fibration factorisation of an algebraic model structure. From this point of view the basis of the construction is a functorial version of the following simple and well known trick. Suppose we are given a fibration $f: X \rightarrow Y$ in a model structure. A path object on f is a factorisation of the diagonal map $\Delta_f: X \rightarrow X \times_Y X$ as a weak equivalence $r: X \rightarrow Pf$ followed by a fibration $p: Pf \rightarrow X \times_Y X$. A very good path object is the same, but where r is required to be a trivial cofibration, not just a weak equivalence. If we are given a path object, then we can use it to produce a very good path object, by factorising the weak equivalence r as a

cofibration Cr followed by a trivial fibration $F^t r$, as in the diagram below.

$$\begin{array}{ccc}
 X & \xrightarrow{\Delta} & X \times_Y X \\
 Cr \downarrow & \searrow r & \nearrow p \\
 Mr & \xrightarrow{F^t r} & Pf
 \end{array}$$

By the 3-for-2 property Cr is a weak equivalence, and so a trivial cofibration, and $p \circ F^t r$ is fibration, making Mr a very good path object.

We will see a more complete proof (in comparison to [20]) that the resulting structure can be used to implement identity types. For this we will use a notion of stable functorial very good path object due to Van den Berg and Garner [24]. This has the advantage that it can be used to implement identity types in a relatively direct way with less reliance on general coherence results (the advantages of this approach will be discussed further in section 3). However in order to satisfy this definition it is no longer sufficient to work in a model structure. Instead we build on the notion of *algebraic model structure* due to Riehl [16]. We expand on Riehl's definition by adding a structured notion of weak equivalence and show how to use such structures to construct stable functorial choices of very good path object. We then adapt the straightforward proof above to a functorial version using this definition.

It is however non trivial to satisfy this definition of algebraic model structure with structured weak equivalence. Because of this, we will also give a more direct proof that the same construction yields identity types in BCH cubical sets, and many other categories.

In a separate paper the author will prove that BCH cubical sets do in fact satisfy this definition of algebraic model structure with structured weak equivalences. This will use a result due to Sattler [18], together with Huber's proof that the universe of small types is fibrant [10], combined with an argument using Grothendieck fibrations and other observations. Moreover, further results by Sattler (currently unpublished) suggest this extra structure can be found in a wide variety of categories, including CCHM cubical sets.

An earlier draft of this work was circulated online, and for reference remains available at [19]. The main change is that the original draft contained some rather messy arguments based on the concrete definition of the relevant awfs's in 01-substitution sets. In contrast, the results here will use an approach developed by Gambino and Sattler in [7] (and following some suggestions by Peter Lumsdaine). This yields a much clearer proof and much more general result, although essentially following the same outline as the original. The definition of ams with structured weak equivalences has been slightly generalised, but as we will see it turns out to not be essentially different to the original version. The earlier draft also contained an unproved claim that 01-substitution sets are an algebraic model structure with structured weak equivalences, which as mentioned above will now appear in a separate paper.

The construction of identity types in the Cohen-Coquand-Huber-Mörtberg (CCHM) cubical set model [6] was inspired by this work via the earlier draft and correspondence with the author. This included some simplifications discovered by Cohen, Coquand, Huber and Mörtberg that apply to CCHM cubical sets. This definition was generalised to a large class of models by Orton and Pitts

in [14] and by Van den Berg and Frumin in [23]. To be clear however, these simplifications do not always apply, and for instance do not include the original BCH cubical set model, which will be covered by the results in this paper. In section 6 we will give an abstract view of the the CCHM definition and see how it relates to the definition given here.

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2 Review of Algebraic Weak Factorisation Systems and Model Structures

Definition 2.1. Let \mathbb{C} be a category and let $i : U \rightarrow V$ and $f : X \rightarrow Y$ be morphisms in \mathbb{C} . We write $i \pitchfork f$ and say i has the left lifting property with respect to f and f has the right lifting property with respect to i if the following holds. For every commutative square of the form

$$\begin{array}{ccc} U & \longrightarrow & X \\ i \downarrow & & \downarrow f \\ V & \longrightarrow & Y \end{array}$$

there is a diagonal map j as below, making the two triangles commute.

$$\begin{array}{ccc} U & \longrightarrow & X \\ i \downarrow & \nearrow j & \downarrow f \\ V & \longrightarrow & Y \end{array}$$

Definition 2.2. Let \mathbb{C} be a category and \mathcal{M} a class of maps in \mathbb{C} . We define

$$\mathcal{M}^{\pitchfork} = \{f \mid (\forall i \in \mathcal{M}) i \pitchfork f\} \quad (1)$$

$$\pitchfork \mathcal{M} = \{i \mid (\forall f \in \mathcal{M}) i \pitchfork f\} \quad (2)$$

Definition 2.3. Let \mathbb{C} be a category. A *weak factorisation system* on \mathbb{C} consists of classes of maps \mathcal{C} and \mathcal{F} such that $\mathcal{C} = \pitchfork \mathcal{F}$ and $\mathcal{F} = \mathcal{C}^{\pitchfork}$ and every morphism g in \mathbb{C} factors as $g = f \circ i$ with $i \in \mathcal{C}$ and $f \in \mathcal{F}$.

Definition 2.4 (Quillen). Let \mathbb{C} be a category. A *model structure* on \mathbb{C} consists of two weak factorisation systems $(\mathcal{C}, \mathcal{F})$ and $(\mathcal{C}^t, \mathcal{F})$, together with a class of morphisms \mathcal{W} such that the following hold.

1. $\mathcal{C}^t = \mathcal{C} \cap \mathcal{W}$
2. $\mathcal{F}^t = \mathcal{F} \cap \mathcal{W}$

3. (3-for-2) If $f: X \rightarrow Y$, $g: Y \rightarrow Z$ and $h := g \circ f$, and any two maps out of f , g and h belong to \mathcal{W} , then so does the third.

Definition 2.5. Let \mathbb{C} be a category. A *functorial factorisation* on \mathbb{C} consists of a functor $\mathbb{C}^2 \rightarrow \mathbb{C}^3$ which is a section to the composition functor $\mathbb{C}^3 \rightarrow \mathbb{C}^2$. We will usually write out a functorial factorisation as three separate components L, K, R as follows. E.g., if f is an object of \mathbb{C}^2 (i.e. a morphism in \mathbb{C}) we might write the factorisation as

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow Lf & \nearrow Rf \\ & & Kf \end{array}$$

Definition 2.6 (Grandis, Tholen). Let \mathbb{C} be a category and (L, R) a functorial factorisation on \mathbb{C} . Note that L is an endofunctor on \mathbb{C}^2 and can be made into a copointed endofunctor in a canonical way. Dually, R can be made into a pointed endofunctor. An *algebraic weak factorisation system* on \mathbb{C} consists of a functorial factorisation together with a comultiplication map $\Sigma : L \rightarrow L^2$ making L into a comonad and a multiplication map $\Pi : R^2 \rightarrow R$ making R into a monad. Furthermore, the canonical map $LR \rightarrow RL$ is a distributive law.

We will write the category of coalgebras over the comonad as $L\text{-Map}$ and the category of algebras over the monad as $R\text{-Map}$. In many cases it is difficult (or impossible) to show that a map satisfies the comultiplication law required to be an L -coalgebra. For this reason we will usually work with the category of coalgebras over the underlying copointed endofunctor of L . We will write this category as $L\text{-map}$. For example $L\text{-map}$ is always closed under retracts whereas $L\text{-Map}$ is not.

The dual issue for R -algebras does not cause problems in practice, because of the following proposition.

Proposition 2.7. *Suppose that (L, R) is a cofibrantly generated awfs on a category \mathbb{C} . Then given a algebra structure for the underlying pointed endofunctor for R on a map f in \mathbb{C} , we can functorially assign f the structure of an algebra over the monad R .*

Proof. See e.g. [16, Lemma 2.30] □

For R , we will only every work over the category of (monad) algebras, $R\text{-Map}$. The reason is that for cubical sets (as in [2] and [6]) the definition of Kan fibration is already fixed and used in the interpretation of type theory, and in practice $R\text{-Map}$ corresponds more closely to these definitions.

There is also a special case where we can do the same for left maps, as we'll see in proposition 5.10.

Algebraic model structures were developed by Riehl in [16]. Before giving Riehl's definition, we first define a weaker version that will play an important role.

Definition 2.8. A *pre algebraic model structure* (pre-ams) consists of two awfs's (C^t, F) and (C, F^t) together with a morphism of awfs's $\xi: (C^t, F) \rightarrow (C, F^t)$. We refer to the morphism ξ as the *comparison map* of the pre-ams.

Definition 2.9 (Riehl). An *algebraic model structure* consists of a pre algebraic model structure on a complete and cocomplete category together with a class of maps \mathcal{W} such that if \mathcal{C} is the class of maps that admit a (copointed endofunctor) C -coalgebra structure, \mathcal{F} is the class of maps that admit a (pointed endofunctor) F -algebra structure, then $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ is a model structure.

3 Identity Types in an Awfs

In this section we will review a description of the semantics for identity types due to Van den Berg and Garner [24, Section 3]. We first talk about two related well known issues with the implementation of identity types that arise in the description of identity types by Awodey and Warren, and are elegantly resolved by the Van den Berg-Garner definition.

3.1 Coherence for J -Terms

It is a well known issue in type theory that great care needs to be taken to ensure the interpretation of type theory into categorical semantics is correct, due to so called coherence issues. This was noticed and then solved by Hofmann in [9] for the interpretation of extensional type theory into a locally cartesian closed category. Essentially the issue is as follows.

In most formulations of models of type theory, such as categories with families (CwFs), one needs to have a notion of substitution for types and terms, and furthermore the substitution needs to be strict in the following sense. If we are given morphisms of contexts $\sigma: \Xi \rightarrow \Delta$ and $\tau: \Delta \rightarrow \Gamma$, and a type $\Gamma \vdash X$, then we need to ensure that $X[\tau][\sigma]$ is strictly equal to $X[\tau \circ \sigma]$ (as types in context Ξ). Similarly for terms.

For the interpretation of extensional type theory in a locally cartesian closed category, this is an issue for interpreting types, but the interpretation of terms is not a problem.

On the other hand when Awodey and Warren developed the interpretation of identity types using very good path objects in [1], there is a coherence issue for terms. Specifically, as Awodey and Warren explain in [1, Section 4.1], they do not ensure that J terms are preserved by substitution. The reason is that J terms are interpreted as diagonal fillers of certain lifting problems. In a weak factorisation system, we are only guaranteed that at least one filler exists for every lifting problem of a trivial cofibration against a fibration. So there's no reason to expect the different choices of fillers to agree with each other under substitution.

It is possible to fix this issue using general coherence theorems, such as local universes (developed by Lumsdaine and Warren in [13]) or using a universe of small types, as used for simplicial sets in [11]. However, the Van den Berg-Garner interpretation allows us to deal with this problem in a much more direct way, with less reliance on general coherence results, and which can be used directly in an existing CwF with less work.

3.2 The Computational Meaning of Transport

In more computationally minded approaches to homotopy type theory, there is an emphasis on the computational meaning of transport, which is in turn strongly connected with the J terms.

For example, this was noticed early on by Harper and Licata in [12], and by Bezem, Coquand and Huber in [2], but also plays an important part in more recent developments.

The issue is as follows. Suppose we are given a type A in context Γ . For simplicity, say that Γ consists of a single type C . Suppose further we are given $c_0, c_1 : C$ and also a term p of type $\text{Id}_C(c_0, c_1)$. Then we need to show how we can take a term a of type $A(c_0)$ and then compute a new term of type $A(c_1)$. Using univalence we show that there are non trivial instances of this problem. We take C to be the universe of small types, $A(x)$ is defined to be the type $\text{El}(x)$, and p is an identity between two types constructed from an equivalence using univalence. Then in order to compute what the transport should be, we need to recover the computational information from the equivalence that we put in.

The solution is that whenever we interpret a type $\Gamma \vdash A$ in a model, we include all the computational information we need about how to compute transport in A for paths in Γ . The Van den Berg-Garner approach allows us to clearly see the necessary structure in an abstract way which is conceptually very similar to the Awodey-Warren approach. Namely, we work in a setting where it is natural to view fibrations not as a class of maps but as algebras over a monad. The computational information we need to associate with a type is precisely contained in an algebra structure over the monad.

3.3 The Van den Berg-Garner Interpretation of Identity Types

The key part of the Van den Berg-Garner interpretation is that instead of a weak factorisation system, they use an *algebraic weak factorisation system*¹.

With an awfs (L, R) , it is natural to view fibrations not as just a class of maps (as is the case for wfs's) but instead as a category of algebras over the monad R . Meanwhile trivial cofibrations are best viewed as coalgebras. Given a map m together with coalgebra structure and a map f with algebra structure, f not only has the right lifting property against m , but using the structures, we have a choice of diagonal filler. Furthermore, given a morphism of coalgebras and a morphism of algebras, we also get compatible diagonal fillers.

Then, as Van den Berg and Garner show in [24, Section 2], we can use this to define a type category where types are implemented as R -algebras, and substitution is implemented as pullbacks that preserve R -algebra structure. Then the algebra structure contains the computational information that we need to implement transport.

Finally, Van den Berg and Garner implement identity types in the type category using the following definition². Observe that we require not just that certain maps are fibrations and trivial cofibrations, but that we are given a choice

¹The exact formulation used by Van den Berg and Garner is not quite an awfs, but a slightly weaker notion.

²We change the terminology to fit better with other ideas in this paper.

of algebra and coalgebra structure. This allows one to give an explicit definition of diagonal fillers and thereby of the interpretation of J -terms. Furthermore, we require functoriality with respect to trivial cofibrations and fibrations everywhere. This ensures that the choice of diagonal fillers, and so of J terms is stable under substitution.

Definition 3.1 (Van den Berg, Garner). 1. A *choice of very good path objects* consists of an assignment to every F -map $f : X \rightarrow Y$ a factorisation

$$X \xrightarrow{r_f} P(f) \xrightarrow{p_f} X \times_Y X \quad (3)$$

of the diagonal $\Delta : X \rightarrow X \times_Y X$ together with an C^t -coalgebra structure on r_f and an F -algebra structure on p_f .

2. A choice of very good path objects is *functorial* if the assignment of (3) provides the action of objects of a functor $F\text{-Map} \rightarrow F\text{-Map} \times_{\mathbb{C}} C^t\text{-map}$.
3. A choice of very good path objects is *stable* when every map of F -algebras whose underlying square is a pullback makes the following square given by functoriality a pullback

$$\begin{array}{ccc} P(f) & \xrightarrow{P(h,k)} & P(f') \\ p_f \downarrow & & p_{f'} \downarrow \\ X \times_Y X & \longrightarrow & X' \times_{Y'} X' \end{array}$$

4. The awfs is *Frobenius* if to every square

$$\begin{array}{ccc} f^* X & \xrightarrow{\bar{f}} & X \\ \bar{i} \downarrow & & i \downarrow \\ Z & \xrightarrow{f} & Y \end{array}$$

together with an F -algebra structure on f and C^t -coalgebra structure on i we have assigned an C^t -coalgebra structure on \bar{i} . It is *functorially Frobenius* if this assignment gives rise to a functor $F\text{-Map} \times_{\mathbb{C}} C^t\text{-map} \rightarrow C^t\text{-map}$.

5. A *homotopy theoretic model of identity types* is a finitely complete category \mathbb{C} together with an awfs that is functorially Frobenius and has a stable functorial choice of very good path objects.

Theorem 3.2 (Van den Berg, Garner). *Every homotopy theoretic model of identity types gives rise to a model of type theory with identity types.*

Proof. See [24, Section 3.3]. □

Remark 3.3. *We leave it for future work to check that theorem 3.2 can be proved constructively.*

4 Identity Types in an Algebraic Model Structure

In this section we give our first construction of a stable functorial choice of very good path objects. This is an intuitively clear proof, based on a simple trick that can be carried out in any model structure. However, showing that BCH cubical sets satisfy the conditions needed in order to apply the theorem is difficult, and in fact will appear in a separate paper. We include the theorem here anyway, since it illustrates the main idea which will be used in section 5, where the results can be easily applied to cubical sets.

4.1 Structured Weak Equivalences

Definition 4.1. Suppose we are given a pre-ams $\xi : (C^t, F) \rightarrow (C, F^t)$ on a category \mathbb{C} .

Suppose further we are given a category $W\text{-Map}$ together with a faithful functor $W\text{-Map} \rightarrow \mathbb{C}^2$. We refer to objects in $W\text{-Map}$ as *structured weak equivalences*. Given f in \mathbb{C}^2 , we say a *weak equivalence structure on f* is an object in the preimage of f . Given objects f and g of \mathbb{C}^2 together with weak equivalence structures on f and g , we say a morphism in \mathbb{C}^2 (i.e. a commutative square) is a *morphism of weak equivalences* if it is the image of a morphism in $W\text{-Map}$ between structured weak equivalences.

Definition 4.2. Suppose we are given a pre-ams $\xi : (C^t, F) \rightarrow (C, F^t)$ on a category \mathbb{C} together with a faithful functor $W\text{-Map} \rightarrow \mathbb{C}^2$. A *functorial 3-for-2 operator* is the following. Given morphisms f_1, f_2, f_3 such that $f_3 = f_2 \circ f_1$ and suppose for $i \neq j \in \{1, 2, 3\}$ we are given weak equivalence structures on f_i and f_j , then writing k for the remaining element of $\{1, 2, 3\}$ we have assigned a weak equivalence structure on f_k . Furthermore, these assignments are functorial, in the following sense. Suppose we are given a commutative diagram as below.

$$\begin{array}{ccccc} U & \xrightarrow{f_1} & V & \xrightarrow{f_2} & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{g_1} & Y & \xrightarrow{g_2} & Z \end{array}$$

Let $f_3 := f_2 \circ f_1$ and $g_3 := g_2 \circ g_1$ and write α_1 for the left hand square α_2 for the right hand square and α_3 for the big rectangle. If $i \neq j \in \{1, 2, 3\}$ and we are given weak equivalence structures on f_i, g_i, f_j and g_j such that α_i and α_j are morphisms of structured weak equivalences, then α_k is a morphism between the weak equivalence structures we have assigned on f_k and g_k .

Definition 4.3. An *ams with structured weak equivalences* is a pre-ams on a finitely complete and cocomplete category \mathbb{C} together with a category $W\text{-Map}$, a faithful functor $W\text{-Map} \rightarrow \mathbb{C}$ and the following:

1. A functorial 3-for-2 operator.
2. Given a weak equivalence structure and a C -coalgebra structure on each map f a choice of C^t -coalgebra structure on f which is the action on objects of a functor $C\text{-map} \times_{\mathbb{C}^2} W\text{-Map} \rightarrow C^t\text{-map}$.

3. Given a weak equivalence structure and a F -algebra structure on a map f a choice of F^t -algebra structure which is the action on objects of a functor $F\text{-Map} \times_{\mathbb{C}^2} W\text{-Map} \rightarrow F^t\text{-Map}$.
4. Given a C^t -coalgebra structure on each map f , a choice of weak equivalence structure on f which is the action on objects of a functor $C^t\text{-map} \rightarrow W\text{-Map}$.
5. Given an F^t -algebra structure on each map f , a choice of weak equivalence structure on f which is the action on objects of a functor $F^t\text{-Map} \rightarrow W\text{-Map}$.

Remark 4.4. *We don't assume the existence of any additional structure on $W\text{-Map}$. Recently Bourke has shown in [4] that in some natural situations weak equivalences can be viewed as algebras over a monad. However, he also showed that this is not the case e.g. for simplicial sets with the Kan model structure.*

4.2 Some Remarks on Ams's with Structured Weak Equivalences

4.2.1 Pointwise Ams's with Structured Weak Equivalences

Given an ordinary model structure on a category \mathbb{C} there are three main ways to define a model structure on a functor category $\mathbb{C}^{\mathcal{A}}$, under suitable conditions. These are the projective model structure, injective model structure and the Reedy model structure.

We will see however (following a suggestion by Emily Riehl) that there is another option inherent in the definition of ams with structured weak equivalence, that is different to the standard constructions for ordinary model structures. This construction does not require any additional assumptions on the underlying ams with structured weak equivalences, or on \mathcal{A} or \mathbb{C} .

Suppose that we are given an ams with structured weak equivalences on a category \mathbb{C} , and another category \mathcal{A} . We will define a new ams with structured weak equivalences on the functor category $\mathbb{C}^{\mathcal{A}}$.

First recall that given an awfs on \mathbb{C} we can define the *pointwise awfs* on $\mathbb{C}^{\mathcal{A}}$.

Definition 4.5. Suppose we are given an awfs (L, R) on a category \mathbb{C} . The *pointwise awfs* on $\mathbb{C}^{\mathcal{A}}$ is the awfs $(L_{\mathcal{A}}, R_{\mathcal{A}})$ defined as follows. Note that we need to define functors $(\mathbb{C}^{\mathcal{A}})^2 \rightarrow (\mathbb{C}^{\mathcal{A}})^2$, however it suffices to instead define functors $\mathcal{A} \times (\mathbb{C}^2)^{\mathcal{A}} \rightarrow \mathbb{C}^2$. We define $L_{\mathcal{A}}$ by composition of L with the evaluation map $\mathcal{A} \times (\mathbb{C}^2)^{\mathcal{A}} \rightarrow \mathbb{C}^2$, and define $R_{\mathcal{A}}$ by composition of R with evaluation. We similarly define multiplication and comultiplication pointwise. Namely, μ needs to be a natural transformation $R_{\mathcal{A}}^2 \rightarrow R_{\mathcal{A}}$. Hence for each $f \in (\mathbb{C}^2)^{\mathcal{A}}$, we need μ_f to be a natural transformation from $R_{\mathcal{A}}^2(f) \rightarrow R_{\mathcal{A}}(f)$. Hence, for each $A \in \mathcal{A}$, we need a map $\mu_{f,A}: R^2(f(A)) \rightarrow R(f(A))$. We take $\mu_{f,A}$ to be $\mu_{f(A)}$. Comultiplication is defined similarly. Naturality and the other required equalities follow from the corresponding conditions on (L, R) .

Proposition 4.6. *Let (L, R) be an awfs on a category \mathbb{C} , and \mathcal{A} a small category. Suppose we are given a morphism f in the functor category $\mathbb{C}^{\mathcal{A}}$. Note that we can view f as a functor $f: \mathcal{A} \rightarrow \mathbb{C}^2$. Then $R_{\mathcal{A}}$ -algebra structures on f*

naturally correspond to functors $\alpha: \mathcal{A} \rightarrow R\text{-Map}$ in the following commutative diagram.

$$\begin{array}{ccc} & R\text{-Map} & \\ & \nearrow \alpha & \downarrow \\ \mathcal{A} & \xrightarrow{f} & \mathbb{C}^2 \end{array}$$

Dually for $L_{\mathcal{A}}$ -coalgebra structures.

Proof. This is straightforward to check. \square

Definition 4.7. Given an ams $\xi: (C, F^t) \rightarrow (C^t, F)$ with structured weak equivalences $W\text{-Map}$ and a small category \mathcal{A} , we define the pointwise ams with structured weak equivalences as follows. We define $(C_{\mathcal{A}}, F_{\mathcal{A}}^t)$ and $(C_{\mathcal{A}}^t, F_{\mathcal{A}})$ to be the pointwise awfs's. The comparison map is also defined pointwise.

We define the category $W\text{-Map}_{\mathcal{A}}$ to be the functor category $W\text{-Map}^{\mathcal{A}}$. To define the functor $W\text{-Map}_{\mathcal{A}} \rightarrow (\mathbb{C}^{\mathcal{A}})^2$, we can instead define a functor $\mathcal{A} \times W\text{-Map}_{\mathcal{A}} \rightarrow \mathbb{C}^2$. We take this to be evaluation followed by the map $W\text{-Map} \rightarrow \mathbb{C}^2$.

Proposition 4.8. *We can define the necessary functors to make definition 4.7 an ams with structured weak equivalences.*

Proof. These are once again defined pointwise. For illustration, we just consider the functor $C\text{-map}_{\mathcal{A}} \times_{\mathbb{C}^{\mathcal{A} \times 2}} W\text{-Map}_{\mathcal{A}} \rightarrow F^t\text{-Map}_{\mathcal{A}}$. Note that this amounts to constructing a functor $\mathcal{A} \times C\text{-map}^{\mathcal{A}} \times_{\mathbb{C}^{\mathcal{A} \times 2}} W\text{-Map}^{\mathcal{A}} \rightarrow F^t\text{-Map}$. However, the evaluation maps give us a functor $\mathcal{A} \times C\text{-map}^{\mathcal{A}} \times_{\mathbb{C}^{\mathcal{A} \times 2}} W\text{-Map}^{\mathcal{A}} \rightarrow C\text{-map} \times_{\mathbb{C}^2} W\text{-Map}$. We can then compose this with the map we are given from $C\text{-map} \times_{\mathbb{C}^2} W\text{-Map}$ to $F^t\text{-Map}$ to get the required structure. \square

4.2.2 An Explicit Definition for $W\text{-Map}$

We now show that without loss of generality we can assume $W\text{-Map}$ is given by the following explicit definition³.

Proposition 4.9. *If we are given an ams with structured weak equivalences on a category \mathbb{C} , then there is another ams with structured weak equivalences, with the same underlying model structure, such that $W\text{-Map}$ is defined as the pullback below.*

$$\begin{array}{ccc} W\text{-Map} & \longrightarrow & F^t\text{-Map} \\ \downarrow \lrcorner & & \downarrow \\ \mathbb{C}^2 & \xrightarrow{F} & \mathbb{C}^2 \end{array} \quad (4)$$

Proof. Assume that we are given an ams with structured weak equivalences given by $W\text{-Map}$. Write $W\text{-Map}'$ for the category defined as in (4). Note that to show $W\text{-Map}'$ also gives structured weak equivalences, it suffices to show that we can construct functors from $W\text{-Map}'$ to $W\text{-Map}$ and from $W\text{-Map}$ to $W\text{-Map}'$ over \mathbb{C}^2 .

³In an earlier draft of this paper this was taken as the definition of $W\text{-Map}$.

We first construct the functor $W\text{-Map}' \rightarrow W\text{-Map}$. Suppose we are given a map f and an F^t -algebra structure on Ff . We can use this to assign Ff the structure of a $W\text{-Map}$ -weak equivalence. Since $C^t f$ is given the structure of a C^t -coalgebra using the comultiplication map, we can also assign it a $W\text{-Map}$ -weak equivalence structure. Finally, we use the functorial 3-for-2 operator to assign a $W\text{-Map}$ -weak equivalence structure to f , the composition of $C^t f$ and Ff .

We now construct the functor $W\text{-Map} \rightarrow W\text{-Map}'$. Suppose we are given a map f with $W\text{-Map}$ -weak equivalence structure. We factor f as $C^t f$ followed by Ff , using the awfs (C^t, F) . We then also have a $W\text{-Map}$ -weak equivalence structure on $C^t f$. Hence we can use the functorial 3-for-2 operator to assign Ff a $W\text{-Map}$ -weak equivalence structure. We then have both a F -algebra structure and a $W\text{-Map}$ -weak equivalence structure on Ff , which gives us an F^t -algebra structure on Ff . But we have now given f the structure of a $W\text{-Map}'$ -structured weak equivalence. \square

Note that dually, we could also choose $W\text{-Map}$ to be defined by the pullback below.

$$\begin{array}{ccc} W\text{-Map} & \longrightarrow & C^t\text{-map} \\ \downarrow \lrcorner & & \downarrow \\ \mathbb{C}^2 & \xrightarrow{C} & \mathbb{C}^2 \end{array}$$

Finally, we note that in this case we can drop one of the required functions from the definition of ams with structured weak equivalences.

Proposition 4.10. *Suppose we are given a pre-ams $\xi: (C^t, F) \rightarrow (C, F^t)$, that $W\text{-Map}$ is the category defined as in proposition 4.9 and (C, F^t) is cofibrantly generated. Then we have in any case a functor $F\text{-Map} \times_{\mathbb{C}^2} W\text{-Map} \rightarrow F^t\text{-Map}$.*

Proof. Note that an F -algebra structure on a map f witnesses it as the retract of Ff . Hence, if we are given an F^t -algebra structure on Ff , we can assign f also the structure of an F^t -algebra. Since we defined this using a retract, in general it might be only a pointed endofunctor algebra. However, since (C, F^t) is cofibrantly generated, we can correct this to obtain an algebra structure over the monad. \square

4.3 Constructing Very Good Path Objects from Path Objects

We now use an ams with structured weak equivalences to define a weaker version of very good path objects (definition 3.1) where we replace the requirement of r_f being a trivial cofibration to just a weak equivalence. We will then show how to use this to construct very good path objects.

Definition 4.11. 1. A *choice of path objects* consists of an assignment to every F -map $f: X \rightarrow Y$ a factorisation

$$X \xrightarrow{r_f} P(f) \xrightarrow{p_f} X \times_Y X \quad (5)$$

of the diagonal $\Delta: X \rightarrow X \times_Y X$ together with a weak equivalence structure on r_f and an F -algebra structure on p_f .

2. A choice of path objects is *functorial* if the assignment of (5) provides the action of objects of a functor $F\text{-Map} \rightarrow F\text{-Map} \times_{\mathbb{C}} W\text{-Map}$.
3. A choice of path objects is *stable* when every map of F -algebras whose underlying square is a pullback makes the following square given by functoriality a pullback

$$\begin{array}{ccc} P(f) & \xrightarrow{P(h,k)} & P(f') \\ p_f \downarrow & & \downarrow p_{f'} \\ X \times_Y X & \longrightarrow & X' \times_{Y'} X' \end{array}$$

Definition 4.12. We say an awfs (L, R) is *pullback stable*, if for every pullback square of the form below,

$$\begin{array}{ccc} U & \longrightarrow & X \\ f \downarrow & \lrcorner & \downarrow g \\ V & \longrightarrow & Y \end{array}$$

the square below given by functoriality is also a pullback.

$$\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ Rf \downarrow & & \downarrow Rg \\ V & \longrightarrow & Y \end{array}$$

Theorem 4.13. Suppose we are given an awfs with structured weak equivalences $\xi: (C^t, F) \rightarrow (C, F^t)$ where (C, F^t) is pullback stable.

Then given a stable functorial choice of path objects we can construct a stable functorial choice of very good path objects.

Proof. Suppose that we are given a choice of path objects. That is, we are given for each fibration f a factorisation

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times_Y X \\ & \searrow r_f & \nearrow p_f \\ & Pf & \end{array}$$

together with weak equivalence on r_f and R -algebra structure on p . Then we apply the (C, F^t) factorisation to r_f to extend the diagram as follows.

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times_Y X \\ Cr_f \downarrow & \searrow r_f & \nearrow p_f \\ Mr_f & \xrightarrow{F^t r_f} & Pf \end{array}$$

Then we may use the functorial 3-for-2 operator to construct a weak equivalence structure on Cr_f from the weak equivalence structures on r_f and $F^t r_f$. Since Cr is a cofibration, we can construct from this a C^t -coalgebra structure on Cr .

Furthermore we can produce an R -algebra structure on $F^t r$ using the comparison map, and then assign an F -algebra structure to $p_f \circ F^t r_f$ by composing the structures on $F^t r_f$ and p_f .

All of the above constructions can be done functorially, so the following is a functorial choice of very good path objects.

$$\begin{array}{ccc}
 X & \xrightarrow{\Delta} & X \times_Y X \\
 & \searrow^{Cr_f} & \nearrow^{p_f \circ F^t r_f} \\
 & & Mr_f
 \end{array}$$

Now we just need to check that this construction satisfies stability. Assume that we are given a pullback square as below.

$$\begin{array}{ccc}
 X & \longrightarrow & X' \\
 \downarrow f & \lrcorner & \downarrow f' \\
 Y & \longrightarrow & Y'
 \end{array} \tag{6}$$

We first check that the square below is a pullback.

$$\begin{array}{ccc}
 X \times_Y X & \longrightarrow & X' \times_{Y'} X' \\
 \downarrow & & \downarrow \\
 Y & \longrightarrow & Y'
 \end{array} \tag{7}$$

To this end, consider the following commutative cube.

$$\begin{array}{ccccc}
 & & X \times_Y X & \longrightarrow & X' \times_{Y'} X' \\
 & \swarrow & \downarrow & & \downarrow \\
 X & \longrightarrow & X & \longrightarrow & X' \\
 \downarrow & & \downarrow & & \downarrow \\
 Y & \longrightarrow & Y & \longrightarrow & Y'
 \end{array}$$

The front face is a pullback by our assumption and the left and right faces are pullbacks by definition. Hence the back face is also a pullback. But the bottom face is a pullback once again by our assumption. Hence the composition of the back face and bottom face is a pullback, but this is precisely (7), as required.

Now consider the diagram

$$\begin{array}{ccc}
 X & \longrightarrow & X' \\
 \downarrow & & \downarrow \\
 Pf & \longrightarrow & Pf' \\
 \downarrow & & \downarrow \\
 X \times_Y X & \longrightarrow & X' \times_{Y'} X' \\
 \downarrow & & \downarrow \\
 Y & \longrightarrow & Y'
 \end{array} \tag{8}$$

We have just checked that the lower square is a pullback. The middle square is also a pullback since we assumed P is stable. The entire rectangle is also a pullback by assumption (it is precisely (6)). We deduce that the upper square is also a pullback.

Finally consider the following diagram.

$$\begin{array}{ccc}
 Mr_f & \longrightarrow & Mr_{f'} \\
 \downarrow R^t r_f & & \downarrow R^t r_{f'} \\
 Pf & \longrightarrow & Pf' \\
 \downarrow p_f & & \downarrow p_{f'} \\
 X \times_Y X & \longrightarrow & X' \times_{Y'} X'
 \end{array} \tag{9}$$

Since the upper square in (8) is a pullback and (C, F^t) preserves pullbacks, we deduce that the upper square in (9) is a pullback. The lower square is also a pullback by the assumption that P is stable. Hence the entire rectangle is a pullback. But this is precisely what we need to show that the very good path objects we defined before are stable. \square

5 Some Sufficient Conditions for the Existence of Identity Types

In this section we give a direct proof that identity types can be constructed in certain categories. We will follow the same construction as in section 4.3. Instead of an ams, we only work with a pre-ams satisfying certain axioms, based on those considered by Gambino and Sattler in [7]. These axioms are much easier to show for the examples we consider than the construction of a 3-for-2 operator.

We note that throughout this section, part of the work lies in adapting arguments based on wfs's to stronger functorial versions. Under suitable conditions it is possible to save work by doing this automatically using pointwise awfs's or (as the author will show in a future paper) category indexed family fibrations. However, for now we work directly with the functorial versions, to give a clearer picture of the objects and maps involved in the construction. Since all the constructions involved are fairly simple, this does not cause too much difficulty.

We first review some necessary background material.

5.1 The Leibniz Construction

The Leibniz construction is a well known construction in homotopical algebra. See e.g. [17, Construction 11.1.7] for a standard reference. It was first applied to the semantics of homotopy type theory, and in particular CCHM cubical sets by Gambino and Sattler in [7]. The idea is that given a monoidal category, (\mathbb{C}, \otimes) , we can give \mathbb{C}^2 also the structure of a monoidal category using pushout product $\hat{\otimes}$. Furthermore, if we are given right adjoints to $- \otimes X$ for each X in \mathbb{C} , we can produce also right adjoints to $-\hat{\otimes}f$ for each f in \mathbb{C}^2 using pullback hom.

Definition 5.1. Let (\mathbb{C}, \otimes) be a monoidal category with pushouts. The *pushout product* is the monoidal product $\hat{\otimes}$ defined on \mathbb{C}^2 as follows. Given $f: U \rightarrow V$ and $g: X \rightarrow Y$, we define $f \hat{\otimes} g$ as the map given by the universal property of the pushout below.

$$\begin{array}{ccc}
 U \otimes X & \xrightarrow{f \otimes X} & V \otimes X \\
 U \otimes g \downarrow & & \downarrow \\
 U \otimes Y & \xrightarrow{\quad} & \cdot \\
 & \searrow f \hat{\otimes} g & \downarrow V \otimes g \\
 & & V \otimes Y
 \end{array}$$

$\swarrow f \otimes Y$

Definition 5.2. Let (\mathbb{C}, \otimes) be a monoidal category with pushouts and pullbacks. Suppose that for each X , $- \otimes X$ has a right adjoint $\text{hom}(X, -)$. Then for each map f , $f \hat{\otimes} -$ has a right adjoint, $\hat{\text{hom}}(f, -)$ referred to as *pullback hom*, which is defined explicitly as the map given by the universal property of the pullback below. Let $f: U \rightarrow V$ and $g: X \rightarrow Y$.

$$\begin{array}{ccc}
 \text{hom}(V, X) & \xrightarrow{\text{hom}(V, g)} & \text{hom}(V, Y) \\
 \text{hom}(f, g) \searrow & & \downarrow \text{hom}(f, Y) \\
 \cdot & \xrightarrow{\quad} & \text{hom}(U, Y) \\
 \text{hom}(f, X) \searrow & & \downarrow \text{hom}(U, g) \\
 \text{hom}(U, X) & \xrightarrow{\quad} & \text{hom}(U, Y)
 \end{array}$$

5.2 The Conditions

Definition 5.3. A monoidal product \otimes is *affine* if the unit of the monoidal product is a terminal object.

Remark 5.4. For any affine monoidal product, we can define projection maps $X \otimes Y \rightarrow X$ and $X \otimes Y \rightarrow Y$.

Our basic set up is a monoidal category (\mathbb{C}, \otimes) satisfying the following conditions.

1. \mathbb{C} is finitely complete and finitely cocomplete.
2. \otimes is an affine and symmetric monoidal product that preserves colimits.
3. $\delta_i: 1 \rightarrow \mathbb{I}$ for $i = 0, 1$ is an interval object.
4. $- \otimes \mathbb{I}$ has a right adjoint, which we denote P .
5. (\mathbb{C}, F^t) is pullback stable.
6. Axioms 5.5, 5.6 and 5.7 below.

Axiom 5.5. If we are given a cofibration m , then we can also give $m \hat{\otimes} [\delta_0, \delta_1]$ the structure of a cofibration, and this assignment is functorial in m .

Note that pullback hom might not be defined in general, but we do know that $-\hat{\otimes}\delta_i$ has a right adjoint given by $\hat{\text{hom}}(\delta_i, -)$ using P .

Axiom 5.6. Given an F algebra structure on a map f , we can define in a functorial way, an F^t algebra structure on $\hat{\text{hom}}(\delta_i, f)$ for $i = 0, 1$.

Axiom 5.7. Given an F algebra structure on a map f , we can define in a functorial way, an F algebra structure on $\hat{\text{hom}}([\delta_0, \delta_1], f)$.

5.3 Some Useful Propositions

Before giving some examples, we prove a couple of propositions that will be useful for verifying the examples do satisfy the axioms, and later for the theorem itself.

Proposition 5.8. *Suppose that a pre-ams $\xi: (C^t, F) \rightarrow (C, F^t)$ satisfies axiom 5.6. Then given a cofibration m , we can assign $m\hat{\otimes}\delta_i$ the structure of a trivial cofibration.*

Proof. This follows from the adjunction between pushout product and pullback hom. \square

As remarked in section 1, when working with cofibrations, we usually use the category $C\text{-map}$ of copointed endofunctor coalgebras. However, in order to construct the comparison map it is useful to instead work over the category $C\text{-Map}$ of comonad coalgebras. Because of this, we will aim towards a lemma constructing a functor $C\text{-map} \rightarrow C\text{-Map}$ over \mathbb{C}^2 using the assumption that (C, F^t) is pullback stable.

Lemma 5.9. *“Every retract of a monomorphism is a pullback.” More formally, suppose we are given diagram as below, where $k \circ h = 1_X$ and $m \circ l = 1_Y$, and that g (and so also f) is a monomorphism.*

$$\begin{array}{ccccc}
 X & \xrightarrow{h} & Z & \xrightarrow{k} & X \\
 f \downarrow & & g \downarrow & & f \downarrow \\
 Y & \xrightarrow{l} & W & \xrightarrow{m} & Y
 \end{array} \tag{10}$$

Then the left hand square in (10) is a pullback.

Proof. Suppose we are given a commutative diagram as in the solid lines below. We need to show there is a unique map t as in the dotted line below, making the diagram commute.

$$\begin{array}{ccccc}
 A & & & & \\
 & \searrow & & & \\
 & & X & \xrightarrow{h} & Z \\
 & & f \downarrow & & g \downarrow \\
 & & Y & \xrightarrow{l} & W \\
 & \swarrow & & & \\
 & & & &
 \end{array}$$

Note however that uniqueness easily follows from the fact that f is monic. Hence we only have to show existence.

We take t to be $k\text{op}$. It is straightforward to verify that the resulting diagram commutes. \square

Proposition 5.10. *Suppose that (C, F^t) is pullback stable and that every cofibration is a monomorphism. Then there is a functor that takes a copointed endofunctor coalgebra structure on a map f , and returns a comonad coalgebra structure on the same map. That is, we construct a functor $C\text{-map} \rightarrow C\text{-Map}$ over \mathbb{C}^2 .*

Proof. Suppose that f has the structure of a coalgebra over the underlying copointed endofunctor of (C, F^t) . The coalgebra structure witnesses f as a retract of Cf . Hence by lemma 5.9 the same diagram witnesses f as a pullback of Cf . Since (C, F^t) is pullback stable we can pullback the C -coalgebra structure on Cf to obtain a C -coalgebra structure on f .

This is clearly functorial. \square

5.4 Examples

5.4.1 Gambino-Sattler Axioms

As stated above, these axioms are based on those of Gambino and Sattler in [7, Section 7].

We can recover a similar definition to Gambino and Sattler's as follows. We add the requirement that (C, F^t) is algebraically cofibrantly generated, by a diagram $M: J \rightarrow \mathbb{C}^2$ that we refer to as the *generating cofibrations*. We then further require that (C^t, F) is cofibrantly generated by the coproduct of the two diagrams of the form $M \hat{\otimes} \delta_i$ for $i = 0, 1$. Instead of assuming axioms 5.6 and 5.7. We will keep assuming that axiom 5.5 holds.

In this set up, the awfs (C^t, F) is by definition cofibrantly generated by the pushout product of generating cofibrations with the endpoint inclusions δ_i . Observe that using the adjunction between pushout product and pullback exponential, this automatically gives us axiom 5.6. Furthermore, we also get axiom 5.7. We again use the adjunction between pushout product and pullback exponential to instead check the dual definition for generating trivial cofibrations. Any generating trivial cofibration is of the form $m \hat{\otimes} \delta_i$ where m is a generating cofibration. Then by symmetry of the monoidal product, we have the following isomorphism.

$$(m \hat{\otimes} \delta_i) \hat{\otimes} [\delta_0, \delta_1] \cong (m \hat{\otimes} [\delta_0, \delta_1]) \hat{\otimes} \delta_i \quad (11)$$

Combining this with axiom 5.5, for each generating trivial cofibration $m \hat{\otimes} \delta_i$, we can assign $(m \hat{\otimes} \delta_i) \hat{\otimes} [\delta_0, \delta_1]$ with the structure of a trivial cofibration. Moreover, noting that (11) is part of a natural isomorphism and using the functorial part of 5.5, we can ensure that a generating morphism of trivial cofibrations is sent to a morphism of trivial cofibrations. This then gives us axiom 5.7.

If we follow Gambino and Sattler in assuming that the generating cofibrations are functorially closed under pushout product with endpoint inclusion, one can construct the comparison map of the pre-ams using Riehl's observation in [16, Remark 3.6] that it suffices to show that one can functorially assign the generating trivial cofibrations with cofibration structures, and also using proposition 5.10.

There are, however still a few minor differences with the Gambino-Sattler axioms. We work with symmetric monoidal products and an interval rather than

functorial cylinders. Axiom 5.5 does not seem to follow from [7, Definition 7.1]. Note however that it does hold in the examples of simplicial sets and (CCHM) cubical sets appearing in [7, Example 7.2]. More generally, using the axiom that $-\hat{\otimes}\delta_i$ preserves cofibrations, it follows from the additional assumption that cofibrations are functorially closed under binary union. This assumption was added, for instance by Sattler in [18, Definition 3.2], and also appears in the Orton-Pitts axioms [14].

5.4.2 BCH Cubical Sets and 01-Substitution Sets

BCH cubical sets were introduced by Bezem, Coquand and Huber in [2], and further developed by Huber in [10]. They were later still further developed by Bezem, Coquand and Huber in [3], where they showed how to interpret the univalence axiom in the model. In [15] Pitts showed that this category of cubical sets is equivalent to a category based on nominal sets, denoted 01-substitution sets. Since this paper is only concerned with structure that is preserved up to isomorphism by equivalences of categories, we can freely switch back and forth between the two presentations.

The monoidal product we use is *separated product*. This is naturally defined in 01-substitution sets, where it has the same definition as the separated product in nominal sets. The right adjoint to $-\otimes\mathbb{I}$ exists and can be explicitly described in terms of name abstraction. See [10, Section 2.4] for a detailed description by Huber.

In [21, Section 7.5.3] the author showed that trivial fibrations can be viewed as cofibrantly generated in two senses. They can be viewed as cofibrantly generated in Garner’s sense by boundary inclusions together with a uniformity condition (which in loc. cit. is referred to as *cofibrantly generated with respect to the category indexed families fibration*). This is essentially the same as the definition by Bezem, Coquand and Huber in [3]. Trivial fibrations can also be viewed as cofibrantly generated with respect to the codomain fibration. The latter description can be used to show that the awfs (C, F^t) of cofibrations and trivial fibrations is stable under pullback if it exists. In fact, using [21, Corollary 7.5.5] we can show that (C, F^t) is *strongly fibred* with respect to the codomain fibration. This means that it forms part of a fibred awfs over the codomain fibration in which the restriction to each fibre category is pullback stable. This leaves the problem of actually constructing the awfs (C, F^t) , which can be done using [22, Theorem 6.14], together with the observation that the generating cofibrations are locally decidable, or by using Garner’s small object argument [8]. The awfs can also be constructed directly in a similar manner to [20], followed by direct verification of pullback stability. See [3] for such a construction (or [19]). Note that n -dimensional boundary inclusions can be defined as the pushout product of n copies of $[\delta_0, \delta_1]$. It follows that the generating cofibrations are closed under pushout product with $[\delta_0, \delta_1]$, which then ensures that axiom 5.5 is satisfied.

We define the interval object to be the same as in [2, Section 6.1].

We define the awfs (C^t, F) of trivial cofibrations and fibrations to be cofibrantly generated by pushout product of a generating cofibration with an endpoint inclusion δ_i , following the same construction as in section 5.4.1. Note that the pushout product of a boundary inclusion with δ_i gives us the standard open box in direction i . Hence this gives the same definition of fibration as given

by Huber in [10, Remark 3.9] or by the author in [20, Section 5]. This defines the awfs (C^t, F) uniquely up to isomorphism. In order to show that (C^t, F) actually exists, one can either apply Garner's small object argument, or give a direct description as in [20]. Once again this definition automatically gives us axioms 5.6 and 5.7.

This time constructing the comparison map is a little trickier. However, it can be done using the following lemma.

Lemma 5.11. *Suppose that we are given a finitely cocomplete affine monoidal category (\mathbb{C}, \otimes) together with an awfs (L, R) and an interval object $\delta_0, \delta_1: 1 \rightarrow \mathbb{I}$.*

If we are given an L -coalgebra structure on a map m and an L -coalgebra structure on $[\delta_0, \delta_1] \hat{\otimes} m$, then we can produce an L -coalgebra structure on $\delta_i \hat{\otimes} m$ for $i = 0, 1$.

Moreover, this assignment is functorial, in the sense that it is the action on objects of a functor $L\text{-Map} \times L\text{-Map} \rightarrow L\text{-Map}$.

Proof. We will just do the case $i = 0$, the other case being similar. Roughly the idea is that if we are given an n dimensional open box, we can find a filler by first building the top lid (which is $n - 1$ dimensional), and then filling the resulting boundary of the n -cube. We will show how $\delta_0 \hat{\otimes} m$ can be built up from cofibrations.

Say that $m: A \rightarrow B$ and write D for the domain of $\delta_0 \hat{\otimes} m$. By definition, D is given by the following pushout diagram.

$$\begin{array}{ccc} 1 \otimes A & \xrightarrow{\delta_0 \otimes A} & \mathbb{I} \otimes A \\ 1 \otimes m \downarrow & & \downarrow \\ 1 \otimes B & \longrightarrow & D \end{array}$$

Now if L is the domain of $[\delta_0, \delta_1] \hat{\otimes} m$, then note that we have a canonical map $i: D \rightarrow C$ given by the inclusion $\iota_1: 1 \rightarrow 2$. This gives us a commutative square of the form below.

$$\begin{array}{ccccc} 1 \otimes A & \xrightarrow{\iota_1} & \mathbb{I} \otimes A & \longrightarrow & D \\ 1 \otimes m \downarrow & & & & \downarrow \\ 1 \otimes B & \xrightarrow{\iota_1 \otimes B} & 2 \otimes B & \longrightarrow & C \end{array}$$

One can verify by diagram chase that in fact this square is a pushout. We think of this as gluing the missing lid to the n -dimensional open box to make it into the boundary of the n -dimensional cube. Further diagram chasing shows that in fact $\delta_0 \hat{\otimes} m$ factors as the map $D \rightarrow C$ followed by $[\delta_0, \delta_1] \hat{\otimes} m$. Informally, we visualise this as including the n -dimension open box into the n -cube, by first including it into the boundary, and then including the boundary into the n -cube.

However, we have now exhibited $\delta_0 \hat{\otimes} m$ as a pushout of a generating cofibration, m composed with $[\delta_0, \delta_1] \hat{\otimes} m$, which is also a generating cofibration. Both pushout and composition preserve L -coalgebra structure, so we obtain an L -coalgebra structure on $\delta_0 \hat{\otimes} m$.

Functoriality is tedious but straightforward to verify. \square

Theorem 5.12. *Suppose that $M: J \rightarrow \mathbf{01Sub}^2$ is the generating diagram of cofibrations, defined as above, using boundary inclusions. Then $\delta_0 \hat{\otimes} M$ and $\delta_1 \hat{\otimes} M$ factor through the forgetful functor from C -coalgebras to $\mathbf{01Sub}^2$.*

Proof. As explained above, we can show how to functorially assign $[\delta_0, \delta_1] \hat{\otimes} M$ with the structure of a C -coalgebra. However, we can now apply lemma 5.11. \square

In this setting it is also possible to view axioms 5.5, 5.6 and 5.7 in a more geometric fashion. Axiom 5.5 corresponds to the fact that given a boundary of an n dimensional cube, we obtain a new boundary of an $n + 1$ dimensional cube by taking the product with the interval to get a tube, and then pasting on both ends of the tube. As stated above, axiom 5.6 follows from the way we define generating trivial cofibrations, in which a generating trivial cofibration is an open box, which can be seen as a tube with only one of the ends pasted on. Finally axiom 5.5 corresponds to the fact that if we are given an open box we obtain a new open box when we take the product with the interval, and then paste on both sides of the prism, while leaving the top open.

5.4.3 Van den Berg-Frumin/Orton-Pitts Axioms

In [21, Section 7.5.2] the author gave a definition related to a class of structures considered by Orton and Pitts in [14] and by Van den Berg and Frumin in [23]. This presentation is closest to that of Van den Berg and Frumin, which in turn is based on the Gambino-Sattler definition above. We work over a locally cartesian closed category \mathbb{C} with finite colimits, disjoint coproducts, and an interval object $\delta_0, \delta_1: 1 \rightarrow \mathbb{I}$. In order to construct the awfs's (C, F^t) and (C^t, F) we will assume that \mathbb{C} satisfies one of the ‘‘codomain fibred’’ versions of the small object argument developed by the author in [22]. For example, it suffices that \mathbb{C} is a topos with natural number object and satisfies WISC. Examples of such structures include CCHM cubical sets, as defined in [6] and simplicial sets. For now, we don't assume that the interval object has connections, although we will see in section 6.1 that in that case we can simplify the argument.

In this setup we start with an awfs (C, F^t) which is cofibrantly generated with respect to the codomain fibration by a family of maps of the form below.

$$\begin{array}{ccc} 1 & \xrightarrow{\top} & \Sigma \\ & \searrow \top & \nearrow \parallel \\ & & \Sigma \end{array}$$

In this case we automatically get pullback stability, as for BCH cubical sets. For now we don't assume that the generating cofibrations are closed under composition, although we'll see later in section 6.2 that in this case the argument simplifies. We do however, assume that cofibrations are closed under finite union and that both endpoint inclusions δ_i are cofibrations. This ensures we get axiom 5.5.

We again define (C^t, F) as cofibrantly generated by pushout product with a cofibration and an endpoint inclusion. In contrast to BCH cubical sets, in this case (C^t, F) will also be cofibrantly generated with respect to the codomain fibration on \mathbb{C} . For this to work smoothly, instead of working with arbitrary monoidal products as before, we only consider cartesian product, which easily

extends to a fibred monoidal product over the codomain fibration, which then ensures pushout product is also fibred [21, Section 6]. We then define (C^t, F) to be the awfs cofibrantly generated by the coproduct of the following two families of maps.

$$\begin{array}{ccc} \mathbb{I} +_1 \Sigma & \xrightarrow{\delta_0 \hat{\times} \top} & \mathbb{I} \times \Sigma \\ & \searrow & \swarrow \\ & \Sigma & \end{array} \quad \begin{array}{ccc} \mathbb{I} +_1 \Sigma & \xrightarrow{\delta_1 \hat{\times} \top} & \mathbb{I} \times \Sigma \\ & \searrow & \swarrow \\ & \Sigma & \end{array}$$

See [21, Section 7.5.2] for more details. Note that we again easily obtain axioms 5.6 and 5.7 from this definition.

5.5 Proof of Existence of Identity Types

First note that we can define a functorial choice of factorisations of diagonal maps using the following well known construction, usually referred to as *mapping path space*.

Given a map $f: X \rightarrow Y$, we define $P(f)$ to be given by the pullback below, where the bottom map $Y \rightarrow P(Y)$ corresponds to the projection $Y \otimes \mathbb{I} \rightarrow Y$ under the adjunction.

$$\begin{array}{ccc} P(f) & \longrightarrow & P(X) \\ \downarrow \lrcorner & & \downarrow \\ Y & \longrightarrow & P(Y) \end{array}$$

When f is clear from the context, we will also write $P(f)$ as $P_Y(X)$.

Note that we have evident maps $r: X \rightarrow P_Y(X)$ and projections $p_0, p_1: P_Y(X) \rightarrow X$ over Y .

One can verify by diagram chase that $p_f := \langle p_0, p_1 \rangle: P_Y(X) \rightarrow X \times_Y X$ can be viewed as a pullback of $\hat{\text{hom}}(f, [\delta_0, \delta_1])$ (see [11, Proposition 2.3.3] for the analogous statement in simplicial sets). We can therefore assign it the structure of a fibration by first applying axiom 5.7 to give $\hat{\text{hom}}(f, [\delta_0, \delta_1])$ the structure of a fibration, and then assigning p_f the unique fibration structure preserved by the pullback.

We construct the choice of factorisations $\text{Id}_Y(X)$ using (C, F^t) in the preams together with P , exactly like in theorem 4.13. It remains to show that this does give us a stable functorial choice of very good path objects.

We will use a structured version of the definitions of homotopy and strong deformation retract defined below. This is based on the non-structured version used e.g. by Gambino and Sattler in [7, Remark 4.2].

Definition 5.13. Suppose we are given morphisms $f, g: X \rightarrow Y$. A *structured homotopy* from f to g is a map $h: X \otimes \mathbb{I} \rightarrow Y$ fitting into the following

commutative diagram.

$$\begin{array}{ccc}
 X & & Y \\
 & \searrow^{f} & \\
 & X \otimes \delta_0 & \\
 & & X \otimes \mathbb{I} \xrightarrow{h} Y \\
 & \nearrow_{X \otimes \delta_1} & \\
 X & & Y \\
 & \nearrow_{g} &
 \end{array}$$

Suppose we are given maps $f, g: X \rightarrow Y$, $f', g': X' \rightarrow Y'$, together with a structured homotopy h from f to g and a structured homotopy h' from f' to g' , and maps $k: X \rightarrow X'$ and $l: Y \rightarrow Y'$. We say k and l *preserve the structured homotopies* if the following equations hold.

1. $l \circ f = f' \circ k$
2. $l \circ g = g' \circ k$
3. $h' \circ (k \otimes \mathbb{I}) = l \circ h$

Definition 5.14. A (structured) *strong deformation retract* from X to Y consists of the following.

1. A morphism $f: X \rightarrow Y$.
2. A morphism $s: Y \rightarrow X$.
3. A structured homotopy h from $s \circ f$ to 1_X .

The maps are required to satisfy the following equalities.

1. $f \circ s = 1_Y$
2. $h \circ (s \otimes \mathbb{I}) = s \circ \pi_0$ (where π_0 is the projection $X \otimes \mathbb{I} \rightarrow X$)

Suppose we are given strong deformation retracts (f, s, h) and (f', s', h') , together with a map $k: X \rightarrow X'$ and $l: Y \rightarrow Y'$. We say k and l *preserve the structured strong deformation retracts* if they satisfy the following equalities.

1. $l \circ f = f' \circ k$
2. $k \circ s = s' \circ l$
3. k and l preserve the structured homotopies h and h'

Lemma 5.15. *Suppose we are given a fibration $f: X \rightarrow Y$. Then we can give each projection $e_i: P_Y X \rightarrow X$ the structure of a trivial fibration. Moreover this is functorial, in the sense that given a morphism of fibrations, the commutative square derived from the functoriality of $P_Y X$ is a morphism of trivial fibrations.*

Proof. It is straightforward to exhibit e_i as a pullback of $\widehat{\text{hom}}(\delta_i, f)$, which has the structure of a trivial fibration by axiom 5.6. We assign e_i the unique trivial fibration structure that makes the pullback a morphism of trivial fibrations. \square

Lemma 5.16. *Suppose we are given maps $r: A \rightarrow B$ and $i: B \rightarrow A$ such that $i \circ r = 1_A$, together with the structure of a cofibration on r and the structure of a trivial fibration on i . Then we can extend r and i to a structured strong deformation retract.*

Also, this is functorial in the sense that every commutative square that preserves the cofibration and trivial fibration structures leads to a commutative square preserving the strong deformation retract structures.

Proof. We will define a lifting problem of $r \hat{\otimes} [\delta_0, \delta_1]$ against i . By axiom 5.5, we will be able to find a diagonal filler.

The domain of $r \hat{\otimes} [\delta_0, \delta_1]$ is by definition the pushout below.

$$\begin{array}{ccc} A \otimes 2 & \xrightarrow{A \otimes [\delta_0, \delta_1]} & A \otimes \mathbb{I} \\ r \otimes \mathbb{I} \downarrow & & \downarrow \\ B \otimes 2 & \longrightarrow & A \otimes \mathbb{I} +_{A \otimes 2} B \otimes 2 \end{array}$$

To define a map $A \otimes \mathbb{I} +_{A \otimes 2} B \otimes 2 \rightarrow B$ is therefore to define a map $B \otimes 2 \rightarrow B$ and a map $A \otimes \mathbb{I} \rightarrow B$ ensuring that the two maps $A \otimes 2 \rightarrow B$ agree.

Note that since \otimes is affine and preserves colimits, we have $B \otimes 2 \cong B + B$. We can therefore define a map $B \otimes 2 \rightarrow B$ as $[r \circ i, 1_B]$.

We define the map $A \otimes \mathbb{I} \rightarrow B$ to be $r \circ \pi_0$.

This then gives us the following lifting problem, where the top morphism is defined as above.

$$\begin{array}{ccc} A \otimes \mathbb{I} + B \otimes 2 & \longrightarrow & B \\ r \hat{\otimes} [\delta_0, \delta_1] \downarrow & & \downarrow i \\ B \otimes \mathbb{I} & \xrightarrow{i \circ \pi_0} & A \end{array}$$

We take $h: B \otimes \mathbb{I} \rightarrow B$ to be the diagonal filler given by the cofibration structure on $r \hat{\otimes} [\delta_0, \delta_1]$ (which in turn is given by the cofibration structure on r via axiom 5.5) together with the trivial fibration structure on i .

We need to check that $h: B \otimes \mathbb{I} \rightarrow B$ does witness that r is a strong deformation retract. The upper triangle identity ensures that $h \circ B \otimes \delta_0 = r \circ i$, that $h \circ B \otimes \delta_1 = 1_B$, and that $h \circ r \otimes \mathbb{I} = r \circ \pi_0$. Hence, h does indeed witness that r is a strong deformation retract.

Functoriality follows from axiom 5.5 together with diagram chasing. \square

Lemma 5.17. *The map $Cr: X \rightarrow \text{Id}_Y(X)$ has the structure of a strong deformation retract.*

Furthermore, this is functorial in the following sense. If we are given a morphism of fibrations from $f: X \rightarrow Y$ to $f': X' \rightarrow Y'$, then the commutative square given by the functoriality of Id preserves the strong deformation retract structure.

Proof. Note that Cr has a retract given by $i := e_0 \circ F^t r$. Note that e_0 is given the structure of a trivial fibration by lemma 5.15, and F^t also has the structure of a trivial fibration. By composing these, we give i also the structure of a trivial fibration.

We now apply lemma 5.16. \square

Lemma 5.18. *Suppose that we are given a cofibration $t: A \rightarrow B$ with the structure of a strong deformation retract. Then we can assign t the structure of a trivial cofibration, and moreover this can be done functorially.*

Proof. This is essentially a functorial version of a special case of [7, Lemma 4.3], but for completeness we write out the details below.

We write out the structure of a strong deformation retract as a map i such that $i \circ t = 1_A$ and the map $h: B \otimes \mathbb{I} \rightarrow B$ in the diagram below.

$$\begin{array}{ccc}
 B & & \\
 \searrow^{B \otimes \delta_0} & & \nearrow^{t \circ i} \\
 & B \otimes \mathbb{I} & \xrightarrow{h} B \\
 \nearrow^{B \otimes \delta_1} & & \searrow^{1_B} \\
 B & &
 \end{array} \tag{12}$$

We also have that $i \circ h = i \circ \pi_0$ and $h \circ t \otimes \mathbb{I} = t \circ \pi_0$.

We can now exhibit t as a retract of $t \hat{\otimes} \delta_0$ in the diagram below. Note that we can show that the right hand square really does commute using the upper part of (12) together with the identity $h \circ t \otimes \mathbb{I} = t \circ \pi_0$. The upper horizontal composition is trivially the identity. We show that the lower horizontal composition is the identity on B by using the lower half of (12).

$$\begin{array}{ccccc}
 A & \xrightarrow{A \otimes \delta_1} & B +_A A & \xrightarrow{[i, \pi_0]} & A \\
 \downarrow t & & \downarrow t \hat{\otimes} \delta_0 & & \downarrow t \\
 B & \xrightarrow{B \otimes \delta_1} & B \otimes \mathbb{I} & \xrightarrow{h} & B
 \end{array}$$

However, this allows us to assign t the structure of a trivial cofibration from that of $t \hat{\otimes} \delta_0$, which we construct using proposition 5.8 together with axiom 5.6. \square

Theorem 5.19. *The objects $\text{Id}_Y(X)$ can be given the structure of a stable functorial choice of very good path objects.*

Proof. We need to show how to give the map $Cr: X \rightarrow \text{Id}_Y(X)$ the structure of a trivial cofibration. We showed in lemma 5.17 how to give it the structure of a strong deformation retract. In any case it has the structure of a cofibration, and so by lemma 5.18 we can assign it the structure of a trivial cofibration.

We next show how to assign a fibration structure to the map $\text{Id}_Y(X) \rightarrow X \times_Y X$ defined as the composition of $F^t r: \text{Id}_Y(X) \rightarrow P_Y X$ with the original map $p_f: P_Y X \rightarrow X \times_Y X$. We know that $F^t r$ has the structure of a trivial fibration, so using the comparison map $\xi: (C^t, F) \rightarrow (C, F^t)$ we can assign it also the structure of a fibration. In general in an awfs fibrations can be composed functorially, giving us a fibration structure on $p_f \circ F^t r$, as required.

Finally, the proof of stability is exactly the same as in theorem 4.13, using the assumptions that P is stable and that (C, F^t) is pullback stable. \square

6 Two Simplifications in CCHM Cubical Sets

The first version of this work was largely specific to 01-substitution sets, although it was fairly clear that the main ideas should generalise to other situations. At this time Cohen, Coquand, Huber and Mörtberg were already using the newer definition of cubical sets, which now appears in [6]. There was some discussion between Thierry Coquand, Simon Huber and the author on how to translate the ideas into the new definition of cubical sets. During this discussion, Coquand noticed that in fact in this specific situation, a simplified definition of identity type can be used, where it is easier to give explicit definitions of the objects and maps involved. This simplified version was the one used for the definition of identity types in [6]. The same construction was applied to a wide class of models by Orton and Pitts in [14]. Another variant of that construction was used by Van den Berg and Frumin in [23] and in fact the presentation here will be much closer to the Van den Berg-Frumin version.

6.1 Using Connections for the Strong Deformation Retract Structure

The first observation was that for constructing the strong deformation retract structure one can exploit the fact that CCHM cubical sets include connections to get a more explicit definition that does not require the map f to be a fibration.

We will give an explanation of this in the lemmas below. Following Gambino and Sattler [7], we note that in fact all we need is that the interval object \mathbb{I} has connections, in the form of two maps $c_i: I \otimes I \rightarrow I$ for $i = 0, 1$ satisfying appropriate equalities.

Lemma 6.1. *Suppose that we are given maps $\mathbb{I} \otimes \mathbb{I} \rightarrow \mathbb{I}$ giving connections on \mathbb{I} satisfy the conditions given by Sattler in [18, Section 3.2]. Then $X \rightarrow P_Y X$ always has the structure of a strong deformation retract (even if f is not a fibration).*

Proof. As Gambino and Sattler point out in [7, Section 2], the connections on \mathbb{I} give P the structure of a functorial cylinder in the opposite category. It follows that the maps $X \rightarrow P_Y(X)$ are strong deformation retracts by [7, Remark 4.2] noting that the extra conditions added in [18, Section 3.2] imply that this holds for P_Y for all Y rather than just P . \square

To give the map $X \rightarrow \text{Id}_Y(X)$ the structure of a strong deformation retract, we use the lemma below to “lift” the strong deformation retract structure on the map $X \rightarrow P_Y(X)$.

Lemma 6.2. *Suppose we are given a diagram as below.*

$$\begin{array}{ccc}
 & & B' \\
 & \nearrow r' & \downarrow f \\
 A & \xrightarrow{r} & B
 \end{array}$$

Suppose further that r is given the structure of a strong deformation retract of the form $i: B \rightarrow A$ and $h: B \otimes \mathbb{I} \rightarrow B$, that f is given the structure of a trivial fibration, and that r' is also given the structure of a cofibration. Then we can

assign r' the structure of a strong deformation retract of the form $i': B' \rightarrow A$ and $h': B' \otimes \mathbb{I} \rightarrow \mathbb{I}$, satisfying the following commutative diagrams.

$$\begin{array}{ccc}
& B' & \\
i' \swarrow & & \downarrow f \\
A & \xleftarrow{i} & B
\end{array}
\quad
\begin{array}{ccc}
B' \otimes \mathbb{I} & \xrightarrow{h'} & B' \\
f \otimes \mathbb{I} \downarrow & & \downarrow f \\
B \otimes \mathbb{I} & \xrightarrow{h} & B
\end{array}
\tag{13}$$

Furthermore this assignment is functorial.

Proof. First note that the first diagram in (13) tells us that we have to have $i' = i \circ f$, so we take this for the definition of i' . Then we already see that $i' \circ r' = 1_A$.

Next, following the same outline as in the proof of lemma 5.16, we will define a lifting problem of $r' \hat{\otimes} [\delta_0, \delta_1]$ against f .

We again note that to define a map $A \otimes \mathbb{I} +_{A \otimes 2} B' \otimes 2 \rightarrow B'$ is therefore to define a map $B' \otimes 2 \rightarrow B'$ and a map $A \otimes \mathbb{I} \rightarrow B'$ ensuring that the two maps $A \otimes 2 \rightarrow B'$ agree.

In fact we define both of these exactly the same as in lemma 5.16. Namely, we define the map $B' \otimes 2 \rightarrow B'$ to be given by $[\delta_0, \delta_1]$ via the isomorphism $B' \otimes 2 \cong B' + B'$, and we define the map $A \otimes \mathbb{I} \rightarrow B'$ to be $r' \circ \pi_0$.

We define the lower map $B' \otimes \mathbb{I} \rightarrow A$ to be $h \circ f \otimes \mathbb{I}$. We then take h' to be the diagonal filler given by the cofibration structure on $r' \hat{\otimes} [\delta_0, \delta_1]$, as below.

$$\begin{array}{ccc}
A \otimes \mathbb{I} + B' \otimes 2 & \xrightarrow{\quad} & B' \\
r' \hat{\otimes} [\delta_0, \delta_1] \downarrow & \nearrow h' & \downarrow f \\
B' \otimes \mathbb{I} & \xrightarrow{f \otimes \mathbb{I}} B \otimes \mathbb{I} \xrightarrow{h} & A
\end{array}$$

Exactly the same as for lemma 5.16, the upper triangle ensures that h' is a homotopy from $r' \circ i'$ to $1_{B'}$, and that $h' \circ r' \otimes \mathbb{I} = r' \circ \pi_0$. Hence, this does give a strong deformation retract.

The lower triangle tells us that the square in (13) commutes.

Functoriality is again the same as for lemma 5.16. \square

Remark 6.3. We can in fact recover lemma 5.16 as a special case of lemma 6.2, by taking $A = B$ and r to be 1_A , which trivially has the structure of a strong deformation retract.

We can now give an alternative proof of lemma 5.17. We split it into two steps. First show that the reflexivity map $X \rightarrow P_Y(X)$ is a strong deformation retract using lemma 6.1, and then lift this structure to the map $X \rightarrow \text{Id}_Y(X)$ using lemma 6.2.

This version of the proof makes essential use of the connections on \mathbb{I} , but now applies to any map f , without needing any fibration structure on f .

6.2 Avoiding a Transfinite Construction

Note that the definition of the cofibrantly generated awfs (C, F^t) according to Garner's small object argument [8] involves a transfinite construction. The

second observation, by Coquand, was that in fact a transfinite construction is not necessary, and one can give a much simpler definition that suffices for constructing identity types.

The key point is that there is a much simpler awfs (C_1, F_1^t) such that (C, F^t) is algebraically free on the underlying lawfs of (C_1, F_1^t) . One way of understanding (C_1, F_1^t) is as an internal version of step-one of Garner's small object argument. See [21, Section 7.5.2] for more precise explanation of this. As observed by Gambino and Sattler it can also be viewed as a partial map classifier. See [7, Remark 9.5] for that description. See also the description by Van den Berg and Frumin in [23]. In any case this construction gives an lawfs, and as Bourke and Garner show in [5], extending an lawfs to an awfs corresponds precisely to giving a natural way of composing C_1 -coalgebras⁴. So, the reason that we can do this is that in CCHM cubical sets the generating cofibrations can be composed (see [21, Section 7.5.2] for more detail). As noticed by Sattler (and as explained in [7, Remark 9.5]), although (C, F^t) is algebraically free on the underlying lawfs of (C_1, F_1^t) , these are definitely different awfs's.

We will now show that in general in this situation we can instead use (C_1, F_1^t) to construct the identity types.

For convenience, we will continue to assume that we are given a pre-ams $(C^t, F) \rightarrow (C, F^t)$. However, we observe that now neither awfs is being used to construct objects, but only to define the categories of (trivial) cofibrations and fibrations. It is possible to use this idea to rephrase the results to work without those awfs's at all. Indeed the proofs in [6], [14] and [23] do not use any transfinite construction.

Lemma 6.4. *Suppose that (C_1, F_1^t) is an lawfs and $\zeta: (C_1, F_1^t) \rightarrow (C, F^t)$ is a morphism of lawfs's witnessing that (C, F^t) is algebraically free on (C_1, F_1^t) . Suppose further that $\mu: (F_1^t)^2 \rightarrow F^t$ is a natural transformation making F_1^t into a monad. Then for each f we can assign C_1f the structure of a cofibration and $F_1^t f$ the structure of a trivial fibration.*

Moreover, this assignment is functorial in f .

Proof. First note that the morphism ζ gives us a canonical map from C_1 -coalgebras to C -coalgebras commuting with the forgetful functors.

We apply this functor to the canonical C_1 -coalgebra structure on C_1f given by comultiplication to give it the structure of a C -coalgebra.

Similarly, ζ gives a morphism from F^t -algebras to F_1^t -algebras commuting with the forgetful functor. By the definition of algebraic freeness this functor is an isomorphism, and so its inverse is a functor from F_1^t -algebras (in the pointed endofunctor sense) to F^t -algebras (in the monad sense).

Next, note that we can assign $F_1^t f$ the structure of an F_1 -algebra using the multiplication μ . We then forget that the algebra respects the multiplication to get an algebra over F_1^t as a pointed endofunctor, and then apply the functor above to give it the structure of an F^t -algebra. \square

Theorem 6.5. *Suppose that $\xi: (C^t, F) \rightarrow (C, F^t)$ is a pre-ams satisfying the conditions in section 5, that (C, F^t) is algebraically free on an lawfs (C_1, F_1^t) and that we are given a multiplication map μ making F_1^t a monad.*

⁴Here we mean coalgebras over the comonad C_1 .

For each $f: X \rightarrow Y$, define $\text{Id}_Y(X)$ to be given by the (C_1, F_1^t) factorisation of the map $X \rightarrow P_Y(X)$ as in the diagram below.

$$\begin{array}{ccc}
 & \text{Id}_Y(X) & \\
 C_1 r \nearrow & & \searrow F_1^t r \\
 X & \xrightarrow{r} & P_Y(X)
 \end{array}$$

Then $\text{Id}_Y(X)$ can be equipped with structure of functorial choice of very good path objects.

Proof. We apply lemma 6.4 to give $C_1 r$ the structure of a C -coalgebra and $F_1^t r$ the structure of a F^t -algebra. We then continue with exactly the same proof as in section 5.5 \square

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