SCHUR ALGEBRAS AND QUANTUM SYMMETRIC PAIRS WITH UNEQUAL PARAMETERS

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Abstract. We study the (quantum) Schur algebras of type B/C corresponding to the Hecke algebras with unequal parameters. We prove that the Schur algebras afford a stabilization construction in the sense of Beilinson-Lusztig-MacPherson that constructs a multiparameter upgrade of the quantum symmetric pair coideal subalgebras of type AIII/AIV with no black nodes. We further obtain the canonical basis of the Schur/coideal subalgebras, at the specialization associated to any weight function. These bases are the counterparts of Lusztig's bar-invariant basis for Hecke algebras with unequal parameters. In the appendix we provide an algebraic version of a type D Beilinson-Lusztig-MacPherson construction which is first introduced by Fan-Li from a geometric viewpoint.

Contents

1. Introduction

1.1. **Background.** The quantum groups introduced by Drinfeld and Jimbo have played a central role in representation theory and many other branches of mathematics. Equally important are Lusztig's modified (or idempotented) quantum groups (cf. [Lu93]) that admit the canonical bases, which are analogs of the Kazhdan-Lusztig bases for the Hecke algebras. In [BLM90], a geometric construction of the modified quantum group $\dot{\mathbf{U}}(\mathfrak{gl}_n)$ is given by Beilinson-Lusztig-MacPherson. Their construction is now referred as the BLM or stabilization construction after a stabilization property of the family of the (quantum) Schur algebras of type A. In this paper, by a (equal-parameter)[1] *stabilization construction of type* X we mean a construction of an algebra $\dot{\mathbf{K}}_n^X$ over $\mathbb{Z}[v, v^{-1}]$ such that

(1) There is a family of quantum Schur algebras $\mathbf{S}_{n,d}^X$, which are the centralizing algebras to the action of the Hecke algebra $\mathbf{H}_d^X$ of type X_d, for all n, d;
(2) The family $\{\mathbf{S}_{n,d}^X \mid d \in \mathbb{N}\}$ admits a stabilization property, namely, the algebra $\dot{\mathbf{K}}_n^X = \underset{\infty \leftarrow d}{\mathrm{Stab}}\, \mathbf{S}_{n,d}^X$ is well-defined. As a consequence, there is a basis of $\dot{\mathbf{K}}_n^X$ that is compatible with the Kazhdan-Lusztig bases for $\mathbf{H}_d^X$, and the canonical bases of $\mathbf{S}_{n,d}^X$ for all d.

The stabilization constructions have been developed for classical type and for certain affine type (see Table 1 for the references) – there are geometric approaches using partial flags and counting over finite fields developed; while there also are algebraic approaches in the framework of the Hecke algebras using combinatorics on Coxeter groups.

[1] Our goal

Table 1. Known BLM/stabilization constructions

type	finite A	finite B/C	finite D	affine A	affine C
geometric	[BLM90]	[BKLW18]	[FL15]	[Lu99]	[FL^{3}Wa]
algebraic	[DDPW08]	?	?	[DF15]	[FL^{3}Wb]

We remark that the algebraic approach for finite type B/C is more or less a special case for affine type C; while the algebraic approach for type D will be given in the appendix of this present paper.

The stabilization construction in general produces not the Drinfeld-Jimbo's quantum groups but Letzter-Kolb's quantum symmetric pairs (cf. [Le02, Ko14]). For example, the stabilization constructions of type A and B/C lead to the quantum symmetric pairs of type AIII/IV with no black nodes.

1.2. **A new direction.** A recent work by Bao-Wang-Watanabe brings to the author's attention that a multiparameter Schur duality (cf. [BWW18]) plays a governing role among the Schur dualities of classical type. They also introduce a multiparameter upgrade of quantum symmetric pairs of type AIII/AIV with no black nodes.

While it is unclear how to proceed a geometric approach with unequal parameters since dimension counting does not make sense in an obvious way, an algebraic/combinatorial approach seems viable. The goal of this article is to provide a stabilization construction with respect to the Schur duality with unequal parameters in *loc. cit.* We show that the multiparameter stabilization algebras constructed are the coideal subalgebras appearing in the quantum symmetric pairs of type AIII/AIV with no black nodes. As an application, we construct, for the first time, the canonical bases for the type B/C Schur algebras with unequal parameters associated to any weight function, using Lusztig's bar-invariant basis [Lu03] with unequal parameters.

The following diagram explains briefly the connection between the stabilization construction of type B/C for equal and unequal parameters (here $\mathbf{c} = \gcd(\mathbf{L}(s_0), \mathbf{L}(s_1))$, and there are two distinct cases where $\bullet$ can be replaced by $\imath$ or $\jmath$):

Table 2. Relation between Schur duality of type B/C at various specializations

$$\begin{array}{ll}
\mathbb{S}^{\bullet}_{n,d} \curvearrowright \mathbb{V}^{\otimes d} \curvearrowleft \mathbb{H}_d & \text{over } \mathbb{Z}[u^{\pm 1}, v^{\pm 1}] \\
\Downarrow \text{ specialization at } u = \mathbf{v}^{\mathbf{L}(s_0)}, v = \mathbf{v}^{\mathbf{L}(s_1)} & \\
\mathbb{S}^{\bullet,\mathbf{L}}_{n,d} \curvearrowright \mathbb{V}^{\otimes d}_{\mathbf{L}} \curvearrowleft \mathbb{H}^{\mathbf{L}}_d & \text{over } \mathbb{Z}[\mathbf{v}^{\pm \mathbf{c}}] \\
\Downarrow \text{ specialization at } u = v = \mathbf{v} \text{ (i.e., } \mathbf{L} = \ell) & \\
\mathbf{S}^{\bullet}_{n,d} \curvearrowright \mathbf{V}^{\otimes d} \curvearrowleft \mathbf{H}_d & \text{over } \mathbb{Z}[\mathbf{v}^{\pm 1}]
\end{array}$$

At the specialization $u = 1$, the Hecke algebra contains the type D Hecke algebra over $\mathbb{Z}(v^{\pm 1})$ as a proper subalgebra. Hence the multiparameter Schur duality yields a weak Schur duality of type D which is used in [Bao17] to formulate the Kazhdan-Lusztig theory for classical and super type D. The very duality also appears in [ES18] as a piece of a larger skew Howe duality of the quantum symmetric pair coideal subalgebra with itself.

1.3. **Unequal parameters.** While the organization of this paper follows closely to the (equal-parameter) affine type C construction [FL^{3}Wb], the technical lemmas therein do not generalize naively. Below we mention some notable difficulties working with unequal parameters.

The first difficulty comes to dealing with the combinatorics of (type B/C) quantum numbers with two parameters. The key observation here is that the (equal-parameter) quantum numbers/factorials used in the BLM-type constructions arise from the (equal-parameter) Poincare polynomials corresponding to the Weyl groups. Hence, we compute the multiparameter upgrade for the type B/C Poincare polynomials (cf. Lemma 2.3.1), and then extract from it a type B/C quantum factorial (2.3.3) with two parameters.

The second difficulty arises in constructing a standard basis of $\mathbb{S}^{\jmath}_{n,d}$. For the equal-parameter case such a basis element $[A]$ is obtained by multiplying a $\mathbf{v}$-power to the evident basis e_A; while for unequal parameters, it is not obvious how to define a multiplier $u^{\bullet}v^{\bullet}$ that specializes to the original $\mathbf{v}$-power. We solve this problem by reducing it to getting an explicit formula (cf. Lemma 4.1.2) for the leading coefficient under the bar map. For the equal-parameter case the formula is obtained using certain identities on the dual Kazhdan-Lusztig basis due to Curtis. However, there are no multiparameter Kazhdan-Lusztig basis known to us (yet). Hence, we take a detour via Lusztig's bar-invariant basis c_w with unequal parameters and have successfully define a standard basis that affords the entire stabilization process.

Finally, we remark that there is an unexpected behavior for our multiparameter monomial bases – the basis elements are not bar-invariant, unlike the (equal-parameter) monomial basis elements. As a result, we can only show the existence of canonical bases for Schur algebras at certain specialization (see Section 4.4).

1.4. **Organization and main results.** Throughout the article the algebras are over the ground ring

$$\mathbb{A} = \mathbb{Z}[u^{\pm 1}, v^{\pm 1}]$$

(u, v are independent indeterminants) and its specializations.

We first start with the case $\bullet = \jmath$. In Section 2 we recall combinatorial properties of Weyl groups of type B/C in terms of permutation matrices. We characterize a matrix set $\Xi_{n,d}$ (see (2.2.2)) associated to certain double coset representatives. We also introduce the multiparameter quantum numbers of type B/C corresponding to the Poincare polynomials. In Section 3 we introduce the Schur algebra $\mathbb{S}^{\jmath}_{n,d}$ (see (3.1.6)) with an evident basis $\{e_A \mid A \in \Xi_{n,d}\}$. In Section 4 we introduce a standard basis $\{[A] \mid A \in \Xi_{n,d}\}$ (see (4.2.3)), and we show that, using Lusztig's basis c_w for the Hecke algebras with unequal parameters, it satisfies a unitriangular condition under the bar involution. The first main result is the following multiparameter upgrade of the multiplication formulas in [BKLW18]:

Theorem A (Theorem 4.2.3)**.** *Let $A, B \in \Xi_{n,d}$ and $B - b(E_{h,h+1} + E_{-h,-h-1})$ is diagonal. Let $\gamma^C_{B,A} \in \mathbb{A}$ be such that $[B][A] = \sum_C \gamma^C_{B,A}[C] \in \mathbb{S}^{\jmath}_{n,d}$. The explicit formula and the vanishing criterion for $\gamma^C_{B,A}$ are computed.*

The multiplication formula plays an essential step towards constructing a monomial basis in the sense that a stabilization property (4.3.3) holds.

Theorem B (Proposition 4.3.1, Theorem 4.4.1)**.** *There exists a monomial basis $\{m_A\}$ for the Schur algebra $\mathbb{S}^{\jmath}_{n,d}$ over $\mathbb{A}$. Consequently, at a specialization associated to a weight function $\mathbf{L}$, there exists a canonical basis $\{\{A\}^{\mathbf{L}}\}$ for $\mathbb{S}^{\jmath,\mathbf{L}}_{n,d}$.*

In Section 5 we show that the stabilization procedure along the line of Beilinson-Lusztig-MacPherson applies to the family of Schur algebras $\{\mathbb{S}^{\jmath}_{n,d} \mid d \geqslant 1\}$ with a fixed n, which leads to the construction of stabilization algebra $\dot{\mathbb{K}}^{\jmath}_n$ (cf. Corollary 5.1.3) together with its canonical basis.

Theorem C (Theorem 5.2.2)**.** *There exists a monomial basis $\{m_A\}$ for the stabilization algebra $\dot{\mathbb{K}}^{\jmath}_n$. As a corollary, there exists a canonical basis $\{\{A\}^{\mathbf{L}}\}$ for $\dot{\mathbb{K}}^{\jmath}_n$ at a specialization associated to a weight function $\mathbf{L}$.*

Section 6 is dedicated to the counterparts of Theorems B and C for the case $\bullet = \imath$ (see Theorems 6.2.2 and 6.3.8). In Section 7 we show that the stabilization algebras coincide with the $\mathfrak{gl}$-variants $\mathbb{U}^{\jmath}, \mathbb{U}^{\imath}$ of the multiparameter quantum symmetric pair coideal subalgebras studied by Bao-Wang-Watanabe in [BWW18] (referred as $\mathbf{U}^{\jmath}, \mathbf{U}^{\imath}$ therein). The argument is made bypassing the idempotented (or modified) quantum algebras.

Theorem D (Theorems 7.2.1 and 7.3.2)**.** *There are algebra isomorphisms $\dot{\mathbb{K}}^{\jmath}_n \simeq \dot{\mathbb{U}}^{\jmath}, \dot{\mathbb{K}}^{\imath}_n \simeq \dot{\mathbb{U}}^{\imath}$.*

In the appendix we provide an algebraic version of a type D Beilinson-Lusztig-MacPherson construction which is first introduced by Fan-Li from a geometric viewpoint.

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2. Combinatorics on Weyl groups

2.1. **Weyl groups as permutation groups.** Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Fix $N, n, D, d \in \mathbb{N}$ such that

$$N = 2n + 1, D = 2d + 1. \tag{2.1.1}$$

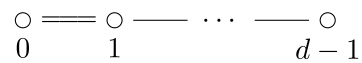
Let $\mathrm{Perm}(X)$ be the group of permutations on a set X. Let (W, S) be the Coxeter system of type B/C by

$$W = \{g \in \mathrm{Perm}([-d, d]) \mid g(-i) = -g(i)\}, \quad S = \{s_0, \ldots, s_{d-1}\}, \tag{2.1.2}$$

where

$$s_0 = (-1, 1), \quad s_i = (i, i+1)(-i, -i-1) \quad (1 \leqslant i < d). \tag{2.1.3}$$

In particular, $g(0) = 0$ for any $g \in W$. The corresponding Coxeter diagram is as below:



Since that any $g \in W$ is uniquely determined by $(g(1), \ldots, g(d))$, we use the two-line/one-line notations (referred as the window notation in [BB05])

$$g \equiv \left| \begin{matrix} 1, & \ldots & , d \\ g(1), & \ldots & , g(d) \end{matrix} \right|_{\mathfrak{c}} \equiv |g(1), \cdots, g(d)|_{\mathfrak{c}}. \tag{2.1.4}$$

Let $\ell : W \to \mathbb{N}$ be the length function on W. We introduce a truncated length function $\ell_{\mathfrak{c}} : W \to \mathbb{N}$ such that $\ell_{\mathfrak{c}}(g)$ equals to the total number of s_0's in a reduced expression of g. The function $\ell_{\mathfrak{c}}$ is well-defined since it is the weight function (cf. [Lu03]) determined by $\ell_{\mathfrak{c}}(s_0) = 1, \ell_{\mathfrak{c}}(s_i) = 0$ for $i \geqslant 1$. We set $\ell_{\mathfrak{a}} = \ell - \ell_{\mathfrak{c}}$.

Lemma 2.1.1. *For $g \in W$, we have*

$$\ell_{\mathfrak{c}}(g) = \frac{1}{2}\sharp \left\{ (i,j) \in [1,d] \times \{0\} \Big|_{\substack{i<j \\ g(i)>g(j)}} \text{ or } {}_{\substack{i>j \\ g(i)<g(j)}} \right\}, \tag{2.1.5}$$

$$\ell_{\mathfrak{a}}(g) = \frac{1}{2}\sharp \left\{ (i,j) \in [1,d] \times ([-d,d] - \{0\}) \Big|_{\substack{i<j \\ g(i)>g(j)}} \text{ or } {}_{\substack{i>j \\ g(i)<g(j)}} \right\}. \tag{2.1.6}$$

$$\ell(g) = \frac{1}{2}\sharp \left\{ (i,j) \in [1,d] \times [-d,d] \Big|_{\substack{i<j \\ g(i)>g(j)}} \text{ or } {}_{\substack{i>j \\ g(i)<g(j)}} \right\}. \tag{2.1.7}$$

Proof. It follows by an easy induction that $\ell_{\mathfrak{c}}(g) = \sharp\{i \in [1,d] \mid g(i) < 0\}$, which yields to (2.1.5) by a direct calculation. The formula (2.1.7) for $\ell(g)$ is equivalent to the formula [BB05, (8.2)]. Then there comes the formula (2.1.6) by $\ell_{\mathfrak{a}}(g) = \ell(g) - \ell_{\mathfrak{c}}(g)$. □

Remark 2.1.2. The expressions in Lemma 2.1.1 are not the most straight-forward. There are simpler ones, for example, $\ell_{\mathfrak{a}} = \mathrm{inv} + \mathrm{neg}$ and $\ell_{\mathfrak{c}} = \mathrm{neg}$ following the convention in [BB05]. We will see in Lemma 2.2.2 the advantage of choosing such symmetrized expressions. See also [FL^{3}Wb, Appendix A] for similar symmetrized length formulas for finite and affine classical types.

Denote the set of weak compositions of d of $n+1$ parts by

$$\Lambda_{n,d} = \{\lambda = (\lambda_n, \ldots, \lambda_1, 2\lambda_0 + 1, \lambda_1, \ldots, \lambda_n) \in \mathbb{N}^{2n+1} \mid \textstyle\sum_{i=0}^n \lambda_i = d\}. \tag{2.1.8}$$

For any $\lambda \in \Lambda_{n,d}$ and integer $i \in [-n, n]$, we define integer intervals R_i^λ by

$$R_i^\lambda = \begin{cases} [\lambda_0 + \sum\limits_{1 \leqslant j < i} \lambda_j + 1, \lambda_0 + \sum\limits_{1 \leqslant j \leqslant i} \lambda_j] & \text{if } 0 < i \leqslant n; \\ [-\lambda_0, \lambda_0] & \text{if } i = 0; \\ -R_{-i}^\lambda & \text{if } -n \leqslant i < 0. \end{cases} \tag{2.1.9}$$

For any subset $X \subset [-d,d]$, let $\mathrm{Stab}(X)$ be the stabilizer of X in W. A parabolic subgroup of W must be of the form

$$W_\lambda = \bigcap_{i=0}^{n} \mathrm{Stab}(R_i^\lambda), \quad \text{for some } \lambda \in \Lambda_{n,d}. \tag{2.1.10}$$

Precisely, W_λ is the parabolic subgroup of W generated by $S - \{s_{\lambda_0}, s_{\lambda_0+\lambda_1}, \dots, s_{d-\lambda_n}\}$. Denote the set of shortest right coset representatives for $W_\lambda \backslash W$ by

$$\mathscr{D}_\lambda = \{w \in W \mid \ell(wg) = \ell(w) + \ell(g) \text{ for all } w \in W_\lambda\} \tag{2.1.11}$$

$$= \{w \in W \mid w^{-1} \text{ is order-preserving on all } R_i^\lambda\}. \tag{2.1.12}$$

Denote the set of minimal length double coset representatives for $W_\lambda \backslash W / W_\mu$ by

$$\mathscr{D}_{\lambda\mu} = \mathscr{D}_\lambda \cap \mathscr{D}_\mu^{-1}. \tag{2.1.13}$$

In the following we collect some standard results for Coxeter groups from [DDPW08, Proposition 4.16, Lemma 4.17 and Theorem 4.18].

Lemma 2.1.3. *Let $\lambda, \mu \in \Lambda_{n,d}$ and $g \in \mathscr{D}_{\lambda\mu}$.*

(a) *There exists $\delta \in \Lambda_{n',d}$ for some n' such that $W_\delta = g^{-1} W_\lambda g \cap W_\mu$.*
(b) *The map $W_\lambda \times (\mathscr{D}_\delta \cap W_\mu) \to W_\lambda g W_\mu$ sending (x,y) to xgy is a bijection; moreover, we have $\ell(xgy) = \ell(x) + \ell(g) + \ell(y)$.*
(c) *The map $W_\delta \times (\mathscr{D}_\delta \cap W_\mu) \to W_\mu$ sending (x,y) to xy is a bijection; moreover, we have $\ell(x)+\ell(y) = \ell(xy)$.*

An essential step in deriving the multiplication formula is to understand the set $\mathscr{D}_\delta \cap W_\mu$, which we will see in Section 3.2.

2.2. **Set-valued matrices.** Let

$$\Theta_{N,D} := \left\{ (a_{ij})_{-n \leqslant i,j \leqslant n} \in \mathrm{Mat}_{N\times N}(\mathbb{N}) \;\middle|\; \sum_{ij} a_{ij} = D \right\}, \quad \Theta_N = \bigcup_{D \in 2\mathbb{N}+1} \Theta_{N,D}. \tag{2.2.1}$$

Note that the columns/rows of such a matrix are indexed by $[-n,n]$ instead of $[1,N]$. Let

$$\Xi_{n,d} := \left\{ (a_{ij}) \in \Theta_{N,D} \;\middle|\; \begin{array}{l} a_{00} \in 2\mathbb{Z}+1, \\ a_{ij} = a_{-i,-j} \quad \text{for all} \quad i,j \end{array} \right\}, \quad \Xi_n = \bigcup_{d \in \mathbb{N}} \Xi_{n,d}. \tag{2.2.2}$$

For $A = (a_{ij}) \in \Xi_{n,d}$ we define a matrix $A^{\mathcal{P}} = (A^{\mathcal{P}}_{ij})$ to be the unique set-valued matrix satisfying:

(P0) The sets $(A^{\mathcal{P}}_{ij})_{ij}$ partition $[-d,d]$;
(P1) $|A^{\mathcal{P}}_{ij}| = a_{ij}$ for all i,j;
(P2) Every element in $A^{\mathcal{P}}_{ij}$ is smaller than any element in $A^{\mathcal{P}}_{xy}$ if $(i,j) < (x,y)$ in the lexicographical order (i.e., $(i,j) < (x,y)$ if and only if $i < x$ or $(i = x, j < y)$).

In words, the set-valued matrix $A^{\mathcal{P}}$ is obtained by filling integers from $-d$ to d into the entries $A^{\mathcal{P}}_{ij}$ row-by-row, top-to-bottom. For $T \in \Theta_N$, we define its row sum vector $\mathrm{row}(T) = (\mathrm{row}(T)_k)_{k=-n}^n$ and column sum vector $\mathrm{col}(T) = (\mathrm{col}(T)_k)_{k=-n}^n$ by

$$\mathrm{row}(T)_k = \sum_{-n \leqslant j \leqslant n} t_{kj} \quad \text{and} \quad \mathrm{col}(T)_k = \sum_{-n \leqslant i \leqslant n} t_{ik}. \tag{2.2.3}$$

Lemma 2.2.1. *The following map is bijective:*

$$\kappa : \bigsqcup_{\lambda,\mu \in \Lambda_{n,d}} \{\lambda\} \times \mathscr{D}_{\lambda\mu} \times \{\mu\} \to \Xi_{n,d}, \quad \kappa(\lambda, g, \mu) = (|R_i^\lambda \cap g R_j^\mu|)_{ij}. \tag{2.2.4}$$

Moreover, the inverse is given by $\kappa^{-1}(A) = (\mathrm{row}(A), g_A, \mathrm{col}(A))$, where g_A is the permutation sending k to the k-th number in the column-reading of $A^{\mathcal{P}}$ (see Example 2.2.3 below).

Proof. The surjectivity follows from $\kappa(\mathrm{row}(A), g_A, \mathrm{col}(A)) = A$ ($\forall A \in \Xi_{n,d}$) by a direct calculation.

For injectivity, we assume $\kappa(\lambda, g, \mu) = A = \kappa(\lambda', g', \mu')$. Then $\lambda = \lambda' = \mathrm{row}(A)$ and $\mu = \mu' = \mathrm{col}(A)$ and hence $g, g' \in \mathscr{D}_{\lambda\mu}$. It follows from $|R_i^\lambda \cap gR_j^\mu| = |R_i^\lambda \cap g'R_j^\mu|$ ($\forall i, j \in [-n, n]$) that $g = w_{(\lambda)} g' w_{(\mu)}$ for some $w_{(\lambda)} \in W_\lambda, w_{(\mu)} \in W_\mu$. Therefore $g = g'$ since they are both minimal double coset representatives in $W_\lambda \backslash W / W_\mu$. □

Thanks to Lemma 2.2.1, we define length functions $\ell, \ell_{\mathfrak{c}}, \ell_{\mathfrak{a}}$ on $\Xi_{n,d}$ by

$$\ell(A) = \ell(g), \quad \ell_{\mathfrak{c}}(A) = \ell_{\mathfrak{c}}(g), \quad \ell_{\mathfrak{a}}(A) = \ell_{\mathfrak{a}}(g) \quad (\text{for } A = \kappa(\lambda, g, \mu)). \tag{2.2.5}$$

We define index subsets of type A/C by the following:

$$I_{\mathfrak{a}} = (\{0\} \times [1, n]) \sqcup ([1, n] \times [-n, n]), \quad I_{\mathfrak{c}} = I_{\mathfrak{a}} \sqcup \{(0, 0)\}. \tag{2.2.6}$$

For $(i, j) \in I_{\mathfrak{c}}$, we set

$$a_{ij}^\natural = \begin{cases} \frac{1}{2}(a_{ij} - 1) & \text{if } (i, j) = (0, 0); \\ a_{ij} & \text{otherwise.} \end{cases} \tag{2.2.7}$$

There is an alternative length formula in terms of products of matrix entries as below.

Lemma 2.2.2. *Recall $a_{ij}^\natural$ from (2.2.7). The (truncated) length functions of A are given by*

$$\ell(A) = \frac{1}{2}\Big(\sum_{(i,j)\in I_{\mathfrak{c}}} \Big(\sum_{\substack{x<i \\ y>j}} + \sum_{\substack{x>i \\ y<j}} \Big) a_{ij}^\natural a_{xy} \Big), \quad \ell_{\mathfrak{c}}(A) = \frac{1}{2}\Big(\sum_{\substack{0<x \\ 0>y}} + \sum_{\substack{0>x \\ 0<y}} \Big) a_{xy}, \tag{2.2.8}$$

$$\ell_{\mathfrak{a}}(A) = \frac{1}{2}\Big(\sum_{(i,j)\in I_{\mathfrak{c}}} \Big(\sum_{\substack{x<i \\ y>j}} + \sum_{\substack{x>i \\ y<j}} \Big) a_{ij}^{\natural\natural} a_{xy} \Big), \tag{2.2.9}$$

where $a_{00}^{\natural\natural} = a_{00}^\natural - 1 = \frac{1}{2}(a_{00} - 3)$ and $a_{ij}^{\natural\natural} = a_{ij}$ if $(i, j) \in I_{\mathfrak{a}}$.

Proof. These three formulas are paraphrases of those in Lemma 2.1.1. □

Let $A = \kappa(\lambda, g, \mu) \in \Xi_{n,d}$. We define a signed weak composition as below:

$$\delta(A) = (a_{nn}, \ldots, \ldots, \ldots, a_{00}^\natural, a_{10}, \ldots, a_{n0}, a_{-n,1}, a_{-n+1,1}, \ldots, a_{n1}, \ldots, \ldots, a_{-n,n}, a_{-n+1,n}, \ldots, a_{nn}). \tag{2.2.10}$$

A direct computation shows that $\delta(A)$ is indeed a weak composition δ in Lemma 2.1.3(a).

Example 2.2.3. Let $A = \left[\begin{smallmatrix} 1 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 3 & 1 \end{smallmatrix}\right]$. We have

$$\mathrm{row}(A) = (5, 3, 5), \quad \mathrm{col}(A) = (3, 7, 3), \quad A^{\mathcal{P}} = \begin{bmatrix} \{-6\} & \{-5,-4,-3\} & \{-2\} \\ \{-1\} & \{0\} & \{1\} \\ \{2\} & \{3,4,5\} & \{6\} \end{bmatrix}.$$

Column-reading of $A^{\mathcal{P}}$ gives us a sequence $-6, -1, 2, -5, -4, -3, 0, 3, 4, 5, -2, 1, 6$, and hence g_A is the permutation

$$g_A = |3, 4, 5, -2, 1, 6|_{\mathfrak{c}} = s_1 s_0 s_2 s_1 s_3 s_2 s_4 s_3.$$

Indeed, we have

$$\begin{aligned} \ell(A) &= \frac{1}{2}\Big(a_{00}^\natural(1+1) + a_{01}(0+4) + a_{1,-1}(6+0) + a_{10}(2+0) + a_{11}(0+0) \Big) \\ &= \frac{1}{2}(0 + 4 + 6 + 6 + 0) = 8, \\ \ell_{\mathfrak{c}}(A) &= \frac{1}{2}(a_{1,-1} + a_{-1,1}) = 1, \\ \ell_{\mathfrak{a}}(A) &= \frac{1}{2}\Big(a_{00}^{\natural\natural}(2) + a_{01}(4) + a_{1,-1}(6) + a_{10}(2) + a_{11}(0) \Big) = 7. \end{aligned}$$

Furthermore, $\delta(A) = (1, 1, 1, 3, 0, 3, 1, 1, 1)$.

2.3. **Quantum combinatorics.** We denote the quantum v-number by

$$[a] = \frac{v^{2a}-1}{v^2-1} \quad (a \in \mathbb{Z}). \tag{2.3.1}$$

We denote the type-A quantum v-factorials by, for $t \in \mathbb{N}$, $A = (a_{ij}) \in \Theta_N$,

$$[t]! = \prod_{k=1}^{t}[k], \quad [A]! = \prod_{-n\leqslant i,j\leqslant n}[a_{ij}]!. \tag{2.3.2}$$

The type-B/C analogues are defined by, for $t \in \mathbb{N}$, $A = (a_{ij}), B = (b_{ij}) \in \Xi_n$,

$$[2t]_{\mathfrak{c}} = [t](u^2v^{2(t-1)}+1), \quad [t]_{\mathfrak{c}}^! = \prod_{k=1}^{t}[2k]_{\mathfrak{c}}, \quad [A]_{\mathfrak{c}}^! = [a_{00}^{\natural}]_{\mathfrak{c}}^! \prod_{(i,j)\in I_{\mathfrak{a}}}[a_{ij}]!. \tag{2.3.3}$$

In particular, the specialization of $[2t]_{\mathfrak{c}}$ at $u = v$ is $[t](1 + v^{2t}) = [2t]$. Furthermore, we set, for any $a \in \mathbb{Z}$ and $b \in \mathbb{N}$,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \prod_{i=1}^{b}\frac{v^{2(a-i+1)}-1}{v^{2i}-1}.$$

Lemma 2.3.1. *Let $A = \kappa(\mu, g, \nu)$, and let $\delta = \delta(A)$. Then* $\sum_{w\in W_\delta} u^{2\ell_{\mathfrak{c}}(w)}v^{2\ell_{\mathfrak{a}}(w)} = [A]_{\mathfrak{c}}^!$.

Proof. Let $W_d^{\mathfrak{c}}$ be the Weyl group of type C_d.

Recall δ in (2.2.10). We have $W_\delta \simeq W^{\mathfrak{c}}_{a_{00}^{\natural}} \times \prod_{(i,j)\in I_{\mathfrak{a}}} \mathfrak{S}_{a_{ij}}$. For each $w \in \mathfrak{S}_{a_{ij}}$ we have $\ell_{\mathfrak{c}}(w) = 0, \ell_{\mathfrak{a}}(w) = \ell(w)$, and hence

$$\sum_{w\in\mathfrak{S}_{a_{ij}}} u^{2\ell_{\mathfrak{c}}(w)}v^{2\ell_{\mathfrak{a}}(w)} = \sum_{w\in\mathfrak{S}_{a_{ij}}} v^{2\ell(w)} = [a_{ij}]^!. \tag{2.3.4}$$

Thus

$$\sum_{w\in W_\delta} u^{2\ell_{\mathfrak{c}}(w)}v^{2\ell_{\mathfrak{a}}(w)} = \Big(\sum_{w\in W^{\mathfrak{c}}_{a_{00}^{\natural}}} u^{2\ell_{\mathfrak{c}}(w)}v^{2\ell_{\mathfrak{a}}(w)}\Big)\prod_{(i,j)\in I_{\mathfrak{a}}}[a_{ij}]!.$$

It suffices to show that

$$\sum_{w\in W_d^{\mathfrak{c}}} u^{2\ell_{\mathfrak{c}}(w)}v^{2\ell_{\mathfrak{a}}(w)} = [d]_{\mathfrak{c}}^!. \tag{2.3.5}$$

Let $\lambda = (0, \dots, 0, 1, 2d-1, 1, 0, \dots, 0) \in \Lambda_{n,d}$. We have $W_\lambda \simeq W_{d-1}^{\mathfrak{c}}$, and hence

$$\sum_{w\in W_d^{\mathfrak{c}}} u^{2\ell_{\mathfrak{c}}(w)}v^{2\ell_{\mathfrak{a}}(w)} = \Big(\sum_{w\in W_{d-1}^{\mathfrak{c}}} u^{2\ell_{\mathfrak{c}}(w)}v^{2\ell_{\mathfrak{a}}(w)}\Big)\Big(\sum_{w\in \mathscr{D}_\lambda} u^{2\ell_{\mathfrak{c}}(w)}v^{2\ell_{\mathfrak{a}}(w)}\Big). \tag{2.3.6}$$

By (2.1.11), $g \in \mathscr{D}_\lambda$ if and only if g^{-1} is order-preserving on $[-d + 1, d - 1]$. Hence,

$$\mathscr{D}_\lambda = \left\{ |i_1, \cdots, i_{d-1}, \pm j|_{\mathfrak{c}}^{-1} \;\middle|\; \begin{matrix} [1,d] = \{j\} \sqcup \{i_1, \dots i_{d-1}\}, \\ i_1 < \dots < i_{d-1} \end{matrix} \right\}. \tag{2.3.7}$$

Consequently, we have

$$\sum_{w\in \mathscr{D}_\lambda} u^{2\ell_{\mathfrak{c}}(w)}v^{2\ell_{\mathfrak{a}}(w)} = [d](1 + u^2v^{2(d-1)}) = [2d]_{\mathfrak{c}}. \tag{2.3.8}$$

Therefore, (2.3.5) follows from a downward iteration. The Lemma is proved. □

3. Schur algebras

3.1. **Schur algebras.** The Hecke algebra $\mathbb{H} = \mathbb{H}(W)$ over $\mathbb{A}$ is an algebra with a basis $\{T_g \mid g \in W\}$ satisfying

$$T_w T_{w'} = T_{ww'} \quad \text{if } \ell(ww') = \ell(w) + \ell(w'), \tag{3.1.1}$$

$$(T_{s_0} + 1)(T_{s_0} - u^2) = 0, \tag{3.1.2}$$

$$(T_s + 1)(T_s - v^2) = 0 \quad \text{for } s \in S - \{s_0\}. \tag{3.1.3}$$

For any subset $X \subset W$ and for $\lambda \in \Lambda_{n,d}$ (2.1.8), set

$$T_X = \sum_{w \in X} T_w, \quad T^g_{\lambda\mu} = T_{(W_\lambda)g(W_\mu)}, \quad x_\lambda = T^{\mathbb{1}}_{\lambda\lambda} = T_{W_\lambda}, \tag{3.1.4}$$

where $\mathbb{1}$ is the identity element of W.

Lemma 3.1.1. *If $w \in W_\lambda$, then $T_w x_\lambda = u^{2\ell_{\mathfrak{c}}(w)} v^{2\ell_{\mathfrak{a}}(w)} x_\lambda = x_\lambda T_w$.*

Proof. This reduces to the case when $w = s \in S$. It then follows from the Hecke relation (3.1.1). □

For $\lambda, \mu \in \Lambda_{n,d}$ and $g \in \mathscr{D}_{\lambda\mu}$, we consider a right $\mathbb{H}$-linear map $\phi^g_{\lambda\mu} \in \mathrm{Hom}_{\mathbb{H}}(x_\mu \mathbb{H}, \mathbb{H})$, sending x_μ to $T^g_{\lambda\mu}$. Thanks to Lemma 2.1.3(b), we have $T^g_{\lambda\mu} = x_\lambda T_g T_{\mathscr{D}_\delta \cap W_\mu}$ for some $\delta \in \Lambda_{n',d}$, and hence we have constructed a right $\mathbb{H}$-linear map

$$\phi^g_{\lambda\mu} \in \mathrm{Hom}_{\mathbb{H}}(x_\mu \mathbb{H}, x_\lambda \mathbb{H}), \qquad T^{\mathbb{1}}_{\mu\mu} \mapsto T^g_{\lambda\mu}. \tag{3.1.5}$$

The *Schur algebra* $\mathbb{S}^{\jmath}_{n,d}$ is defined as the following $\mathbb{A}$-algebra

$$\mathbb{S}^{\jmath}_{n,d} = \mathrm{End}_{\mathbb{H}}\Big(\bigoplus_{\lambda \in \Lambda_{n,d}} x_\lambda \mathbb{H} \Big) = \bigoplus_{\lambda,\mu \in \Lambda_{n,d}} \mathrm{Hom}_{\mathbb{H}}(x_\mu \mathbb{H}, x_\lambda \mathbb{H}). \tag{3.1.6}$$

Thanks to Lemma 2.2.1, for $A = \kappa(\lambda, g, \mu)$ we define

$$e_A = \phi^g_{\lambda\mu}. \tag{3.1.7}$$

A formal argument as in [Du92, G97] is applicable to our setting and gives us the following:

Lemma 3.1.2. *The set $\{e_A \mid A \in \Xi_{n,d}\}$ forms an $\mathbb{A}$-basis of $\mathbb{S}^{\jmath}_{n,d}$.*

For $T = (t_{ij}) \in \Theta_N$, let $\mathrm{diag}(T) = (\delta_{ij} t_{ij}) \in \Theta_N$ and denote its centro-symmetrizer by

$$T^\theta = (t^\theta_{ij}), \quad \text{where} \quad t^\theta_{ij} = t_{ij} + t_{-i,-j}. \tag{3.1.8}$$

We remark that $T^\theta \notin \Xi_n$ since t^θ_{00} is even. A matrix $B \in \Xi_{n,d}$ is called a *Chevalley matrix* if

$$B - \mathrm{diag}(B) = b E^\theta_{h,h+1}, \quad (b \in \mathbb{N}, -n \leqslant h < n). \tag{3.1.9}$$

An easy consequence of Lemma 2.2.2 is that $g_B = \mathbb{1}$ if B is Chevalley. We assume from now on that B is a Chevalley matrix, and we fix $B = \kappa(\lambda, \mathbb{1}, \mu)$, $A = \kappa(\mu, g, \nu)$. Recall $[A]^!_{\mathfrak{c}}$ from (2.3.3). We have the following identity.

Lemma 3.1.3. $x_\mu T_g x_\nu = [A]^!_{\mathfrak{c}}\, e_A(x_\nu)$.

Proof. Let $\delta = \delta(A)$. By Lemma 2.1.3(c), we have $x_\nu = x_\delta T_{\mathscr{D}_\delta \cap W_\nu}$, and hence

$$x_\mu T_g x_\nu = x_\mu T_g x_\delta T_{\mathscr{D}_\delta \cap W_\nu} = \sum_{w \in W_\delta} x_\mu T_g T_w T_{\mathscr{D}_\delta \cap W_\nu}. \tag{3.1.10}$$

By Lemma 2.1.3(a), $w \in g^{-1} W_\mu g \cap W_\nu \subset W_\nu$ and hence $T_g T_w = T_{gw}$ since $g \in \mathscr{D}_{\mu\nu} \subset \mathscr{D}_\nu^{-1}$. Moreover, we have $gw = w'g$ for some $w' \in W_\mu$. Since $g \in \mathscr{D}_{\mu\nu} \subset \mathscr{D}_\mu$, we have

$$\ell(g) + \ell(w) = \ell(gw) = \ell(w'g) = \ell(w') + \ell(g) \tag{3.1.11}$$

and therefore $\ell(w') = \ell(w)$. Moreover, note that $\ell_{\mathfrak{c}}$ is a well-defined weight function (cf. [Lu03]) determined by $\ell(s_0) = 1$ and $\ell(s_i) = 0$ $(i \geqslant 1)$. Counting the number of s_0 appeared in a reduced form of

$gw = w'g$, we have $\ell_{\mathfrak{c}}(gw) = \ell_{\mathfrak{c}}(g) + \ell_{\mathfrak{c}}(w)$ and $\ell_{\mathfrak{c}}(w'g) = \ell_{\mathfrak{c}}(w') + \ell_{\mathfrak{c}}(g)$ by (3.1.11). Thus $\ell_{\mathfrak{c}}(w) = \ell_{\mathfrak{c}}(w')$ (and hence $\ell_{\mathfrak{a}}(w) = \ell_{\mathfrak{a}}(w')$). Finally, we have

$$\sum_{w\in W_\delta} x_\mu T_{gw} = \sum_{w\in W_\delta} x_\mu T_{w'}T_g = \sum_{w\in W_\delta} u^{2\ell_{\mathfrak{c}}(w)}v^{2\ell_{\mathfrak{a}}(w)}x_\mu T_g = [A]^!_{\mathfrak{c}} x_\mu T_g, \tag{3.1.12}$$

where the second equality follows from Lemma 3.1.1, while the third equality follows from Lemma 2.3.1. The rest follows by the definition $e_A(x_\nu) = x_\mu T_g T_{\mathscr{D}_\delta\cap W_\nu}$. □

3.2. Multiplication formulas $\mathscr{D}_\delta \cap W_\mu$.

Lemma 3.2.1. *Fix $B = \kappa(\lambda, \mathbb{1}, \mu)$, $A = \kappa(\mu, g, \nu)$ and let $\delta = \delta(B)$. Let y^w be the shortest double coset representative for $W_\lambda wgW_\nu$, and set $A^w = \kappa(\lambda, y^w, \nu)$. Then*

$$e_Be_A = \sum_{w\in\mathscr{D}_\delta\cap W_\mu} \frac{[A^w]^!_{\mathfrak{c}}}{[A]^!_{\mathfrak{c}}}(u^2)^{\ell_{\mathfrak{c}}(w)+\ell_{\mathfrak{c}}(g)-\ell_{\mathfrak{c}}(y^w)}(v^2)^{\ell_{\mathfrak{a}}(w)+\ell_{\mathfrak{a}}(g)-\ell_{\mathfrak{a}}(y^w)}e_{A^w}. \tag{3.2.1}$$

Proof. By Lemma 3.1.3 and (3.1.5) (which implies $e_B(x_\mu) = x_\lambda T_{\mathscr{D}_\delta\cap W_\mu}$) we see that

$$e_Be_A(x_\nu) = e_B\Big(\frac{1}{[A]^!_{\mathfrak{c}}}x_\mu T_g x_\nu\Big) = \frac{1}{[A]^!_{\mathfrak{c}}}e_B(x_\mu)T_g x_\nu = \frac{1}{[A]^!_{\mathfrak{c}}}x_\lambda T_{\mathscr{D}_\delta\cap W_\mu}T_g x_\nu. \tag{3.2.2}$$

Since $g \in \mathscr{D}_{\mu\nu} \subset \mathscr{D}_\mu$, so $T_wT_g = T_{wg}$ for all $w \in \mathscr{D}_\delta \cap W_\mu \subset W_\mu$. For $w \in \mathscr{D}_\delta \cap W_\mu$, there exists $w_\lambda \in W_\lambda, w_\nu \in W_\nu$ such that $wg = w_\lambda y^w w_\nu$. Moreover, we have

$$\ell(wg) = \ell(w) + \ell(g) = \ell(w_\lambda) + \ell(y^w) + \ell(w_\nu). \tag{3.2.3}$$

Thus, we have

$$x_\lambda T_{wg}x_\nu = x_\lambda T_{w_\lambda}T_{y^w}T_{w_\nu}x_\nu = (u^2)^{\ell_{\mathfrak{c}}(w_\lambda)+\ell_{\mathfrak{c}}(w_\nu)}(v^2)^{\ell_{\mathfrak{a}}(w_\lambda)+\ell_{\mathfrak{a}}(w_\nu)}x_\lambda T_{y^w}x_\nu. \tag{3.2.4}$$

Combining the (3.2.2), (3.2.4) and applying Lemma 3.1.3 on $x_\lambda T_{y^w}x_\nu$, we have

$$e_Be_A(x_\nu) = \frac{1}{[A]^!_{\mathfrak{c}}}\sum_{w\in\mathscr{D}_\delta\cap W_\mu} x_\lambda T_{wg}x_\nu = \sum_{w\in\mathscr{D}_\delta\cap W_\mu}\frac{[A^w]^!_{\mathfrak{c}}}{[A]^!_{\mathfrak{c}}}(u^2)^{\ell_{\mathfrak{c}}(wg)-\ell_{\mathfrak{c}}(y^w)}(v^2)^{\ell_{\mathfrak{a}}(wg)-\ell_{\mathfrak{a}}(y^w)}e_{A^w}(x_\nu). \tag{3.2.5}$$

The lemma follows from (3.2.3). □

Proposition 3.2.2. *Suppose that $A, B, C \in \Xi_{n,d}$ and $h \in [1, n]$.*

(1) If $B - bE^\theta_{h,h-1}$ is diagonal, $\mathrm{col}(B) = \mathrm{row}(A)$*, then*

$$e_Be_A = \sum_t v^{2\sum_{k<l}t_la_{h,k}}\prod_{l=-n}^{n}\begin{bmatrix} a_{h,l}+t_l \\ t_l\end{bmatrix}e_{\check{A}_{t,h}}, \tag{3.2.6}$$

where $t = (t_i)_{-n\leqslant i\leqslant n} \in \mathbb{N}^N$ with $\sum_{i=-n}^n t_i = b$ such that $\begin{cases} t_i \leqslant a_{h-1,i} & \text{if } h>1; \\ t_i + t_{-i} \leqslant a_{h-1,i} & \text{if } h=1,\end{cases}$ *and*

$$\check{A}_{t,h} = A + \sum_{l=-n}^n t_lE^\theta_{h,l} - \sum_{l=-n}^n t_lE^\theta_{h-1,l}.$$

(2) Suppose $C - cE^\theta_{h-1,h}$ is diagonal and $\mathrm{col}(C) = \mathrm{row}(A)$*. If $h \neq 1$, then*

$$e_Ce_A = \sum_t v^{2\sum_{k>l}t_la_{h-1,k}}\prod_{l=-n}^{n}\begin{bmatrix} a_{h-1,l}+t_l \\ t_l\end{bmatrix}e_{\hat{A}_{t,h}}, \tag{3.2.7}$$

where $t = (t_i)_{-n\leqslant i\leqslant n} \in \mathbb{N}^N$ with $\sum_{i=-n}^n t_i = c$ such that $t_i \leqslant a_{h,i}$, and

$$\hat{A}_{t,h} = A - \sum_{l=-n}^n t_lE^\theta_{h,l} + \sum_{l=-n}^n t_lE^\theta_{h-1,l}.$$

If $h=1$, then

$$e_C e_A = \sum_t u^{2\sum_{l<0} t_l} v^{2\sum_{k>l} a_{0,k}t_l + 2\sum_{l<k<-l} t_l t_k + \sum_{l<0} t_l(t_l-3)} \frac{[a^\natural_{0,0}+t_0]^!_{\mathfrak{c}}}{[a^\natural_{0,0}]^!_{\mathfrak{c}}[t_0]!} \prod_{l=1}^n \frac{[a_{0,l}+t_l+t_{-l}]!}{[a_{0,l}]![t_l]![t_{-l}]!} e_{\widehat{A}_{t,1}}, \tag{3.2.8}$$

where $t=(t_i)_{-n\leqslant i\leqslant n}\in\mathbb{N}^N$ with $\sum_{i=-n}^n t_i = c$ such that $t_i \leqslant a_{1,i}$.

Proof. For Part (1), we only present the proof for the most complicated case $h=1$. Let $\delta=\delta(B)$ and take any $t=(t_i)_{-n\leqslant i\leqslant n}\in\mathbb{N}^N$ as in the assumptions. Among those $w\in\mathscr{D}_\delta\cap W_\mu$ such that $A^w=\breve{A}_{t,1}$, there is a unique shortest element w_t with

$$\ell(w_t)=\sum_{k>l}(a_{0,k}-t_k)t_l - \sum_{l<k<-l} t_l t_k - \frac{1}{2}\sum_{l<0} t_l(t_l-1). \tag{3.2.9}$$

In particular, we have

$$\ell_{\mathfrak{c}}(w_t)=\sum_{l<0} t_l, \quad \ell_{\mathfrak{a}}(w_t)=\sum_{k>l}(a_{0,k}-t_k)t_l - \sum_{l<k<-l} t_l t_k - \frac{1}{2}\sum_{l<0} t_l(t_l+1). \tag{3.2.10}$$

By a combinatorial argument, we calculate that

$$\sum_{\substack{w\in\mathscr{D}_\delta\cap W_\mu,\\ A^w=\breve{A}_{t,1}}} u^{2\ell_{\mathfrak{c}}(w)} v^{2\ell_{\mathfrak{a}}(w)} = u^{2\ell_{\mathfrak{c}}(w_t)} v^{2\ell_{\mathfrak{a}}(w_t)} \left(\sum_{x+y=t_0} \begin{bmatrix} a^\natural_{00} \\ x \end{bmatrix}\begin{bmatrix} a^\natural_{00}-x \\ y \end{bmatrix} u^{2x}(v^2)^{\frac{x(x-1)}{2}+x(a^\natural_{00}-t_0)}\right) \prod_{l=1}^n \begin{bmatrix} a_{0l} \\ t_l \end{bmatrix}\begin{bmatrix} a_{0l}-t_l \\ t_{-l} \end{bmatrix}.$$

Note that

$$\begin{aligned}
&\sum_{x+y=t_0} \begin{bmatrix} a^\natural_{0,0} \\ x \end{bmatrix}\begin{bmatrix} a^\natural_{0,0}-x \\ y \end{bmatrix} u^{2x}(v^2)^{\frac{x(x-1)}{2}+x(a^\natural_{0,0}-t_0)} \\
&= \begin{bmatrix} a^\natural_{0,0} \\ t_0 \end{bmatrix} \sum_{x=0}^{t_0} \begin{bmatrix} t_0 \\ x \end{bmatrix} v^{x(x-1)}(uv^{a^\natural_{0,0}-t_0})^{2x} \overset{(\diamondsuit)}{=} \begin{bmatrix} a^\natural_{0,0} \\ t_0 \end{bmatrix} \prod_{i=1}^{t_0}(1+v^{2(i-1)}u^2v^{2(a^\natural_{0,0}-t_0)}) = \frac{[a^\natural_{0,0}]^!_{\mathfrak{c}}}{[a^\natural_{0,0}-t_0]^!_{\mathfrak{c}}[t_0]!},
\end{aligned}$$

where $(\diamondsuit)$ is due to the quantum binomial theorem $\sum_{x=0}^m \begin{bmatrix} m \\ x \end{bmatrix} v^{x(x-1)}z^x = \prod_{i=0}^{m-1}(1+v^{2i}z)$. Therefore

$$\sum_{w\in\mathscr{D}_\delta\cap W_\mu, A^w=\breve{A}_{t,1}} u^{2\ell_{\mathfrak{c}}(w)} v^{2\ell_{\mathfrak{a}}(w)} = u^{2\ell_{\mathfrak{c}}(w_t)} v^{2\ell_{\mathfrak{a}}(w_t)} \frac{[a^\natural_{0,0}]^!_{\mathfrak{c}}}{[a^\natural_{0,0}-t_0]^!_{\mathfrak{c}}[t_0]!} \prod_{l=1}^n \begin{bmatrix} a_{0,l} \\ t_l \end{bmatrix}\begin{bmatrix} a_{0,l}-t_l \\ t_{-l} \end{bmatrix}. \tag{3.2.11}$$

Furthermore, it follows from Lemma 2.2.2 that

$$\ell_{\mathfrak{c}}(A)-\ell_{\mathfrak{c}}(\breve{A}_{t,1}) = -\sum_{l<0} t_l \tag{3.2.12}$$

$$\ell_{\mathfrak{a}}(A)-\ell_{\mathfrak{a}}(\breve{A}_{t,1}) = \sum_{k<l} t_l a_{1,k} - \sum_{k>l}(a_{0,k}-t_k)t_l + \sum_{l<k<-l} t_l t_k + \frac{1}{2}\sum_{l<0} t_l(t_l+1). \tag{3.2.13}$$

Part (1) then follows from combining (3.2.1), (3.2.10)–(3.2.13). For Part (2), we only present a proof for the most complicated case that $h=1$. Let $\delta=\delta(C)$ and take any $t=(t_i)_{-n\leqslant i\leqslant n}\in\mathbb{N}^N$ as in the assumptions. Among those $w\in\mathscr{D}_\delta\cap W_\mu$ such that $A^w=\widehat{A}_{t,1}$, there is a shortest element w_t with

$$\ell_{\mathfrak{c}}(w_t)=0 \quad \text{and} \quad \ell_{\mathfrak{a}}(w_t)=\sum_{k<l} t_l(a_{1,k}-t_k). \tag{3.2.14}$$

Direct computation yields to the following identities:

$$\sum_{\substack{w\in\mathscr{D}_\delta\cap W_\mu,\\ A^w=\widehat{A}_{t,1}}} u^{2\ell_{\mathfrak{c}}(w)} v^{2\ell_{\mathfrak{a}}(w)} = u^{2\ell_{\mathfrak{c}}(w_t)} v^{2\ell_{\mathfrak{a}}(w_t)} \prod_{l=-n}^n \begin{bmatrix} a_{1,l} \\ t_l \end{bmatrix} = v^{2\sum_{k<l} t_l(a_{1,k}-t_k)} \prod_{l=-n}^n \begin{bmatrix} a_{1,l} \\ t_l \end{bmatrix}, \tag{3.2.15}$$

$$\ell_{\mathfrak{c}}(A)-\ell_{\mathfrak{c}}(\widehat{A}_{t,1}) = \sum_{l<0} t_l, \tag{3.2.16}$$

$$\ell_{\mathfrak{a}}(A) - \ell_{\mathfrak{a}}(\widehat{A}_{t,1}) = \sum_{k>l} a_{0,k} t_l - \sum_{k<l} t_l(a_{1,k} - t_k) + \sum_{l<k<-l} t_l t_k + \frac{1}{2}\sum_{l<0} t_l(t_l - 3). \tag{3.2.17}$$

Part (2) then follows from combining (3.2.1), (3.2.14)–(3.2.17). □

Remark 3.2.3. These explicit formulas match the ones in [BKLW18] (resp. the unsigned ones in [FL15]) if we specialize $u = v$ (resp. $u = 1$).

4. Canonical bases

4.1. **The bar involution.** There is an $\mathbb{A}$-algebra involution $^- : \mathbb{H} \to \mathbb{H}$, which sends $u \mapsto u^{-1}, v \mapsto v^{-1}, T_w \mapsto T_{w^{-1}}^{-1}$, for all $w \in W$. In particular, we have, for $s \in S - \{s_0\}$,

$$\overline{T_s} = v^{-2} T_s + v^{-2} - 1, \quad \overline{T_{s_0}} = u^{-2} T_{s_0} + u^{-2} - 1. \tag{4.1.1}$$

For $\lambda, \mu \in \Lambda_{n,d}$ (see (2.1.8)), let $g^+_{\lambda\mu}$ be the longest element in the double coset $W_\lambda g W_\mu$ for $g \in \mathscr{D}_{\lambda\mu}$, and let $w_\circ^\mu = \mathbb{1}^+_{\mu\mu}$ be the longest element in the parabolic subgroup $W_\mu = W_\mu \mathbb{1} W_\mu$. The lemma below is standard (cf. [DDPW08, Corollary 4.19]).

Lemma 4.1.1. *Let $A = \kappa(\lambda, g, \mu)$, $\delta = \delta(A)$. Then:*

(a) *$g^+_{\lambda\mu} = w_\circ^\lambda g w_\circ^\delta w_\circ^\mu$, and $\ell(g^+_{\lambda\mu}) = \ell(w_\circ^\lambda) + \ell(g) - \ell(w_\circ^\delta) + \ell(w_\circ^\mu)$.*

(b) *$W_\lambda g W_\mu = \{w \in W \mid g \leqslant w \leqslant g^+_{\lambda\mu}\}$.*

Following [KL79], denote by $\{C'_w\}$ the Kazhdan-Lusztig $\mathbb{Z}[v, v^{-1}]$-basis of the Hecke algebra $\mathbb{H}|_{u=v}$ characterized by Conditions (C1)–(C2) below:

(C1) C'_w is bar-invariant;

(C2) $C'_w = v^{-\ell(w)} \sum_{y \leqslant w} P_{yw}(v) T_y$.

Here $\leqslant$ is the (strong) Bruhat order, and P_{yw} is the Kazhdan-Lusztig polynomial satisfying that $P_{ww} = 1$ and $P_{yw} \in \mathbb{Z}[v^2]$ with $\deg_v P_{yw} \leqslant \ell(w) - \ell(y) - 1$ for $y < w$. Recall $T^g_{\lambda\mu}$ from (3.1.4) and denote

$$C^g_{\lambda\mu} = C'_{g^+_{\lambda\mu}} \quad (g \in \mathscr{D}_{\lambda\mu}, \lambda, \mu \in \Lambda_{n,d}). \tag{4.1.2}$$

Following [Cur85], let $\mathbf{H}_{\lambda\mu}$ be the $\mathbb{Z}[v, v^{-1}]$-submodule of $\mathbb{H}|_{u=v}$ with basis $\{T^g_{\lambda\mu}\}_{g \in \mathscr{D}_{\lambda\mu}}$. It is shown in *loc. cit.* that $\{C^g_{\lambda\mu}\}_{g \in \mathscr{D}_{\lambda\mu}}$ also forms a bar-invariant basis of $\mathbf{H}_{\lambda\mu}$.

It is shown in [Lu03, §5] that, for any weight function $\mathbf{L} : W \to \mathbb{N}$, there exists a bar-invariant basis $\{C^{\mathbf{L}}_w\}$ (referred as c_w therein) at the specialization $u = \mathbf{v}^{\mathbf{L}(s_0)}, v = \mathbf{v}^{\mathbf{L}(s_1)}$, given by

$$C^{\mathbf{L}}_w = u^{-\ell_{\mathfrak{c}}(w)} v^{-\ell_{\mathfrak{a}}(w)} \sum_{y \leqslant w} p_{y,w}(\mathbf{v}) \; T_y|_{u=\mathbf{v}^{\mathbf{L}(s_0)}, v=\mathbf{v}^{\mathbf{L}(s_1)}}, \tag{4.1.3}$$

where $p_{y,w}(\mathbf{v})$ is an analogue of Kazhdan-Lusztig polynomial. For $\lambda, \mu \in \Lambda_{n,d}$, let $\mathbb{H}_{\lambda\mu}$ be the $\mathbb{Z}[u^{\pm 2}, v^{\pm 2}]$-submodule of $\mathbb{H}$ with basis $\{T^g_{\lambda\mu}\}_{g \in \mathscr{D}_{\lambda\mu}}$. It follows from [CIK72, Lemma 2.10] and Lemma 3.1.1 that $\mathbb{H}_{\lambda\mu}$ can be characterized as below:

$$\mathbb{H}_{\lambda\mu} = \left\{ h \in \mathbb{H} \;\middle|\; \begin{array}{l} T_w h = u^{2\ell_{\mathfrak{c}}(w)} v^{2\ell_{\mathfrak{a}}(w)} h, (\forall w \in W_\lambda), \\ h T_{w'} = u^{2\ell_{\mathfrak{c}}(w')} v^{2\ell_{\mathfrak{a}}(w')} h, (\forall w' \in W_\mu) \end{array} \right\}. \tag{4.1.4}$$

Below we show that the bar involution is closed on $\mathbb{H}_{\lambda\mu}$ although lacking of bar-invariant basis.

Lemma 4.1.2. *Let $A = \kappa(\lambda, g, \mu)$. Then $\overline{T^g_{\lambda\mu}} \in \mathbb{H}_{\lambda\mu}$. In particular,*

$$\overline{T^g_{\lambda\mu}} \in u^{-2\ell_{\mathfrak{c}}(g^+_{\lambda\mu})} v^{-2\ell_{\mathfrak{a}}(g^+_{\lambda\mu})} T^g_{\lambda\mu} + \sum_{\substack{y \in \mathscr{D}_{\lambda\mu} \\ y < g}} \mathbb{Z}[u^{\pm 2}, v^{\pm 2}] T^y_{\lambda\mu}. \tag{4.1.5}$$

Moreover, $u^{-\ell_{\mathfrak{c}}(w_\circ^\mu)} v^{-\ell_{\mathfrak{a}}(w_\circ^\mu)} x_\mu$ is bar-invariant.

Proof. First, we show that $\overline{x_\nu} \in \mathbb{A}x_\nu$ for all $\nu \in \Lambda_{n,d}$ via bar-invariant basis $C^{\mathbf{L}}_w$. Let $\mathbb{H}^{\mathbf{L}}_{\lambda\mu}$ be the specialization of $\mathbb{H}_{\lambda\mu}$ at $u = \mathbf{v}^{\mathbf{L}(s_0)}, v = \mathbf{v}^{\mathbf{L}(s_1)}$. From (4.1.4), a direct calculation shows that $C^{\mathbf{L}}_{w^\nu_\circ} \in \mathbb{H}^{\mathbf{L}}_{\nu\nu}$ and hence

$$C^{\mathbf{L}}_{w^\nu_\circ} = u^{-\ell_{\mathfrak{c}}(w^\nu_\circ)} v^{-\ell_{\mathfrak{a}}(w^\nu_\circ)} \sum_{y \leqslant w^\nu_\circ} p_{y,w^\nu_\circ} T_y\big|_{u=\mathbf{v}^{\mathbf{L}(s_0)},v=\mathbf{v}^{\mathbf{L}(s_1)}} \in \sum_{g \in \mathscr{D}_{\nu\nu}} \mathbb{Z}(\mathbf{v}^{\pm\mathbf{L}(s_0)}, \mathbf{v}^{\pm\mathbf{L}(s_1)})\, T^g_{\nu\nu}\big|_{u=\mathbf{v}^{\mathbf{L}(s_0)},v=\mathbf{v}^{\mathbf{L}(s_1)}}. \tag{4.1.6}$$

Upon comparing coefficients, we obtain

$$C^{\mathbf{L}}_{w^\nu_\circ} = u^{-\ell_{\mathfrak{c}}(w^\nu_\circ)} v^{-\ell_{\mathfrak{a}}(w^\nu_\circ)}\, T^{\mathbb{1}}_{\nu\nu}\big|_{u=\mathbf{v}^{\mathbf{L}(s_0)},v=\mathbf{v}^{\mathbf{L}(s_1)}}. \tag{4.1.7}$$

Note that $x_\nu = T^{\mathbb{1}}_{\nu\nu}$. Hence, for any weight function $\mathbf{L}$, we have

$$(\overline{x_\nu} - u^{-2\ell_{\mathfrak{c}}(w^\nu_\circ)} v^{-2\ell_{\mathfrak{a}}(w^\nu_\circ)} x_\nu)\Big|_{u=\mathbf{v}^{\mathbf{L}(s_0)},v=\mathbf{v}^{\mathbf{L}(s_1)}} = 0. \tag{4.1.8}$$

Therefore $\overline{x_\nu} = u^{-2\ell_{\mathfrak{c}}(w^\nu_\circ)} v^{-2\ell_{\mathfrak{a}}(w^\nu_\circ)} x_\nu$. We now show that $\overline{T^g_{\lambda\mu}} \in \mathbb{H}_{\lambda\mu}$. By Lemma 3.1.3, we have $T^g_{\lambda\mu} \in \mathbb{Z}[u^{\pm 2}, v^{\pm 2}]x_\lambda T_g x_\mu$, and hence

$$\overline{T^g_{\lambda\mu}} \in \mathbb{Z}[u^{\pm 2}, v^{\pm 2}]\overline{x_\lambda}\,\overline{T_g}\,\overline{x_\mu} = \sum_{z \leqslant g} \mathbb{Z}[u^{\pm 2}, v^{\pm 2}] x_\lambda T_z x_\mu. \tag{4.1.9}$$

Similar to (3.2.3), we have $x_\lambda T_z x_\mu \in \mathbb{Z}[u^{\pm 2}, v^{\pm 2}]x_\lambda T_y x_\mu$ for some $y \in \mathscr{D}_{\lambda\mu}$ such that $y \leqslant z$. Finally, we have $\overline{T^g_{\lambda\mu}} \in \sum_{y \in \mathscr{D}_{\lambda\mu}} \mathbb{Z}[u^{\pm 2}, v^{\pm 2}]x_\lambda T_y x_\mu \subseteq \mathbb{H}_{\lambda\mu}$. The leading coefficient is obtained by a lengthy calculation which we omit. □

The bar involution $^-$ on $\mathbb{S}^{\jmath}_{n,d}$ is defined as follows: for each $f \in \mathrm{Hom}_{\mathbb{H}}(x_\mu\mathbb{H}, x_\lambda\mathbb{H})$, let $\overline{f} \in \mathrm{Hom}_{\mathbb{H}}(x_\mu\mathbb{H}, x_\lambda\mathbb{H})$ be the $\mathbb{H}$-linear map which sends x_μ to $\overline{f(\overline{x_\mu})}$.

4.2. **A standard basis in $\mathbb{S}^{\jmath}_{n,d}$.** We define, for $A \in \Xi_{n,d}$, the (truncated) generalized length functions of A by

$$\widehat{\ell}(A) = \frac{1}{2}\Big(\sum_{(i,j) \in I_{\mathfrak{c}}} \Big(\sum_{\substack{x \leqslant i \\ y > j}} + \sum_{\substack{x \geqslant i \\ y < j}} \Big) a^{\natural}_{ij} a_{xy} \Big), \quad \widehat{\ell}_{\mathfrak{c}}(A) = \frac{1}{2}\Big(\sum_{\substack{0 \leqslant x \\ 0 > y}} + \sum_{\substack{0 \geqslant x \\ 0 < y}} \Big) a_{xy}, \tag{4.2.1}$$

$$\widehat{\ell}_{\mathfrak{a}}(A) = \widehat{\ell}(A) - \widehat{\ell}_{\mathfrak{c}}(A) = \frac{1}{2}\Big(\sum_{(i,j) \in I_{\mathfrak{c}}} \Big(\sum_{\substack{x \leqslant i \\ y > j}} + \sum_{\substack{x \geqslant i \\ y < j}} \Big) a^{\natural\natural}_{ij} a_{xy} \Big), \tag{4.2.2}$$

where $a^{\natural\natural}_{00} = \frac{1}{2}(a_{00} - 3)$ and $a^{\natural\natural}_{ij} = a_{ij}$ if $(i,j) \in I_{\mathfrak{a}}$. We shall see in Proposition 4.2.2 that $\widehat{\ell}_{\mathfrak{a}}(A), \widehat{\ell}_{\mathfrak{c}}(A) \in \mathbb{N}$.

Remark 4.2.1. The function $\widehat{\ell}$ counts the dimension of the generalized Schubert variety associated to the matrix A (cf. [FL^{3}Wb, Appendix A]), and is equal to the length of A when A is a permutation matrix (that is when the associated variety is a genuine Schubert variety).

Set

$$[A] = u^{-\widehat{\ell}_{\mathfrak{c}}(A)} v^{-\widehat{\ell}_{\mathfrak{a}}(A)} e_A. \tag{4.2.3}$$

The set $\{[A] \mid A \in \Xi_{n,d}\}$ forms an $\mathbb{A}$-basis of $\mathbb{S}^{\jmath}_{n,d}$, which we call the *standard basis*. For $A \in \Xi_n$, we let

$$\sigma_{ij}(A) = \sum_{x \leqslant i, y \geqslant j} a_{xy}. \tag{4.2.4}$$

Now we define a partial order $\leqslant_{\mathrm{alg}}$ on Ξ_n by letting, for $A, B \in \Xi_n$,

$$A \leqslant_{\mathrm{alg}} B \Leftrightarrow \mathrm{row}(A) = \mathrm{row}(B),\ \mathrm{col}(A) = \mathrm{col}(B),\ \text{and } \sigma_{ij}(A) \leqslant \sigma_{ij}(B), \forall i < j. \tag{4.2.5}$$

We denote $A <_{\mathrm{alg}} B$ if $A \leqslant_{\mathrm{alg}} B$ and $A \neq B$.

Proposition 4.2.2. *Let $A = \kappa(\lambda, g, \mu) \in \Xi_{n,d}$. Then we have $\overline{[A]} \in [A] + \sum_{B <_{\mathrm{alg}} A} \mathbb{A}[B]$.*

Proof. By the finite type analogue of [FL³Wb, Proposition 5.3], we have

$$\widehat{\ell}_{\mathfrak{c}}(A) = \ell_{\mathfrak{c}}(g^+_{\lambda\mu}) - \ell_{\mathfrak{c}}(w^\mu_\circ), \quad \widehat{\ell}_{\mathfrak{a}}(A) = \ell_{\mathfrak{a}}(g^+_{\lambda\mu}) - \ell_{\mathfrak{a}}(w^\mu_\circ). \tag{4.2.6}$$

Hence,

$$[A](u^{-\ell_{\mathfrak{c}}(w^\mu_\circ)}v^{-\ell_{\mathfrak{a}}(w^\mu_\circ)}x_\mu) = u^{-\ell_{\mathfrak{c}}(g^+_{\lambda\mu})}v^{-\ell_{\mathfrak{a}}(g^+_{\lambda\mu})}T^g_{\lambda\mu}. \tag{4.2.7}$$

Thus, by Lemma 4.1.2, the map $\overline{[A]}$ is determined by

$$\overline{[A]}(u^{-\ell_{\mathfrak{c}}(w^\mu_\circ)}v^{-\ell_{\mathfrak{a}}(w^\mu_\circ)}x_\mu) = u^{\ell_{\mathfrak{c}}(g^+_{\lambda\mu})}v^{\ell_{\mathfrak{a}}(g^+_{\lambda\mu})}\overline{T^g_{\lambda\mu}} \in u^{-\ell_{\mathfrak{c}}(g^+_{\lambda\mu})}v^{-\ell_{\mathfrak{a}}(g^+_{\lambda\mu})}T^g_{\lambda\mu} + \sum_{y<g}\mathbb{A}T^y_{\lambda\mu}. \tag{4.2.8}$$

We note that $[\kappa(\lambda, y, \mu)](x_\mu) \in \mathbb{A}T^y_{\lambda\mu}$. An induction on $\ell(g)$ shows that

$$\overline{[A]} \in [A] + \sum_{y\in\mathscr{D}_{\lambda\mu}, y<g} \mathbb{A}\,[\kappa(\lambda, y, \mu)]. \tag{4.2.9}$$

A finite type analogue of [FL³Wb, Corollary 5.5] shows that $\kappa(\lambda, y, \mu) <_{\rm alg} A$ if $y < g$. We conclude the statement. ☐

Let us reformulate the multiplication formula for $\mathbb{S}^\jmath_{n,d}$ (Proposition 3.2.2) in terms of the standard basis.

Theorem 4.2.3. *Suppose that $A, B, C \in \Xi_{n,d}$ and $h \in [1, n]$.*

(1) If $B - bE^\theta_{h,h-1}$ is diagonal, $\mathrm{col}(B) = \mathrm{row}(A)$, then

$$[B][A] = \sum_t u^{-\delta_{h,1}\sum_{l>0}t_l}v^{\beta(t)}\prod_{l=-n}^{n}\overline{\begin{bmatrix} a_{h,l}+t_l \\ t_l \end{bmatrix}}[\check{A}_{t,h}], \tag{4.2.10}$$

where t is summed over as in Propsition 3.2.2 (1), and

$$\beta(t) = \sum_{k\leqslant l}t_la_{h,k} - \sum_{k<l}t_l(a_{h-1,k}-t_k) + \delta_{h,1}\Big(\sum_{-l<k<l}t_lt_k + \sum_{l>0}\frac{t_l(t_l+3)}{2}\Big). \tag{4.2.11}$$

(2) Suppose $C - cE^\theta_{h-1,h}$ is diagonal and $\mathrm{col}(C) = \mathrm{row}(A)$. If $h \neq 1$ then

$$[C][A] = \sum_t v^{\beta'(t)}\prod_{l=-n}^{n}\overline{\begin{bmatrix} a_{h-1,l}+t_l \\ t_l \end{bmatrix}}[\widehat{A}_{t,h}], \tag{4.2.12}$$

where t is summed over as in Propsition 3.2.2 (2), and

$$\beta'(t) = \sum_{k\geqslant l}t_la_{h-1,k} - \sum_{k>l}t_l(a_{h,k}-t_k). \tag{4.2.13}$$

If $h = 1$ then

$$[C][A] = \sum_t u^{\sum_{l\leqslant 0}t_l}v^{\beta''(t)}\overline{\left(\frac{[a^\natural_{0,0}+t_0]^!_{\mathfrak{c}}}{[a^\natural_{0,0}]^!_{\mathfrak{c}}[t_0]^!}\prod_{l=1}^{n}\frac{[a_{0,l}+t_l+t_{-l}]^!}{[a_{0,l}]^![t_l]^![t_{-l}]^!}\right)}[\widehat{A}_{t,1}], \tag{4.2.14}$$

where

$$\beta''(t) = \sum_{k\geqslant l}t_la_{0,k} - \sum_{k>l}t_l(a_{1,k}-t_k) + \sum_{l<k\leqslant -l}t_lt_k + \sum_{l\leqslant 0}\frac{t_l(t_l-3)}{2}. \tag{4.2.15}$$

Proof. For Part (1), by Proposition 3.2.2, we have

$$[B][A] = \sum_t u^{\widehat{\ell}_{\mathfrak{c}}(\check{A}_{t,h})-\widehat{\ell}_{\mathfrak{c}}(A)-\widehat{\ell}_{\mathfrak{c}}(B)}v^{\widehat{\ell}_{\mathfrak{a}}(\check{A}_{t,h})-\widehat{\ell}_{\mathfrak{a}}(A)-\widehat{\ell}_{\mathfrak{a}}(B)+2\sum_{k<l}t_la_{h,k}+2\sum_l t_la_{h,l}}\prod_{l=-n}^{n}\overline{\begin{bmatrix} a_{h,l}+t_l \\ t_l \end{bmatrix}}[\check{A}_{t,h}].$$

Part (1) concludes by combining the following identities via direct computation:

$$\widehat{\ell}_{\mathfrak{c}}(B) = 0, \quad \widehat{\ell}_{\mathfrak{a}}(B) = bb_{h,h} = \sum_{l,k}t_la_{h,k}, \quad \widehat{\ell}_{\mathfrak{c}}(\check{A}_{t,h}) - \widehat{\ell}_{\mathfrak{c}}(A) = -\delta_{h,1}\sum_{l>0}t_l,$$

$$\widehat{\ell}_{\mathfrak{a}}(\check{A}_{t,h}) - \widehat{\ell}_{\mathfrak{a}}(A) = \sum_{k>l} t_l a_{h,k} - \sum_{k<l} t_l (a_{h-1,k} - t_k) + \delta_{h,1}(\sum_{-l<k<l} t_l t_k + \sum_{l>0} \frac{t_l(t_l+3)}{2}).$$

For Part (2), we only present the most complicated case that $h = 1$. A direct computation shows that

$$\frac{[a^{\natural}_{0,0} + t_0]^!_{\mathfrak{c}}}{[a^{\natural}_{0,0}]^!_{\mathfrak{c}}[t_0]!} \prod_{l=1}^{n} \frac{[a_{0,l} + t_l + t_{-l}]!}{[a_{0,l}]![t_l]![t_{-l}]!} = u^{2t_0} v^{\sum_l (2a_{0,l}t_l + t_l t_{-l}) - 3t_0} \overline{\left(\frac{[a^{\natural}_{0,0} + t_0]^!_{\mathfrak{c}}}{[a^{\natural}_{0,0}]^!_{\mathfrak{c}}[t_0]!} \prod_{l=1}^{n} \frac{[a_{0,l} + t_l + t_{-l}]!}{[a_{0,l}]![t_l]![t_{-l}]!} \right)}. \quad (4.2.16)$$

Part (2) follows from combining (4.2.16) and the calculation below:

$$\widehat{\ell}_{\mathfrak{c}}(C) = c = \sum_l t_l, \quad \widehat{\ell}_{\mathfrak{a}}(C) = \sum_{l,k} t_l a_{0,k} + \frac{c(c-3)}{2}, \quad \widehat{\ell}_{\mathfrak{c}}(\widehat{A}_{t,1}) - \widehat{\ell}_{\mathfrak{c}}(A) = \sum_{l>0} t_l,$$

$$\widehat{\ell}_{\mathfrak{a}}(\widehat{A}_{t,1}) - \widehat{\ell}_{\mathfrak{a}}(A) = \sum_{k<l} t_l a_{0,k} - \sum_{l<k} t_l (a_{1,k} - t_k) + (\sum_{-l<k<l} t_l t_k + \sum_{l>0} \frac{t_l(t_l-3)}{2}).$$

□

4.3. **A monomial basis in $\mathbb{S}^{\jmath}_{n,d}$.** Thanks to Remark 3.2.3, we can use results in [BKLW18] freely when we specialize $u = v$. For $A \in \Xi_{n,d}$, we can use the algorithm in [BKLW18, Theorem 3.10] with the fixed order therein to produce a unique family of Chevalley matrices $\{A^{(1)}, \dots, A^{(x)}\}$ in $\Xi_{n,d}$ for some $x = x(A) \in \mathbb{N}$. At the specialization $u = v$, a unitriangular relation is satisfied:

$$[A^{(1)}] \cdots [A^{(x)}]\Big|_{u=v} = [A] + \textstyle\sum_{B <_{\rm alg} A} \mathbb{A}[B]|_{u=v}\,. \quad (4.3.1)$$

Denote the product of the corresponding elements in $\mathbb{S}^{\jmath}_{n,d}$ by

$$m_A = [A^{(1)}] \cdots [A^{(x)}] \in \mathbb{S}^{\jmath}_{n,d}. \quad (4.3.2)$$

Let I be the identity matrix. Since the algorithm in [BKLW18, Theorem 3.10] produces matrices $A^{(1)}, \dots, A^{(x)}$ according to mainly the off-diagonal matrices of A and then determine the diagonal entries of these $A^{(i)}$ by the row and column sums, we have that $x(A) = x(A + pI)$ and $(A + pI)^{(i)} = A^{(i)} + pI$ for all $p \in 2\mathbb{N}$, i.e.,

$$m_{A+pI} = [A^{(1)} + pI] \cdots [A^{(x)} + pI]. \quad (4.3.3)$$

Proposition 4.3.1. *For $A \in \Xi_{n,d}$ the element $m_A \in \mathbb{S}^{\jmath}_{n,d}$ has the following property:*

$$m_A = [A] + \sum_{B <_{\rm alg} A} \mathbb{A}[B]. \quad (4.3.4)$$

Moreover, $\{m_A\}_{A \in \Xi_{n,d}}$ form a basis of $\mathbb{S}^{\jmath}_{n,d}$, which we call the monomial basis.

Proof. A direct proof can be pursued using the multiplication formulas (Proposition 4.2.3), similar to the proofs of [BKLW18, Theorem 3.10] and [FL15, Theorem 4.6.3]. Here we offer a simpler proof by combining [BKLW18, Theorem 3.10] and [FL15, Theorem 4.6.3] as below: now

$$m_A = u^{\alpha(A)} v^{\beta(A)} [A] + \sum_{B <_{\rm alg} A} \mathbb{A}[B], \quad \text{for some} \quad \alpha(A), \beta(A) \in \mathbb{N}$$

It follows from [BKLW18, Theorem 3.10] (resp. [FL15, Theorem 4.6.3]) that $v^{\alpha(A)} v^{\beta(A)} = 1$ (resp. $1^{\alpha(A)} v^{\beta(A)} = 1$), which forces that $u^{\alpha(A)} v^{\beta(A)} = 1$ and hence (4.3.4) holds. Hence the transition matrix from $\{m_A \mid A \in \Xi_{n,d}\}$ to the standard basis $\{[A] \mid A \in \Xi_{n,d}\}$ is unital triangular. Therefore $\{m_A \mid A \in \Xi_{n,d}\}$ form a basis of $\mathbb{S}^{\jmath}_{n,d}$. □

Remark 4.3.2. The monomial basis acts as an intermediate step toward constructing canonical basis in the one-parameter case. Moreover, the two-parameter stabilization procedure is made possible thanks to the property (4.3.3) of monomial basis.

4.4. **The canonical basis at the specialization.** For any weight function $\mathbf{L}$, let $\mathbf{c} = \gcd(\mathbf{L}(s_0), \mathbf{L}(s_1))$. We show that the specialization of $\mathbb{S}^{\jmath}_{n,d}$ at $u = \mathbf{v}^{\mathbf{L}(s_0)}, v = \mathbf{v}^{\mathbf{L}(s_1)}$ admits canonical basis with respect to $\mathbf{v}^{\mathbf{c}}$. For $A \in \Xi_{n,d}$, let $[A]^{\mathbf{L}}$ (and $m^{\mathbf{L}}_A$, resp.) be the standard basis (and monomial basis, resp.) of the specialization of $\mathbb{S}^{\jmath}_{n,d}$ at $u = \mathbf{v}^{\mathbf{L}(s_0)}, v = \mathbf{v}^{\mathbf{L}(s_1)}$. It follows from (4.2.9) and (4.3.4) that the following unitriangular relations hold:

$$\overline{[A]^{\mathbf{L}}} \in [A]^{\mathbf{L}} + \sum_{B <_{\rm alg} A} \mathbb{Z}[\mathbf{v}^{\mathbf{c}}, \mathbf{v}^{-\mathbf{c}}][B]^{\mathbf{L}}, \tag{4.4.1}$$

$$\overline{m^{\mathbf{L}}_A} = m^{\mathbf{L}}_A \in [A]^{\mathbf{L}} + \sum_{B <_{\rm alg} A} \mathbb{Z}[\mathbf{v}^{\mathbf{c}}, \mathbf{v}^{-\mathbf{c}}][B]^{\mathbf{L}}. \tag{4.4.2}$$

If A is diagonal, set $\{A\}^{\mathbf{L}} = [A]^{\mathbf{L}}$. Arguing inductively on the partial order $\leqslant_{\rm alg}$ and using a standard argument (cf. [Lu93, 24.2.1]) there exists a unique element $\{A\}^{\mathbf{L}} \in \mathbb{S}^{\jmath}_{n,d}$ such that

$$\overline{\{A\}^{\mathbf{L}}} = \{A\}^{\mathbf{L}} \in [A]^{\mathbf{L}} + \sum_{B <_{\rm alg} A} \mathbf{v}^{-\mathbf{c}}\mathbb{Z}[\mathbf{v}^{-\mathbf{c}}][B]^{\mathbf{L}}. \tag{4.4.3}$$

Let $\mathbb{S}^{\jmath,\mathbf{L}}_{n,d}$ be the specialization of $\mathbb{S}^{\jmath}_{n,d}$ at $u = \mathbf{v}^{\mathbf{L}(s_0)}, v = \mathbf{v}^{\mathbf{L}(s_1)}$.

Theorem 4.4.1. *There exists a canonical basis $\{\{A\}^{\mathbf{L}} \mid A \in \Xi_{n,d}\}$ for $\mathbb{S}^{\jmath,\mathbf{L}}_{n,d}$, which is characterized by the property* (4.4.3)*.*

5. Stabilization algebra $\dot{\mathbb{K}}^{\jmath}_n$

In this section, we shall establish a stabilization property for the family of Schur algebras $\mathbb{S}^{\jmath}_{n,d}$ as d varies, which leads to a quantum algebra $\dot{\mathbb{K}}^{\jmath}_n$.

5.1. **A BLM-type stabilization.** Let

$$\tilde{\Xi}_n = \left\{ (a_{ij})_{-n \leqslant i,j \leqslant n} \in \mathrm{Mat}_{N\times N}(\mathbb{Z}) \middle| \begin{matrix} a_{-i,-j} = a_{ij} (\forall i,j), \\ a_{xy} \in \mathbb{N} (\forall x \neq y), a_{00} \in 2\mathbb{Z}+1 \end{matrix} \right\}. \tag{5.1.1}$$

Extending the partial ordering $\leqslant_{\rm alg}$ for Ξ_n, we define a partial ordering $\leqslant_{\rm alg}$ on $\tilde{\Xi}_n$ using the same recipe (4.2.5). For each $A \in \tilde{\Xi}_n$ and $p \in 2\mathbb{N}$, we write

$${}_pA = A + pI \in \tilde{\Xi}_n. \tag{5.1.2}$$

Then ${}_pA \in \Xi_n$ for even $p \gg 0$. Let π be an indeterminate (independent of u, v), and $\mathcal{R}_1$ be the subring of $\mathbb{Q}(u,v)[\pi, \pi^{-1}]$ generated by, for $a \in \mathbb{Z}, k \in \mathbb{Z}_{>0}$,

$$r^{(1)}_{a,k}, \quad r^{(2)}_{a,k}, \quad v^a, \quad \text{and} \quad u^a, \tag{5.1.3}$$

where

$$r^{(1)}_{a,k}(u,v,\pi) = \prod_{i=1}^{k} \frac{v^{-2(a-i)}\pi^2 - 1}{v^{-2i} - 1}, \tag{5.1.4}$$

$$r^{(2)}_{a,k}(u,v,\pi) = \prod_{i=1}^{k} \frac{(u^{-2}v^{-2(a-1-i)}\pi + 1)(v^{-2(a-i)}\pi - 1)}{v^{-2i} - 1}. \tag{5.1.5}$$

Let $\mathcal{R}_2$ be the subring of $\mathbb{Q}(u,v)[\pi, \pi^{-1}]$ generated by, for $a \in \mathbb{Z}, k \in \mathbb{Z}_{>0}$,

$$r^{(1)}_{a,k}, \quad \overline{r}^{(1)}_{a,k}, \quad r^{(2)}_{a,k}, \quad \overline{r}^{(2)}_{a,k}, \quad v^a, \quad \text{and} \quad u^a. \tag{5.1.6}$$

We extend the bar-involution to $\mathcal{R}_2$ by requiring $\overline{\pi} = \pi^{-1}$.

Proposition 5.1.1. *Let $A_1, \ldots, A_f \in \tilde{\tilde{\Xi}}_n$ be such that* $\mathrm{col}(A_i) = \mathrm{row}(A_{i+1})$ *for all i. Then there exists matrices $Z_1, \ldots, Z_m \in \tilde{\tilde{\Xi}}_n$ and $\zeta_i(u, v, \pi) \in \mathcal{R}_1$ such that for even integer $p \gg 0$,*

$$[{}_pA_1][{}_pA_2] \cdots [{}_pA_f] = \sum_{i=1}^{m} \zeta_i(u, v, v^{-p})[{}_pZ_i]. \tag{5.1.7}$$

Proof. We assume first that $f = 2$ and A_1 is such that $A_1 - bE^{\theta}_{h,h-1}$ is diagonal for some $h \in [1, n]$ and some $b \geqslant 0$. Let $A_2 = A = (a_{ij})$. For each $t = (t_i)_{-n \leqslant i \leqslant n} \in \mathbb{N}^N$, we define

$$\zeta_t(u, v, \pi) = u^{-\delta_{h,1} \sum_{l>0} t_l} v^{\beta(t)} \overline{\prod_{h \neq l \in [-n,n]} \begin{bmatrix} a_{h,l} + t_l \\ t_l \end{bmatrix} \prod_{i=1}^{t_h} \frac{v^{-2(a_{h,h}+t_h-i+1)}\pi^2 - 1}{v^{-2i} - 1}} \in \mathcal{R}_1$$

where $\beta(t)$ is defined in (4.2.11). Though $\beta(t)$ depends on A, it is invariant if A is replaced by ${}_pA$. Therefore we have the following formula for large enough even p by (4.2.10):

$$[{}_pA_1][{}_pA] = \sum_t \zeta_t(u, v, v^{-p})[{}_p\breve{A}_{t,h}].$$

The statement holds in this case.

We next assume that $f = 2$ and A_1 is such that $A_1 - cE^{\theta}_{h-1,h}$ is diagonal for some $h \in [1, n]$ and some $c \geqslant 0$. Let $A_2 = A = (a_{ij})$. Recall $\beta'(t)$ and $\beta''(t)$ in (4.2.13) and (4.2.15), respectively. If $h \neq 1$, for each $t = (t_i)_{-n \leqslant i \leqslant n} \in \mathbb{N}^N$, we define

$$\zeta_t(u, v, \pi) = v^{\beta'(t)} \overline{\prod_{h-1 \neq l \in [-n,n]} \begin{bmatrix} a_{h-1,l} + t_l \\ t_l \end{bmatrix} \prod_{i=1}^{t_h} \frac{v^{-2(a_{h-1,h-1}+t_{h-1}-i+1)}\pi^2 - 1}{v^{-2i} - 1}} \in \mathcal{R}_1;$$

If $h = 1$, we define

$$\begin{aligned} \zeta_t(u, v, \pi) = & u^{\sum_{l \leqslant 0} t_l} v^{\beta''(t)} \overline{\left(\prod_{l=1}^{n} \frac{[a_{0,l} + t_l + t_{-l}]!}{[a_{0,l}]![t_l]![t_{-l}]!} \right)} \\ & \cdot \prod_{i=1}^{t_0} \frac{(u^{-2} v^{-2(a^{\natural}_{00}+t_0-1-i)}\pi + 1)(v^{-2(a^{\natural}_{00}+t_0-i)}\pi - 1)}{v^{-2i} - 1} \in \mathcal{R}_1. \end{aligned}$$

It is clear that both $\beta'(t)$ and $\beta''(t)$ are invariant if A is replaced by ${}_pA$. Therefore the following formula holds for large enough even p by (4.2.12):

$$[{}_pA_1][{}_pA] = \sum_t \zeta_t(u, v, v^{-p})[{}_p\widehat{A}_{t,h}].$$

Hence the proposition is verified in the present case.

Using induction on f, we know that the proposition holds for general f in the case where $A_1, \ldots, A_f$ are Chevalley matrices (i.e. of one of the two types considered above). It follows from (4.3.2) and (4.3.4) that for any $A \in \Xi_{n,d}$, there exists Chevalley matrices $B_1, B_2, \ldots, B_M$ such that

$$[B_1][B_2] \cdots [B_M] = [A] + \text{lower terms}.$$

Then we can prove the proposition by using induction on $\Psi(A) = \sum_{i<j} \sigma_{ij}(A)$. We omit the subsequent argument here since it is totally as the same as those for [BLM90, Proposition 4.2]. □

By an argument identical with [BLM90, Proposition 4.3], we obtain below the stabilization of bar involution by allowing extra coefficients as seen in (5.1.6).

Proposition 5.1.2. *For any $A \in \tilde{\tilde{\Xi}}_n$, there exist matrices $T_1, \ldots, T_s \in \tilde{\tilde{\Xi}}_n$ and $\tau_i(u, v, \pi) \in \mathcal{R}_2$ such that, for even integer $p \gg 0$,*

$$\overline{[{}_pA]} = \sum_{i=1}^{s} \tau_i(u, v, v^{-p})[{}_pT_i]. \tag{5.1.8}$$

Let $\dot{\mathbb{K}}^{\jmath}_n$ be the free $\mathbb{A}$-module with an $\mathbb{A}$-basis given by the symbols $[A]$ for $A \in \tilde{\Xi}_n$ (which will be called a standard basis of $\dot{\mathbb{K}}^{\jmath}_n$). By Propositions 5.1.1–5.1.2 and applying a specialization at $\pi = 1$ (note that $\zeta_i(u, v, 1) \in \mathbb{A}$), we have the following corollary.

Corollary 5.1.3. *There is a unique associative $\mathbb{A}$-algebra structure on $\dot{\mathbb{K}}^{\jmath}_n$ with multiplication given by*

$$[A_1][A_2]\cdots[A_f] = \begin{cases} \sum_{i=1}^m \zeta_i(u,v,1)[Z_i] & \text{if } \mathrm{col}(A_i) = \mathrm{row}(A_{i+1}) \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the map $^{-}\colon \dot{\mathbb{K}}^{\jmath}_n \to \dot{\mathbb{K}}^{\jmath}_n$ *given by* $\overline{[A]} = \sum_{i=1}^s \tau_i(u,v,1)[T_i]$ *is an $\mathbb{A}$-linear involution.*

The following multiplication formula in $\dot{\mathbb{K}}^{\jmath}_n$ follows directly from Theorem 4.2.3 by the stabilization construction.

Proposition 5.1.4. *Let $A, B, C \in \tilde{\Xi}_n$ and $h \in [1, n]$.*

(1) If $B - bE^{\theta}_{h,h-1}$ is diagonal and $\mathrm{col}(B) = \mathrm{row}(A)$, *then*

$$[B][A] = \sum_t u^{-\delta_{h,1}\sum_{l>0} t_l} v^{\beta(t)} \overline{\prod_{l=-n}^{n} \begin{bmatrix} a_{h,l}+t_l \\ t_l \end{bmatrix}} [\check{A}_{t,h}], \tag{5.1.9}$$

where $t = (t_i)_{-n\leqslant i\leqslant n} \in \mathbb{N}^N$ with $\sum_{i=-n}^n t_i = b$ such that

$$\begin{cases} t_i \leqslant a_{h-1,i} & \text{if } i+1 \neq h > 1; \\ t_i + t_{-i} \leqslant a_{0,i} & \text{if } h = 1, i \neq 0, \end{cases}$$

(2) Suppose $C - cE^{\theta}_{h-1,h}$ is diagonal and $\mathrm{col}(C) = \mathrm{row}(A)$.
If $h \neq 1$ then

$$[C][A] = \sum_t v^{\beta'(t)} \overline{\prod_{l=-n}^{n} \begin{bmatrix} a_{h-1,l}+t_l \\ t_l \end{bmatrix}} [\hat{A}_{t,h}], \tag{5.1.10}$$

where $t = (t_i)_{-n\leqslant i\leqslant n} \in \mathbb{N}^N$ with $\sum_{i=-n}^n t_i = c$ such that $t_i \leqslant a_{h,i}$ if $i \neq h$.
If $h = 1$ then

$$[C][A] = \sum_t u^{\sum_{l\leqslant 0} t_l} v^{\beta''(t)} \overline{\left(\frac{\prod_{k=a^{\natural}_{00}+1}^{a^{\natural}_{00}+t_0} [k](u^2 v^{2(k-1)}+1)}{\prod_{k=1}^{t_0}[k]} \prod_{l=1}^{n} \frac{[a_{0,l}+t_l+t_{-l}]!}{[a_{0,l}]![t_l]![t_{-l}]!} \right)} [\hat{A}_{t,1}], \tag{5.1.11}$$

where $t = (t_i)_{-n\leqslant i\leqslant n} \in \mathbb{N}^N$ with $\sum_{i=-n}^n t_i = c$ such that $t_i \leqslant a_{1,i}$ if $i \neq 1$.

5.2. **Monomial and canonical bases for $\dot{\mathbb{K}}^{\jmath}_n$.** The proposition below follows from Proposition 4.3.1 by the stabilization construction.

Proposition 5.2.1. *For any $A \in \tilde{\Xi}_n$, there exist Chevalley matrices $A^{(1)}, \ldots, A^{(x)}$ in $\tilde{\Xi}_n$ satisfying* $\mathrm{row}(A^{(1)}) = \mathrm{row}(A)$, $\mathrm{col}(A^{(x)}) = \mathrm{col}(A)$, $\mathrm{col}(A^{(i)}) = \mathrm{row}(A^{(i+1)})$ *for* $1 \leqslant i \leqslant x-1$

$$[A^{(1)}][A^{(2)}]\cdots[A^{(x)}] \in [A] + \sum_{B <_{\mathrm{alg}} A} \mathbb{A}[B] \ \in \dot{\mathbb{K}}^{\jmath}_n. \tag{5.2.1}$$

By abuse of notation, we denote the product in $\dot{\mathbb{K}}^{\jmath}_n$ by

$$m_A = [A^{(1)}][A^{(2)}]\cdots[A^{(x)}] \in \dot{\mathbb{K}}^{\jmath}_n. \tag{5.2.2}$$

Hence $\{m_A \mid A \in \tilde{\Xi}_n\}$ forms a basis for $\dot{\mathbb{K}}^{\jmath}_n$ (called a *monomial basis*). Similar to Section 4.4, we define, by abuse of notation, elements $[A]^{\mathbf{L}}, m_A^{\mathbf{L}}, \{A\}^{\mathbf{L}}$ to be the according basis elements of $\dot{\mathbb{K}}^{\jmath}_n$ at the specialization $u = \mathbf{v}^{\mathbf{L}(s_0)}, v = \mathbf{v}^{\mathbf{L}(s_1)}$.

Theorem 5.2.2. *There exists a canonical basis $\dot{\mathfrak{B}} = \{\{A\}^{\mathbf{L}} \mid A \in \Xi_{n,d}\}$ for $\dot{\mathbb{K}}^{\jmath}_n$ at the specialization $u = \mathbf{v}^{\mathbf{L}(s_0)}, v = \mathbf{v}^{\mathbf{L}(s_1)}$, which is characterized by the property* (4.4.3).

6. A different stabilization algebra $\dot{\mathbb{K}}^\imath_n$

In this section we formulate a variant of Schur algebras and their corresponding stabilization algebras. We construct the distinguished bases of these algebras. Recall $N = 2n+1$.

6.1. $\imath$-Schur algebras. Recall $\Xi_{n,d}$ from (2.2.2). Let

$$\Xi^\imath = \{A \in \Xi_{n,d} \mid \mathrm{row}(A)_0 = 1 = \mathrm{col}(A)_0\}. \tag{6.1.1}$$

Recall $\Lambda_{n,d}$ (2.1.8). Let

$$\Lambda^\imath_{n,d} = \{\lambda = (\lambda_n, \ldots, \lambda_1, 1, \lambda_1, \ldots, \lambda_n) \in \Lambda_{n,d}\}.$$

The lemma below is the $\imath$-analog of Lemma 2.2.1, which follows by a similar argument.

Lemma 6.1.1. *The map $\kappa^\imath : \bigsqcup_{\lambda,\mu\in\Lambda^\imath_{n,d}} \{\lambda\} \times \mathscr{D}_{\lambda\mu} \times \{\mu\} \longrightarrow \Xi^\imath$ sending (λ, g, μ) to $(|R^\lambda_i \cap gR^\mu_j|)$ is a bijection.*

Now we define the *$\imath$-Schur algebra* as

$$\mathbb{S}^\imath_{n,d} = \mathrm{End}_{\mathbb{H}}\big(\bigoplus_{\lambda\in\Lambda^\imath_{n,d}} x_\lambda \mathbb{H}\big). \tag{6.1.2}$$

By definition the algebra $\mathbb{S}^\imath_{n,d}$ is naturally a subalgebra of $\mathbb{S}^\jmath_{n,d}$. Moreover, both $\{e_A \mid A \in \Xi^\imath\}$ and $\{[A] \mid A \in \Xi^\imath\}$ are bases of $\mathbb{S}^\imath_{n,d}$ as a free $\mathbb{A}$-module.

6.2. Monomial and canonical bases for $\mathbb{S}^\imath_{n,d}$.

Proposition 6.2.1. *For each $A \in \Xi^\imath$, we have $m_A \in \mathbb{S}^\imath_{n,d}$. Hence the set $\{m_A \mid A \in \Xi^\imath\}$ forms an $\mathbb{A}$-basis of $\mathbb{S}^\imath_{n,d}$. Furthermore, we have $m_A \in [A] + \sum_{B\in\Xi^\imath, B<_{\mathrm{alg}}A} \mathbb{A}[B]$.*

Proof. It follows from [BKLW18, Proposition 5.6] thanks to Remark 3.2.3. □

Theorem 6.2.2. *At the specialization $u = \mathbf{v}^{\mathbf{L}}(s_0), v = \mathbf{v}^{\mathbf{L}}(s_1)$, there is a canonical basis $\mathfrak{B}^\imath_{n,d} = \{\{A\}^{\mathbf{L}} \mid A \in \Xi^\imath\}$ of $\mathbb{S}^\imath_{n,d}$ such that $\overline{\{A\}^{\mathbf{L}}} = \{A\}^{\mathbf{L}}$ and $\{A\}^{\mathbf{L}} \in [A]^{\mathbf{L}} + \sum_{B\in\Xi^\imath, B<_{\mathrm{alg}}A} \mathbf{v}^{-\mathbf{c}}\mathbb{Z}[\mathbf{v}^{-\mathbf{c}}][B]^{\mathbf{L}}$. Moreover, we have $\mathfrak{B}^\imath_{n,d} = \mathfrak{B}^\jmath_{n,d} \cap \mathbb{S}^\imath_{n,d}$.*

Proof. The first half statement on the canonical basis follows by Proposition 6.2.1 and a standard argument (cf. [Lu93, 24.2.1]). The second half statement follows from the uniqueness characterization of the canonical basis $\mathfrak{B}^\imath_{n,d}$. □

6.3. Stabilization algebra of type $\imath$. We define two subsets of $\tilde{\Xi}_n$ (5.1.1) as follows:

$$\tilde{\Xi}^<_n = \{A = (a_{ij}) \in \tilde{\Xi}_n \mid a_{00} < 0\}, \quad \tilde{\Xi}^>_n = \{A = (a_{ij}) \in \tilde{\Xi}_n \mid a_{00} > 0\}. \tag{6.3.1}$$

For any matrix $A \in \tilde{\Xi}_n$ and $p \in 2\mathbb{N}$, we define

$$_{\check{p}}A = A + p(I - E^{00}). \tag{6.3.2}$$

Lemma 6.3.1. *For $A_1, A_2, \ldots, A_f \in \tilde{\Xi}^>_n$, there exists $\mathcal{Z}_i \in \tilde{\Xi}^>_n$ and $\zeta^\imath_i(u, v, \pi) \in \mathcal{R}_1$ such that for all even integers $p \gg 0$, we have an identity in $\mathbb{S}^\jmath_{n,d}$ of the form:*

$$[_{\check{p}}A_1][_{\check{p}}A_2] \ldots [_{\check{p}}A_f] = \sum_{i=1}^m \zeta^\imath_i(u, v, v^{-p})[_{\check{p}}\mathcal{Z}_i].$$

Proof. The proof is similar to the proof of Proposition 5.1.1 where $_pA = A + pI$ is used instead of $_{\check{p}}A$. □

Consequently, the vector space $\dot{\mathbb{K}}^>_n$ over $\mathbb{A}$ spanned by the symbols $[A]$, for $A \in \tilde{\Xi}^>_n$, is a stabilization algebra whose multiplicative structure is given by (with $f = 2$; associativity follows from $f = 3$):

$$[A_1][A_2]\cdots[A_f] = \begin{cases} \sum_{i=1}^m \zeta^\imath_i(u, v, 1)[\mathcal{Z}_i] & \text{if } \mathrm{col}(A_i) = \mathrm{row}(A_{i+1}) \ \forall i, \\ 0 & \text{otherwise.} \end{cases} \tag{6.3.3}$$

Precisely, we have the following multiplication formulas for Chevalley generators in $\dot{\mathbb{K}}^>_n$.

Proposition 6.3.2. *Let $A, B, C \in \widetilde{\Xi}_n^{>}$ and $h \in [1, n]$.*

(1) If $B - bE^{\theta}_{h,h-1}$ is diagonal and $\mathrm{col}(B) = \mathrm{row}(A)$*, then*

$$[B][A] = \sum_t u^{-\delta_{h,1}\sum_{l>0} t_l} v^{\beta(t)} \overline{\prod_{l=-n}^{n} \begin{bmatrix} a_{h,l} + t_l \\ t_l \end{bmatrix}} [\breve{A}_{t,h}], \tag{6.3.4}$$

where $t = (t_i)_{-n \leqslant i \leqslant n} \in \mathbb{N}^N$ with $\sum_{i=-n}^{n} t_i = b$ such that

$$\begin{cases} t_i \leqslant a_{h-1,i} & \text{if } i+1 \neq h > 1; \\ t_i + t_{-i} \leqslant a_{0,i} & \text{if } h = 1, \forall i, \end{cases}$$

(2) Suppose $C - cE^{\theta}_{h-1,h}$ is diagonal and $\mathrm{col}(C) = \mathrm{row}(A)$*.*
If $h \neq 1$ then

$$[C][A] = \sum_t v^{\beta'(t)} \overline{\prod_{l=-n}^{n} \begin{bmatrix} a_{h-1,l} + t_l \\ t_l \end{bmatrix}} [\widehat{A}_{t,h}], \tag{6.3.5}$$

where $t = (t_i)_{-n \leqslant i \leqslant n} \in \mathbb{N}^N$ with $\sum_{i=-n}^{n} t_i = c$ such that $t_i \leqslant a_{h,i}$ if $i \neq h$.
If $h = 1$ then

$$[C][A] = \sum_t u^{\sum_{l \leqslant 0} t_l} v^{\beta''(t)} \overline{\left(\frac{\prod_{k=a^{\natural}_{00}+1}^{a^{\natural}_{00}+t_0} [k](u^2 v^{2(k-1)} + 1)}{\prod_{k=1}^{t_0} [k]} \prod_{l=1}^{n} \frac{[a_{0,l} + t_l + t_{-l}]!}{[a_{0,l}]![t_l]![t_{-l}]!} \right)} [\widehat{A}_{t,1}], \tag{6.3.6}$$

where $t = (t_i)_{-n \leqslant i \leqslant n} \in \mathbb{N}^N$ with $\sum_{i=-n}^{n} t_i = c$ such that $t_i \leqslant a_{1,i}$ if $i \neq 1$.

By arguments entirely analogous to those for Corollary 5.1.3 and Theorem 5.2.2, $\dot{\mathbb{K}}_n^{>}$ admits a (stabilizing) bar involution, $\dot{\mathbb{K}}_n^{>}$ admits a monomial basis $\{m_A \mid A \in \widetilde{\Xi}_n^{>}\}$, and a canonical basis $\dot{\mathfrak{B}}^{\jmath,>}$. Let $\dot{\mathbb{K}}_n^{\imath}$ be the $\mathbb{A}$-submodule of $\dot{\mathbb{K}}_n^{>}$ generated by $\{[A] \mid A \in \widetilde{\Xi}^{\imath}\}$, where

$$\widetilde{\Xi}^{\imath} = \{A \in \widetilde{\Xi}_n^{>} \mid \mathrm{col}(A)_0 = \mathrm{row}(A)_0 = 1\}. \tag{6.3.7}$$

The goal of this subsection is to realize $\dot{\mathbb{K}}_n^{\imath}$ as a subquotient of $\dot{\mathbb{K}}_n^{\jmath}$ with compatible bases by following [BKLW18, Appendix A]. It follows from (6.3.7) that $\dot{\mathbb{K}}_n^{\imath}$ is a subalgebra of $\dot{\mathbb{K}}_n^{>}$. Since the bar-involution on $\dot{\mathbb{K}}_n^{>}$ restricts to an involution on $\dot{\mathbb{K}}_n^{\imath}$, we reach the following conclusion.

Lemma 6.3.3. *The set $\dot{\mathbb{K}}_n^{\imath} \cap \dot{\mathfrak{B}}^{\jmath,>}$ forms a canonical basis of $\dot{\mathbb{K}}_n^{\imath}$.*

The submodule of $\dot{\mathbb{K}}_n^{\jmath}$ spanned by $[A]$ for $A \in \widetilde{\Xi}^{\imath}$ is not a subalgebra. This is why we need a somewhat different stabilization above to construct the canonical basis for $\dot{\mathbb{K}}_n^{\imath}$. We shall see below the stabilization above is related to the stabilization used earlier. Define $\mathbb{J}$ to be the $\mathbb{A}$-submodule of $\dot{\mathbb{K}}_n^{\jmath}$ spanned by $[A]$ for all $A \in \widetilde{\Xi}_n^{<}$.

Lemma 6.3.4. *The submodule $\mathbb{J}$ is a two-sided ideal of $\dot{\mathbb{K}}_n^{\jmath}$.*

Proof. We note that $\mathbb{J}$ is clearly invariant under the anti-involution for $\dot{\mathbb{K}}_n^{\jmath}$ below:

$$[A] \mapsto u^{-\widehat{\ell}_{\mathfrak{c}}(A) + \widehat{\ell}_{\mathfrak{c}}({}^tA)} v^{-\widehat{\ell}_{\mathfrak{a}}(A) + \widehat{\ell}_{\mathfrak{a}}({}^tA)} [{}^tA]. \tag{6.3.8}$$

Hence the claim that $\mathbb{J}$ is a left ideal of $\dot{\mathbb{K}}_n^{\jmath}$ is equivalent to that $\mathbb{J}$ is a right ideal of $\dot{\mathbb{K}}_n^{\jmath}$. We shall show that $\mathbb{J}$ is a left ideal of $\dot{\mathbb{K}}_n^{\jmath}$. To that end, it suffices to show that $[B][A] \in \mathbb{J}$ for arbitrary $A \in \widetilde{\Xi}_n^{<}$ and $B \in \widetilde{\Xi}_n$ such that $B - bE_{h,h-1}$ or $B - bE_{h-1,h}$ is diagonal for some $h \in [1, n]$ and $b \geqslant 0$. Thanks to the multiplication formulas in Proposition 5.1.4, unless the case of $B - bE^{\theta}_{0,1}$ being diagonal, the $(0, 0)$-entry of the terms arising in $[B][A]$ never exceeds $a_{0,0}$. Thus $[B][A] \in \mathbb{J}$ in these cases.

Consider the case that $B - bE^{\theta}_{0,1}$ is diagonal. Recall the formula (6.3.6). If the $(0,0)$-entry $a_{0,0} + 2t_0$ of the term $[\widehat{A}_{t,1}]$ is positive, then the coefficient of this term must be zero since

$$\frac{\prod_{k=a^{\natural}_{00}+1}^{a^{\natural}_{00}+t_0}[k](u^2v^{2(k-1)}+1)}{\prod_{k=1}^{t_0}[k]} = 0,$$

because of $a^{\natural}_{00} + 1 \leqslant 0 < a^{\natural}_{00} + t_0$. Therefore, we always have $[B][A] \in \mathbb{J}$. □

Lemma 6.3.5. *If $A \in \widetilde{\Xi}^{<}_{n}$ then $m_A \in \mathbb{J}$.*

Proof. The proof is as the same as the one of [BKLW18, Lemma A.6 (1)]. □

Recall $\dot{\mathbb{K}}^{\jmath}_{n}$ admits a canonical basis of $\dot{\mathfrak{B}}$ at the specialization $u = \mathbf{v}^{\mathbf{L}(s_0)}, v = \mathbf{v}^{\mathbf{L}(s_1)}$ from Theorem 5.2.2.

Theorem 6.3.6. *The ideal $\mathbb{J}$ admits a monomial basis $\{m_A \mid A \in \widetilde{\Xi}^{<}_{n}\}$. Moreover, its specialization at $u = \mathbf{v}^{\mathbf{L}(s_0)}, v = \mathbf{v}^{\mathbf{L}(s_1)}$ (denoted by $\mathbb{J}^{\mathbf{L}}$) has a canonical basis $\dot{\mathfrak{B}} \cap \mathbb{J}^{\mathbf{L}} = \{\{A\}^{\mathbf{L}} \mid A \in \widetilde{\Xi}^{<}_{n}\}$.*

Proof. The first statement follows from the above lemma directly. Since $m_A = [A] +$ lower terms, we know that $\mathbb{J}^{\mathbf{L}}$ is bar invariant. Thus $\mathbb{J}^{\mathbf{L}}$ does admit a canonical bases parameterized by $A \in \widetilde{\Xi}^{<}_{n}$, which should be $\dot{\mathfrak{B}} \cap \mathbb{J}^{\mathbf{L}} = \{\{A\}^{\mathbf{L}} \mid A \in \widetilde{\Xi}^{<}_{n}\}$ by the uniqueness of canonical basis. □

Proposition 6.3.7. *The following statements hold:*

(a) *The quotient algebra $\dot{\mathbb{K}}^{\jmath}_{n}/\mathbb{J}$ admits a monomial basis $\{m_A + \mathbb{J} \mid A \in \widetilde{\Xi}^{>}_{n}\}$.*
(b) *The specialization at $u = \mathbf{v}^{\mathbf{L}(s_0)}, v = \mathbf{v}^{\mathbf{L}(s_1)}$ of the quotient algebra $\dot{\mathbb{K}}^{\jmath}_{n}/\mathbb{J}$ admits a canonical basis $\{\{A\}^{\mathbf{L}} + \mathbb{J}^{\mathbf{L}} \mid A \in \widetilde{\Xi}^{>}_{n}\}$.*
(c) *The map $\sharp : \dot{\mathbb{K}}^{\jmath}_{n}/\mathbb{J} \to \dot{\mathbb{K}}^{>}_{n}$ sending $[A] + \mathbb{J} \mapsto [A]$ is an isomorphism of $\mathbb{A}$-algebras, which matches the corresponding monomial bases. It also matches the corresponding canonical bases at the specialization $u = \mathbf{v}^{\mathbf{L}(s_0)}, v = \mathbf{v}^{\mathbf{L}(s_1)}$.*

Proof. Parts (a) and (b) follow directly from Theorem 6.3.6. Below we prove the Part (c). Knowing that the map $\sharp$ is a linear isomorphism, we need to verify it is an algebraic homomorphism. Comparing the multiplication formulas for $\dot{\mathbb{K}}^{\jmath}_{n}$ in Proposition 5.1.4 with the ones for $\dot{\mathbb{K}}^{>}_{n}$ in Proposition 6.3.2, we can see that the structure constants with respect to the Chevalley generators for $\dot{\mathbb{K}}^{\jmath}_{n}/\mathbb{J}$ are as the same as those for $\dot{\mathbb{K}}^{>}_{n}$. Therefore $\sharp$ is an algebraic homomorphism.

Since $\sharp$ matches the Chevalley generators, it matches the corresponding monomial bases. We also obtain that $\sharp$ commutes with the bar involution. Notice that the partial orders $<_{\mathrm{alg}}$ are compatible, hence $\sharp$ also matches the corresponding canonical bases at the specialization $u = \mathbf{v}^{\mathbf{L}(s_0)}, v = \mathbf{v}^{\mathbf{L}(s_1)}$. □

We summarize Lemma 6.3.3 and Proposition 6.3.7 above as follows.

Theorem 6.3.8. *As an $\mathbb{A}$-algebra, $\dot{\mathbb{K}}^{\imath}_{n}$ is isomorphic to a subquotient of $\dot{\mathbb{K}}^{\jmath}_{n}$, with compatible standard, monomial basis. They have compatible canonical bases at the specialization $u = \mathbf{v}^{\mathbf{L}(s_0)}, v = \mathbf{v}^{\mathbf{L}(s_1)}$.*

Let $\dot{\mathbb{K}}^{\jmath,1}_{n}$ be the $\mathbb{A}$-submodule of $\dot{\mathbb{K}}^{\jmath}_{n}$ spanned by $[A]$ where $A \in \widetilde{\Xi}_n$ with $\mathrm{row}(A)_0 = \mathrm{col}(A)_0 = 1$. It is clear that $\dot{\mathbb{K}}^{\jmath,1}_{n}$ is a subalgebra of $\dot{\mathbb{K}}^{\jmath}_{n}$. Let $\mathbb{J}^1 = \mathbb{J} \cap \dot{\mathbb{K}}^{\jmath,1}_{n}$, i.e.

$$\mathbb{J}^1 = \mathrm{span}_{\mathbb{A}}\{[A] \mid A \in \widetilde{\Xi}_n, \mathrm{row}(A)_0 = \mathrm{col}(A)_0 = 1, a_{00} < 0\}.$$

Imitating the argument in [BKLW18, §A.3], we have the following.

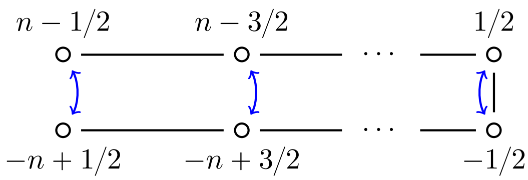
Proposition 6.3.9.

(a) *The monomial basis of $\dot{\mathbb{K}}^{\jmath}_{n}$ restricts to the monomial basis of $\dot{\mathbb{K}}^{\jmath,1}_{n}$; the monomial basis of $\dot{\mathbb{K}}^{\jmath,1}_{n}$ restricts to the monomial basis of $\mathbb{J}^1$. So does the canonical basis at the specialization $u = \mathbf{v}^{\mathbf{L}(s_0)}, v = \mathbf{v}^{\mathbf{L}(s_1)}$.*
(b) *The quotient $\mathbb{A}$-subalgebra $\dot{\mathbb{K}}^{\jmath,1}_{n}/\mathbb{J}^1$ admits a monomial basis $\{m_A + \mathbb{J}^1 \mid A \in \widetilde{\Xi}^{\imath}\}$. It also admits a canonical basis $\{\{A\}^{\mathbf{L}} + \mathbb{J}^{1,\mathbf{L}} \mid A \in \widetilde{\Xi}^{\imath}\}$ at the specialization $u = \mathbf{v}^{\mathbf{L}(s_0)}, v = \mathbf{v}^{\mathbf{L}(s_1)}$, where $\mathbb{J}^{1,\mathbf{L}} = \mathbb{J}^1|_{u=\mathbf{v}^{\mathbf{L}(s_0)},v=\mathbf{v}^{\mathbf{L}(s_1)}}$.*

(c) *There is an* $\mathbb{A}$*-algebra isomorphism* $\dot{\mathbb{K}}_n^{\jmath,1}/\mathbb{J}^1 \cong \dot{\mathbb{K}}_n^{\imath}$*, which matches the corresponding monomial bases. It also matches the corresponding canonical basis at the specialization* $u = \mathbf{v}^{\mathbf{L}(s_0)}, v = \mathbf{v}^{\mathbf{L}(s_1)}$.

7. Quantum symmetric pairs

7.1. **The quantum symmetric pair** $(\mathbb{U}, \mathbb{U}^\jmath)$**.** We start with the quantum symmetric pairs of type AIII/AIV without fixed points nor black nodes, associated with the following Satake diagram:



Note that we use half integers for the index set following the convention in [BW13]. Set

$$\mathbb{I}_{2n} = \left\{-n+\tfrac{1}{2}, -n+\tfrac{3}{2}, \dots, n-\tfrac{1}{2}\right\} \quad \text{and} \quad \mathbb{I}_n^\jmath = \left\{\tfrac{1}{2}, \tfrac{3}{2}, \dots, n-\tfrac{1}{2}\right\}. \tag{7.1.1}$$

Let $\mathbb{U} = \mathbb{U}(\mathfrak{gl}_{2n+1})$ be the algebra over $\mathbb{Q}(u,v)$ generated by E_i, F_i, $(i \in \mathbb{I}_{2n})$ and D_a, $(a \in [-n,n])$ subject to the following relations, for $i,j \in \mathbb{I}_{2n}, a,b \in [-n,n]$:

$$D_a D_a^{-1} = D_a^{-1} D_a = 1, \quad D_a D_b = D_b D_a, \tag{7.1.2}$$

$$D_a E_j D_a^{-1} = v^{\delta_{a,j-\frac{1}{2}} - \delta_{a,j+\frac{1}{2}}} E_j, \quad D_a F_j D_a^{-1} = v^{-\delta_{a,j-\frac{1}{2}} + \delta_{a,j+\frac{1}{2}}} F_j, \tag{7.1.3}$$

$$E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{v - v^{-1}}, \tag{7.1.4}$$

$$E_i^2 E_j + E_j E_i^2 = (v+v^{-1}) E_i E_j E_i, \quad F_i^2 F_j + F_j F_i^2 = (v+v^{-1}) F_i F_j F_i, \qquad (|i-j|=1), \tag{7.1.5}$$

$$E_i E_j = E_j E_i, \quad F_i F_j = F_j F_i, \qquad (|i-j|>1). \tag{7.1.6}$$

(Here and below $K_i := D_{i-\frac{1}{2}} D_{i+\frac{1}{2}}^{-1}$.)

Let $\mathbb{U}^\jmath = \mathbb{U}^\jmath(\mathfrak{gl}_{2n+1})$ be the $\mathbb{Q}(u,v)$-algebra with generators

$$e_i, f_i, \quad (i \in \mathbb{I}_n^\jmath), \quad d_a^{\pm 1} \quad (0 \leqslant a \leqslant n),$$

subject to the following relations, for $i \in \mathbb{I}_n^\jmath, a,b \in [0,n]$:

$$d_a d_a^{-1} = 1 = d_a^{-1} d_a, \quad d_a d_b = d_b d_a, \tag{7.1.7}$$

$$d_0 e_{\frac{1}{2}} d_0^{-1} = v^2 e_{\frac{1}{2}}, \quad d_0 f_{\frac{1}{2}} d_0^{-1} = v^{-2} f_{\frac{1}{2}}, \tag{7.1.8}$$

$$d_a e_j d_a^{-1} = v^{\delta_{a,j-\frac{1}{2}} - \delta_{a,j+\frac{1}{2}}} e_j, \quad d_a f_j d_a^{-1} = v^{-\delta_{a,j-\frac{1}{2}} + \delta_{a,j+\frac{1}{2}}} f_j, \qquad ((a,j) \neq (0,\tfrac{1}{2})), \tag{7.1.9}$$

$$e_i f_j - f_j e_i = \delta_{i,j} \frac{k_i - k_i^{-1}}{v - v^{-1}}, \qquad ((i,j) \neq (\tfrac{1}{2}, \tfrac{1}{2})), \tag{7.1.10}$$

$$e_i e_j = e_j e_i, \quad f_i f_j = f_j f_i, \qquad (|i-j|>1), \tag{7.1.11}$$

$$e_i^2 e_j + e_j e_i^2 = (v+v^{-1}) e_i e_j e_i, \quad f_i^2 f_j + f_j f_i^2 = (v+v^{-1}) f_i f_j f_i, \qquad (|i-j|=1), \tag{7.1.12}$$

$$e_{\frac{1}{2}}^2 f_{\frac{1}{2}} + f_{\frac{1}{2}} e_{\frac{1}{2}}^2 = (v+v^{-1}) \left(e_{\frac{1}{2}} f_{\frac{1}{2}} e_{\frac{1}{2}} - e_{\frac{1}{2}} (uvk_{\frac{1}{2}} + u^{-1}v^{-1}k_{\frac{1}{2}}^{-1}) \right), \tag{7.1.13}$$

$$f_{\frac{1}{2}}^2 e_{\frac{1}{2}} + e_{\frac{1}{2}} f_{\frac{1}{2}}^2 = (v+v^{-1}) \left(f_{\frac{1}{2}} e_{\frac{1}{2}} f_{\frac{1}{2}} - (uvk_{\frac{1}{2}} + u^{-1}v^{-1}k_{\frac{1}{2}}^{-1}) f_{\frac{1}{2}} \right). \tag{7.1.14}$$

(Here $k_i = d_{i-\frac{1}{2}} d_{i+\frac{1}{2}}^{-1}$, $(i \neq \frac{1}{2})$, and $k_{\frac{1}{2}} = v^{-1} d_0 d_1^{-1}$.)

It is known in [BWW18, §4.1] that there is a $\mathbb{Q}(u,v)$-algebra homomorphism $\mathbb{U}^\jmath \to \mathbb{U}$ given by, for $i \in \mathbb{I}_n^\jmath - \{\frac{1}{2}\}$, and for $1 \leqslant a \leqslant n$,

$$\begin{aligned} &d_0 \mapsto v^{-1}D_0^2, \quad e_i \mapsto E_i + F_{-i}K_i^{-1} \quad e_{\frac{1}{2}} \mapsto E_{\frac{1}{2}} + u^{-1}F_{-\frac{1}{2}}K_{\frac{1}{2}}^{-1}, \\ &d_a \mapsto D_aD_{-a}, \quad f_i \mapsto E_{-i} + K_{-i}^{-1}F_i, \quad f_{\frac{1}{2}} \mapsto E_{-\frac{1}{2}} + uK_{-\frac{1}{2}}^{-1}F_{\frac{1}{2}}. \end{aligned} \tag{7.1.15}$$

Remark 7.1.1. The (multiparameter) quantum symmetric pairs $(\mathbb{U}, \mathbb{U}^\jmath)$ in this paper are the $\mathfrak{gl}$-variant of the quantum symmetric pairs in [BWW18].

7.2. **Isomorphism $\dot{\mathbb{U}}^\jmath \simeq \dot{\mathbb{K}}_n^\jmath$.** Following [Lu93, §23.1], it is routine to define the modified quantum algebra $\dot{\mathbb{U}}^\jmath$ from $\mathbb{U}^\jmath$. Let $\widetilde{\Xi}_n^{\mathrm{diag}}$ be the set of all diagonal matrices in $\widetilde{\Xi}_n$. Denote by $\lambda = \mathrm{diag}(\lambda_{-n}, \lambda_{-n+1}, \dots, \lambda_n)$ a diagonal matrix in $\widetilde{\Xi}_n^{\mathrm{diag}}$. For $\lambda, \lambda' \in \widetilde{\Xi}_n^{\mathrm{diag}}$, we set

$$_\lambda\mathbb{U}^\jmath_{\lambda'} = \mathbb{U}^\jmath \Big/ \left(\sum_{a=0}^n (d_a - v^{\lambda_a})\mathbb{U}^\jmath + \sum_{a=0}^n \mathbb{U}^\jmath(d_a - v^{\lambda'_a}) \right). \tag{7.2.1}$$

The modified quantum algebra $\dot{\mathbb{U}}^\jmath$ is defined by

$$\dot{\mathbb{U}}^\jmath = \bigoplus_{\lambda,\lambda' \in \widetilde{\Xi}_n^{\mathrm{diag}}} {}_\lambda\dot{\mathbb{U}}^\jmath_{\lambda'}. \tag{7.2.2}$$

Let $1_\lambda = p_{\lambda,\lambda}(1)$, where $p_{\lambda,\lambda} : \mathbb{U}^\jmath \to {}_\lambda\dot{\mathbb{U}}^\jmath_\lambda$ is the canonical projection. Thus the unit of $\mathbb{U}^\jmath$ is replaced by a collection of orthogonal idempotents 1_λ in $\dot{\mathbb{U}}^\jmath$. It is clear that

$$\dot{\mathbb{U}}^\jmath = \sum_{\lambda \in \widetilde{\Xi}_n^{\mathrm{diag}}} \mathbb{U}^\jmath 1_\lambda = \sum_{\lambda \in \widetilde{\Xi}_n^{\mathrm{diag}}} 1_\lambda \mathbb{U}^\jmath.$$

For $\lambda \in \widetilde{\Xi}_n^{\mathrm{diag}}$ and $i \in \mathbb{I}_n^\jmath$, we use the following short-hand notations:

$$\lambda + \alpha_i = \lambda + E^\theta_{i-\frac{1}{2},i-\frac{1}{2}} - E^\theta_{i+\frac{1}{2},i+\frac{1}{2}}, \quad \lambda - \alpha_i = \lambda - E^\theta_{i-\frac{1}{2},i-\frac{1}{2}} + E^\theta_{i+\frac{1}{2},i+\frac{1}{2}}. \tag{7.2.3}$$

We also define, for $r \in \mathbb{N}$,

$$[\![r]\!] = \frac{v^r - v^{-r}}{v - v^{-1}}. \tag{7.2.4}$$

A multiparameter version of [BKLW18, Proposition 4.6] gives a presentation of $\dot{\mathbb{U}}^\jmath$ as a $\mathbb{Q}(u,v)$-algebra generated by the symbols, for $i \in \mathbb{I}_n^\jmath, \lambda \in \widetilde{\Xi}_n^{\mathrm{diag}}$,

$$1_\lambda, \quad e_i1_\lambda, \quad 1_\lambda e_i, \quad f_i1_\lambda, \quad 1_\lambda f_i,$$

subject to the following relations, for $i, j \in \mathbb{I}_n^\jmath, \lambda, \mu \in \widetilde{\Xi}_n^{\mathrm{diag}}$, $x, y \in \{1, e_i, e_j, f_i, f_j\}$:

$$x1_\lambda 1_\mu y = \delta_{\lambda,\mu} x1_\lambda y, \tag{7.2.5}$$

$$e_i1_\lambda = 1_{\lambda+\alpha_i}e_i, \quad f_i1_\lambda = 1_{\lambda-\alpha_i}f_i, \tag{7.2.6}$$

$$e_i1_\lambda f_j = f_j1_{\lambda+\alpha_i+\alpha_j}e_i, \qquad (i \neq j), \tag{7.2.7}$$

$$(e_if_i - f_ie_i)1_\lambda = \left[\!\!\left[\lambda_{i-\frac{1}{2}} - \lambda_{i+\frac{1}{2}}\right]\!\!\right] 1_\lambda, \qquad (i \neq \tfrac{1}{2}), \tag{7.2.8}$$

$$e_ie_j1_\lambda = e_je_i1_\lambda, \quad f_if_j1_\lambda = f_jf_i1_\lambda, \qquad (|i-j| > 1), \tag{7.2.9}$$

$$(e_i^2e_j + e_je_i^2)1_\lambda = [\![2]\!]\, e_ie_je_i1_\lambda, \quad (f_i^2f_j + f_jf_i^2)1_\lambda = [\![2]\!]\, f_if_jf_i1_\lambda, \qquad (|i-j| = 1), \tag{7.2.10}$$

$$([\![2]\!]\, e_{\frac{1}{2}}f_{\frac{1}{2}}e_{\frac{1}{2}} - e_{\frac{1}{2}}^2f_{\frac{1}{2}} - f_{\frac{1}{2}}e_{\frac{1}{2}}^2)1_\lambda = [\![2]\!]\,(uv^{\lambda_0-\lambda_1} + u^{-1}v^{-\lambda_0+\lambda_1})e_{\frac{1}{2}}1_\lambda, \tag{7.2.11}$$

$$([\![2]\!]\, f_{\frac{1}{2}}e_{\frac{1}{2}}f_{\frac{1}{2}} - f_{\frac{1}{2}}^2e_{\frac{1}{2}} - e_{\frac{1}{2}}f_{\frac{1}{2}}^2)1_\lambda = [\![2]\!]\,(uv^{\lambda_0-\lambda_1-3} + u^{-1}v^{-\lambda_0+\lambda_1+3})f_{\frac{1}{2}}1_\lambda. \tag{7.2.12}$$

Here and below we always write $x_11_{\lambda^1}x_21_{\lambda^2}\cdots x_k1_{\lambda^k} = x_1x_2\cdots x_k1_{\lambda^k}$, if the product is not zero; in this case such $\lambda^1, \lambda^2, \dots, \lambda^{k-1}$ are all uniquely determined by λ^k.

For $\forall i \in \mathbb{I}_n^\jmath, \lambda \in \widetilde{\Xi}_n^{\mathrm{diag}}$, write

$$\mathbf{e}_i1_\lambda = [\lambda - E^\theta_{i+\frac{1}{2},i+\frac{1}{2}} + E^\theta_{i-\frac{1}{2},i+\frac{1}{2}}] \in \dot{\mathbb{K}}_n^\jmath \quad \text{and} \quad \mathbf{f}_i1_\lambda = [\lambda - E^\theta_{i-\frac{1}{2},i-\frac{1}{2}} + E^\theta_{i+\frac{1}{2},i-\frac{1}{2}}] \in \dot{\mathbb{K}}_n^\jmath.$$

Set ${}_{\mathbb{Q}}\dot{\mathbb{K}}^{\jmath}_n = \mathbb{Q}(u,v) \otimes_{\mathbb{A}} \dot{\mathbb{K}}^{\jmath}_n$.

Theorem 7.2.1. *There is an isomorphism of $\mathbb{Q}(u,v)$-algebras $\aleph : \dot{\mathbb{U}}^{\jmath} \to {}_{\mathbb{Q}}\dot{\mathbb{K}}^{\jmath}_n$ such that, for $\forall i \in \mathbb{I}^{\jmath}_n, \lambda \in \widetilde{\Xi}^{\mathrm{diag}}_n$,*

$$e_i 1_\lambda \mapsto \mathbf{e}_i 1_\lambda, \quad f_i 1_\lambda \mapsto \mathbf{f}_i 1_\lambda, \quad 1_\lambda \mapsto [\lambda].$$

Proof. A direct computation using Theorem 4.2.3 shows that relations (7.2.5)-(7.2.12) also hold if we replace e_i, f_i's by $\mathbf{e}_i, \mathbf{f}_i$'s. Here we only present details for (7.2.11) regarding $\mathbf{e}_{\frac12} 1_\lambda$ and $\mathbf{f}_{\frac12} 1_\lambda$ as follows:

$$\begin{aligned}
\mathbf{e}^2_{\frac12}\mathbf{f}_{\frac12} 1_\lambda &= u^{-2}v^{-2\lambda_0-\lambda_1+4}[2](e_{\lambda-2E^\theta_{1,1}+2E^\theta_{0,1}-E^\theta_{0,0}+E^\theta_{1,0}} + [\lambda_0-1]_{\mathfrak{c}} e_{\lambda-E^\theta_{1,1}+E^\theta_{0,1}}),\\
\mathbf{f}_{\frac12}\mathbf{e}^2_{\frac12} 1_\lambda &= u^{-2}v^{-2\lambda_0-\lambda_1+2}[2](e_{\lambda-2E^\theta_{1,1}+E^\theta_{0,1}+E^\theta_{1,-1}} + e_{\lambda-2E^\theta_{1,1}+2E^\theta_{0,1}-E^\theta_{0,0}+E^\theta_{1,0}} + [\lambda_1-1]e_{\lambda-E^\theta_{1,1}+E^\theta_{0,1}}),\\
\mathbf{e}_{\frac12}\mathbf{f}_{\frac12}\mathbf{e}_{\frac12} 1_\lambda &= u^{-2}v^{-2\lambda_0-\lambda_1+3}(e_{\lambda-2E^\theta_{1,1}+E^\theta_{0,1}+E^\theta_{1,-1}} + [2]e_{\lambda-2E^\theta_{1,1}+2E^\theta_{0,1}-E^\theta_{0,0}+E^\theta_{1,0}}\\
&\quad + (u^2v^{2\lambda_0-2} + [\lambda_0-1]_{\mathfrak{c}}v^2 + [\lambda_1])e_{\lambda-E^\theta_{1,1}+E^\theta_{0,1}}),
\end{aligned}$$

Combining the identities above, we get $(\llbracket 2\rrbracket\, \mathbf{e}_{\frac12}\mathbf{f}_{\frac12}\mathbf{e}_{\frac12} - \mathbf{e}^2_{\frac12}\mathbf{f}_{\frac12} - \mathbf{f}_{\frac12}\mathbf{e}^2_{\frac12})1_\lambda = \llbracket 2\rrbracket\,(uv^{\lambda_0-\lambda_1} + u^{-1}v^{-\lambda_0+\lambda_1})\mathbf{e}_{\frac12}1_\lambda$. That is, $\aleph$ is indeed an algebra homomorphism.

We also know that $\aleph$ is a linear isomorphism. The argument is almost as the same as that for the case of specialization at $u = v$, which can be found in the proof of [BKLW18, Theorem 4.7]. Therefore $\aleph$ is an isomorphism of $\mathbb{Q}(u,v)$-algebras. ☐

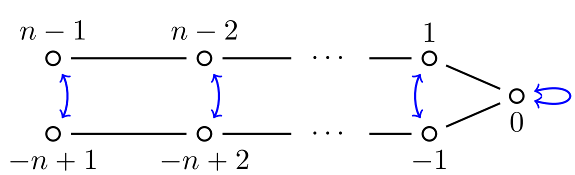
It has been shown in [BWW18, Lemma 4.1] that there exists a unique $\mathbb{Q}$-linear bar involution on $\mathbb{U}^{\jmath}$ such that $\overline{u} = u^{-1}, \overline{v} = v^{-1}, \overline{d_a} = d_a^{-1}$ $(0 \leqslant a \leqslant n), \overline{e_i} = e_i, \overline{f_i} = f_i$ $(i \in \mathbb{I}^{\jmath}_n)$. This bar involution on $\mathbb{U}^{\jmath}$ induces a compatible bar involution on $\dot{\mathbb{U}}^{\jmath}$, denoted also by $^-$, fixing all the generators 1_λ, e_i1_λ, f_i1_λ.

Note that $\mathbf{e}_i1_\lambda$, $\mathbf{f}_i1_\lambda$, $[\lambda]$ are bar invariant elements in $\dot{\mathbb{K}}^{\jmath}_n$, which implies that the isomorphism $\aleph$ intertwines the bar involution on $\dot{\mathbb{U}}^{\jmath}$ and on ${}_{\mathbb{Q}}\dot{\mathbb{K}}^{\jmath}_n$.

Set ${}_{\mathbb{A}}\dot{\mathbb{U}}^{\jmath} = \aleph^{-1}(\dot{\mathbb{K}}^{\jmath}_n)$. It is an $\mathbb{A}$-subalgebra of $\dot{\mathbb{U}}^{\jmath}$. We have the following result.

Proposition 7.2.2. *The integral form ${}_{\mathbb{A}}\dot{\mathbb{U}}^{\jmath}$ is a free $\mathbb{A}$-submodule of $\dot{\mathbb{U}}^{\jmath}$. It is stable under the bar involution.*

7.3. **The quantum symmetric pair** $(\mathbb{U}, \mathbb{U}^{\imath})$**.** Below we formulate the counterparts of Sections 7.1–7.2. The proofs are very similar and will often be omitted. We now work on quantum symmetric pairs of type AIII with fixed points associated with the Satake diagram below:



Let $\mathbb{U} = \mathbb{U}(\mathfrak{gl}_{2n})$ be the algebra over $\mathbb{Q}(u,v)$ generated by E_i, F_i, $(i \in [-n+1, n-1])$ and D_a, $(a \in [-n+1, n])$ subject to the following relations, for $i, j \in [-n+1, n-1], a, b \in [-n+1, n]$:

$$D_aD_a^{-1} = D_a^{-1}D_a = 1, \quad D_aD_b = D_bD_a, \tag{7.3.1}$$

$$D_aE_jD_a^{-1} = v^{\delta_{a,j}-\delta_{a,j+1}}E_j, \quad D_aF_jD_a^{-1} = v^{-\delta_{a,j}+\delta_{a,j+1}}F_j, \tag{7.3.2}$$

$$E_iF_j - F_jE_i = \delta_{i,j}\frac{K_i - K_i^{-1}}{v - v^{-1}}, \tag{7.3.3}$$

$$E_i^2E_j + E_jE_i^2 = (v+v^{-1})E_iE_jE_i, \quad F_i^2F_j + F_jF_i^2 = (v+v^{-1})F_iF_jF_i, \qquad (|i-j| = 1), \tag{7.3.4}$$

$$E_iE_j = E_jE_i, \quad F_iF_j = F_jF_i, \qquad (|i-j| > 1). \tag{7.3.5}$$

(Here and below $K_i := D_iD_{i+1}^{-1}$.)

Let $\mathbb{U}^\imath = \mathbb{U}^\imath(\mathfrak{gl}_{2n})$ be the $\mathbb{Q}(u,v)$-algebra with generators

$$t, \quad e_i, \quad f_i \quad (i \in [1,n-1]), \quad d_a^{\pm 1} \quad (a \in [1,n]),$$

subject to the following relations, for $i,j \in [1,n-1], a,b \in [1,n]$:

$$d_a d_a^{-1} = 1 = d_a^{-1} d_a, \quad d_a d_b = d_b d_a, \tag{7.3.6}$$

$$d_a t d_a^{-1} = t, \quad d_a e_j d_a^{-1} = v^{\delta_{a,j}-\delta_{a,j+1}} e_j, \quad d_a f_j d_a^{-1} = v^{-\delta_{a,j}+\delta_{a,j+1}} f_j, \tag{7.3.7}$$

$$e_i f_j - f_j e_i = \delta_{i,j} \frac{k_i - k_i^{-1}}{v - v^{-1}}, \tag{7.3.8}$$

$$e_i e_j = e_j e_i, \quad f_i f_j = f_j f_i, \qquad (|i-j| > 1), \tag{7.3.9}$$

$$e_i^2 e_j + e_j e_i^2 = (v+v^{-1}) e_i e_j e_i, \quad f_i^2 f_j + f_j f_i^2 = (v+v^{-1}) f_i f_j f_i, \qquad (|i-j| = 1), \tag{7.3.10}$$

$$e_i t = t e_i, \quad f_i t = t f_i, \qquad (i \neq 1), \tag{7.3.11}$$

$$t^2 e_1 + e_1 t^2 = (v+v^{-1}) t e_1 t + e_1, \quad e_1^2 t + t e_1^2 = (v+v^{-1}) e_1 t e_1, \tag{7.3.12}$$

$$t^2 f_1 + f_1 t^2 = (v+v^{-1}) t f_1 t + f_1, \quad f_1^2 t + t f_1^2 = (v+v^{-1}) f_1 t f_1. \tag{7.3.13}$$

(Here $k_i = d_i d_{i+1}^{-1}$.)

It has been known in [BWW18, §2.1] that there is a $\mathbb{Q}(u,v)$-algebra homomorphism $\mathbb{U}^\imath \to \mathbb{U}$ given by, for $i \in [1,n-1]$, and for $a \in [1,n]$,

$$\begin{aligned} d_a &= D_a D_{-a}, & t &= E_0 + vF_0 K_0^{-1} + \tfrac{u-u^{-1}}{v-v^{-1}} K_0^{-1}, \\ e_i &= E_i + F_{-i} K_i^{-1}, & f_i &= E_{-i} + K_{-i}^{-1} F_i. \end{aligned} \tag{7.3.14}$$

Remark 7.3.1. It was observed in [Le99, BWW18] that the parameter $\omega \in \mathbb{Q}(u,v)$ in the embedding $t = E_0 + vF_0 K_0^{-1} + \omega K_0^{-1}$, is irrelevant to the presentation of the algebra $\mathbb{U}^\imath$.

Let ${}^\imath\widetilde{\Xi}_n^{\mathrm{diag}}$ be the set of all diagonal matrices in $\widetilde{\Xi}^\imath$. Denote by $\lambda = \mathrm{diag}(\lambda_{-n}, \ldots, \lambda_{-1}, 1, \lambda_1, \ldots, \lambda_n)$ a diagonal matrix in ${}^\imath\widetilde{\Xi}_n^{\mathrm{diag}}$. We define the modified algebra $\dot{\mathbb{U}}^\imath$ similarly to the construction of $\dot{\mathbb{U}}^\jmath$ as follows:

$$\dot{\mathbb{U}}^\imath = \bigoplus_{\lambda,\lambda' \in {}^\imath\widetilde{\Xi}_n^{\mathrm{diag}}} {}_\lambda\dot{\mathbb{U}}^\imath_{\lambda'} = \sum_{\lambda \in {}^\imath\widetilde{\Xi}_n^{\mathrm{diag}}} \mathbb{U}^\imath 1_\lambda = \sum_{\lambda \in {}^\imath\widetilde{\Xi}_n^{\mathrm{diag}}} 1_\lambda \mathbb{U}^\imath,$$

where ${}_\lambda\mathbb{U}^\imath_{\lambda'} = \mathbb{U}^\imath / \left(\sum_{a=1}^n (d_a - v^{\lambda_a}) \mathbb{U}^\imath + \sum_{a=1}^n \mathbb{U}^\imath (d_a - v^{\lambda'_a}) \right)$ and $1_\lambda \in {}_\lambda\mathbb{U}^\imath_\lambda$ is the canonical projection image of the unit of $\mathbb{U}^\imath$.

For $\lambda \in {}^\imath\widetilde{\Xi}_n^{\mathrm{diag}}$ and $i \in [1,n-1]$, we use the following short-hand notations:

$$\lambda + \alpha_i = \lambda + E^\theta_{ii} - E^\theta_{i+1,i+1}, \quad \lambda - \alpha_i = \lambda - E^\theta_{ii} + E^\theta_{i+1,i+1}. \tag{7.3.15}$$

We thus obtain a presentation of $\dot{\mathbb{U}}^\imath$ as a $\mathbb{Q}(u,v)$-algebra generated by the symbols, for $i \in [1,n-1], \lambda \in {}^\imath\widetilde{\Xi}_n^{\mathrm{diag}}$,

$$1_\lambda, \quad t1_\lambda, \quad 1_\lambda t, \quad e_i 1_\lambda, \quad 1_\lambda e_i, \quad f_i 1_\lambda, \quad 1_\lambda f_i,$$

subject to the following relations, for $i,j \in [1,n-1], \lambda,\mu \in {}^\imath\widetilde{\Xi}_n^{\mathrm{diag}}$, $x,y \in \{1, e_i, e_j, f_i, f_j, t\}$:

$$x 1_\lambda 1_\mu y = \delta_{\lambda,\mu} x 1_\lambda y, \tag{7.3.16}$$

$$e_i 1_\lambda = 1_{\lambda+\alpha_i} e_i, \quad f_i 1_\lambda = 1_{\lambda-\alpha_i} f_i, \quad t 1_\lambda = 1_\lambda t, \tag{7.3.17}$$

$$e_i 1_\lambda f_j = f_j 1_{\lambda+\alpha_i+\alpha_j} e_i, \qquad (i \neq j), \tag{7.3.18}$$

$$(e_i f_i - f_i e_i) 1_\lambda = [\![\lambda_i - \lambda_{i+1}]\!] 1_\lambda, \tag{7.3.19}$$

$$e_i e_j 1_\lambda = e_j e_i 1_\lambda, \quad f_i f_j 1_\lambda = f_j f_i 1_\lambda, \qquad (|i-j| > 1), \tag{7.3.20}$$

$$(e_i^2 e_j + e_j e_i^2) 1_\lambda = [\![2]\!] e_i e_j e_i 1_\lambda, \quad (f_i^2 f_j + f_j f_i^2) 1_\lambda = [\![2]\!] f_i f_j f_i 1_\lambda, \qquad (|i-j| = 1), \tag{7.3.21}$$

$$f_i t 1_\lambda = t f_i 1_\lambda, \quad e_i t 1_\lambda = t e_i 1_\lambda \qquad (i \neq 1), \tag{7.3.22}$$

$$(t^2 f_1 + f_1 t^2) 1_\lambda = ([\![2]\!] t f_1 t + f_1) 1_\lambda, \quad (f_1^2 t + t f_1^2) 1_\lambda = [\![2]\!] f_1 t f_1 1_\lambda, \tag{7.3.23}$$

$$(t^2 e_1 + e_1 t^2) 1_\lambda = ([\![2]\!] t e_1 t + e_1) 1_\lambda, \quad (e_1^2 t + t e_1^2) 1_\lambda = [\![2]\!] e_1 t e_1 1_\lambda. \tag{7.3.24}$$

For $i \in [1, n-1], \lambda \in {}^{\imath}\widetilde{\Xi}_n^{\mathrm{diag}}$, write

$$\mathbf{e}_i 1_\lambda = [\lambda - E^\theta_{i+1,i+1} + E^\theta_{i,i+1}], \qquad \mathbf{f}_i 1_\lambda = [\lambda - E^\theta_{i,i} + E^\theta_{i+1,i}],$$
$$\mathbf{t} 1_\lambda = [\lambda - E^\theta_{1,1} + E^\theta_{-1,1}] + v^{-\lambda_1} \frac{u-u^{-1}}{v-v^{-1}}[\lambda]. \tag{7.3.25}$$

Set ${}_{\mathbb{Q}}\dot{\mathbb{K}}_n^{\jmath} = \mathbb{Q}(u,v) \otimes_{\mathbb{A}} \dot{\mathbb{K}}_n^{\imath}$.

Theorem 7.3.2. *There is an isomorphism of $\mathbb{Q}(u,v)$-algebras $\aleph : \dot{\mathbb{U}}^\imath \to {}_{\mathbb{Q}}\dot{\mathbb{K}}_n^\imath$ such that, for all $i \in [1, n-1], \lambda \in \widetilde{\Xi}_n^{\mathrm{diag}}$,*

$$t1_\lambda \mapsto \mathbf{t}1_\lambda, \quad e_i 1_\lambda \mapsto \mathbf{e}_i 1_\lambda, \quad f_i 1_\lambda \mapsto \mathbf{f}_i 1_\lambda, \quad 1_\lambda \mapsto [\lambda].$$

Proof. By a direct computation using Theorem 4.2.3 one can show that the relations (7.3.16)-(7.3.24) for t, e_i, f_i's also hold for $\mathbf{t}, \mathbf{e}_i, \mathbf{f}_i$'s. Hence $\aleph$ is a homomorphism of $\mathbb{Q}(u,v)$-algebras. Here we only present the details for the first relation in (7.3.24) as follows. Note that as an element in $\dot{\mathbb{K}}_n^{>}$,

$$\mathbf{t}1_\lambda = \mathbf{f}_0 \mathbf{e}_0 1_\lambda - \frac{u^{-1}v^{\lambda_1} - uv^{-\lambda_1}}{v - v^{-1}} 1_\lambda, \tag{7.3.26}$$

where $\mathbf{e}_0 1_\lambda = [\lambda - E^\theta_{1,1} + E^\theta_{0,1}]$ and $\mathbf{f}_0 1_{\lambda + E^\theta_{0,0} - E^\theta_{1,1}} = [\lambda - E^\theta_{1,1} + E^\theta_{1,0}] \in \dot{\mathbb{K}}_n^{>}$. Moreover, we have

$$\mathbf{t}^2 1_\lambda = [\![2]\!]\, u^{-1} v^{-2\lambda_1+2} \frac{u-u^{-1}}{v-v^{-1}} e_{\lambda - E^\theta_{1,1} + E^\theta_{-1,1}} + \left(v^{-2\lambda_1+2}[\lambda_1] + v^{-2\lambda_1} \frac{(u-u^{-1})^2}{(v-v^{-1})^2} \right) e_\lambda$$
$$+ u^{-2} v^{-2\lambda_1+2} [2] e_{\lambda - 2E^\theta_{1,1} + 2E^\theta_{-1,1}}.$$

Hence

$$\mathbf{e}_1 \mathbf{t}^2 1_\lambda = [\![2]\!]\, u^{-1} v^{-3\lambda_1+2} \frac{u-u^{-1}}{v-v^{-1}} e_{\lambda - E^\theta_{1,1} + E^\theta_{-1,1} - E^\theta_{2,2} + E^\theta_{1,2}}$$
$$+ \left(v^{-3\lambda_1+2}[\lambda_1] + v^{-3\lambda_1} \frac{(u-u^{-1})^2}{(v-v^{-1})^2} \right) e_{\lambda - E^\theta_{2,2} + E^\theta_{1,2}} + u^{-2} v^{-3\lambda_1+2} [2] e_{\lambda - 2E^\theta_{1,1} + 2E^\theta_{-1,1} - E^\theta_{2,2} + E^\theta_{1,2}},$$

and

$$\mathbf{t}^2 \mathbf{e}_1 1_\lambda = [\![2]\!]\, u^{-1} v^{-3\lambda_1} \frac{u-u^{-1}}{v-v^{-1}} (e_{\lambda - E^\theta_{1,1} + E^\theta_{-1,1} - E^\theta_{2,2} + E^\theta_{1,2}} + e_{\lambda - E^\theta_{2,2} + E^\theta_{-1,2}})$$
$$+ \left(v^{-3\lambda_1}[\lambda_1 + 1] + v^{-3\lambda_1-2} \frac{(u-u^{-1})^2}{(v-v^{-1})^2} \right) e_{\lambda - E^\theta_{2,2} + E^\theta_{1,2}}$$
$$+ u^{-2} v^{-3\lambda_1} [2] (e_{\lambda - 2E^\theta_{1,1} + 2E^\theta_{-1,1} - E^\theta_{2,2} + E^\theta_{1,2}} + e_{\lambda - E^\theta_{1,1} + E^\theta_{-1,1} - E^\theta_{2,2} + E^\theta_{-1,2}}).$$

Finally, using (7.3.26) again, we compute that

$$\mathbf{t}\mathbf{e}_1\mathbf{t} 1_\lambda = [\![2]\!]\, u^{-1} v^{-3\lambda_1+1} \frac{u-u^{-1}}{v-v^{-1}} e_{\lambda - E^\theta_{1,1} + E^\theta_{-1,1} - E^\theta_{2,2} + E^\theta_{1,2}} + u^{-1} v^{-3\lambda_1} \frac{u-u^{-1}}{v-v^{-1}} e_{\lambda - E^\theta_{2,2} + E^\theta_{-1,2}}$$
$$+ \left(v^{-\lambda_1} \frac{(1 - v^{-2\lambda_1})(v + v^{-1})}{v - v^{-1}} + v^{-3\lambda_1-1} \frac{(u-u^{-1})^2}{(v-v^{-1})^2} \right) e_{\lambda - E^\theta_{2,2} + E^\theta_{1,2}}$$
$$+ u^{-2} v^{-3\lambda_1+1} e_{\lambda - E^\theta_{1,1} + E^\theta_{-1,1} - E^\theta_{2,2} + E^\theta_{-1,2}} + u^{-2} v^{-3\lambda_1+1} [2] e_{\lambda - 2E^\theta_{1,1} + 2E^\theta_{-1,1} - E^\theta_{2,2} + E^\theta_{1,2}}.$$

Combining the identities above, we see that indeed $(\mathbf{t}^2\mathbf{e}_1 + \mathbf{e}_1\mathbf{t}^2)1_\lambda = ([\![2]\!]\, \mathbf{t}\mathbf{e}_1 t + \mathbf{e}_1)1_\lambda$.

An argument similar to the proof of [BKLW18, Theorem A.15] also shows $\aleph$ is a linear isomorphism. Therefore $\aleph$ is an isomorphism of $\mathbb{Q}(u,v)$-algebras. □

Thanks to [BWW18, Lemma 2.1], we know there exists a unique $\mathbb{Q}$-algebra bar involution on $\dot{\mathbb{U}}^\imath$ such that $\overline{u} = u^{-1}, \overline{v} = v^{-1}, \overline{d_a} = d_a^{-1}$ $(a \in [1,n]), \overline{e_i} = e_i, \overline{f_i} = f_i$ $(i \in [1, n-1]), \overline{t} = t$. This bar involution on $\mathbb{U}^\imath$ induces a compatible bar involution on $\dot{\mathbb{U}}^\imath$, denoted also by $^-$, fixing all the generators 1_λ, $e_i 1_\lambda$, $f_i 1_\lambda, t$.

Set ${}_{\mathbb{A}}\dot{\mathbb{U}}^\imath = \aleph^{-1}(\dot{\mathbb{K}}_n^\imath)$. It is an $\mathbb{A}$-subalgebra of $\dot{\mathbb{U}}^\imath$.

Proposition 7.3.3. *The integral form* ${}_{\mathbb{A}}\dot{\mathbb{U}}^{\imath}$ *is a free* $\mathbb{A}$*-submodule of* $\dot{\mathbb{U}}^{\imath}$*. It is stable under the bar involution.*

Remark 7.3.4. Theorem 5.2.2 (resp. Theorem 6.3.6) provides a canonical basis for the modified form of $\mathbb{U}^{\jmath}$ (resp. $\mathbb{U}^{\imath}$) at the specialization $u = \mathbf{v}^{\mathbf{L}(s_0)}, v = \mathbf{v}^{\mathbf{L}(s_1)}$. A general theory of canonical bases for quantum symmetric pairs with parameters of arbitrary finite type was developed in [BW16].

Appendix A. An algebraic approach to Schur algebras of type D

As we mentioned in Section 1.2, at the specialization $u = 1$ the multiparameter Schur duality yields a weak Schur duality of type D that is used in [Bao17] to formulate the Kazhdan-Lusztig theory for classical and super type D. These algebras $\mathbb{S}^{\bullet}_{n,d}|_{u=1}$ ($\bullet = \imath$ or $\jmath$), however, are not the Schur algebras introduced in [FL15]. While bases of Schur algebras of finite type A/B/C and affine type A/C can be parametrized by a matrix set (cf. $\Xi_{n,d}$ in 2.2.2), for finite type D Fan and Li showed that a matrix set is not enough – a notion of signed matrices that indexes a larger algebra is needed. From a geometric point of view, this reflects the fact that there are two connected components for the maximal isotropic Grassmannian associated to SO$(2d)$. In this appendix, we provide an algebraic approach to Fan-Li's construction parallel to our multiparameter results. The arguments are very similar to the multiparameter counterpart, so we will omit the easy proofs in this appendix.

A.1. **Weyl groups of type D.** Fix $d \in \mathbb{N}$, and we set set

$$J_d = \{-d, \ldots, -1, 1, \ldots, d\}. \tag{A.1.1}$$

Let $W_{\mathbf{D}}$ be the Weyl group of type $\mathbf{D}_d$. It is known (c.f. [BB05]) that $W_{\mathbf{D}}$ can be identified as a permutation subgroup of J_d which consists of those permutations g satisfying that

$$^{\#}\{i \in J_d \mid i > 0, g(i) < 0\} \in 2\mathbb{N}, \quad g(-i) = -g(i) \quad (1 \leqslant i \leqslant d).$$

Let $S_{\mathbf{D}} = \{\varsigma_0, \varsigma_1, \ldots, \varsigma_{d-1}\}$, where $\varsigma \in W_{\mathbf{D}}$ are given by the following products of transpositions:

$$\varsigma_0 = (1, -2)(2, -1) \quad \text{and} \quad \varsigma_i = (i, i+1)(-i-1, -i) \quad \text{for } i = 1, \ldots, d.$$

It is also known (see [BB05, (8.18),(8.19)]) that $(W_{\mathbf{D}}, S_{\mathbf{D}})$ is a Coxeter group associated with the length function as below:

Lemma A.1.1. *The length of $g \in W_{\mathbf{D}}$ is given by*

$$\ell(g) = {}^{\#}\{(i, j) \in J_d^2 \mid |i| < j, g(i) > g(j)\}.$$

A.2. **Signed compositions.** Fix $n \in \mathbb{N}$. Recall that (2.1.8) first $\Lambda_{n,d}$ is the set of weak compositions of d into $n + 1$ parts. Set

$$\Lambda^0 = \{\lambda \in \Lambda_{n,d} \mid \lambda_0 > 0\} \times \{0\}, \quad \Lambda^\epsilon = \{\lambda \in \Lambda_{n,d} \mid \lambda_0 = 0\} \times \{\epsilon\}, \quad (\epsilon = + \text{ or } -). \tag{A.2.1}$$

In below we abbreviate $(\lambda, \alpha) \in \Lambda^\alpha$ by λ^α where $\alpha \in \{0, +, -\}$. We further set

$$\Lambda_{\mathbf{D}} = \Lambda^0 \sqcup \Lambda^+ \sqcup \Lambda^-. \tag{A.2.2}$$

Elements in $\Lambda_{\mathbf{D}}$ will be called *signed compositions*. Recall that $\lambda_{0,i} = \lambda_0 + \lambda_1 + \cdots + \lambda_i$ for $i \in [0, n], \lambda \in \Lambda_{n,d}$. We define positive integer intervals associated to λ^α by

$$R_i^{\lambda^0} = \begin{cases} [-\lambda_0, \lambda_0] \backslash \{0\} & \text{if } i = 0; \\ [\lambda_{0,i-1} + 1, \lambda_{0,i}] & \text{if } i \in [1, n], \end{cases} \tag{A.2.3}$$

$$R_i^{\lambda^+} = \begin{cases} \varnothing & \text{if } i = 0; \\ [1, \lambda_1] & \text{if } i = 1; \\ [\lambda_1 + 1, \lambda_{0,i}] & \text{if } i \in [2, n], \end{cases} \qquad R_i^{\lambda^-} = \begin{cases} \varnothing & \text{if } i = 0; \\ \{-1, 2, \ldots, \lambda_1\} & \text{if } i = 1; \\ [\lambda_1 + 1, \lambda_{0,i}] & \text{if } i \in [2, n]. \end{cases} \tag{A.2.4}$$

For $-n \leqslant i \leqslant 1$, we set $R_i^{\lambda^\alpha} = \{-x | x \in R_{-i}^{\lambda^\alpha}\}$. We remark that the sets $\{R_i^{\lambda^\alpha}\}_{i \in [-n,n]}$ partition the set J_d.

For any $\lambda^\alpha \in \Lambda_{\mathbf{D}}$, let W_{λ^α} be the parabolic subgroup of $W_{\mathbf{D}}$ generated by

$$\begin{cases} S_{\mathbf{D}} \backslash \{\varsigma_{\lambda_0}, \varsigma_{\lambda_{0,1}}, \ldots, \varsigma_{\lambda_{0,n-1}}\} & \text{if } \alpha = 0, \\ S_{\mathbf{D}} \backslash \{\varsigma_0, \varsigma_{\lambda_{0,1}}, \ldots, \varsigma_{\lambda_{0,n-1}}\} & \text{if } \alpha = +, \\ S_{\mathbf{D}} \backslash \{\varsigma_1, \varsigma_{\lambda_{0,1}}, \ldots, \varsigma_{\lambda_{0,n-1}}\} & \text{if } \alpha = -. \end{cases} \tag{A.2.5}$$

Denote by Stab(X) the stabilizer of J_d in $W_{\mathbf{D}}$, for any $X \subset J_d$.

Lemma A.2.1. *For any $\lambda^\alpha \in \Lambda_{\mathbf{D}}$, we have $W_{\lambda^\alpha} = \bigcap_{i=0}^n \text{Stab}(R_i^{\lambda^\alpha})$.*

Denote the set of minimal length right coset representatives of W_{λ^α} in $W_{\mathbf{D}}$ by

$$\mathscr{D}_{\lambda^\alpha} = \big\{g \in W_{\mathbf{D}} \mid \ell(wg) = \ell(w) + \ell(g), \forall w \in W_{\lambda^\alpha}\big\}. \tag{A.2.6}$$

Hence, the set $\mathscr{D}_{\lambda^\alpha\mu^\beta} = \mathscr{D}_{\lambda^\alpha} \cap \mathscr{D}_{\mu^\beta}^{-1}$ is the set of minimal length double coset representatives for $W_{\lambda^\alpha}\backslash W_{\mathbf{D}}/W_{\mu^\beta}$.

Lemma A.2.2. *Let $g \in W_{\mathbf{D}}$ and $\lambda^\alpha \in \Lambda_{\mathbf{D}}$.*

(a) *If $\alpha = \pm$, then $g \in \mathscr{D}_{\lambda^\alpha}$ if and only if g^{-1} is order-preserving on $R_i^{\lambda^\alpha}$, for all $i \in [1,n]$;*
(b) *If $\alpha = 0$, then $g \in \mathscr{D}_{\lambda^\alpha}$ if and only if g^{-1} is order-preserving on $R_i^{\lambda^\alpha}$ for all $i \in [1,n]$ and*

$$g^{-1}(-2) < g^{-1}(1) < g^{-1}(2) < \cdots < g^{-1}(\lambda_0).$$

By a similar argument for [DDPW08, Proposition 4.16, Lemma 4.17 and Theorem 4.18], we have the following facts.

Proposition A.2.3. *Let $\lambda^\alpha, \mu^\beta \in \Lambda_{\mathbf{D}}$ and $g \in \mathscr{D}_{\lambda^\alpha\mu^\beta}$.*

(a) *There is a weak composition $\delta = \delta(\lambda^\alpha, g, \mu^\beta) \in \Lambda_{n',d}$ for some n' such that $W_{\delta^\beta} = g^{-1}W_{\lambda^\alpha}g \cap W_{\mu^\beta}$.*
(b) *The map $W_{\lambda^\alpha} \times (\mathscr{D}_\delta \cap W_{\mu^\beta}) \to W_{\lambda^\alpha}gW_{\mu^\beta}$ sending (x,y) to xgy is a bijection; moreover, we have $\ell(xgy) = \ell(x) + \ell(g) + \ell(y)$.*
(c) *The map $(\mathscr{D}_\delta \cap W_{\mu^\beta}) \times W_\delta \to W_{\mu^\beta}$ sending (x,y) to xy is a bijection; moreover, we have $\ell(x) + \ell(y) = \ell(xy)$.*

A.3. **Schur algebras.** The Hecke algebra $\mathbf{H} = \mathbf{H}(W_{\mathbf{D}})$ over $\mathbf{A} = \mathbb{Z}[v, v^{-1}]$ is an $\mathbf{A}$-algebra with basis $\{T_g \mid g \in W_{\mathbf{D}}\}$ satisfying that

$$\begin{aligned} &T_wT_{w'} = T_{ww'} && \text{if } \ell(ww') = \ell(w) + \ell(w'),\\ &(T_s+1)(T_s - v^2) = 0, && \text{for } s \in S_{\mathbf{D}}. \end{aligned}$$

For any finite subset $X \subset W_{\mathbf{D}}$ and for $\lambda^\alpha \in \Lambda_{\mathbf{D}}$, set

$$T_X = \sum_{w\in X} T_w \quad \text{and} \quad x_{\lambda^\alpha} = T_{W_{\lambda^\alpha}}. \tag{A.3.1}$$

For $\lambda^\alpha, \mu^\beta \in \Lambda_{\mathbf{D}}$ and $g \in \mathscr{D}_{\lambda^\alpha\mu^\beta}$, we consider a right $\mathbf{H}$-linear map $\phi^g_{\lambda^\alpha\mu^\beta} \in \mathrm{Hom}_{\mathbf{H}}(x_{\mu^\beta}\mathbf{H}, \mathbf{H})$, sending x_{μ^β} to $T_{W_{\lambda^\alpha}gW_{\mu^\beta}}$. Thanks to Proposition A.2.3 (b), we have $T_{W_\lambda gW_\mu} = x_\lambda T_g T_{\mathscr{D}_\delta \cap W_\mu}$ for some $\delta \in \Lambda_{n',d}$, and hence we have constructed a right $\mathbf{H}$-linear map

$$\phi^g_{\lambda^\alpha\mu^\beta} \in \mathrm{Hom}_{\mathbf{H}}(x_{\mu^\beta}\mathbf{H}, x_{\lambda^\alpha}\mathbf{H}), \qquad x_{\mu^\beta} \mapsto T_{W_{\lambda^\alpha}gW_{\mu^\beta}} = x_{\lambda^\alpha}T_gT_{\mathscr{D}_\delta\cap W_{\mu^\beta}}. \tag{A.3.2}$$

We define the *Schur algebra* $\mathbf{S}_{n,d}$ *of type* $\mathbf{D}$ as

$$\mathbf{S}_{n,d} = \mathrm{End}_{\mathbf{H}}\Big(\bigoplus_{\lambda^\alpha\in\Lambda_{\mathbf{D}}} x_{\lambda^\alpha}\mathbf{H}\Big) = \bigoplus_{\lambda^\alpha,\mu^\beta\in\Lambda_{\mathbf{D}}} \mathrm{Hom}_{\mathbf{H}}(x_{\mu^\beta}\mathbf{H}, x_{\lambda^\alpha}\mathbf{H}). \tag{A.3.3}$$

Introduce the following subset of $\Lambda_{\mathbf{D}} \times W_{\mathbf{D}} \times \Lambda_{\mathbf{D}}$:

$$\mathscr{D}_{n,d} = \bigsqcup_{\lambda^\alpha,\mu^\beta\in\Lambda_{\mathbf{D}}} \{\lambda^\alpha\} \times \mathscr{D}_{\lambda^\alpha\mu^\beta} \times \{\mu^\beta\}. \tag{A.3.4}$$

Lemma A.3.1. *The set $\{\phi^g_{\lambda^\alpha\mu^\beta} \mid (\lambda^\alpha, g, \mu^\beta) \in \mathscr{D}_{n,d}\}$ forms an $\mathbf{A}$-basis of $\mathbf{S}_{n,d}$.*

A.4. **Signed matrices.** From now on, we fix

$$N = 2n+1, \qquad D = 2d.$$

Notice that D is even and is different from the convention (2.1.1). Set

$$\Xi = \Big\{A = (a_{ij})_{-n\leqslant i,j\leqslant n} \in \mathrm{Mat}_{N\times N}(\mathbb{N}) \;\Big|\; a_{-i,-j} = a_{ij}, \forall i,j \in [-n,n]; \textstyle\sum_{i,j=-n}^n a_{ij} = D\Big\}. \tag{A.4.1}$$

Recall row(T) and col(T) in (2.2.3), we set

$$\begin{aligned}\Xi^0 &= \{A \in \Xi \mid \mathrm{row}(A)_0 > 0 \text{ and } \mathrm{col}(A)_0 > 0\} \times \{0\},\\ \Xi^+ &= \{A \in \Xi \mid \mathrm{row}(A)_0 = 0 \text{ or } \mathrm{col}(A)_0 = 0\} \times \{+\},\\ \Xi^- &= \{A \in \Xi \mid \mathrm{row}(A)_0 = 0 \text{ or } \mathrm{col}(A)_0 = 0\} \times \{-\}.\end{aligned} \tag{A.4.2}$$

In below we abbreviate $(A, \alpha) \in \Xi^\alpha$ by A^α where $\alpha \in \{0, +, -\}$. We further set

$$\Xi_{\mathbf{D}} = \Xi^0 \sqcup \Xi^+ \sqcup \Xi^-, \tag{A.4.3}$$

whose elements are called *signed matrices.* Define a sign map sgn : $\{0, +, -\}^2 \to \{0, +, -\}$ by

$$\mathrm{sgn}(\alpha, \beta) = \begin{cases} 0, & \text{if } (\alpha, \beta) = (0, 0);\\ +, & \text{if } (\alpha, \beta) = (0, +), (+, 0), (+, +), (+, -);\\ -, & \text{if } (\alpha, \beta) = (0, -), (-, 0), (-, -), (-, +).\end{cases} \tag{A.4.4}$$

Define a map $\kappa : \mathscr{D}_{n,d} \to \Xi_{\mathbf{D}}$ by $\kappa(\lambda^\alpha, g, \mu^\beta) = \Big(|R_i^{\lambda^\alpha} \cap gR_j^{\mu^\beta}|\Big)^{\mathrm{sgn}(\alpha,\beta)}$.

Lemma A.4.1. *The map $\kappa : \mathscr{D}_{n,d} \to \Xi_{\mathbf{D}}$ is a bijection.*

For each $\mathcal{A} = \kappa(\lambda^\alpha, g, \mu^\beta) \in \Xi_{\mathbf{D}}$, we write $e_{\mathcal{A}} = \phi^g_{\lambda^\alpha\mu^\beta}$, and hence $\{e_{\mathcal{A}} \mid A \in \Xi\}$ forms a basis of $\mathbf{S}_{n,d}$. For any $A = (a_{ij}) \in \Xi$, we set

$$a'_{ij} = \begin{cases} \frac{1}{2}a_{00} & \text{if } (i,j) = (0,0);\\ a_{ij} & \text{otherwise,}\end{cases} \quad \text{and} \quad a''_{ij} = \begin{cases} a_{00} - 1 & \text{if } (i,j) = (0,0);\\ a_{ij} & \text{otherwise.}\end{cases} \tag{A.4.5}$$

Let $I^+ = (\{0\} \times [0, n]) \sqcup ([1, n] \times [-n, n])$ be the index set corresponding to the "positive half part" of matrices in Ξ.

Lemma A.4.2. *If $A^{\mathrm{sgn}(\alpha,\beta)} = \kappa(\lambda^\alpha, g, \mu^\beta) \in \Xi_{\mathbf{D}}$ where $A = (a_{ij}) \in \Xi$, then the length of $g \in W_{\mathbf{D}}$ is*

$$\ell(g) = \frac{1}{2}\left(\sum_{(i,j)\in I^+}\left(\sum_{x>i, y<j} + \sum_{x<i, y>j}\right) a'_{ij}a''_{xy}\right). \tag{A.4.6}$$

In particular, the length is independent of the sign $\mathrm{sgn}(\alpha, \beta)$. Thus we write, for $\mathcal{A} = A^{\mathrm{sgn}(\alpha,\beta)} = \kappa(\lambda^\alpha, g, \mu^\beta) \in \Xi_{\mathbf{D}}$,

$$\ell(A) = \ell(g) \quad \text{or} \quad \ell(\mathcal{A}) = \ell(g) \tag{A.4.7}$$

For each signed matrix $\mathcal{A} = A^{\mathrm{sgn}(\alpha,\beta)} = \kappa(\lambda^\alpha, g, \mu^\beta) \in \Xi_{\mathbf{D}}$ with $A = (a_{ij}) \in \Xi$, we introduce the following notations:

$$\begin{aligned}&\mathrm{sgn}(\mathcal{A}) = \mathrm{sgn}(\alpha, \beta), \quad s_l(\mathcal{A}) = \alpha, \quad s_r(\mathcal{A}) = \beta,\\ &\mathrm{row}(\mathcal{A}) = \mathrm{row}(A), \quad \mathrm{col}(\mathcal{A}) = \mathrm{col}(A), \quad p(\mathcal{A}) = \begin{cases} - & \text{if } \sum_{i<0, j>0} a_{ij} \text{ is odd;}\\ + & \text{otherwise,}\end{cases}\\ &\mathcal{A} \pm B = A \pm B, \quad \text{for any } N \times N \text{ matrix } B.\end{aligned} \tag{A.4.8}$$

Note that $\mathcal{A} \pm B$ is a matrix instead of a signed matrix. The following lemmas follows immediately from definition.

Lemma A.4.3. *Let $\mathcal{A} = \kappa(\lambda^\alpha, g, \mu^\beta) \in \Xi_{\mathbf{D}}$, then $p(\mathcal{A}) = +$ (resp. $-$) if and only if $g(1) > 0$ (resp. < 0).*

Lemma A.4.4. *For a signed matrix $\mathcal{A} \in \Xi_{\mathbf{D}}$, we have*

$$s_l(\mathcal{A}) = \begin{cases} 0 & \text{if } \mathrm{row}(\mathcal{A})_0 > 0;\\ \mathrm{sgn}(\mathcal{A}) & \text{if } \mathrm{row}(\mathcal{A})_0 = 0,\end{cases} \quad s_r(\mathcal{A}) = \begin{cases} 0 & \text{if } \mathrm{col}(\mathcal{A})_0 > 0;\\ -\mathrm{sgn}(\mathcal{A}) & \text{if } \mathrm{col}(\mathcal{A})_0 = \mathrm{row}(\mathcal{A})_0 = 0, p(\mathcal{A}) = -;\\ \mathrm{sgn}(\mathcal{A}) & \text{otherwise.}\end{cases} \tag{A.4.9}$$

Let $\mathcal{A} = \kappa(\lambda^\alpha, g, \mu^\beta) \in \Xi_{\mathbf{D}}$. We define a signed weak composition as below:

$$\delta(\mathcal{A}) = (\frac{a_{00}}{2}, a_{10}, \ldots, a_{n0}, a_{-n,1}, a_{-n+1,1}, \ldots, a_{n1}, \ldots, \ldots, a_{-n,n}, a_{-n+1,n}, \ldots, a_{nn})^\beta. \tag{A.4.10}$$

A direct computation shows that $\delta(\mathcal{A})$ is indeed a weak composition δ in Proposition A.2.3(a).

Proposition A.4.5. *Let $\mathcal{A} = \kappa(\lambda^\alpha, g, \mu^\beta) \in \Xi_{\mathbf{D}}$. Then $W_{\delta(\mathcal{A})} = g^{-1} W_{\lambda^\alpha} g \cap W_{\mu^\beta}$.*

We define type D quantum factorials by

$$[0]^!_{\mathfrak{d}} = [2]^!_{\mathfrak{d}} = 1, \quad [2k]^!_{\mathfrak{d}} = [k][2][4]\cdots[2(k-1)], \quad (k \geqslant 2).$$

We further define, for $A = (a_{ij}) \in \Xi$,

$$[A]^!_{\mathfrak{d}} = [a_{0,0}]^!_{\mathfrak{d}} \prod_{(i,j)\in I^+\backslash\{(0,0)\}} [a_{ij}]!. \tag{A.4.11}$$

We write $[\mathcal{A}]^!_{\mathfrak{d}} = [A]^!_{\mathfrak{d}}$ if $\mathcal{A} = A^{\mathrm{sgn}\mathcal{A}}$. The type D quantum factorials are defined in the sense that the following identity on the Poincare polynomial for $W_{\delta(\mathcal{A})}$ holds:

Lemma A.4.6. *For any $\mathcal{A} = A^\alpha \in \Xi_{\mathbf{D}}$ with $A = (a_{ij})$, we have $\sum_{w\in W_{\delta(\mathcal{A})}} v^{2\ell(w)} = [A]^!_{\mathfrak{d}}$.*

A.5. **Multiplication formulas.** The proofs of Lemma A.5.1–A.5.3 are very similar to their counterparts (Lemma 3.1.3, (3.2.2) and Lemma 3.2.1) so we omit.

Lemma A.5.1. *Let $\mathcal{A} = \kappa(\lambda^\alpha, g, \mu^\beta)$ for $\lambda^\alpha, \mu^\beta \in \Lambda_{\mathbf{D}}, g \in \mathscr{D}_{\lambda^\alpha\mu^\beta}$. Then $x_{\lambda^\alpha} T_g x_{\mu^\beta} = [A]^!_{\mathfrak{d}}\, e_{\mathcal{A}}(x_{\mu^\beta})$.*

Lemma A.5.2. *Let $\mathcal{B} = \kappa(\lambda^\alpha, g_1, \mu^\beta)$ and $\mathcal{A} = \kappa(\mu^\beta, g_2, \nu^\gamma)$, where $\lambda^\alpha, \mu^\beta, \nu^\gamma \in \Lambda_{\mathbf{D}}$, $g_1 \in \mathscr{D}_{\lambda^\alpha\mu^\beta}$, and $g_2 \in \mathscr{D}_{\mu^\beta\nu^\gamma}$. Write $\delta = \delta(\mathcal{A})$. Then we have $e_{\mathcal{B}} e_{\mathcal{A}}(x_{\nu^\gamma}) = \frac{1}{[A]^!_{\mathfrak{d}}} x_{\lambda^\alpha} T_{g_1} T_{(\mathscr{D}_\delta \cap W_{\mu^\beta}) g_2} x_{\nu^\gamma}$.*

Lemma A.5.3. *Let $\mathcal{B} = \kappa(\lambda^\alpha, 1, \mu^\beta), \mathcal{A} = \kappa(\mu^\beta, g, \nu^\gamma)$. Let $y^{(w)}$ be the shortest double coset representative for $W_\lambda w g W_\nu$, and let $\mathcal{A}^{(w)} = \kappa(\lambda^\alpha, y^{(w)}, \nu^\gamma)$. Then*

$$e_{\mathcal{B}} e_{\mathcal{A}} = \sum_{w\in\mathscr{D}_\delta\cap W_{\mu^\beta}} v^{2(\ell(w)+\ell(g)-\ell(y^{(w)}))} \frac{[\mathcal{A}^{(w)}]^!_{\mathfrak{d}}}{[A]^!_{\mathfrak{d}}} e_{\mathcal{A}^{(w)}}$$

In the multiplication formulas below, we regard $e_{\mathcal{A}} = 0$ if $\mathcal{A} \notin \Xi_{\mathbf{D}}$.

Proposition A.5.4. *Suppose that $\mathcal{A} = A^{\mathrm{sgn}(\mathcal{A})}, \mathcal{B}, \mathcal{C} \in \Xi_{\mathbf{D}}$ and $h \in [1, n]$. Let $\Gamma_r = \{t = (t_i)_{-n\leqslant i\leqslant n} \in \mathbb{N}^N \mid \sum_{i=-n}^n t_i = r\}$.*

(1) If $h \neq 1$, $\mathcal{B} - rE^\theta_{h,h-1}$ is diagonal, $\mathrm{col}(\mathcal{B}) = \mathrm{row}(\mathcal{A})$, and $s_r(\mathcal{B}) = s_l(\mathcal{A})$, then

$$e_{\mathcal{B}} e_{\mathcal{A}} = \sum_{t\in\Gamma_r} v^{2\sum_{k<p} t_p a_{h,k}} \prod_{p=-n}^n \begin{bmatrix} a_{h,p} + t_p \\ t_p \end{bmatrix} e_{\check{\mathcal{A}}_{t,h}}, \tag{A.5.1}$$

where $\check{\mathcal{A}}_{t,h} = (A + t_p E^\theta_{h,p} - t_p E^\theta_{h-1,p}, \mathrm{sgn}(s_l(\mathcal{B}), s_r(\mathcal{A})))$, $s_l(\check{\mathcal{A}}_{t,h}) = s_l(\mathcal{B})$ and $s_r(\check{\mathcal{A}}_{t,h}) = s_r(\mathcal{A})$.

(2) If $\mathcal{B} - rE^\theta_{1,0}$ is diagonal, $\mathrm{col}(\mathcal{B}) = \mathrm{row}(\mathcal{A})$, and $s_r(\mathcal{B}) = s_l(\mathcal{A})$, then

$$e_{\mathcal{B}} e_{\mathcal{A}} = \sum_{t\in\Gamma_r} v^{2\sum_{k<p} t_p a_{1,k}} (1 + (1 - \delta_{r,\frac{1}{2}\mathrm{row}(\mathcal{A})_0})(1 - \delta_{a'_{0,0},0})\delta_{a'_{0,0},t_0}) \prod_{p=-n}^n \begin{bmatrix} a_{1,p} + t_p \\ t_p \end{bmatrix} e_{\check{\mathcal{A}}_{t,1}}. \tag{A.5.2}$$

(3) If $h \neq 1$, $\mathcal{C} - rE^\theta_{h-1,h}$ is diagonal, $\mathrm{col}(\mathcal{C}) = \mathrm{row}(\mathcal{A})$, and $s_r(\mathcal{C}) = s_l(\mathcal{A})$, then

$$e_{\mathcal{C}} e_{\mathcal{A}} = \sum_{t\in\Gamma_r} v^{2\sum_{k>p} t_p a_{h-1,k}} \prod_{p=-n}^n \begin{bmatrix} a_{h-1,p} + t_p \\ t_p \end{bmatrix} e_{\widehat{\mathcal{A}}_{t,h}}, \tag{A.5.3}$$

where $\widehat{\mathcal{A}}_{t,h} = (A - t_p E^\theta_{h,p} + t_p E^\theta_{h-1,p}, \mathrm{sgn}(s_l(\mathcal{C}), s_r(\mathcal{A})))$, $s_l(\widehat{\mathcal{A}}_{t,h}) = s_l(\mathcal{C})$ and $s_r(\widehat{\mathcal{A}}_{t,h}) = s_r(\mathcal{A})$.

(4) If $\mathcal{C} - rE^{\theta}_{0,1}$ *is diagonal,* $\mathrm{col}(\mathcal{C}) = \mathrm{row}(\mathcal{A})$, *and* $s_r(\mathcal{C}) = s_l(\mathcal{A})$, *then*

$$e_{\mathcal{C}} e_{\mathcal{A}} = \sum_{t\in\Gamma_r} v^{2\sum_{k>p} a_{0,k}t_p + 2\sum_{p<k<-p} t_pt_k + \sum_{p<0} t_p(t_p-1)} \frac{[a_{0,0}+2t_0]^!_{\mathfrak{d}}}{[a_{0,0}]^!_{\mathfrak{d}}[t_0]!} \prod_{p=1}^{n} \frac{[a_{0,p}+t_p+t_{-p}]!}{[a_{0,p}]![t_p]![t_{-p}]!} e_{\widehat{\mathcal{A}}_{t,1}}. \tag{A.5.4}$$

Proof. Here we only prove Parts (2) and (4) while omitting the easier parts (1) and (3). For Part (2), let $\mathcal{A} = \kappa(\mu^\beta, g_2, \nu^\gamma)$, and let $\delta = \delta(\mathcal{B})$. Take any $t \in \Gamma_r$, we consider two cases: $r < \frac{1}{2}\mathrm{row}(\mathcal{A})_0$ or $r = \frac{1}{2}\mathrm{row}(\mathcal{A})_0$.

Case 1: $r < \frac{1}{2}\mathrm{row}(\mathcal{A})_0$: Let w_t be the minimal length element in the set $\{w \in \mathscr{D}_\delta \cap W_{\mu^\beta} \mid \mathcal{A}^{(w)} = \check{\mathcal{A}}_{t,1}\}$. A direct computation shows that its length is give by

$$\begin{aligned} \ell(w_t) &= \sum_{\substack{k>p\geqslant 0 \\ \text{or} \\ k\geqslant -p>0}} t_p(a_{0,k}-t_k) + \textstyle\sum_{|k|<-p} t_p(a_{0,k}-t_k-t_{-k}) - \sum_{p<0} \frac{(t_p+1)t_p}{2} \\ &= \textstyle\sum_{k>p}(a_{0,k}-t_k)t_p - \sum_{p<k<-p} t_pt_k - \frac{1}{2}\sum_{p<0} t_p(t_p+1). \end{aligned} \tag{A.5.5}$$

By a combinatorial argument, we calculate that

$$\sum_{\substack{w\in\mathscr{D}_\delta\cap W_{\mu^\beta}, \\ \mathcal{A}^{(w)}=\widehat{\mathcal{A}}_{t,1}}} v^{2\ell(w)} = v^{2\ell(w_t)} \left(\sum_{x+y=t_0} \begin{bmatrix} a'_{0,0} \\ x \end{bmatrix} \begin{bmatrix} a'_{0,0}-x \\ y \end{bmatrix} (v^2)^{\frac{x(x-1)}{2}+x(a'_{0,0}-t_0)} \right) \prod_{p=1}^{n} \begin{bmatrix} a_{0,p} \\ t_p \end{bmatrix} \begin{bmatrix} a_{0,p}-t_p \\ t_{-p} \end{bmatrix}. \tag{A.5.6}$$

Note that

$$\begin{aligned} \sum_{x+y=t_0} \begin{bmatrix} a'_{0,0} \\ x \end{bmatrix} \begin{bmatrix} a'_{0,0}-x \\ y \end{bmatrix} (v^2)^{\frac{x(x-1)}{2}+x(a'_{0,0}-t_0)} &= \begin{bmatrix} a'_{0,0} \\ t_0 \end{bmatrix} \sum_{x=0}^{t_0} \begin{bmatrix} t_0 \\ x \end{bmatrix} v^{x(x-1)} (v^{a'_{0,0}-t_0})^{2x} \\ &\overset{(\diamondsuit)}{=} \begin{bmatrix} a'_{0,0} \\ t_0 \end{bmatrix} \prod_{i=1}^{t_0} (1 + v^{2(i-1)} v^{2(a'_{0,0}-t_0)}) \\ &= (1 + (1-\delta_{a'_{0,0},0})\delta_{a'_{0,0},t_0}) \frac{[a_{0,0}]^!_{\mathfrak{d}}}{[a_{0,0}-2t_0]^!_{\mathfrak{d}}[t_0]!}, \end{aligned} \tag{A.5.7}$$

where $(\diamondsuit)$ is due to the quantum binomial theorem $\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} v^{r(r-1)} x^r = \prod_{k=0}^{n-1}(1+v^{2k}x)$. Hence

$$\sum_{w\in\mathscr{D}_\delta\cap W_{\mu^\beta}, \mathcal{A}^{(w)}=\widehat{\mathcal{A}}_{t,1}} v^{2\ell(w)} = v^{2\ell(w_t)} \frac{[a_{0,0}]^!_{\mathfrak{d}}}{[a_{0,0}-2t_0]^!_{\mathfrak{d}}[t_0]!} \prod_{p=1}^{n} \begin{bmatrix} a_{0,p} \\ t_p \end{bmatrix} \begin{bmatrix} a_{0,p}-t_p \\ t_{-p} \end{bmatrix}. \tag{A.5.8}$$

Moreover, using (A.4.6), we obtain

$$\begin{aligned} \ell(\mathcal{A}) - \ell(\widehat{\mathcal{A}}_{t,1}) &= -\sum_{k>p}(a_{0,k}-t_k)t_p + \frac{1}{2}\sum_{p<0} t_p + \sum_{k<p} t_pa_{1,k} + \frac{1}{2}\sum_{k<-p} t_pt_k \\ &= \sum_{k<p} t_pa_{1,k} - \sum_{k>p}(a_{0,k}-t_k)t_p + \sum_{p<k<-p} t_pt_k + \frac{1}{2}\sum_{p<0} t_p(t_p+1). \end{aligned} \tag{A.5.9}$$

Combining Lemma A.5.3, (A.5.5), (A.5.8) and (A.5.9), we obtain that, if $r < \frac{1}{2}\mathrm{row}(\mathcal{A})_0$,

$$e_{\mathcal{B}} e_{\mathcal{A}} = \sum_{t\in\Gamma_r} v^{2\sum_{k<p} t_pa_{1,k}} (1 + (1-\delta_{a'_{0,0},0})\delta_{a'_{0,0},t_0}) \prod_{p=-n}^{n} \begin{bmatrix} a_{1,p}+t_p \\ t_p \end{bmatrix} e_{\check{\mathcal{A}}_{t,1}}.$$

Case 2: $r = \frac{1}{2}\mathrm{row}(\mathcal{A})_0$: In this case, each term $e_{\check{\mathcal{A}}_{t,1}} = 0$ unless $a_{0,p} = t_p + t_{-p}$ for all $p \in [-n, n]$. (Particularly, $a'_{0,0} = t_0$.) For the non-vanishing terms, we have

$$\sum_{w\in\mathscr{D}_\delta\cap W_{\mu^\beta}, \mathcal{A}^{(w)}=\widehat{\mathcal{A}}_{t,1}} v^{2\ell(w)} = v^{2\ell(w_t)} \left(\sum_{x} \begin{bmatrix} a'_{0,0} \\ x \end{bmatrix} (v^2)^{\frac{x(x-1)}{2}} \right) \prod_{p=1}^{n} \begin{bmatrix} a_{0,p} \\ t_p \end{bmatrix},$$

where x runs over all integers such that $0 \leqslant x \leqslant a'_{0,0}$ and $x + \sum_{p<0} t_p \in 2\mathbb{N}$. Note that

$$\sum_{a'_{0,0} \geqslant x \in 2\mathbb{N}} \begin{bmatrix} a'_{0,0} \\ x \end{bmatrix} v^{x(x-1)} = \sum_{a'_{0,0} \geqslant x \in 2\mathbb{N}+1} \begin{bmatrix} a'_{0,0} \\ x \end{bmatrix} v^{x(x-1)} = \prod_{i=1}^{a'_{0,0}-1} (1 + v^{2i}).$$

Hence

$$\sum_{w \in \mathscr{D}_\delta \cap W_{\mu^\beta}, \mathcal{A}^{(w)} = \widehat{\mathcal{A}}_{t,1}} v^{2\ell(w)} = v^{2\ell(w_t)} \prod_{i=1}^{a'_{0,0}-1} (1 + v^{2i}) \prod_{p=1}^{n} \begin{bmatrix} a_{0,p} \\ t_p \end{bmatrix}. \tag{A.5.10}$$

Combining Lemma A.5.3, (A.5.5), (A.5.9) and (A.5.10), we obtain, if $r = \frac{1}{2}\mathrm{row}(\mathcal{A})_0$,

$$e_{\mathcal{B}} e_{\mathcal{A}} = \sum_{t \in \Gamma_r} v^{2\sum_{k<p} t_p a_{1,k}} \prod_{p=-n}^{n} \begin{bmatrix} a_{1,p} + t_p \\ t_p \end{bmatrix} e_{\check{\mathcal{A}}_{t,1}}.$$

Part (2) concludes.

For Part (4), Let $\mathcal{A} = \kappa(\mu^\beta, g_2, \nu^\gamma)$, $\delta = \delta(\mathcal{C})$ and take any $t \in \Gamma_r$. Let w_t be the shortest element in the set $\{w \in \mathscr{D}_\delta \cap W_{\mu^\beta} \mid \mathcal{A}^{(w)} = \widehat{\mathcal{A}}_{t,1}\}$. Its length is given by

$$\sum_{w \in \mathscr{D}_\delta \cap W_{\mu^\beta}, \mathcal{A}^{(w)} = \widehat{\mathcal{A}}_{t,1}} v^{2\ell(w)} = v^{2\ell(w_t)} \prod_{p=-n}^{n} \begin{bmatrix} a_{1,p} \\ t_p \end{bmatrix} = v^{2\sum_{k<p} t_p(a_{1,k} - t_k)} \prod_{p=-n}^{n} \begin{bmatrix} a_{1,p} \\ t_p \end{bmatrix}. \tag{A.5.11}$$

Moreover, using (A.4.6), we obtain

$$\begin{aligned} \ell(\mathcal{A}) - \ell(\widehat{\mathcal{A}}_{t,1}) &= \sum_{k>p} a_{0,k} t_p - \frac{1}{2}\sum_{p<0} t_p - \sum_{k<p} t_p(a_{1,k} - t_k) + \frac{1}{2}\sum_{k<-p} t_p t_k \\ &= \sum_{k>p} a_{0,k} t_p - \sum_{k<p} t_p(a_{1,k} - t_k) + \sum_{p<k<-p} t_p t_k + \frac{1}{2}\sum_{p<0} t_p(t_p - 1). \end{aligned} \tag{A.5.12}$$

Combining Lemma A.5.3,(A.5.11) and (A.5.12), we finally get that

$$\begin{aligned} e_{\mathcal{C}} e_{\mathcal{A}} &= \sum_{t \in \Gamma_r} v^{2\sum_{k>p} a_{0,k} t_p + 2\sum_{p<k<-p} t_p t_k + \sum_{p<0} t_p(t_p-1)} \left(\prod_{p=-n}^{n} \begin{bmatrix} a_{1,p} \\ t_p \end{bmatrix} \frac{[a_{1,p} - t_p]!}{[a_{1,p}]!} \right) \\ &\quad \cdot \left(\frac{[a_{0,0} + 2t_0]^!_{\mathfrak{d}}}{[a_{0,0}]^!_{\mathfrak{d}}} \prod_{p=1}^{n} \frac{[a_{0,p} + t_p + t_{-p}]!}{[a_{0,p}]!} \right) e_{\widehat{\mathcal{A}}_{t,1}} \\ &= \sum_{t \in \Gamma_r} v^{2\sum_{k>p} a_{0,k} t_p + 2\sum_{p<k<-p} t_p t_k + \sum_{p<0} t_p(t_p-1)} \frac{[a_{0,0} + 2t_0]^!_{\mathfrak{d}}}{[a_{0,0}]^!_{\mathfrak{d}} [t_0]!} \prod_{p=1}^{n} \frac{[a_{0,p} + t_p + t_{-p}]!}{[a_{0,p}]![t_p]![t_{-p}]!} e_{\widehat{\mathcal{A}}_{t,1}}. \end{aligned}$$

□

Take $r = 1$ in Proposition A.5.4, we have the following corollary.

Corollary A.5.5. *Suppose that $\mathcal{A} = A^{\mathrm{sgn}(\mathcal{A})}, \mathcal{B}, \mathcal{C} \in \Xi_{\mathbf{D}}$ and $h \in [1, n]$.*

(1) If $h \neq 1$, $\mathcal{B} - E^\theta_{h,h-1}$ is diagonal, $\mathrm{col}(\mathcal{B}) = \mathrm{row}(\mathcal{A})$, and $s_r(\mathcal{B}) = s_l(\mathcal{A})$, then

$$e_{\mathcal{B}} e_{\mathcal{A}} = \sum_{p=-n}^{n} v^{2\sum_{k<p} a_{h,k}} [a_{h,p} + 1] e_{\mathcal{A}_p}, \tag{A.5.13}$$

where $\mathcal{A}_p = (A + E^\theta_{h,p} - E^\theta_{h-1,p}, \mathrm{sgn}(s_l(\mathcal{B}), s_r(\mathcal{A})))$.

(2) If $\mathcal{B} - E^\theta_{1,0}$ is diagonal, $\mathrm{col}(\mathcal{B}) = \mathrm{row}(\mathcal{A})$, and $s_r(\mathcal{B}) = s_l(\mathcal{A})$, then

$$e_{\mathcal{B}} e_{\mathcal{A}} = \sum_{p \neq 0} v^{2\sum_{k<p} a_{1,k}} [a_{1,p} + 1] e_{\mathcal{A}_p} + v^{2\sum_{k<0} a_{1,k}} (2 - \delta_{2,\mathrm{row}(\mathcal{A})_0}) [a_{1,0} + 1] e_{\mathcal{A}_0}. \tag{A.5.14}$$

(3) If $h \neq 1$, $\mathcal{C} - E^{\theta}_{h-1,h}$ is diagonal, $\mathrm{col}(\mathcal{C}) = \mathrm{row}(\mathcal{A})$, *and* $s_r(\mathcal{C}) = s_l(\mathcal{A})$, *then*

$$e_{\mathcal{C}} e_{\mathcal{A}} = \sum_{p=-n}^{n} v^{2\sum_{k>p} a_{h-1,k}} [a_{h-1,p}+1] e_{\mathcal{A}(h,p)}, \tag{A.5.15}$$

where $\mathcal{A}(h,p) = (A - E^{\theta}_{h,p} + E^{\theta}_{h-1,p}, \mathrm{sgn}(s_l(\mathcal{C}), s_r(\mathcal{A})))$.

(4) If $\mathcal{C} - E^{\theta}_{0,1}$ is diagonal, $\mathrm{col}(\mathcal{C}) = \mathrm{row}(\mathcal{A})$, *and* $s_r(\mathcal{C}) = s_l(\mathcal{A})$, *then*

$$e_{\mathcal{C}} e_{\mathcal{A}} = \sum_{p\neq 0} v^{2\sum_{k>p} a_{0,k}} [a_{0,p}+1] e_{\mathcal{A}(1,p)} + v^{2\sum_{k>0} a_{0,k}} \left([a_{0,0}+1] + (1-\delta_{0,a_{0,0}}) v^{a_{0,0}}\right) e_{\mathcal{A}(1,0)}. \tag{A.5.16}$$

Remark A.5.6. The multiplication formulas with $e_{\mathcal{A}}$ (Proposition A.5.4 and Corollary A.5.5) match Fan-Li's multiplication formulas ([FL15, Proposition 4.3.2 and Corollary 4.3.4].) with $e^{\mathrm{geo}}_{\mathcal{A}}$, via the following correspondence:

$$e_{\mathcal{A}} \to \begin{cases} \frac{1}{2} e^{\mathrm{geo}}_{\mathcal{A}}, & \text{if } a_{0,0}=0,\ \mathrm{row}(A)_0 \neq 0 \text{ and } \mathrm{col}(A)_0 \neq 0; \\ e^{\mathrm{geo}}_{\mathcal{A}} & \text{otherwise.} \end{cases} \tag{A.5.17}$$

Remark A.5.7. An immediate application of the multiplication formulas is to demonstrate a stabilization property for $\{\mathbf{S}_{n,d} \mid d \in \mathbb{N}\}$, and further construct an algebra $\mathcal{K}_n$ so that the multiplication rules on $\mathcal{K}_n$ are compatible with the rules on any $\mathbf{S}_{n,d}$. The algebras $\mathcal{K}_n$ have been introduced by Fan and Li in *loc. cit.*

A.6. **Schur duality.** Let $\mathfrak{g}$ be the simple Lie algebra of type $\mathbf{D}_d$, and let ρ be the half sum of the positive roots of $\mathfrak{g}$. It was mentioned in a framework [LW17] that $\Lambda_{\mathbf{D}}$ can be viewed as the set of orbits of W on a (truncated) ρ-shifted weight lattice of $\mathfrak{g}$. Then the v-tensor space $\bigoplus_{\lambda^\alpha \in \Lambda_{\mathbf{D}}} x_{\lambda^\alpha}\mathbf{H}$ can be viewed as the quantum version of the Grothendieck groups of the category $\mathcal{O}$ of $\mathfrak{g}$-modules.

This picture is also valid when $\Lambda_{\mathbf{D}}$ is replaced by its subset. Each subset $\Lambda_f \subset \Lambda_{\mathbf{D}}$ corresponds to a Schur algebra

$$\mathbf{S}_f = \mathrm{End}_{\mathbf{H}}\Big(\bigoplus_{\lambda \in \Lambda_f} x_\lambda \mathbf{H}.\Big)$$

A Schur duality is also obtained in *loc. cit.* for each pair $(\mathbf{S}_f, \mathbf{H})$ on the tensor space $\oplus_{\lambda\in\Lambda_f} x_\lambda \mathbf{H}$.

Remark A.6.1. If $\Lambda_f = \Lambda^+ \sqcup \Lambda^-$, then $\mathbf{S}_f$ is the algebra $\mathcal{S}^m$ in [FL15, §6.1]. The stabilization procedure affords a different quantum algebra $\mathcal{K}^m$ in *loc. cit.*

Remark A.6.2. Fan and Li told the authors in private conversations that they have also been aware of the Schur algebra $\mathbf{S}_f$ and the related Schur duality for $\Lambda_f = \Lambda^+$ or $\Lambda^0 \sqcup \Lambda^+$ although they did not write it down.

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