

NONCOMMUTATIVE WEIL CONJECTURE

GONÇALO TABUADA

ABSTRACT. In this article, following an insight of Kontsevich, we extend the famous Weil conjecture (as well as the strong form of the Tate conjecture) from the realm of algebraic geometry to the broad noncommutative setting of dg categories. As a first application, we prove the noncommutative Weil conjecture (and the noncommutative strong form of the Tate conjecture) in the following cases: twisted schemes, Calabi-Yau dg categories associated to hypersurfaces, noncommutative gluings of schemes, root stacks, (twisted) global orbifolds, connective dg algebras, and finite-dimensional dg algebras. As a second application, we provide an alternative noncommutative proof of Weil's original conjecture (which avoids the involved tools used by Deligne) in the cases of intersections of two quadrics and linear sections of determinantal varieties. Finally, we extend also the classical theory of L -functions (as well as the corresponding conjectures of Tate and Beilinson) from the realm of algebraic geometry to the broad noncommutative setting of dg categories. Among other applications, this leads to an alternative noncommutative proof of a celebrated convergence result of Serre.

1. STATEMENT OF RESULTS: ZETA FUNCTIONS

Let $k = \mathbb{F}_q$ be a finite field of characteristic p , $W(k)$ the ring of p -typical Witt vectors of k , and $K := W(k)_{1/p}$ the fraction field of $W(k)$. Given a smooth proper k -scheme X of dimension d , recall that its zeta function is defined as the formal power series $Z(X; t) := \exp(\sum_{n \geq 1} \#X(\mathbb{F}_{q^n}) \frac{t^n}{n}) \in \mathbb{Q}[[t]]$, where $\exp(t) := \sum_{n \geq 0} \frac{t^n}{n!}$. In the same vein, given an integer $0 \leq w \leq 2d$, consider the formal power series $Z_w(X; t) := \det(\text{id} - t\text{Fr}^w | H_{\text{crys}}^w(X))^{-1} \in K[[t]]$, where $H_{\text{crys}}^*(X)$ stands for the crystalline cohomology $H_{\text{crys}}^*(X/W(k)) \otimes_{W(k)} K$ of X , Fr for the Frobenius endomorphism of X , and Fr^w for the induced automorphism of $H_{\text{crys}}^w(X)$. Thanks to the Lefschetz trace formula established by Grothendieck and Berthelot (see [7, Chapitre VII §3.2]), we have the following weight decomposition:

$$(1.1) \quad Z(X; t) = \frac{Z_0(X; t)Z_2(X; t) \cdots Z_{2d}(X; t)}{Z_1(X; t)Z_3(X; t) \cdots Z_{2d-1}(X; t)}.$$

In the late forties, Weil [67] conjectured the following¹:

Conjecture W(X): The eigenvalues of the automorphism Fr^w , with $0 \leq w \leq 2d$, are algebraic numbers and all their complex conjugates have absolute value $q^{\frac{w}{2}}$.

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¹The conjecture W(X) is a modern formulation of Weil's original conjecture; in the late forties crystalline cohomology was not yet developed.

In the particular case of curves, this famous conjecture follows from Weil’s pioneering work [68]. Later, in the seventies, it was proved in full generality by Deligne² [16]. In contrast with Weil’s proof, which uses solely the classical intersection theory of divisors on surfaces, Deligne’s proof makes use of several involved tools such as the theory of monodromy of Lefschetz pencils. The Weil conjecture has numerous applications. For example, when combined with the weight decomposition (1.1), it implies that the polynomials $\det(\mathrm{id} - t\mathrm{Fr}^w | H_{\mathrm{crys}}^w(X))$ have integer coefficients.

Recall that the Hasse-Weil zeta function of X is defined as the (convergent) infinite product $\zeta(X; s) := \prod_{x \in X^{(d)}} (1 - (q^{\deg(x)})^{-s})^{-1}$, with $\mathrm{Re}(s) > d$, where $X^{(d)}$ stands for the set of closed points of X and $\deg(x)$ for the degree of the finite field extension $\kappa(x)/\mathbb{F}_q$. In the same vein, given an integer $0 \leq w \leq 2d$, consider the function $\zeta_w(X; s) := \det(\mathrm{id} - q^{-s}\mathrm{Fr}^w | H_{\mathrm{crys}}^w(X))^{-1}$. It follows from the Weil conjecture that $\zeta(X; s) = Z(X; q^{-s})$, with $\mathrm{Re}(s) > d$, and that $\zeta_w(X; s) = Z_w(X; q^{-s})$, with $\mathrm{Re}(s) > \frac{w}{2}$. Thanks to (1.1), we hence obtain the weight decomposition:

$$(1.2) \quad \zeta(X; s) = \frac{\zeta_0(X; s)\zeta_2(X; s) \cdots \zeta_{2d}(X; s)}{\zeta_1(X; s)\zeta_3(X; s) \cdots \zeta_{2d-1}(X; s)} \quad \mathrm{Re}(s) > d.$$

Note that (1.2) implies automatically that the Hasse-Weil zeta function of X admits a (unique) meromorphic continuation to the entire complex plane.

Remark 1.3 (Periodicity). Note that the Hasse-Weil zeta function of X is periodic in the sense that $\zeta(X; s) = \zeta(X; s + \frac{2\pi i}{\log(q)})$. Similarly, $\zeta_w(X; s) = \zeta_w(X; s + \frac{2\pi i}{\log(q)})$.

Remark 1.4 (Riemann hypothesis). The above conjecture $W(X)$ is usually called the “analogue of the Riemann hypothesis” because it implies that if $z \in \mathbb{C}$ is a pole of $\zeta_w(X; s)$, then $\mathrm{Re}(z) = \frac{w}{2}$. Consequently, if $z \in \mathbb{C}$ is a pole, resp. a zero, of $\zeta(X; s)$, then $\mathrm{Re}(z) \in \{0, 1, \dots, d\}$, resp. $\mathrm{Re}(z) \in \{\frac{1}{2}, \frac{2}{3}, \dots, \frac{2d-1}{2}\}$.

Let \mathcal{A} be a smooth proper k -linear dg category in the sense of Kontsevich; see §5.1. Examples include the (unique) dg enhancements $\mathrm{perf}_{\mathrm{dg}}(X)$ of the categories of perfect complexes $\mathrm{perf}(X)$ of smooth proper k -schemes X (or, more generally, of smooth proper algebraic k -stacks \mathcal{X}); consult [31, 43]. As explained in §6.1 below, the topological periodic cyclic homology group $TP_0(\mathcal{A})_{1/p}$ of \mathcal{A} (this is a finite-dimensional K -vector space), resp. the topological periodic cyclic homology group $TP_1(\mathcal{A})_{1/p}$ of \mathcal{A} , comes equipped with an automorphism F_0 , resp. F_1 , called the “cyclotomic Frobenius”. Following Kontsevich [34], we hence define the *even/odd zeta function of \mathcal{A}* as the following formal power series:

$$\begin{aligned} Z_{\mathrm{even}}(\mathcal{A}; t) &:= \det(\mathrm{id} - tF_0 | TP_0(\mathcal{A})_{1/p})^{-1} \in K[[t]] \\ Z_{\mathrm{odd}}(\mathcal{A}; t) &:= \det(\mathrm{id} - tF_1 | TP_1(\mathcal{A})_{1/p})^{-1} \in K[[t]]. \end{aligned}$$

Weil’s conjecture admits the following noncommutative counterpart:

Conjecture $W_{\mathrm{nc}}(\mathcal{A})$: The eigenvalues of the automorphism F_0 , resp. F_1 , are algebraic numbers and all their complex conjugates have absolute value 1, resp. $q^{\frac{1}{2}}$.

The noncommutative Weil conjecture was originally envisioned by Kontsevich in his seminal talks [37, 38]. The next result relates this conjecture with Weil’s original conjecture:

²Deligne worked with étale cohomology instead. However, as explained by Katz-Messing in [28], Deligne’s results hold similarly in crystalline cohomology. More recently, Kedlaya [30] gave an alternative proof of the Weil conjecture which uses solely p -adic techniques.

Theorem 1.5. *Given a smooth proper k -scheme X , we have the equivalence of conjectures $W_{\text{nc}}(\text{perf}_{\text{dg}}(X)) \Leftrightarrow W(X)$.*

Intuitively speaking, Theorem 1.5 shows that the Weil conjecture belongs not only to the realm of algebraic geometry but also to the broad noncommutative setting of dg categories.

In contrast with the commutative world, the cyclotomic Frobenius is not induced from an endomorphism³ of \mathcal{A} . Consequently, in contrast with the commutative world, it is not known if the polynomials $\det(\text{id} - tF_0|TP_0(\mathcal{A})_{1/p})$ and $\det(\text{id} - tF_1|TP_1(\mathcal{A})_{1/p})$ have integer coefficients (or rational coefficients). Nevertheless, after choosing an embedding $\iota: K \hookrightarrow \mathbb{C}$, we can define the *even/odd Hasse-Weil zeta function of \mathcal{A}* as follows:

$$\begin{aligned}\zeta_{\text{even}}(\mathcal{A}; s) &:= \det(\text{id} - q^{-s}(F_0 \otimes_{K, \iota} \mathbb{C})|TP_0(\mathcal{A})_{1/p} \otimes_{K, \iota} \mathbb{C})^{-1} \\ \zeta_{\text{odd}}(\mathcal{A}; s) &:= \det(\text{id} - q^{-s}(F_1 \otimes_{K, \iota} \mathbb{C})|TP_1(\mathcal{A})_{1/p} \otimes_{K, \iota} \mathbb{C})^{-1}.\end{aligned}$$

Remark 1.6 (Periodicity). Similarly to Remark 1.3, note that the even/odd Hasse-Weil zeta function of \mathcal{A} is periodic of period $\frac{2\pi i}{\log(q)}$.

Remark 1.7 (Noncommutative Riemann hypothesis). Similarly to Remark 1.4, the conjecture $W_{\text{nc}}(\mathcal{A})$ may be called the ‘‘analogue of the noncommutative Riemann hypothesis’’ because it implies that if $z \in \mathbb{C}$ is a pole of $\zeta_{\text{even}}(\mathcal{A}; s)$, resp. $\zeta_{\text{odd}}(\mathcal{A}; s)$, then $\text{Re}(z) = 0$, resp. $\text{Re}(z) = \frac{1}{2}$ (independently of the embedding $\iota: K \hookrightarrow \mathbb{C}$).

The next result follows from (the proof of) Theorem 1.5:

Corollary 1.8. *Given a smooth proper k -scheme X , we have the following factorization $\zeta_{\text{even}}(\text{perf}_{\text{dg}}(X); s) = \prod_{w \text{ even}} \zeta_w(X; s + \frac{w}{2})$ as well as the following factorization $\zeta_{\text{odd}}(\text{perf}_{\text{dg}}(X); s) = \prod_{w \text{ odd}} \zeta_w(X; s + \frac{w-1}{2})$.*

Roughly speaking, Corollary 1.8 shows that the even/odd Hasse-Weil zeta function of $\text{perf}_{\text{dg}}(X)$ may be understood as the ‘‘weight normalization’’ of the product of the Hasse-Weil zeta functions $\zeta_w(X; s)$. This leads to the following result:

Corollary 1.9. *Given a smooth proper k -scheme X and a complex number $z \in \mathbb{C}$, we have the equality $\text{ord}_{s=z} \zeta_{\text{even}}(\text{perf}_{\text{dg}}(X); s) = \sum_{w \text{ even}} \text{ord}_{s=z+\frac{w}{2}} \zeta_w(X; s)$ as well as the equality $\text{ord}_{s=z} \zeta_{\text{odd}}(\text{perf}_{\text{dg}}(X); s) = \sum_{w \text{ odd}} \text{ord}_{s=z+\frac{w-1}{2}} \zeta_w(X; s)$, where $\text{ord}_{s=z} f(s)$ stands for the order of a meromorphic function $f(s)$ at $s = z$.*

l -adic absolute value. Let $l \neq p$ be a prime number. Given a smooth proper k -scheme X of dimension d , it is well-known that the eigenvalues of the automorphisms Fr^w , $0 \leq w \leq 2d$, are algebraic numbers and that all their l -adic conjugates have absolute value 1. Let \mathcal{A} be a smooth proper k -linear dg category. Motivated by the aforementioned fact, Kontsevich also conjectured in [37, 38] the following:

Conjecture $W_{\text{nc}}^l(\mathcal{A})$: The eigenvalues of the automorphisms F_0 and F_1 are algebraic numbers and all their l -adic conjugates have absolute value 1.

The next result (partially) solves Kontsevich’s conjecture:

Theorem 1.10. *Assume that there exists an integer $C \gg 0$ (which depends on \mathcal{A}) such that the eigenvalues of the automorphisms F_0 and F_1 become algebraic integers after multiplication by q^C . Under this assumption, the conjecture $W_{\text{nc}}^l(\mathcal{A})$ holds.*

³Note that in the particular case where \mathcal{A} is a k -algebra A , the Frobenius map $a \mapsto a^q$ is a k -algebra endomorphism if and only if A is commutative.

As explained in Remark 10.3 below, the assumption of Theorem 1.10 holds when $\mathcal{A} = \text{perf}_{\text{dg}}(X)$ with X a smooth proper k -scheme. Consult §2 for further examples.

Functional equation. Thanks to the work of M. Artin and Grothendieck (consult [23] and the references therein), the Hasse-Weil zeta function $\zeta(X; s)$ of a smooth proper k -scheme X of dimension d is known to satisfy the functional equation

$$(1.11) \quad \zeta(X; s) = \pm q^{\chi(X)s} \cdot q^{-\frac{\chi(X)}{2}d} \cdot \zeta(X; d-s),$$

where $\chi(X)$ stands for the Euler characteristic of X . Morally speaking, the equality (1.11) describes a “symmetry” of $\zeta(X; s)$ along the vertical line $\text{Re}(s) = \frac{d}{2}$. This functional equation admits the following noncommutative counterpart:

Theorem 1.12. *Given a smooth proper k -linear dg category \mathcal{A} , we have the following functional equations*

$$\begin{aligned} \zeta_{\text{even}}(\mathcal{A}; s) &= (-1)^{\chi_0(\mathcal{A})} \cdot q^{\chi_0(\mathcal{A})s} \cdot \det(\text{F}_0 \otimes_{K, \iota} \mathbb{C}) \cdot \zeta_{\text{even}}(\mathcal{A}; -s) \\ \zeta_{\text{odd}}(\mathcal{A}; s) &= (-1)^{\chi_1(\mathcal{A})} q^{-\chi_1(\mathcal{A})(1-s)} \cdot \det(\text{F}_1 \otimes_{K, \iota} \mathbb{C}) \cdot \zeta_{\text{odd}}(\mathcal{A}; 1-s), \end{aligned}$$

where $\chi_0(\mathcal{A}) := \dim_K TP_0(\mathcal{A})_{1/p}$ and $\chi_1(\mathcal{A}) := \dim_K TP_1(\mathcal{A})_{1/p}$.

Intuitively speaking, Theorem 1.12 describes a “symmetry” of $\zeta_{\text{even}}(\mathcal{A}; s)$, resp. $\zeta_{\text{odd}}(\mathcal{A}; s)$, along the vertical line $\text{Re}(s) = 0$, resp. $\text{Re}(s) = \frac{1}{2}$.

Corollary 1.13. *When $\mathcal{A} = \text{perf}_{\text{dg}}(X)$, with X a smooth proper k -scheme, the functional equations of Theorem 1.12 reduce to the following functional equations*

$$\begin{aligned} \prod_{w \text{ even}} \zeta_w(X; s + \frac{w}{2}) &= \pm q^{\chi_{\text{even}}(X)s} \cdot \prod_{w \text{ even}} \zeta_w(X; -s + \frac{w}{2}) \\ \prod_{w \text{ odd}} \zeta_w(X; s + \frac{w-1}{2}) &= \pm q^{\chi_{\text{odd}}(X)s} \cdot \prod_{w \text{ odd}} \zeta_w(X; 1-s + \frac{w-1}{2}), \end{aligned}$$

where $\chi_{\text{even}}(X) := \sum_{w \text{ even}} \dim_K H_{\text{crys}}^w(X)$ and $\chi_{\text{odd}}(X) := \sum_{w \text{ odd}} \dim_K H_{\text{crys}}^w(X)$.

Remark 1.14 (Related work). In [56] we developed a general theory of (Hasse-Weil) zeta functions for smooth proper dg categories equipped with an endomorphism. Among other applications, this theory led to a far-reaching noncommutative generalization of the results of Dwork [17] and Grothendieck [23] concerning the rationality and the functional equation of the classical (Hasse-Weil) zeta function.

Strong form of the Tate conjecture. Given a smooth proper k -scheme X of dimension d and an integer $0 \leq i \leq d$, let us write $\mathcal{Z}^i(X)_{\mathbb{Q}/\sim_{\text{num}}}$ for the \mathbb{Q} -vector space of algebraic cycles of codimension i on X up to numerical equivalence.

In the mid sixties, Tate [65] conjectured the following:

Conjecture ST(X): *The order $\text{ord}_{s=j} \zeta(X; s)$ of the Hasse-Weil zeta function $\zeta(X; s)$ at the pole $s = j$, with $0 \leq j \leq d$, is equal to $-\dim_{\mathbb{Q}} \mathcal{Z}^j(X)_{\mathbb{Q}/\sim_{\text{num}}}$.*

This conjecture is usually called the “strong form of the Tate conjecture”. It holds for 0-dimensional schemes, for curves, for abelian varieties of dimension ≤ 3 , and also for K3-surfaces. Besides these cases (and some other cases scattered in the literature), it remains wide open.

Given a smooth proper k -linear dg category \mathcal{A} , recall from §5.3 below the definition of its numerical Grothendieck group $K_0(\mathcal{A})_{\mathbb{Q}/\sim_{\text{num}}}$. The strong form of the Tate conjecture admits the following noncommutative counterpart:

Conjecture $\text{ST}_{\text{nc}}(\mathcal{A})$: The order $\text{ord}_{s=0}\zeta_{\text{even}}(\mathcal{A}; s)$ of the even Hasse-Weil zeta function $\zeta_{\text{even}}(\mathcal{A}; s)$ at the pole $s = 0$ is equal to $-\dim_{\mathbb{Q}}K_0(\mathcal{A})_{\mathbb{Q}/\sim_{\text{num}}}$.

Remark 1.15 (Alternative formulation). Note that, by definition of the even Hasse-Weil zeta function of \mathcal{A} , the order $\text{ord}_{s=0}\zeta_{\text{even}}(\mathcal{A}; s)$ of the even Hasse-Weil zeta function $\zeta_{\text{even}}(\mathcal{A}; s)$ at the pole $s = 0$ agrees with the algebraic multiplicity of the eigenvalue $q^0 = 1$ of the automorphism $F_0 \otimes_{K, \iota} \mathbb{C}$ (or, equivalently, of F_0). Hence, the conjecture $\text{ST}_{\text{nc}}(\mathcal{A})$ may be alternatively formulated as follows: *the algebraic multiplicity of the eigenvalue 1 of F_0 agrees with $\dim_{\mathbb{Q}}K_0(\mathcal{A})_{\mathbb{Q}/\sim_{\text{num}}}$.* This shows, in particular, that the integer $\text{ord}_{s=0}\zeta_{\text{even}}(\mathcal{A}; s)$ is independent of the embedding $\iota: K \hookrightarrow \mathbb{C}$ used in the definition of $\zeta_{\text{even}}(\mathcal{A}; s)$.

Remark 1.16 (Equivalent conjectures). As proved in Theorem 11.3 below, the noncommutative strong form of the Tate conjecture is equivalent to the noncommutative p -version of the Tate conjecture plus the noncommutative standard conjecture of type D . Moreover, when all smooth proper dg categories are considered simultaneously, the noncommutative strong form of the Tate conjecture becomes equivalent to the fully-faithfulness of the enriched topological periodic cyclic homology functor; consult §6.2 and §11.5 below for details.

The next result relates the noncommutative strong form of the Tate conjecture with the strong form of the Tate conjecture:

Theorem 1.17. *Given a smooth proper k -scheme X , we have the equivalence of conjectures $\text{ST}_{\text{nc}}(\text{perf}_{\text{dg}}(X)) \Leftrightarrow \text{ST}(X)$.*

Similarly to Theorem 1.5, Theorem 1.17 shows that the strong form of the Tate conjecture belongs not only to the realm of algebraic geometry but also to the broad noncommutative setting of smooth proper dg categories.

2. APPLICATIONS TO NONCOMMUTATIVE GEOMETRY

Let $k = \mathbb{F}_q$ be a finite field of characteristic p and $l \neq p$ a prime number. In this section, making use of Theorems 1.5, 1.10, and 1.17, we prove the noncommutative Weil conjecture(s) (i.e. $W_{\text{nc}}(-)$ and $W_{\text{nc}}^l(-)$), as well as the noncommutative strong form of the Tate conjecture, in several (interesting) cases.

Twisted schemes. Let X be a smooth proper k -scheme and \mathcal{F} a sheaf of Azumaya algebras over X . Similarly to $\text{perf}_{\text{dg}}(X)$, we can also consider the (smooth proper) dg category $\text{perf}_{\text{dg}}(X; \mathcal{F})$ of perfect complexes of \mathcal{F} -modules.

Theorem 2.1. *We have the following equivalences of conjectures:*

$$W(X) \Leftrightarrow W_{\text{nc}}(\text{perf}_{\text{dg}}(X; \mathcal{F})) \quad \text{ST}(X) \Leftrightarrow \text{ST}_{\text{nc}}(\text{perf}_{\text{dg}}(X; \mathcal{F})).$$

Moreover, the conjecture $W_{\text{nc}}^l(\text{perf}_{\text{dg}}(X; \mathcal{F}))$ holds.

Morally speaking, Theorem 2.1 shows that in what concerns the (noncommutative) Weil conjecture and the (noncommutative) strong form of the Tate conjecture, there is no difference between schemes and twisted schemes.

Calabi-Yau dg categories associated to hypersurfaces. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $\deg(X) \leq n + 1$. As proved by Kuznetsov in [40, Cor. 4.1], we have a semi-orthogonal decomposition:

$$(2.2) \quad \text{perf}(X) = \langle \mathcal{T}(X), \mathcal{O}_X, \dots, \mathcal{O}_X(n - \deg(X)) \rangle.$$

Moreover, the associated dg category $\mathcal{T}_{\text{dg}}(X)$, defined as the dg enhancement of $\mathcal{T}(X)$ induced from $\text{perf}_{\text{dg}}(X)$, is a smooth proper Calabi-Yau dg category of fractional dimension $\frac{(n+1)(\deg(X)-2)}{\deg(X)}$.

Remark 2.3 (Noncommutative K3-surfaces). In the particular case where $n = 5$ and $\deg(X) = 3$, the dg categories $\mathcal{T}_{\text{dg}}(X)$ are usually called “noncommutative K3-surfaces” because they share many of the key properties of the dg categories of perfect complexes of the classical (smooth proper) K3-surfaces. Moreover, Kuznetsov conjectured in [42] that $\mathcal{T}(X)$ is (Fourier-Mukai) equivalent to the category of perfect complexes of a K3-surface if and only if X is rational.

Theorem 2.4. *We have the following equivalences of conjectures:*

$$W(X) \Leftrightarrow W_{\text{nc}}(\mathcal{T}_{\text{dg}}(X)) \quad \text{ST}(X) \Leftrightarrow \text{ST}_{\text{nc}}(\mathcal{T}_{\text{dg}}(X)).$$

Moreover, the conjecture $W_{\text{nc}}^l(\mathcal{T}_{\text{dg}}(X))$ holds.

Similarly to Theorem 2.1, Theorem 2.4 shows that in what concerns the (noncommutative) Weil conjecture and the (noncommutative) strong form of the Tate conjecture, there is no difference between the hypersurface X and the associated Calabi-Yau dg category $\mathcal{T}_{\text{dg}}(X)$.

Noncommutative gluings of schemes. Let X and Y be two smooth proper k -schemes and B a perfect dg $\text{perf}_{\text{dg}}(X)$ - $\text{perf}_{\text{dg}}(Y)$ -bimodule. Following Orlov [49, §3.2], we can consider the gluing $X \ominus_B Y$ of $\text{perf}_{\text{dg}}(X)$ and $\text{perf}_{\text{dg}}(Y)$ via B (Orlov used a different notation). This new dg category is smooth and proper.

Theorem 2.5. *We have the following equivalences of conjectures:*

$$W(X) + W(Y) \Leftrightarrow W_{\text{nc}}(X \ominus_B Y) \quad \text{ST}(X) + \text{ST}(Y) \Leftrightarrow \text{ST}_{\text{nc}}(X \ominus_B Y).$$

Moreover, the conjecture $W_{\text{nc}}^l(X \ominus_B Y)$ holds.

Intuitively speaking, Theorem 2.5 shows that the noncommutative Weil conjecture and the noncommutative strong form of the Tate conjecture are “additive” with respect to gluings. This implies, in particular, that the noncommutative Weil conjecture(s) hold(s) for every noncommutative gluing and that the noncommutative strong form of the Tate conjecture holds for every noncommutative gluing of curves.

Root stacks. Let X be a smooth proper k -scheme, \mathcal{L} a line bundle on X , $\varsigma \in \Gamma(X, \mathcal{L})$ a global section, and $n \geq 1$ an integer. Following Cadman [15, Def. 2.2.1], the associated *root stack* is defined as the following fiber-product

$$\begin{array}{ccc} \mathcal{X} := \sqrt[n]{(\mathcal{L}, \varsigma)/X} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ f \downarrow & & \downarrow \theta_n \\ X & \xrightarrow{(\mathcal{L}, \varsigma)} & [\mathbb{A}^1/\mathbb{G}_m], \end{array}$$

where θ_n stands for the morphism induced by the n^{th} power map on \mathbb{A}^1 and \mathbb{G}_m .

Theorem 2.6. *Assume that the zero locus $D \hookrightarrow X$ of ς is smooth. Under this assumption, we have the following equivalences of conjectures:*

$$W(X) + W(D) \Leftrightarrow W_{\text{nc}}(\text{perf}_{\text{dg}}(\mathcal{X})) \quad \text{ST}(X) + \text{ST}(D) \Leftrightarrow \text{ST}_{\text{nc}}(\text{perf}_{\text{dg}}(\mathcal{X})).$$

Moreover, the conjecture $W_{\text{nc}}^l(\text{perf}_{\text{dg}}(\mathcal{X}))$ holds.

Theorem 2.6 implies that the noncommutative Weil conjecture(s) hold(s) for every root stack and that the noncommutative strong form of the Tate conjecture holds, for example, for all those root stacks whose underlying scheme is a curve.

Global orbifolds. Let G be a finite group of order n , X a smooth proper k -scheme equipped with a G -action, and $\mathcal{X} := [X/G]$ the associated global orbifold.

Theorem 2.7. *Assume that $p \nmid n$ ($\Leftrightarrow 1/n \in k$). Under this assumption, we have the following implications of conjectures (σ is a cyclic subgroup of G):*

$$(2.8) \quad \sum_{\sigma \subseteq G} W(X^\sigma \times \text{Spec}(k[\sigma])) \Rightarrow W_{\text{nc}}(\text{perf}_{\text{dg}}(\mathcal{X}))$$

$$(2.9) \quad \sum_{\sigma \subseteq G} \text{ST}(X^\sigma \times \text{Spec}(k[\sigma])) \Rightarrow \text{ST}_{\text{nc}}(\text{perf}_{\text{dg}}(\mathcal{X})).$$

Under the stronger assumption $n|(q-1)$ ($\Leftrightarrow k$ contains the n^{th} roots of unity), the same implications hold with $X^\sigma \times \text{Spec}(k[\sigma])$ replaced by X^σ . Moreover, the conjecture $W_{\text{nc}}^l(\text{perf}_{\text{dg}}(\mathcal{X}))$ holds.

Theorem 2.7 implies that the noncommutative Weil conjecture(s) hold(s) for every global orbifold. Since $\text{Spec}(k[\sigma])$ is 0-dimensional, it implies moreover that the noncommutative strong form of the Tate conjecture holds, for example, for all those global orbifolds whose underlying scheme is a curve.

Remark 2.10 (McKay correspondence). A famous conjecture of Reid asserts that the category $\text{perf}(\mathcal{X})$ is (Fourier-Mukai) equivalent to the category of perfect complexes of a(ny) crepant resolution Y of the (singular) geometric quotient $X//G$. Whenever this holds, the right-hand sides of (2.8)-(2.9) may be replaced by the conjectures $W(Y)$ and $\text{ST}(Y)$, respectively. Reid's conjecture has been proved in several cases; consult, for example, the work of Bezrukavnikov and Kaledin [8], Bridgeland, King and Reid [14], Kapranov and Vasserot [27], and Kawamata [29].

Twisted global orbifolds. Let G be a finite group of order n , X a smooth proper k -scheme equipped with a G -action, $\mathcal{X} := [X/G]$ the associated global orbifold, and \mathcal{F} a sheaf of Azumaya algebras⁴ over \mathcal{X} . Similarly to $\text{perf}_{\text{dg}}(\mathcal{X})$, we can also consider the dg category $\text{perf}_{\text{dg}}(\mathcal{X}; \mathcal{F})$ of perfect complexes of \mathcal{F} -modules.

Theorem 2.11. *Assume that $n|(q-1)$. Under this assumption, we have the following implications of conjectures*

$$\sum_{\sigma \subseteq G} W(Y_\sigma) \Rightarrow W_{\text{nc}}(\text{perf}_{\text{dg}}(\mathcal{X}; \mathcal{F})) \quad \sum_{\sigma \subseteq G} \text{ST}(Y_\sigma) \Rightarrow \text{ST}_{\text{nc}}(\text{perf}_{\text{dg}}(\mathcal{X}; \mathcal{F})),$$

where Y_σ is a certain σ^\vee -Galois cover of X^σ induced by the restriction of \mathcal{F} to X^σ . Moreover, the conjecture $W_{\text{nc}}^l(\text{perf}_{\text{dg}}(\mathcal{X}; \mathcal{F}))$ holds.

⁴Equivalently, \mathcal{F} is a G -equivariant sheaf of Azumaya algebras over X .

Similarly to Theorem 2.7, Theorem 2.11 implies that the noncommutative Weil conjecture(s) hold(s) for every twisted global orbifold. It implies moreover that the noncommutative strong form of the Tate conjecture holds for all those twisted global orbifolds whose underlying scheme is a curve.

Connective dg algebras. Let A be a smooth proper *connective* dg k -algebra, i.e., $H^i(A) = 0$ for every $i > 0$. Via the Dold-Kan correspondence, it corresponds to a smooth proper simplicial k -algebra. The next result proves the noncommutative Weil conjecture(s) and the noncommutative strong form of the Tate conjecture for this (large) class of dg algebras:

Theorem 2.12. *The conjectures $W_{\text{nc}}(A)$, $W_{\text{nc}}^l(A)$, and $ST_{\text{nc}}(A)$, hold.*

Finite-dimensional dg algebras. Let A be a smooth *finite-dimensional* dg k -algebra, i.e., $\dim_k(A^i) < \infty$ for every $i \in \mathbb{Z}$. The next result proves the noncommutative Weil conjecture(s) and the noncommutative strong form of the Tate conjecture for this (large) class of dg algebras:

Theorem 2.13. *The conjectures $W_{\text{nc}}(A)$, $W_{\text{nc}}^l(A)$, and $ST_{\text{nc}}(A)$, hold.*

3. APPLICATIONS TO COMMUTATIVE GEOMETRY

Let $k = \mathbb{F}_q$ be a finite field of characteristic p . As mentioned in §1, both the Weil conjecture as well as the strong form of the Tate conjecture hold for curves (recall that Weil proved his famous conjecture for curves using solely the classical intersection theory of divisors on surfaces). In this section, making use of Theorems 1.5 and 1.17, we bootstrap these results from curves to intersections of two quadrics and to linear sections of determinantal varieties. This yields an alternative noncommutative proof of the Weil conjecture for all these (higher dimensional) schemes, which avoids the involved tools used by Deligne. It yields moreover a proof of the strong form of the Tate conjecture in new cases.

Intersections of two quadrics. Let $X \subset \mathbb{P}^{n-1}$ be a smooth complete intersection of two quadric hypersurfaces, with $n \geq 4$. The linear span of these two quadrics gives rise to an hypersurface $Q \subset \mathbb{P}^1 \times \mathbb{P}^{n-1}$, and the projection onto the first factor gives rise to a flat quadric fibration $f: Q \rightarrow \mathbb{P}^1$ of relative dimension $n - 2$.

Theorem 3.1. *Assume that all the fibers of f have corank ≤ 1 . Under this assumption, the following holds:*

- (i) *When n is even, the conjectures $W(X)$ and $ST(X)$ hold.*
- (ii) *When n is odd and $p \neq 2$, the conjectures $W(X)$ and $ST(X)$ hold.*

Linear sections of determinantal varieties. Let U_1 and U_2 be two finite-dimensional k -vector spaces of dimensions d_1 and d_2 , respectively, $V := U_1 \otimes U_2$, and $0 < r < d_1$ an integer. Consider the determinantal variety $\mathcal{Z}_{d_1, d_2}^r \subset \mathbb{P}(V)$ defined as the locus of those matrices $U_2 \rightarrow U_1^*$ with rank $\leq r$.

Example 3.2 (Segre varieties). In the particular case where $r = 1$, the determinantal varieties reduce to the classical Segre varieties. Concretely, \mathcal{Z}_{d_1, d_2}^1 is given by the image of Segre homomorphism $\mathbb{P}(U_1) \times \mathbb{P}(U_2) \rightarrow \mathbb{P}(V)$.

In contrast with the Segre varieties, the varieties \mathcal{Z}_{d_1, d_2}^r , with $r \geq 2$, are not smooth. Their singular locus consists of those matrices $U_2 \rightarrow U_1^*$ with rank $< r$, i.e., it agrees with the closed subvarieties $\mathcal{Z}_{d_1, d_2}^{r-1}$. Nevertheless, it is well-known

that \mathcal{Z}_{d_1, d_2}^r admits a canonical resolution of singularities $X := \mathcal{X}_{d_1, d_2}^r \rightarrow \mathcal{Z}_{d_1, d_2}^r$. Dually, consider the variety $\mathcal{W}_{d_1, d_2}^r \subset \mathbb{P}(V^*)$, defined as the locus of those matrices $U_2^* \rightarrow U_1$ with corank $\geq r$, and the associated canonical resolution of singularities $Y := \mathcal{Y}_{d_1, d_2}^r \rightarrow \mathcal{W}_{d_1, d_2}^r$. Finally, given a linear subspace $L \subseteq V^*$, consider the associated linear sections $X_L := X \times_{\mathbb{P}(V)} \mathbb{P}(L^\perp)$ and $Y_L := Y \times_{\mathbb{P}(V^*)} \mathbb{P}(L)$.

Theorem 3.3. *Assume that X_L and Y_L are smooth⁵, and that $\text{codim}(X_L) = \dim(L)$ and $\text{codim}(Y_L) = \dim(L^\perp)$. Under these assumptions (which hold for a generic choice of L), the following holds:*

- (i) *When $\dim(L) = r(d_1 + d_2 - r) - 2$, the conjectures $W(Y_L)$ and $ST(Y_L)$ hold.*
- (ii) *When $\dim(L) = 2 - r(d_1 - d_2 - r)$, the conjectures $W(X_L)$ and $ST(X_L)$ hold.*

Example 3.4 (Segre varieties). Let $r = 1$. Thanks to Theorem 3.3(ii), when $\dim(L) = 3 - d_1 + d_2$, the conjectures $W(X_L)$ and $ST(X_L)$ hold. In all these cases, X_L is a linear section of the Segre variety \mathcal{Z}_{d_1, d_2}^1 . Moreover, $\dim(X_L) = 2d_1 - 5$. Therefore, for example, by letting $d_1 \rightarrow \infty$ and by keeping $\dim(L)$ fixed, we obtain infinitely many examples of smooth projective k -schemes X_L , of arbitrary high dimension, satisfying the Weil conjecture and the strong form of the Tate conjecture. In what concerns the latter conjecture, these examples are, to the best of the author's knowledge, new in the literature.

Example 3.5 (Square matrices). Let $d_1 = d_2$. Thanks to Theorem 3.3(ii), when $\dim(L) = 2 + r^2$, the conjectures $W(X_L)$ and $ST(X_L)$ hold. In all these cases, we have $\dim(X_L) = 2r(d_1 - r) - 3$. Therefore, for example, by letting $d_1 \rightarrow \infty$ and by keeping $\dim(L)$ fixed, we obtain infinitely many examples of smooth projective k -schemes X_L , of arbitrary high dimension, satisfying the Weil conjecture and the strong form of the Tate conjecture. In what concerns the latter conjecture, these examples are, to the best of the author's knowledge, new in the literature.

4. STATEMENT OF RESULTS: L -FUNCTIONS

Let X be a smooth proper \mathbb{Q} -scheme of dimension d . It is well-known that there exists a finite number of primes p_1, \dots, p_m and a smooth proper scheme \mathfrak{X} over $\text{Spec}(\mathbb{Z}[1/p_1, \dots, 1/p_m])$ such that $X \simeq \mathfrak{X} \times_{\text{Spec}(\mathbb{Z}[1/p_1, \dots, 1/p_m])} \text{Spec}(\mathbb{Q})$. In what follows, we will assume that X has *good reduction* at p_1, \dots, p_m , i.e., we will assume that for every $i = 1, \dots, m$ there exists a smooth proper scheme \mathfrak{X}_i over $\text{Spec}(\mathbb{Z}_{(p_i)})$ such that $X \simeq \mathfrak{X}_i \times_{\text{Spec}(\mathbb{Z}_{(p_i)})} \text{Spec}(\mathbb{Q})$. Given a prime $p \neq p_1, \dots, p_m$, let us write $\mathfrak{X}_p := \mathfrak{X} \times_{\text{Spec}(\mathbb{Z}[1/p_1, \dots, 1/p_m])} \text{Spec}(\mathbb{F}_p)$ for the fiber of \mathfrak{X} at p . In the same vein, let us write $\mathfrak{X}_{p_i} := \mathfrak{X}_i \times_{\text{Spec}(\mathbb{Z}_{(p_i)})} \text{Spec}(\mathbb{F}_{p_i})$ for the fiber of \mathfrak{X}_i at p_i . Under these assumptions and notations, recall that the L -function of X may be defined as the Euler product $L(X; s) := \prod_{p \neq p_1, \dots, p_m} \zeta(\mathfrak{X}_p; s) \cdot \prod_{i=1}^m \zeta(\mathfrak{X}_{p_i}; s)$. As proved by Serre in [53, 54], this infinite product converges (absolutely) in the half-plane $\text{Re}(s) > d + 1$. Moreover, $L(X; s)$ is non-zero in this half-plane region.

In the same vein, given an integer $0 \leq w \leq 2d$, consider the associated L -function $L_w(X; s) := \prod_{p \neq p_1, \dots, p_m} \zeta_w(\mathfrak{X}_p; s) \cdot \prod_{i=1}^m \zeta_w(\mathfrak{X}_{p_i}; s)$. Assuming the conjectures $W(\mathfrak{X}_p)$ and $W(\mathfrak{X}_{p_i})$ (which were proved by Deligne in the seventies), Serre proved in [53, 54] that the latter infinite product converges (absolutely) in the half-plane $\text{Re}(s) > \frac{w}{2} + 1$ and, moreover, that it is non-zero in this region.

⁵The linear section X_L is smooth if and only if the linear section Y_L is smooth.

Finally, note that the weight decomposition (1.2) (applied to the fibers \mathfrak{X}_p and \mathfrak{X}_{p_i}) yields the following weight decomposition of L -functions:

$$(4.1) \quad L(X; s) = \frac{L_0(X; s)L_2(X; s) \cdots L_{2d}(X; s)}{L_1(X; s)L_3(X; s) \cdots L_{2d-1}(X; s)} \quad \operatorname{Re}(s) > d + 1.$$

Let \mathcal{A} be a smooth proper \mathbb{Q} -linear dg category. Similarly to the commutative world, there exists a finite number of primes p_1, \dots, p_m and a smooth proper $\mathbb{Z}[1/p_1, \dots, 1/p_m]$ -linear dg category \mathfrak{A} such that \mathcal{A} and $\mathfrak{A} \otimes_{\mathbb{Z}[1/p_1, \dots, 1/p_m]} \mathbb{Q}$ are Morita equivalent. In what follows, we will assume that \mathcal{A} has *good reduction* at p_1, \dots, p_m , i.e., we will assume that for every $i = 1, \dots, m$ there exists a smooth proper $\mathbb{Z}_{(p_i)}$ -linear dg category \mathfrak{A}_i such that \mathcal{A} and $\mathfrak{A} \otimes_{\mathbb{Z}_{(p_i)}} \mathbb{Q}$ are Morita equivalent. Given a prime $p \neq p_1, \dots, p_m$, let us write $\mathfrak{A}_p := \mathfrak{A} \otimes_{\mathbb{Z}[1/p_1, \dots, 1/p_m]}^L \mathbb{F}_p$. In the same vein, let us write $\mathfrak{A}_{p_i} := \mathfrak{A}_i \otimes_{\mathbb{Z}_{(p_i)}}^L \mathbb{F}_{p_i}$. Under these assumptions and notations, we define the *even/odd L -function* of \mathcal{A} as the following Euler product:

$$(4.2) \quad L_{\text{even}}(\mathcal{A}; s) := \prod_{p \neq p_1, \dots, p_m} \zeta_{\text{even}}(\mathfrak{A}_p; s) \cdot \prod_{1 \leq i \leq m} \zeta_{\text{even}}(\mathfrak{A}_{p_i}; s)$$

$$(4.3) \quad L_{\text{odd}}(\mathcal{A}; s) := \prod_{p \neq p_1, \dots, p_m} \zeta_{\text{odd}}(\mathfrak{A}_p; s) \cdot \prod_{1 \leq i \leq m} \zeta_{\text{odd}}(\mathfrak{A}_{p_i}; s).$$

Theorem 4.4. *Assume that the conjectures $W_{\text{nc}}(\mathfrak{A}_p)$ and $W_{\text{nc}}(\mathfrak{A}_{p_i})$ hold. Under these assumptions, the infinite product (4.2), resp. (4.3), converges (absolutely) in the half-plane $\operatorname{Re}(s) > 1$, resp. $\operatorname{Re}(s) > \frac{3}{2}$. Moreover, the L -functions $L_{\text{even}}(\mathcal{A}; s)$ and $L_{\text{odd}}(\mathcal{A}; s)$ are non-zero in these half-plane regions.*

Remark 4.5 (Convergence). Note that the L -function $L_{\text{even}}(\mathcal{A}; s)$, resp. $L_{\text{odd}}(\mathcal{A}; s)$, converges (absolutely) in the half-plane $\operatorname{Re}(s) > 1$, resp. $\operatorname{Re}(s) > \frac{3}{2}$, if and only if the infinite product $\prod_{p \neq p_1, \dots, p_m} \zeta_{\text{even}}(\mathfrak{A}_p; s)$, resp. $\prod_{p \neq p_1, \dots, p_m} \zeta_{\text{odd}}(\mathfrak{A}_p; s)$, converges (absolutely) in the same half-plane. Therefore, Theorem 4.4 holds similarly without the assumption that the conjectures $W_{\text{nc}}(\mathfrak{A}_{p_i})$, $1 \leq i \leq m$, hold.

Remark 4.6 (Alternative proof of Serre's convergence result). By adapting the arguments used in the proof of Theorem 4.4 (consult §15), we present in §16 below an alternative noncommutative proof of Serre's convergence result.

Example 4.7 (Smooth proper schemes). When $\mathcal{A} = \operatorname{perf}_{\text{dg}}(X)$, with X a smooth proper \mathbb{Q} -scheme, we can choose for \mathfrak{A} the dg category $\operatorname{perf}_{\text{dg}}(\mathfrak{X})$ and for \mathfrak{A}_i the dg category $\operatorname{perf}_{\text{dg}}(\mathfrak{X}_i)$. Hence, since the dg categories \mathfrak{A}_p and \mathfrak{A}_{p_i} are Morita equivalent to $\operatorname{perf}_{\text{dg}}(\mathfrak{X}_p)$ and $\operatorname{perf}_{\text{dg}}(\mathfrak{X}_{p_i})$, respectively, we conclude that the factorizations of Corollary 1.8 (applied to the fibers \mathfrak{X}_p and \mathfrak{X}_{p_i}) yield the following factorizations:

$$(4.8) \quad L_{\text{even}}(\operatorname{perf}_{\text{dg}}(X); s) = \prod_{w \text{ even}} L_w(X; s + \frac{w}{2}) \quad \operatorname{Re}(s) > 1$$

$$(4.9) \quad L_{\text{odd}}(\operatorname{perf}_{\text{dg}}(X); s) = \prod_{w \text{ odd}} L_w(X; s + \frac{w-1}{2}) \quad \operatorname{Re}(s) > \frac{3}{2}.$$

Roughly speaking, this shows that the even/odd L -function of $\operatorname{perf}_{\text{dg}}(X)$ may be understood as the “weight normalization” of the product of the L -functions $L_w(X; s)$.

Example 4.10 (Riemann zeta function). In the particular case where $X = \operatorname{Spec}(\mathbb{Q})$, we can choose for \mathfrak{X} the smooth proper scheme $\operatorname{Spec}(\mathbb{Z})$. Consequently, we conclude

from Example 4.7 that the even L -function $L_{\text{even}}(\text{perf}_{\text{dg}}(\text{Spec}(\mathbb{Q})); s)$ agrees with the famous Riemann zeta function $L(\text{Spec}(\mathbb{Q}); s) := \prod_p \frac{1}{1-p^{-s}} = \sum_{n \geq 1} \frac{1}{n^s}$.

Example 4.11 (Dedekind zeta functions). In the particular case where $X = \text{Spec}(F)$, with F a number field, we can choose for \mathfrak{X} the smooth proper scheme $\text{Spec}(\mathcal{O}_F)$ (over $\text{Spec}(\mathbb{Z})$), where \mathcal{O}_F stands for the ring of integers of F . Consequently, we conclude from Example 4.7 that the even L -function $L_{\text{even}}(\text{perf}_{\text{dg}}(\text{Spec}(\mathbb{F})); s)$ agrees with the famous Dedekind zeta function $L(\text{Spec}(F); s) := \prod_p \prod_{\mathcal{P}|p} \frac{1}{1-N(\mathcal{P})^{-s}} = \sum_{\mathcal{I} \subseteq \mathcal{O}_F} \frac{1}{N(\mathcal{I})^s}$, where \mathcal{P} , resp. \mathcal{I} , is a prime ideal, resp. ideal, of the ring of integers \mathcal{O}_K and $N(\mathcal{P})$, resp. $N(\mathcal{I})$, is its norm.

Meromorphic continuation. Let X be a smooth proper \mathbb{Q} -scheme of dimension d . A classical conjecture (see Weil [66]) in the theory of L -functions is the following:

Conjecture M(X): The L -function $L(X; s)$ admits a (unique) meromorphic continuation to the entire complex plane.

In the same vein, given an integer $0 \leq w \leq 2d$, we have the following conjecture:

Conjecture M_w(X): The L -function $L_w(X; s)$ admits a (unique) meromorphic continuation to the entire complex plane.

The latter conjecture holds for 0-dimensional schemes, for abelian varieties with complex multiplication, for varieties of Fermat type, for certain products of modular curves, and also for certain Shimura varieties. Besides these cases (and some other cases scattered in the literature), it remains wide open. Note that thanks to the above weight decomposition (4.1), we have the implication $\sum_{w=0}^{2d} M_w(X) \Rightarrow M(X)$.

Given a smooth proper \mathbb{Q} -linear dg category \mathcal{A} , the above conjecture(s) admits the following noncommutative counterpart:

Conjecture M_{nc}(A): The even L -function $L_{\text{even}}(\mathcal{A}; s)$, resp. the odd L -function $L_{\text{odd}}(\mathcal{A}; s)$, admits a (unique) meromorphic continuation to the entire complex plane.

Remark 4.12. Given a smooth proper \mathbb{Q} -scheme X of dimension d , note that the factorizations (4.8)-(4.9) yield the implication $\sum_{w=0}^{2d} M_w(X) \Rightarrow M_{\text{nc}}(\text{perf}_{\text{dg}}(X))$. In particular, if the conjectures $\{M_w(X)\}_{w=0}^{2d}$ hold, then the above factorizations (4.8)-(4.9) hold in the entire complex plane.

Tate conjecture. Let X be a smooth proper \mathbb{Q} -scheme of dimension d and $0 \leq w \leq 2d$ an even integer. In what follows, we will assume that the conjecture $M_w(X)$ holds. In the mid sixties, Tate [65] conjectured the following:

Conjecture T_w(X): The L -function $L_w(X; s)$ has a unique pole at $s = \frac{w}{2} + 1$.

This conjecture holds for 0-dimensional schemes, for certain abelian varieties with complex multiplication, for certain varieties of Fermat type, for certain products of modular curves, and also for certain Shimura varieties. Besides these cases (and some other cases scattered in the literature), it remains wide open.

Let \mathcal{A} be a smooth proper \mathbb{Q} -linear dg category. In what follows, we will assume that the conjectures $W_{\text{nc}}(\mathfrak{A}_p)$ and $W_{\text{nc}}(\mathfrak{A}_{p_i})$ hold. Recall from Theorem 4.4 that this implies that $L_{\text{even}}(\mathcal{A}; s)$ converges (absolutely) in the half-plane $\text{Re}(s) > 1$. Also, we will assume that the conjecture $M_{\text{nc}}(\mathcal{A})$ holds. Under these assumptions, Tate's conjecture admits the following noncommutative counterpart:

Conjecture T_{nc}(A): The even L -function $L_{\text{even}}(\mathcal{A}; s)$ has a unique pole at $s = 1$.

Remark 4.13. Given a smooth proper \mathbb{Q} -scheme X , note that the factorization (4.8) yields the implication $\sum_{w \text{ even}} T_w(X) \Rightarrow T_{\text{nc}}(\text{perf}_{\text{dg}}(X))$.

Beilinson conjecture. Let X be a smooth proper \mathbb{Q} -scheme of dimension d and $0 \leq w \leq 2d$ an integer. In what follows, we will assume that the conjecture $M_w(X)$ holds. Given an integer $0 \leq i \leq d$, let us write $\mathcal{Z}^i(X)_{\mathbb{Q}/\sim_{\text{rat}}}$ for the \mathbb{Q} -vector space of algebraic cycles of codimension i on X up to rational equivalence, $\mathcal{Z}^i(X)_{\mathbb{Q}}^0/\sim_{\text{rat}}$ for the \mathbb{Q} -subspace of those algebraic cycles which are homologically trivial, and $\mathcal{Z}^i(X)_{\mathbb{Q}/\sim_{\text{hom}}}$ for the \mathbb{Q} -vector space of algebraic cycles of codimension i on X up to homological equivalence⁶. Also, given integers $i, j \in \mathbb{Z}$, let us write $H_{\text{mot}}^i(X; \mathbb{Q}(j))$ for the motivic cohomology groups of X .

In the eighties, Beilinson [3, 4, 5] conjectured the following:

Conjecture $B_w^j(X)$: *The following equalities hold:*

$$\text{ord}_{s=j} L_w(X; s) = \begin{cases} -\dim_{\mathbb{Q}} \mathcal{Z}^{\frac{w}{2}}(X)_{\mathbb{Q}/\sim_{\text{hom}}} & j = \frac{w}{2} + 1 & w \text{ even} \\ \dim_{\mathbb{Q}} H_{\text{mot}}^{w+1}(X; \mathbb{Q}(w+1-j)) & j \leq \frac{w}{2} & w \text{ even} \\ \dim_{\mathbb{Q}} \mathcal{Z}^{\frac{w+1}{2}}(X)_{\mathbb{Q}}^0/\sim_{\text{rat}} & j = \frac{w+1}{2} & w \text{ odd} \\ \dim_{\mathbb{Q}} H_{\text{mot}}^{w+1}(X; \mathbb{Q}(w+1-j)) & j \leq \frac{w-1}{2} & w \text{ odd}. \end{cases}$$

This conjecture holds for 0-dimensional schemes, for certain elliptic curves, for certain varieties of Fermat type, for certain products of modular curves, and also for certain Shimura varieties. Besides these cases (and some other cases scattered in the literature), it remains wide open.

Remark 4.14 (Tate conjecture). In addition to the conjecture $T_w(X)$, the conjecture $B_w^{\frac{w}{2}+1}(X)$, with w even, was also formulated by Tate in [65]. Note the parallelism between the set of conjectures $\{B_w^{\frac{w}{2}+1}(X)\}_{w \text{ even}}$ and the strong form of the Tate conjecture (consult §1).

Remark 4.15 (Birch and Swinnerton-Dyer conjecture). In the particular case where X is an elliptic curve, the Beilinson conjecture $B_1^1(X)$ reduces to the famous Birch and Swinnerton-Dyer conjecture [69], which asserts that the order $\text{ord}_{s=1} L_1(X; s)$ of the L -function $L_1(X; s)$ at the zero $s = 1$ is equal to the rank of $\text{Pic}^0(X)$.

Let \mathcal{A} be a smooth proper \mathbb{Q} -linear dg category. In what follows, we will assume that the conjectures $W_{\text{nc}}(\mathfrak{A}_p)$ and $W_{\text{nc}}(\mathfrak{A}_{p_i})$ hold. Recall from Theorem 4.4 that this implies that the even L -function $L_{\text{even}}(\mathcal{A}; s)$, resp. odd L -function $L_{\text{odd}}(\mathcal{A}; s)$, converges (absolutely) in the half-plane $\text{Re}(s) > 1$, resp. $\text{Re}(s) > \frac{3}{2}$. Also, we will assume that the conjecture $M_{\text{nc}}(\mathcal{A})$ holds. Recall from §5.4 below the definition of the Grothendieck group $K_0(\mathcal{A})_{\mathbb{Q}}$, of the \mathbb{Q} -subspace $K_0(\mathcal{A})_{\mathbb{Q}}^0$ of those Grothendieck classes which are homologically trivial, and of the homological Grothendieck group $K_0(\mathcal{A})_{\mathbb{Q}/\sim_{\text{hom}}}$. Under these definitions, Beilinson's conjecture admits the following noncommutative counterpart(s):

Conjecture $B_{\text{nc}, \text{even}}^j(\mathcal{A})$: *The following equalities hold:*

$$(4.16) \quad \text{ord}_{s=j} L_{\text{even}}(\mathcal{A}; s) = \begin{cases} -\dim_{\mathbb{Q}} K_0(\mathcal{A})_{\mathbb{Q}/\sim_{\text{hom}}} & j = 1 \\ \dim_{\mathbb{Q}} K_1(\mathcal{A})_{\mathbb{Q}} & j = 0 \\ \dim_{\mathbb{Q}} K_3(\mathcal{A})_{\mathbb{Q}} & j = -1. \end{cases}$$

⁶Recall that in characteristic zero all the classical Weil cohomology theories are isomorphic.

Conjecture $B_{\text{nc,odd}}^j(\mathcal{A})$: The following equalities hold:

$$(4.17) \quad \text{ord}_{s=j} L_{\text{odd}}(\mathcal{A}; s) = \begin{cases} \dim_{\mathbb{Q}} K_0(\mathcal{A})_{\mathbb{Q}}^0 & j = 1 \\ \dim_{\mathbb{Q}} K_2(\mathcal{A})_{\mathbb{Q}} & j = 0. \end{cases}$$

The noncommutative Beilinson conjecture(s) was originally envisioned by Kontsevich in his seminal talks [37, 38]. The next result relates this conjecture(s) with Beilinson's original conjecture:

Theorem 4.18. *Given a smooth proper \mathbb{Q} -scheme X , we have the implications:*

$$\sum_{w \text{ even}} B_w^{\frac{w}{2}+1}(X) \Rightarrow B_{\text{nc,even}}^1(\text{perf}_{\text{dg}}(X)) \quad \sum_{w \text{ odd}} B_w^{\frac{w+1}{2}}(X) \Rightarrow B_{\text{nc,odd}}^1(\text{perf}_{\text{dg}}(X)).$$

Assuming the Beilinson-Soulé vanishing conjecture (i.e., $H_{\text{mot}}^i(X; \mathbb{Q}(j)) = 0$ when $i < 0$ and also when $i = 0$ and $j > 0$), we have moreover the implications

$$\sum_{w \text{ even}} B_w^{\frac{w}{2}}(X) \Rightarrow B_{\text{nc,even}}^0(\text{perf}_{\text{dg}}(X)) \quad \sum_{w \text{ even}} B_w^{\frac{w}{2}-1}(X) \Rightarrow B_{\text{nc,even}}^{-1}(\text{perf}_{\text{dg}}(X))$$

as well as the implication $\sum_{w \text{ odd}} B_w^{\frac{w-1}{2}}(X) \Rightarrow B_{\text{nc,odd}}^0(\text{perf}_{\text{dg}}(X))$.

Remark 4.19 (Potential generalization). Let \mathcal{A} be a smooth proper \mathbb{Q} -linear dg category. Motivated by the above noncommutative Beilinson conjecture(s), it is natural to ask if the following equalities should be added to (4.16)-(4.17):

$$(4.20) \quad \text{ord}_{s=j} L_{\text{even}}(\mathcal{A}; s) = \dim_{\mathbb{Q}} K_{1-2j}(\mathcal{A})_{\mathbb{Q}} \quad j \leq -2$$

$$(4.21) \quad \text{ord}_{s=j} L_{\text{odd}}(\mathcal{A}; s) = \dim_{\mathbb{Q}} K_{2-2j}(\mathcal{A})_{\mathbb{Q}} \quad j \leq -1.$$

As explained in Remark 17.20 below, when $\mathcal{A} = \text{perf}_{\text{dg}}(X)$, with X a smooth proper \mathbb{Q} -scheme, the Beilinson conjecture plus the Beilinson-Soulé vanishing conjecture imply that the equality (4.20), resp. (4.21), holds if and only if the motivic cohomology groups $\{H_{\text{mot}}^{2r+2j-1}(X; \mathbb{Q}(r)) \mid d-j+1 < r \leq d-2j\}$, resp. $\{H_{\text{mot}}^{2r+2j-2}(X; \mathbb{Q}(r)) \mid d-j+1 < r \leq d-2j+1\}$, are zero. Unfortunately, to the best of the author's knowledge, nothing is known about these groups.

5. PRELIMINARIES

5.1. Dg categories. For a survey on dg categories, we invite the reader to consult⁷ [31]. Let k be a commutative ring and $(\mathcal{C}(k), \otimes, k)$ the category of (cochain) complexes of k -modules. A (k -linear) *dg category* \mathcal{A} is a category enriched over $\mathcal{C}(k)$ and a *dg functor* $F: \mathcal{A} \rightarrow \mathcal{B}$ is a functor enriched over $\mathcal{C}(k)$. In what follows, we will write $\text{dgc}at(k)$ for the category of small dg categories and dg functors.

Let \mathcal{A} be a dg category. The opposite dg category \mathcal{A}^{op} , resp. category $H^0(\mathcal{A})$, has the same objects as \mathcal{A} and $\mathcal{A}^{\text{op}}(x, y) := \mathcal{A}(y, x)$, resp. $H^0(\mathcal{A})(x, y) := H^0(\mathcal{A}(x, y))$. A *right dg \mathcal{A} -module* is a dg functor $M: \mathcal{A}^{\text{op}} \rightarrow \mathcal{C}_{\text{dg}}(k)$ with values in the dg category of complexes of k -modules. Let $\mathcal{C}(\mathcal{A})$ be the category of right dg \mathcal{A} -modules. Following [31, §3.2], the *derived category $\mathcal{D}(\mathcal{A})$ of \mathcal{A}* is defined as the localization of $\mathcal{C}(\mathcal{A})$ with respect to the objectwise quasi-isomorphisms. In what follows, we will write $\mathcal{D}_c(\mathcal{A})$ for the subcategory of compact objects.

A dg functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is called a *Morita equivalence* if it induces an equivalence between derived categories $\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}(\mathcal{B})$; see [31, §4.6]. As explained in [60, §1.6],

⁷Consult also the pioneering work [11].

the category $\mathrm{dgc}at(k)$ admits a Quillen model structure whose weak equivalences are the Morita equivalences. Let $\mathrm{Hmo}(k)$ be the associated homotopy category.

The *tensor product* $\mathcal{A} \otimes_k \mathcal{B}$ of dg categories is defined as follows: the set of objects of $\mathcal{A} \otimes_k \mathcal{B}$ is the cartesian product of the sets of objects of \mathcal{A} and \mathcal{B} and $(\mathcal{A} \otimes_k \mathcal{B})((x, w), (y, z)) := \mathcal{A}(x, y) \otimes_k \mathcal{B}(w, z)$. As explained in [60, §1.1.1 and §1.6.4], this construction gives rise to a symmetric monoidal structure $- \otimes_k -$ on the category $\mathrm{dgc}at(k)$, which descends $- \otimes_k^{\mathbf{L}} -$ to the homotopy category $\mathrm{Hmo}(k)$.

A *dg \mathcal{A} - \mathcal{B} -bimodule* is a dg functor $B: \mathcal{A} \otimes_k^{\mathbf{L}} \mathcal{B}^{\mathrm{op}} \rightarrow \mathcal{C}_{\mathrm{dg}}(k)$. An example is the dg \mathcal{A} - \mathcal{B} -bimodule ${}_F B: (x, w) \mapsto \mathcal{B}(w, F(x))$ associated to a dg functor $F: \mathcal{A} \rightarrow \mathcal{B}$.

Following [35, 36, 38, 39], a dg category \mathcal{A} is called *smooth* if the dg \mathcal{A} - \mathcal{A} -bimodule ${}_{\mathrm{id}} B$ belongs to the subcategory $\mathcal{D}_c(\mathcal{A}^{\mathrm{op}} \otimes_k^{\mathbf{L}} \mathcal{A})$ and *proper* if all the complexes of k -modules $\mathcal{A}(x, y)$ belong to the subcategory $\mathcal{D}_c(k)$. As explained in [60, Thm. 1.43], the smooth proper dg categories may be (conceptually) characterized as the dualizable objects of the symmetric monoidal category $\mathrm{Hmo}(k)$. Moreover, the dual of a smooth proper dg category \mathcal{A} is its opposite dg category $\mathcal{A}^{\mathrm{op}}$. In what follows, we will write $\mathrm{dgc}at_{\mathrm{sp}}(k)$ and $\mathrm{Hmo}_{\mathrm{sp}}(k)$ for the full symmetric monoidal subcategories of smooth proper dg categories.

5.2. Noncommutative motives. For a book, resp. survey, on noncommutative motives, we invite the reader to consult [60], resp. [58]. Recall from [60, §4.1] the construction of the category of *noncommutative Chow motives* $\mathrm{NChow}(k)_{\mathbb{Q}}$. This category is \mathbb{Q} -linear, additive, idempotent complete, rigid symmetric monoidal⁸, and comes equipped with a (composed) symmetric monoidal functor:

$$U(-)_{\mathbb{Q}}: \mathrm{dgc}at_{\mathrm{sp}}(k) \longrightarrow \mathrm{Hmo}_{\mathrm{sp}}(k)_{\mathbb{Q}} \longrightarrow \mathrm{NChow}(k)_{\mathbb{Q}}.$$

Moreover, given smooth proper dg categories \mathcal{A} and \mathcal{B} , we have an isomorphism:

$$(5.1) \quad \mathrm{Hom}_{\mathrm{NChow}(k)_{\mathbb{Q}}}(U(\mathcal{A})_{\mathbb{Q}}, U(\mathcal{B})_{\mathbb{Q}}) \simeq K_0(\mathcal{A}^{\mathrm{op}} \otimes_k^{\mathbf{L}} \mathcal{B})_{\mathbb{Q}}.$$

Recall from [60, §4.6] the construction of the category of *noncommutative numerical motives* $\mathrm{NNum}(k)_{\mathbb{Q}}$. This category is also \mathbb{Q} -linear, additive, idempotent complete, rigid symmetric monoidal, and comes equipped with a (quotient) \mathbb{Q} -linear symmetric monoidal functor $\mathrm{NChow}(k)_{\mathbb{Q}} \rightarrow \mathrm{NNum}(k)_{\mathbb{Q}}$.

5.3. Numerical Grothendieck group. Let k be a field. Given a smooth proper k -linear dg category \mathcal{A} , recall that its Grothendieck group $K_0(\mathcal{A}) := K_0(\mathcal{D}_c(\mathcal{A}))$ comes equipped with the Euler bilinear pairing $\chi: K_0(\mathcal{A}) \times K_0(\mathcal{A}) \rightarrow \mathbb{Z}$ defined as follows $([M], [N]) \mapsto \sum_n (-1)^n \dim_k \mathrm{Hom}_{\mathcal{D}_c(\mathcal{A})}(M, N[n])$. This pairing is not symmetric neither skew-symmetric. Nevertheless, making use of the notion of Serre functor developed in [12], it can be shown that the associated left and right kernels of χ agree; consult [60, Prop. 4.24]. Hence, the *numerical Grothendieck group* $K_0(\mathcal{A})/\sim_{\mathrm{num}}$ is defined as the quotient of $K_0(\mathcal{A})$ by the kernel of χ . As proved in [56, Thm. 5.1], $K_0(\mathcal{A})/\sim_{\mathrm{num}}$ is a finitely generated free abelian group. In what follows, we will write $K_0(\mathcal{A})_{\mathbb{Q}}/\sim_{\mathrm{num}}$ for the associated finite-dimensional \mathbb{Q} -vector space $K_0(\mathcal{A})/\sim_{\mathrm{num}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Finally, recall from [60, §4.6-§4.7] that, given smooth proper dg categories \mathcal{A} and \mathcal{B} , we have an isomorphism:

$$(5.2) \quad \mathrm{Hom}_{\mathrm{NNum}(k)_{\mathbb{Q}}}(U(\mathcal{A})_{\mathbb{Q}}, U(\mathcal{B})_{\mathbb{Q}}) \simeq K_0(\mathcal{A}^{\mathrm{op}} \otimes_k \mathcal{B})_{\mathbb{Q}}/\sim_{\mathrm{num}}.$$

⁸Recall that a symmetric monoidal category is called *rigid* if all its objects are dualizable.

5.4. Homological Grothendieck group. Let k be a field of characteristic zero. Following [44, §9], periodic cyclic homology gives rise to a \mathbb{Q} -linear functor

$$(5.3) \quad HP_*(-): \text{NChow}(k)_{\mathbb{Q}} \longrightarrow \text{mod}_{\mathbb{Z}}(k[v^{\pm 1}])$$

with values in the category of (degreewise finite-dimensional) \mathbb{Z} -graded $k[v^{\pm 1}]$ -modules, where v is a variable of degree -2 . Therefore, given a smooth proper k -linear dg category \mathcal{A} , by combining the functor (5.3) with the above identification (5.1) (with $\mathcal{A} := k$ and $\mathcal{B} := \mathcal{A}$), we obtain an induced \mathbb{Q} -linear homomorphism $\text{ch}: K_0(\mathcal{A})_{\mathbb{Q}} \rightarrow HP_0(\mathcal{A})$. Under these notations, the *homological Grothendieck group* $K_0(\mathcal{A})_{\mathbb{Q}}/\sim_{\text{hom}}$ is defined as the quotient of $K_0(\mathcal{A})_{\mathbb{Q}}$ by the kernel of ch . In the same vein, $K_0(\mathcal{A})_{\mathbb{Q}}^0$ is defined as the kernel of ch .

6. TOPOLOGICAL PERIODIC CYCLIC HOMOLOGY

Let $k = \mathbb{F}_q$ be a finite field of characteristic p , with $q = p^r$, $W(k)$ the ring of p -typical Witt vectors of k , and $K := W(k)_{1/p}$ the fraction field of $W(k)$. For recent/modern references on topological (periodic) cyclic homology, we invite the reader to consult [25, 48] (and also [1, 10]). Topological periodic cyclic homology gives rise to a symmetric monoidal functor $TP_*(-)_{1/p}: \text{dgcatsp}(k) \rightarrow \text{mod}_{\mathbb{Z}}(K[v^{\pm 1}])$ with values in the category of (degreewise finite-dimensional) \mathbb{Z} -graded $K[v^{\pm 1}]$ -modules, where v is a variable of degree -2 . As explained in [56, Thm. 2.3], this functor yields a \mathbb{Q} -linear symmetric monoidal functor:

$$(6.1) \quad TP_*(-)_{1/p}: \text{NChow}(k)_{\mathbb{Q}} \longrightarrow \text{mod}_{\mathbb{Z}}(K[v^{\pm 1}]).$$

6.1. Cyclotomic Frobenius. Let \mathcal{A} be a smooth proper k -linear dg category. By construction, its topological Hochschild homology $THH(\mathcal{A})$ carries a canonical cyclotomic structure in the sense of [48, §2]. Using the S^1 -action on $THH(\mathcal{A})$, we can consider the spectrum of homotopy orbits $THH(\mathcal{A})_{hS^1}$, the spectrum of homotopy fixed-points $TC^-(\mathcal{A}) := THH(\mathcal{A})^{hS^1}$, and also the Tate construction $TP(\mathcal{A}) := THH(\mathcal{A})^{tS^1}$ in the sense of Greenlees [21]. As explained in [48, Cor. I.4.3], these spectra are related by the following cofiber sequence

$$(6.2) \quad \Sigma THH(\mathcal{A})_{hS^1} \xrightarrow{N} THH(\mathcal{A})^{hS^1} \xrightarrow{\text{can}} THH(\mathcal{A})^{tS^1},$$

where N stands for the norm map. It is well-known that the abelian groups $THH_*(\mathcal{A})$ are k -linear. Hence, after inverting p , we have $\Sigma THH(\mathcal{A})_{hS^1}[1/p] \simeq *$. Consequently, the above cofiber sequence (6.2) leads to a canonical isomorphism:

$$(6.3) \quad \text{can}: TC_*^-(\mathcal{A})_{1/p} \xrightarrow{\simeq} TP_*(\mathcal{A})_{1/p}.$$

It is also well-known that the spectrum $THH(\mathcal{A})$ is a dualizable $THH(k)$ -module spectrum. Thanks to Bökstedt's celebrated computation $THH_*(k) \simeq k[u]$, where u is a variable of degree 2, this implies that the spectrum $THH(\mathcal{A})$ is *bounded below*, i.e., there exists an integer $m \in \mathbb{Z}$ such that $THH_n(\mathcal{A}) = 0$ for every $n < m$. Since the abelian groups $THH_*(\mathcal{A})$ are k -linear, this also implies that the spectrum $THH(\mathcal{A})$ is p -complete. Therefore, as explained in [48, Lem. II.4.2], the cyclotomic structure of $THH(\mathcal{A})$ yields another homomorphism:

$$(6.4) \quad \varphi_p: TC_*^-(\mathcal{A})_{1/p} \longrightarrow TP_*(\mathcal{A})_{1/p}.$$

It follows from [1, Prop. 4.7] that the homomorphism (6.4) is invertible. Hence, let us write $\varphi := \varphi_p \circ \text{can}^{-1}$ for the induced automorphism of $TP_*(\mathcal{A})_{1/p}$. The automorphism φ is not K -linear. Instead, it is semilinear with respect to the

isomorphism $K \xrightarrow{\sim} K$ induced by the Frobenius map $\lambda \mapsto \lambda^p$ on k . Therefore, its r -fold composition $F_* := \varphi^r$ becomes a K -linear automorphism of $TP_*(\mathcal{A})_{1/p}$.

Notation 6.5. The K -linear automorphism F_* is called the *cyclotomic Frobenius*.

The cyclotomic Frobenius F_* is not $K[v^{\pm 1}]$ -linear. Instead, it is semilinear with respect to the \mathbb{Z} -graded K -algebra isomorphism $\tau: K[v^{\pm 1}] \xrightarrow{\sim} K[v^{\pm 1}], v \mapsto qv$. In other words, we have the following commutative squares (for every $n \in \mathbb{Z}$):

$$(6.6) \quad \begin{array}{ccc} TP_n(\mathcal{A})_{1/p} & \xrightarrow[\simeq]{v \cdot -} & TP_{n-2}(\mathcal{A})_{1/p} \\ F_n \downarrow \simeq & & \simeq \downarrow \frac{1}{q} \cdot F_{n-2} \\ TP_n(\mathcal{A})_{1/p} & \xrightarrow[\simeq]{v \cdot -} & TP_{n-2}(\mathcal{A})_{1/p} . \end{array}$$

Consequently, we obtain an induced isomorphism of \mathbb{Z} -graded $K[v^{\pm 1}]$ -modules:

$$F_*: TP_*(\mathcal{A})_{1/p}^\tau := TP_*(\mathcal{A})_{1/p} \otimes_{K[v^{\pm 1}], \tau} K[v^{\pm 1}] \xrightarrow{\simeq} TP_*(\mathcal{A})_{1/p} .$$

Given smooth proper k -linear dg categories \mathcal{A} and \mathcal{B} , we have a natural isomorphism $F_*^{\mathcal{A} \otimes_k \mathcal{B}} \simeq F_*^{\mathcal{A}} \otimes_{K[v^{\pm 1}]} F_*^{\mathcal{B}}$. Therefore, by construction of the category $\text{NChow}(k)_{\mathbb{Q}}$, the assignment $U(\mathcal{A})_{\mathbb{Q}} \mapsto F_*^{\mathcal{A}}$ (parametrized by the smooth proper dg categories \mathcal{A}) yields a \mathbb{Q} -linear symmetric monoidal natural transformation from the functor

$$(6.7) \quad TP_*(-)_{1/p}^\tau: \text{NChow}(k)_{\mathbb{Q}} \longrightarrow \text{mod}_{\mathbb{Z}}(K[v^{\pm 1}])$$

to the above functor (6.1).

Remark 6.8 (Loss of information). The above commutative squares (6.6) show that there is no loss of information in working solely with F_0 and F_1 (as done as §1).

Similarly to §1, given an integer $n \in \mathbb{Z}$ and an embedding $\iota: K \hookrightarrow \mathbb{C}$, we can consider the following Hasse-Weil zeta function:

$$\zeta_n(\mathcal{A}; s) := \det(\text{id} - q^{-s}(F_n \otimes_{K, \iota} \mathbb{C}) | TP_n(\mathcal{A})_{1/p} \otimes_{K, \iota} \mathbb{C})^{-1} .$$

Thanks to the above squares (6.6), we have $\zeta_n(\mathcal{A}; s) = \zeta_0(\mathcal{A}; s + \frac{n}{2})$ when n is even and $\zeta_n(\mathcal{A}; s) = \zeta_1(\mathcal{A}; s + \frac{n-1}{2})$ when n is odd. Consequently, there is no loss of information in working solely with the even/odd Hasse-Weil zeta functions $\zeta_{\text{even}}(\mathcal{A}; s) := \zeta_0(\mathcal{A}; s)$ and $\zeta_{\text{odd}}(\mathcal{A}; s) := \zeta_1(\mathcal{A}; s)$ (as done in §1).

6.2. Enriched topological periodic cyclic homology functor. Let us write $\text{Aut}(K[v^{\pm 1}])^\tau$ for the category of τ -semilinear automorphisms in $\text{mod}_{\mathbb{Z}}(K[v^{\pm 1}])$. Recall that an object of $\text{Aut}(K[v^{\pm 1}])^\tau$ is a pair (V_*, f_*) , where V_* is a (degree-wise finite-dimensional) \mathbb{Z} -graded $K[v^{\pm 1}]$ -module V_* and $f_*: V_*^\tau \xrightarrow{\sim} V_*$ is an automorphism. Thanks to the considerations of §6.1, note that by combining the functor (6.1) with the cyclotomic Frobenius natural transformation, we obtain the following \mathbb{Q} -linear symmetric monoidal functor:

$$(6.9) \quad \text{NChow}(k)_{\mathbb{Q}} \longrightarrow \text{Aut}(K[v^{\pm 1}])^\tau \quad U(\mathcal{A})_{\mathbb{Q}} \mapsto (TP_*(\mathcal{A})_{1/p}, F_*).$$

In what follows, we will call (6.9) the *enriched topological periodic cyclic homology functor*. As explained in §11.5 below, this functor enables an alternative formulation of the noncommutative strong form of the Tate conjecture (when all smooth proper dg categories are considered simultaneously).

7. PROOF OF THEOREM 1.5

Following [18, Thm. 2][57, Thm. 5.2] (this is a result of Scholze), we have natural isomorphisms of finite-dimensional K -vector spaces:

$$(7.1) \quad TP_0(\mathrm{perf}_{\mathrm{dg}}(X))_{1/p} \simeq \bigoplus_{w \text{ even}} H_{\mathrm{crys}}^w(X)$$

$$(7.2) \quad TP_1(\mathrm{perf}_{\mathrm{dg}}(X))_{1/p} \simeq \bigoplus_{w \text{ odd}} H_{\mathrm{crys}}^w(X).$$

Moreover, following [25, §7], the cyclotomic Frobenius F_0 corresponds under the above isomorphism (7.1) to the following automorphism:

$$(7.3) \quad \bigoplus_{w \text{ even}} q^{-\frac{w}{2}} \mathrm{Fr}^w : \bigoplus_{w \text{ even}} H_{\mathrm{crys}}^w(X) \xrightarrow{\simeq} \bigoplus_{w \text{ even}} H_{\mathrm{crys}}^w(X).$$

Similarly, the cyclotomic Frobenius F_1 corresponds under the above isomorphism (7.2) to the following automorphism:

$$(7.4) \quad \bigoplus_{w \text{ odd}} q^{-\frac{w-1}{2}} \mathrm{Fr}^w : \bigoplus_{w \text{ odd}} H_{\mathrm{crys}}^w(X) \xrightarrow{\simeq} \bigoplus_{w \text{ odd}} H_{\mathrm{crys}}^w(X).$$

As usual, let $d := \dim(X)$. Also, given an integer $0 \leq w \leq 2d$, let $\beta_w := \dim_K H_{\mathrm{crys}}^w(X)$ and $\{\lambda_{(w,1)}, \dots, \lambda_{(w,\beta_w)}\}$ be the eigenvalues (with multiplicity) of the automorphism Fr^w . Thanks to (7.3)-(7.4), the eigenvalues of the cyclotomic Frobenius F_0 are given by $\bigcup_{w \text{ even}} \{q^{-\frac{w}{2}} \lambda_{(w,1)}, \dots, q^{-\frac{w}{2}} \lambda_{(w,\beta_w)}\}$ and the eigenvalues of the cyclotomic Frobenius F_1 are given by $\bigcup_{w \text{ odd}} \{q^{-\frac{w-1}{2}} \lambda_{(w,1)}, \dots, q^{-\frac{w-1}{2}} \lambda_{(w,\beta_w)}\}$. Consequently, since the numbers $q^{-\frac{w}{2}}$, with w even, and $q^{-\frac{w-1}{2}}$, with w odd, are rational, we conclude that $W_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X))$ holds if and only if $W(X)$ holds.

Proof of Corollary 1.8. Thanks to (7.3)-(7.4), note that we have the following description of the even/odd Hasse-Weil zeta function:

$$\begin{aligned} \zeta_{\mathrm{even}}(\mathrm{perf}_{\mathrm{dg}}(X); s) &= \prod_{w \text{ even}} \det(\mathrm{id} - q^{-(s+\frac{w}{2})} (\mathrm{Fr}^w \otimes_{K,\iota} \mathbb{C}) | H_{\mathrm{crys}}^w(X) \otimes_{K,\iota} \mathbb{C})^{-1} \\ \zeta_{\mathrm{odd}}(\mathrm{perf}_{\mathrm{dg}}(X); s) &= \prod_{w \text{ odd}} \det(\mathrm{id} - q^{-(s+\frac{w-1}{2})} (\mathrm{Fr}^w \otimes_{K,\iota} \mathbb{C}) | H_{\mathrm{crys}}^w(X) \otimes_{K,\iota} \mathbb{C})^{-1}. \end{aligned}$$

Since the polynomials $\det(\mathrm{id} - t\mathrm{Fr}^w | H_{\mathrm{crys}}^w(X))$ have integer coefficients, we hence conclude that the right-hand sides of the above equalities are equal to the products $\prod_{w \text{ even}} \zeta_w(X; s + \frac{w}{2})$ and $\prod_{w \text{ odd}} \zeta_w(X; s + \frac{w-1}{2})$, respectively.

8. PROOF OF THEOREM 1.10

Since the proof uses some results of §9, we postpone it to §10.

9. PROOF OF THEOREM 1.12

We start by recalling the following general result, whose proof is a simple linear algebra exercise that we leave for the reader.

Lemma 9.1. *Let $\theta: V \otimes_K W \rightarrow K$ a perfect bilinear pairing of finite-dimensional K -vector spaces, f an automorphism of V , g an automorphism of W , and $\lambda \in K$*

a non-zero scalar, making the following diagram commute:

$$\begin{array}{ccc} V \otimes_K W & \xrightarrow{\theta} & K \\ f \otimes_K g \downarrow \simeq & & \simeq \downarrow \lambda \cdot - \\ V \otimes_K W & \xrightarrow{\theta} & K. \end{array}$$

Under these assumptions, we have the following equality of polynomials:

$$\det(\text{id} - t g | W) = \frac{(-1)^{\dim(V)} \lambda^{\dim(V)} t^{\dim(V)}}{\det(f | V)} \cdot \det(\text{id} - \lambda^{-1} t^{-1} f | V).$$

Proposition 9.2. *Given a smooth proper k -linear dg category \mathcal{A} , there exist perfect bilinear pairings θ_0 and θ_1 making the following diagrams commute:*

$$\begin{array}{ccc} TP_0(\mathcal{A}^{\text{op}})_{1/p} \otimes_K TP_0(\mathcal{A})_{1/p} & \xrightarrow{\theta_0} & K \\ F_0 \otimes_K F_0 \downarrow \simeq & & \parallel \\ TP_0(\mathcal{A}^{\text{op}})_{1/p} \otimes_K TP_0(\mathcal{A})_{1/p} & \xrightarrow{\theta_0} & K \end{array} \quad \begin{array}{ccc} TP_1(\mathcal{A}^{\text{op}})_{1/p} \otimes_K TP_1(\mathcal{A})_{1/p} & \xrightarrow{\theta_1} & K \\ F_1 \otimes_K F_1 \downarrow \simeq & & \simeq \downarrow q \cdot - \\ TP_1(\mathcal{A}^{\text{op}})_{1/p} \otimes_K TP_1(\mathcal{A})_{1/p} & \xrightarrow{\theta_1} & K. \end{array}$$

Proof. Recall from §6.1 that the assignment $U(\mathcal{A})_{\mathbb{Q}} \mapsto F_*^{\mathcal{A}}$ (parametrized by the smooth proper dg categories \mathcal{A}) yields a \mathbb{Q} -linear symmetric monoidal natural transformation from the functor (6.7) to the functor (6.1). Recall also from §5.1-§5.2 that $U(\mathcal{A})_{\mathbb{Q}}$ is a dualizable object of the symmetric monoidal category $\text{NChow}(k)_{\mathbb{Q}}$ and that $U(\mathcal{A}^{\text{op}})_{\mathbb{Q}}$ is the dual of $U(\mathcal{A})_{\mathbb{Q}}$. Consequently, by applying the functors (6.7) and (6.1) to the evaluation morphism $U(\mathcal{A}^{\text{op}})_{\mathbb{Q}} \otimes U(\mathcal{A})_{\mathbb{Q}} \rightarrow U(k)_{\mathbb{Q}}$, we obtain the following commutative diagram:

$$(9.3) \quad \begin{array}{ccc} TP_*^{\tau}(\mathcal{A}^{\text{op}})_{1/p}^{\tau} \otimes_{K[v^{\pm 1}]} TP_*^{\tau}(\mathcal{A})_{1/p}^{\tau} & \longrightarrow & TP_*(k)_{1/p}^{\tau} = K[v^{\pm 1}]^{\tau} \\ F_* \otimes_{K[v^{\pm 1}]} F_* \downarrow \simeq & & \simeq \downarrow F_*^k \\ TP_*^{\tau}(\mathcal{A}^{\text{op}})_{1/p} \otimes_{K[v^{\pm 1}]} TP_*^{\tau}(\mathcal{A})_{1/p} & \longrightarrow & TP_*(k)_{1/p} = K[v^{\pm 1}]. \end{array}$$

Note that $TP_n(-)_{1/p}^{\tau} = TP_n(-)_{1/p}$ for every (fixed) integer $n \in \mathbb{Z}$. Hence, we define the left-hand side commutative diagram of Proposition 9.2 as the following composition

$$\begin{array}{ccc} TP_0(\mathcal{A}^{\text{op}})_{1/p}^{\tau} \otimes_K TP_0(\mathcal{A})_{1/p}^{\tau} & \rightarrow & (TP_*^{\tau}(\mathcal{A}^{\text{op}})_{1/p}^{\tau} \otimes_{K[v^{\pm 1}]} TP_*^{\tau}(\mathcal{A})_{1/p}^{\tau})_0 \rightarrow TP_0(k)_{1/p}^{\tau} \\ F_0 \otimes_K F_0 \downarrow \simeq & & (F_* \otimes_{K[v^{\pm 1}]} F_*)_0 \downarrow \simeq & \parallel_{F_0 = \text{id}} \\ TP_0(\mathcal{A}^{\text{op}})_{1/p} \otimes_K TP_0(\mathcal{A})_{1/p} & \rightarrow & (TP_*^{\tau}(\mathcal{A}^{\text{op}})_{1/p} \otimes_{K[v^{\pm 1}]} TP_*^{\tau}(\mathcal{A})_{1/p})_0 \rightarrow TP_0(k)_{1/p}, \end{array}$$

where the left-hand side horizontal morphisms are induced by the monoidal structure of the category $\text{mod}_{\mathbb{Z}}(K[v^{\pm 1}])$ and the right-hand side horizontal morphisms are induced from (9.3). By construction, the horizontal composition(s), denoted by θ_0 , is a perfect bilinear pairing. Similarly, the right-hand side commutative diagram

of Proposition 9.2 is defined as the composition

$$\begin{array}{ccccc}
TP_1(\mathcal{A}^{\text{op}})_{1/p}^\tau \otimes_K TP_1(\mathcal{A})_{1/p}^\tau & \rightarrow & (TP_*(\mathcal{A}^{\text{op}})_{1/p}^\tau \otimes_{K[v^\pm 1]} TP_*(\mathcal{A})_{1/p}^\tau)_2 & \rightarrow & TP_2(k)_{1/p}^\tau \\
\downarrow \text{F}_1 \otimes_K \text{F}_1 \simeq & & \downarrow (\text{F}_* \otimes_{K[v^\pm 1]} \text{F}_*)_2 \simeq & & \downarrow \text{F}_2 = q \cdot \\
TP_1(\mathcal{A}^{\text{op}})_{1/p} \otimes_K TP_1(\mathcal{A})_{1/p} & \rightarrow & (TP_*(\mathcal{A}^{\text{op}})_{1/p} \otimes_{K[v^\pm 1]} TP_*(\mathcal{A})_{1/p})_2 & \rightarrow & TP_2(k)_{1/p},
\end{array}$$

where the left-hand side horizontal morphisms are induced by the monoidal structure of the category $\text{mod}_{\mathbb{Z}}(K[v^\pm 1])$ and the right-hand side horizontal morphisms are induced from (9.3). By construction, the horizontal composition(s), denoted by θ_1 , is a perfect bilinear pairing. \square

By construction of periodic cyclic homology, we have $TP_*(\mathcal{A}^{\text{op}})_{1/p} = TP_*(\mathcal{A})_{1/p}$ (as \mathbb{Z} -graded $K[v^\pm 1]$ -modules) and $\text{F}_*^{\mathcal{A}^{\text{op}}} = \text{F}_*^{\mathcal{A}}$. Therefore, by applying the above general Lemma 9.1 to the perfect bilinear pairings θ_0 and θ_1 of Proposition 9.2, we hence obtain the following equalities of polynomials:

$$\begin{aligned}
\det(\text{id} - t\text{F}_0 | TP_0(\mathcal{A})_{1/p}) &= \frac{(-1)^{\chi_0(\mathcal{A})} t^{\chi_0(\mathcal{A})}}{\det(\text{F}_0 | TP_0(\mathcal{A})_{1/p})} \cdot \det(\text{id} - t^{-1}\text{F}_0 | TP_0(\mathcal{A})_{1/p}) \\
\det(\text{id} - t\text{F}_1^{\mathcal{A}} | TP_1(\mathcal{A})_{1/p}) &= \frac{(-1)^{\chi_1(\mathcal{A})} q^{\chi_1(\mathcal{A})} t^{\chi_1(\mathcal{A})}}{\det(\text{F}_1 | TP_1(\mathcal{A})_{1/p})} \cdot \det(\text{id} - q^{-1}t^{-1}\text{F}_1 | TP_1(\mathcal{A})_{1/p}).
\end{aligned}$$

Now, choose an embedding $\iota: K \hookrightarrow \mathbb{C}$ and replace F_0 and F_1 by $\text{F}_0 \otimes_{K,\iota} \mathbb{C}$ and $\text{F}_1 \otimes_{K,\iota} \mathbb{C}$, respectively. By further replacing t by q^{-s} , and then by passing to the inverse, we hence obtain the sought functional equations of Theorem 1.12.

Proof of Corollary 1.13. Note that the isomorphisms (7.1)-(7.2) imply that $\chi_0(\text{perf}_{\text{dg}}(X)) = \chi_{\text{even}}(X)$ and $\chi_1(\text{perf}_{\text{dg}}(X)) = \chi_{\text{odd}}(X)$. Note also that the descriptions (7.3) and (7.4) of F_0 and F_1 , respectively, lead to the following equalities

$$(9.4) \quad \det(\text{F}_0 \otimes_{K,\iota} \mathbb{C}) = \prod_{w \text{ even}} q^{-\frac{w}{2}\beta_w} \cdot \det(\text{Fr}^w \otimes_{K,\iota} \mathbb{C})$$

$$(9.5) \quad \det(\text{F}_1 \otimes_{K,\iota} \mathbb{C}) = \prod_{w \text{ odd}} q^{-\frac{w-1}{2}\beta_w} \cdot \det(\text{Fr}^w \otimes_{K,\iota} \mathbb{C}),$$

where $\beta_w := \dim_K H_{\text{crys}}^w(X)$. Now, recall, for example from [24, App. C Thm. 4.4], that we have the following equalities

$$(9.6) \quad \det(\text{Fr}^{2d-w} \otimes_{K,\iota} \mathbb{C}) = \frac{q^{d\beta_w}}{\det(\text{Fr}^w \otimes_{K,\iota} \mathbb{C})} \quad 0 \leq w \leq 2d,$$

where $d := \dim(X)$. By combining the equalities (9.6) with the fact that $\beta_w = \beta_{2d-w}$ for every $0 \leq w \leq 2d$, we hence conclude (via a simple computation) that the square of (9.4), resp. (9.5), is equal to 1, resp. to $q^{\chi_{\text{odd}}(X)}$. These considerations imply that the functional equations of Theorem 1.12, with $\mathcal{A} = \text{perf}_{\text{dg}}(X)$, reduce to the following functional equations:

$$\begin{aligned}
\zeta_{\text{even}}(\text{perf}_{\text{dg}}(X); s) &= \pm q^{\chi_{\text{even}}(X)s} \cdot \zeta_{\text{even}}(\text{perf}_{\text{dg}}(X); -s) \\
\zeta_{\text{odd}}(\text{perf}_{\text{dg}}(X); s) &= \pm q^{\chi_{\text{odd}}(X)s} \cdot \zeta_{\text{odd}}(\text{perf}_{\text{dg}}(X); 1-s).
\end{aligned}$$

Consequently, the proof follows now from Corollary 1.8.

10. PROOF OF THEOREM 1.10

Let λ be an eigenvalue of the automorphism F_0 . If, by hypothesis, $q^C \lambda$ is an algebraic integer, then λ is, in particular, an algebraic number. Therefore, it suffices to prove that all the l -adic conjugates of λ have absolute value 1.

As explained in §9, there exists a perfect bilinear pairing θ_0 making the following diagram commute (consult Proposition 9.2 and the subsequent arguments):

$$(10.1) \quad \begin{array}{ccc} TP_0(\mathcal{A})_{1/p} \otimes_K TP_0(\mathcal{A})_{1/p} & \xrightarrow{\theta_0} & K \\ \text{F}_0 \otimes_K \text{F}_0 \downarrow \simeq & & \parallel \\ TP_0(\mathcal{A})_{1/p} \otimes_K TP_0(\mathcal{A})_{1/p} & \xrightarrow{\theta_0} & K. \end{array}$$

Thanks to Lemma 9.1, this implies that whenever λ is an eigenvalue of the automorphism F_0 , $\frac{1}{\lambda}$ is also an eigenvalue of the automorphism F_0 .

Let us write $\mathbb{Q}(\lambda)/\mathbb{Q}$ for the (finite) field extension of \mathbb{Q} generated by λ , $\mathcal{O} \subset \mathbb{Q}(\lambda)$ for the associated ring of integers, and $(q^C \lambda) = \mathfrak{P}_1 \cdots \mathfrak{P}_n$ and $(q^C \frac{1}{\lambda}) = \mathfrak{P}'_1 \cdots \mathfrak{P}'_m$ for the (unique) prime decomposition in \mathcal{O} of the ideals generated by the algebraic integers $q^C \lambda$ and $q^C \frac{1}{\lambda}$, respectively. Since \mathcal{O} is a Dedekind domain, we have the following (unique) prime decomposition:

$$(q^{2C}) = (q^C \lambda)(q^C \frac{1}{\lambda}) = \mathfrak{P}_1 \cdots \mathfrak{P}_n \mathfrak{P}'_1 \cdots \mathfrak{P}'_m.$$

This implies that all the prime ideals $\mathfrak{P}_1, \dots, \mathfrak{P}_n, \mathfrak{P}'_1, \dots, \mathfrak{P}'_m$ lie over $p \in \mathbb{Z}$. Consequently, since $\lambda = (q^C)^{-1}(q^C \lambda)$, we conclude that the absolute value of all the l -adic conjugates of λ is necessarily equal to 1. This finishes the proof in the case of the eigenvalues of F_0 . The proof in the case of the eigenvalues of F_1 is similar. Simply replace (10.1) by the commutative diagram:

$$(10.2) \quad \begin{array}{ccc} TP_1(\mathcal{A})_{1/p} \otimes_K TP_1(\mathcal{A})_{1/p} & \xrightarrow{\theta_1} & K \\ \text{F}_1 \otimes_K \text{F}_1 \downarrow \simeq & & \simeq \downarrow q \cdot - \\ TP_1(\mathcal{A})_{1/p} \otimes_K TP_1(\mathcal{A})_{1/p} & \xrightarrow{\theta_1} & K. \end{array}$$

Similarly to (10.1), (10.2) implies that whenever λ is an eigenvalue of the automorphism F_1 , $\frac{q}{\lambda}$ is also an eigenvalue of the automorphism F_1 .

Remark 10.3 (Smooth proper schemes). Let X be a smooth proper k -scheme of dimension d . Given an integer $0 \leq w \leq 2d$, let $\beta_w := \dim_K H_{\text{crys}}^w(X)$ and $\{\lambda_{(w,1)}, \dots, \lambda_{(w,\beta_w)}\}$ be the eigenvalues (with multiplicities) of the automorphism Fr^w . As explained in the proof of Theorem 1.5, the eigenvalues of the cyclotomic Frobenius F_0 (F_0 is an automorphism of $TP_0(\text{perf}_{\text{dg}}(X))_{1/p}$) are given by $\bigcup_{w \text{ even}} \{q^{-\frac{w}{2}} \lambda_{(w,1)}, \dots, q^{-\frac{w}{2}} \lambda_{(w,\beta_w)}\}$ and the eigenvalues of the cyclotomic Frobenius F_1 are given by $\bigcup_{w \text{ odd}} \{q^{-\frac{w-1}{2}} \lambda_{(w,1)}, \dots, q^{-\frac{w-1}{2}} \lambda_{(w,\beta_w)}\}$. It is well-known that the eigenvalues $\lambda_{(w,1)}, \dots, \lambda_{(w,\beta_w)}$ are algebraic integers. Therefore, by taking $C := d$, we conclude that the eigenvalues of the automorphisms F_0 and F_1 become algebraic integers after multiplication by q^C . In other words, the assumption of Theorem 1.10 holds when $\mathcal{A} = \text{perf}_{\text{dg}}(X)$.

11. NONCOMMUTATIVE STRONG FORM OF THE TATE CONJECTURE

Let $k = \mathbb{F}_q$ be a finite field of characteristic p . In this section we prove that the noncommutative strong form of the Tate conjecture is equivalent to the noncommutative p -version of the Tate conjecture plus the noncommutative standard conjecture of type D . As a byproduct of this equivalence of conjectures, we obtain a proof of Theorem 1.17 and also an alternative formulation of the noncommutative strong form of the Tate conjecture in terms of the enriched topological periodic cyclic homology functor (see §11.5).

11.1. Noncommutative standard conjecture of type D . Let \mathcal{A} be a smooth proper dg category. Similarly to §5.4, by combining the functor (6.1) with the identification (5.1) (with $\mathcal{A} := k$ and $\mathcal{B} := \mathcal{A}$), we obtain an induced \mathbb{Q} -linear homomorphism $\text{ch}: K_0(\mathcal{A})_{\mathbb{Q}} \rightarrow TP_0(\mathcal{A})_{1/p}$ and the associated homological Grothendieck group $K_0(\mathcal{A})_{\mathbb{Q}/\sim_{\text{hom}}}$. Under these notations, the noncommutative standard conjecture of type D asserts the following:

Conjecture $D_{\text{nc}}(\mathcal{A})$: The equality $K_0(\mathcal{A})_{\mathbb{Q}/\sim_{\text{hom}}} = K_0(\mathcal{A})_{\mathbb{Q}/\sim_{\text{num}}}$ holds.

Remark 11.1 (Standard conjecture of type D). Let X be a smooth proper k -scheme. As proved in [57, Thm. 1.1], we have the following equivalence of conjectures $D_{\text{nc}}(\text{perf}_{\text{dg}}(X)) \Leftrightarrow D(X)$, where $D(X)$ stands for the standard conjecture of type D (consult Grothendieck [22] and Kleiman [32, 33]).

11.2. Noncommutative p -version of the Tate conjecture. Let \mathcal{A} be a smooth proper dg category. As proved in [61, Lem. 3.7], the \mathbb{Q} -linear homomorphism $\text{ch}: K_0(\mathcal{A})_{\mathbb{Q}} \rightarrow TP_0(\mathcal{A})_{1/p}$ defined in §11.1 takes values in $TP_0(\mathcal{A})_{1/p}^{\mathbb{F}_0}$. Hence, we can consider the induced K -linear homomorphism $\text{ch}_K: K_0(\mathcal{A})_K \rightarrow TP_0(\mathcal{A})_{1/p}^{\mathbb{F}_0}$ and the associated homological Grothendieck group $K_0(\mathcal{A})_K/\sim_{\text{hom}}$. Under these notations, the noncommutative p -version of the Tate conjecture asserts the following:

Conjecture $T_{\text{nc}}^p(\mathcal{A})$: The homomorphism ch_K is surjective.

Remark 11.2 (p -version of the Tate conjecture). Let X be a smooth proper k -scheme. As proved in [61, Thm. 1.3], we have the following equivalence of conjectures $T_{\text{nc}}^p(\text{perf}_{\text{dg}}(X)) \Leftrightarrow T^p(X)$, where $T^p(X)$ stands for the p -version of the Tate conjecture (consult Milne [47] and Tate [65]).

11.3. Equivalence of conjectures. The next result is of independent interest:

Theorem 11.3. *Given a smooth proper k -linear dg category \mathcal{A} , we have the equivalence of conjectures $ST_{\text{nc}}(\mathcal{A}) \Leftrightarrow T_{\text{nc}}^p(\mathcal{A}) + D_{\text{nc}}(\mathcal{A})$.*

Proof. We start by proving the implication $ST_{\text{nc}}(\mathcal{A}) \Rightarrow T_{\text{nc}}^p(\mathcal{A}) + D_{\text{nc}}(\mathcal{A})$. Recall from Remark 1.15 that if the conjecture $ST_{\text{nc}}(\mathcal{A})$ holds, then the algebraic multiplicity of the eigenvalue 1 of F_0 agrees with the dimension of the \mathbb{Q} -vector space $K_0(\mathcal{A})_{\mathbb{Q}/\sim_{\text{num}}}$. Note also that the geometric multiplicity of the eigenvalue 1 of F_0 , which is always less (or equal) than the algebraic multiplicity, agrees with the dimension of the K -vector space $TP_0(\mathcal{A})_{1/p}^{\mathbb{F}_0}$. In order to prove the conjecture $T_{\text{nc}}^p(\mathcal{A})$, we need then to show that the dimension of the K -vector space $TP_0(\mathcal{A})_{1/p}^{\mathbb{F}_0}$ is less (or equal) than the dimension of the K -vector space $K_0(\mathcal{A})_K/\sim_{\text{hom}}$. This follows

from the following (in)equalities:

$$\begin{aligned}
\dim_K TP_0(\mathcal{A})_{1/p}^{F_0} &= \text{geometric multiplicity of the eigenvalue 1 of } F_0 \\
&\leq \text{algebraic multiplicity of the eigenvalue 1 of } F_0 \\
&= \dim_{\mathbb{Q}} K_0(\mathcal{A})_{\mathbb{Q}/\sim_{\text{num}}} \\
&\leq \dim_{\mathbb{Q}} K_0(\mathcal{A})_{\mathbb{Q}/\sim_{\text{hom}}} = \dim_K K_0(\mathcal{A})_{K/\sim_{\text{hom}}}.
\end{aligned}$$

Similarly, since $\dim_{\mathbb{Q}} K_0(\mathcal{A})_{\mathbb{Q}/\sim_{\text{num}}} \leq \dim_{\mathbb{Q}} K_0(\mathcal{A})_{\mathbb{Q}/\sim_{\text{hom}}}$, in order to prove the conjecture $D_{\text{nc}}(\mathcal{A})$, we need then to show that the dimension of the \mathbb{Q} -vector space $K_0(\mathcal{A})_{\mathbb{Q}/\sim_{\text{hom}}}$ is less (or equal) than the dimension of the \mathbb{Q} -vector space $K_0(\mathcal{A})_{\mathbb{Q}/\sim_{\text{num}}}$. This follows from the following (in)equalities:

$$\begin{aligned}
\dim_{\mathbb{Q}} K_0(\mathcal{A})_{\mathbb{Q}/\sim_{\text{hom}}} &= \dim_K K_0(\mathcal{A})_{K/\sim_{\text{hom}}} \\
&\leq \dim_K TP_0(\mathcal{A})_{1/p}^{F_0} \\
&= \text{geometric multiplicity of the eigenvalue 1 of } F_0 \\
&\leq \text{algebraic multiplicity of the eigenvalue 1 of } F_0 \\
&= \dim_{\mathbb{Q}} K_0(\mathcal{A})_{\mathbb{Q}/\sim_{\text{num}}}.
\end{aligned}$$

We now prove the implication $T_{\text{nc}}^p(\mathcal{A}) + D_{\text{nc}}(\mathcal{A}) \Rightarrow ST_{\text{nc}}(\mathcal{A})$. Note that if both conjectures $T_{\text{nc}}^p(\mathcal{A})$ and $D_{\text{nc}}(\mathcal{A})$ hold, then the geometric multiplicity of the eigenvalue 1 of F_0 is equal to the dimension of the \mathbb{Q} -vector space $K_0(\mathcal{A})_{\mathbb{Q}/\sim_{\text{num}}}$. Hence, in order to prove the conjecture $ST_{\text{nc}}(\mathcal{A})$, it suffices then to show that the geometric multiplicity of the eigenvalue 1 of F_0 agrees with the algebraic multiplicity of the eigenvalue 1 of F_0 . Thanks to the general Lemma 11.5 below, this will follow from the injectivity of the canonical morphism $\epsilon: TP_0(\mathcal{A})_{1/p}^{F_0} \rightarrow (TP_0(\mathcal{A})_{1/p})_{F_0}$ (induced by the identity on $TP_0(\mathcal{A})_{1/p}$). Recall from §5.1-§5.2 that $U(\mathcal{A})_{\mathbb{Q}}$ is a dualizable object of the symmetric monoidal category $\text{NChow}(k)_{\mathbb{Q}}$ and that $U(\mathcal{A}^{\text{op}})_{\mathbb{Q}}$ is the dual of $U(\mathcal{A})_{\mathbb{Q}}$. Consequently, by applying the functor $\text{Hom}_{\text{NChow}(k)_{\mathbb{Q}}}(U(k)_{\mathbb{Q}}, -)$ to the evaluation morphism $U(\mathcal{A}^{\text{op}})_{\mathbb{Q}} \otimes U(\mathcal{A})_{\mathbb{Q}} \rightarrow U(k)_{\mathbb{Q}}$, we obtain (from the symmetric monoidal structure of $\text{NChow}(k)_{\mathbb{Q}}$) a bilinear pairing $\psi: K_0(\mathcal{A}^{\text{op}})_{\mathbb{Q}} \otimes_{\mathbb{Q}} K_0(\mathcal{A})_{\mathbb{Q}} \rightarrow \mathbb{Q}$. Note that since the \mathbb{Q} -linear functor (6.1) is symmetric monoidal, we have the following commutative diagram

$$\begin{array}{ccc}
TP_0(\mathcal{A}^{\text{op}})_{1/p}^{F_0} \otimes_K TP_0(\mathcal{A})_{1/p}^{F_0} & \xrightarrow{\theta_0} & K \\
\text{ch}_K \otimes_K \text{ch}_K \uparrow & & \parallel \\
K_0(\mathcal{A}^{\text{op}})_K \otimes_K K_0(\mathcal{A})_K & \xrightarrow{\psi_K} & K,
\end{array}$$

where θ_0 stands for the perfect bilinear pairing of Proposition 9.2 and ψ_K for the K -linearization of ψ . By adjunction, this yields the induced commutative diagram:

$$(11.4) \quad \begin{array}{ccc}
TP_0(\mathcal{A})_{1/p}^{F_0} & \xrightarrow{\theta_0^{\natural}} & \text{Hom}_K(TP_0(\mathcal{A}^{\text{op}})_{1/p}^{F_0}, K) \\
\text{ch}_K \uparrow & & \downarrow \text{Hom}_K(\text{ch}_K, K) \\
K_0(\mathcal{A})_{K/\sim_{\text{hom}}} & \xrightarrow{\psi_K^{\natural}} & \text{Hom}_K(K_0(\mathcal{A}^{\text{op}})_{K/\sim_{\text{hom}}}, K).
\end{array}$$

Thanks to the left-hand side commutative diagram of Proposition 9.2, the morphism θ_0^{\natural} admits the following factorization:

$$\theta_0^{\natural}: TP_0(\mathcal{A})_{1/p}^{\mathbb{F}_0} \xrightarrow{\epsilon} (TP_0(\mathcal{A})_{1/p})_{\mathbb{F}_0} \longrightarrow \text{Hom}_K(TP_0(\mathcal{A}^{\text{op}})_{1/p}^{\mathbb{F}_0}, K).$$

Using the fact that the left-hand side vertical morphism in (11.4) is surjective (=conjecture $T_{\text{nc}}^p(\mathcal{A})$), we observe that in order to show that the canonical morphism ϵ is injective, it suffices then to show that the morphism ψ_K^{\natural} is injective. As explained in [46, §6], a Grothendieck class $\alpha \in K_0(\mathcal{A})_{\mathbb{Q}}$ is numerically trivial in the sense of §5.3 if and only if $\psi(\beta, \alpha) = 0$ for every $\beta \in K_0(\mathcal{A}^{\text{op}})_{\mathbb{Q}}$. In other words, the numerical Grothendieck group $K_0(\mathcal{A})_{\mathbb{Q}/\sim_{\text{num}}}$ may be identified with the quotient of $K_0(\mathcal{A})_{\mathbb{Q}/\sim_{\text{hom}}}$ by the kernel of θ_0^{\natural} . Therefore, in order to prove that ψ_K^{\natural} is injective, we can then consider the following commutative diagram:

$$\begin{array}{ccc} (K_0(\mathcal{A})_{\mathbb{Q}/\sim_{\text{hom}}})_K & \xlongequal{\quad} & K_0(\mathcal{A})_{K/\sim_{\text{hom}}} \longrightarrow \text{Hom}_K(K_0(\mathcal{A}^{\text{op}})_{K/\sim_{\text{hom}}}, K) \\ \downarrow & & \downarrow \\ (K_0(\mathcal{A})_{\mathbb{Q}/\sim_{\text{num}}})_K & \xlongequal{\quad} & K_0(\mathcal{A})_{K/\sim_{\text{num}}} \end{array} \quad .$$

Since the curved morphism is injective and the vertical morphism(s) is injective (=conjecture $D_{\text{nc}}(\mathcal{A})$), we hence conclude that the morphism ψ_K^{\natural} is also injective. This finishes the proof of Theorem 11.3. \square

Lemma 11.5. *Let $f: V \xrightarrow{\sim} V$ be an automorphism of a finite-dimensional K -vector space V . Under these assumptions, the geometric multiplicity of the eigenvalue 1 of f agrees with the algebraic multiplicity of the eigenvalue 1 of f if and only if the canonical morphism $\epsilon: V^f \rightarrow V_f$ (induced by the identity on V) is injective.*

Proof. Note that the geometric multiplicity of the eigenvalue 1 of f agrees with the algebraic multiplicity of the eigenvalue 1 of f if and only if $\text{Ker}(\text{id} - f) = \text{Ker}((\text{id} - f)^2)$. Hence, the proof follows from the fact that the latter condition is equivalent to the condition $\text{Ker}(\text{id} - f) \cap \text{Im}(\text{id} - f) = \emptyset$, i.e., to the injectivity of the canonical morphism $V^f \rightarrow V_f$. \square

11.4. Proof of Theorem 1.17. As mentioned in Remarks 11.1 and 11.2, we have the following equivalences of conjectures

$$D_{\text{nc}}(\text{perf}_{\text{dg}}(X)) \Leftrightarrow D(X) \quad T_{\text{nc}}^p(\text{perf}_{\text{dg}}(X)) \Leftrightarrow T^p(X).$$

Therefore, by combining Theorem 11.3 (with $\mathcal{A} = \text{perf}_{\text{dg}}(X)$) with the equivalence of conjectures $\text{ST}(X) \Leftrightarrow T^p(X) + D(X)$ established in [47, Thm. 1.11] (consult also [64, Thm. 2.9]), we obtain the sought equivalence $\text{ST}_{\text{nc}}(\text{perf}_{\text{dg}}(X)) \Leftrightarrow \text{ST}(X)$.

Remark 11.6 (Direct proof). A direct proof of the following implication of conjectures $\text{ST}(X) \Rightarrow \text{ST}_{\text{nc}}(\text{perf}_{\text{dg}}(X))$ can be achieved as follows: thanks to Corollary 1.9 and to the factorization (1.2), we have the following equality:

$$\text{ord}_{s=0} \zeta_{\text{even}}(\text{perf}_{\text{dg}}(X); s) = \sum_{0 \leq j \leq d} \text{ord}_{s=j} \zeta(X; s).$$

Therefore, since the numerical Grothendieck group $K_0(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}/\sim_{\text{num}}}$ is isomorphic to the direct sum $\bigoplus_{i=0}^d \mathcal{Z}^i(X)_{\mathbb{Q}/\sim_{\text{num}}}$ (consult [59, Prop. 1.7(i)]), we conclude that the conjecture $\text{ST}_{\text{nc}}(\text{perf}_{\text{dg}}(X))$ follows from $\text{ST}(X)$.

11.5. Alternative formulation. The next result, of independent result, provides an alternative formulation of the noncommutative strong form of the Tate conjecture (when all smooth proper dg categories are considered simultaneously):

Proposition 11.7. *The enriched topological periodic cyclic homology functor (6.9) induces the following K -linear symmetric monoidal fully-faithful functor*

$$(11.8) \quad \text{NNum}(k)_K \longrightarrow \text{Aut}(K[v^{\pm 1}])^\tau \quad U(\mathcal{A})_K \mapsto (TP_*(\mathcal{A})_{1/p}, F_*)$$

if and only if the conjecture $\text{ST}_{\text{nc}}(\mathcal{A})$ holds for every smooth proper dg category \mathcal{A} .

Proof. By construction, every object of $\text{NNum}(k)_K$ is a direct summand of an object of the form $U(\mathcal{A})_K$ with \mathcal{A} a smooth proper k -linear dg category; see [60, §4.6]. This implies that the functor (6.9) descends to the category of noncommutative numerical motives $\text{NNum}(k)_K$ if and only if the conjecture $\text{D}_{\text{nc}}(\mathcal{A})$ holds for every smooth proper dg category \mathcal{A} . Given smooth proper dg categories \mathcal{B} and \mathcal{C} , note that by definition of the category $\text{Aut}(K[v^{\pm 1}])^\tau$, we have an isomorphism:

$$\text{Hom}_{\text{Aut}(K[v^{\pm 1}])^\tau}((TP_*(\mathcal{B})_{1/p}, F_*), (TP_*(\mathcal{C})_{1/p}, F_*)) \simeq TP_0(\mathcal{B}^{\text{op}} \otimes_k \mathcal{C})_{1/p}^{\text{F}_0}.$$

This implies that the K -linear symmetric monoidal faithful functor (11.8) is moreover full if and only if the conjecture $\text{T}_{\text{nc}}^p(\mathcal{A})$ holds for every smooth proper dg category \mathcal{A} . Consequently, the proof follows now from Theorem 11.3. \square

12. PROOF OF THEOREMS 2.1, 2.4-2.7, AND 2.11-2.13

Let $\text{NM} \in \text{NChow}(k)_\mathbb{Q}$ be a noncommutative Chow motive. Note that similarly to smooth proper dg categories, we can formulate the conjectures $\text{W}_{\text{nc}}(\text{NM})$, $\text{W}_{\text{nc}}^l(\text{NM})$, $\text{ST}_{\text{nc}}(\text{NM})$, $\text{D}_{\text{nc}}(\text{NM})$, and $\text{T}_{\text{nc}}^p(\text{NM})$. Moreover, a proof similar to the one of Theorem 11.3 yields the equivalence $\text{ST}_{\text{nc}}(\text{NM}) \Leftrightarrow \text{T}_{\text{nc}}^p(\text{NM}) + \text{D}_{\text{nc}}(\text{NM})$.

In the particular case where $\text{NM} = U(\mathcal{A})_\mathbb{Q}$, with \mathcal{A} a smooth proper k -linear dg category, the aforementioned conjectures reduce to $\text{W}_{\text{nc}}(\mathcal{A})$, $\text{W}_{\text{nc}}^l(\mathcal{A})$, $\text{ST}_{\text{nc}}(\mathcal{A})$, $\text{D}_{\text{nc}}(\mathcal{A})$, and $\text{T}_{\text{nc}}^p(\mathcal{A})$, respectively.

Proposition 12.1. *The conjectures $\text{W}_{\text{nc}}(-)$, $\text{W}_{\text{nc}}^l(-)$, $\text{ST}_{\text{nc}}(-)$, $\text{D}_{\text{nc}}(-)$, $\text{T}_{\text{nc}}^p(-)$, are stable under direct sums and direct summands of noncommutative Chow motives.*

Proof. The stability under direct sums is clear. The stability under direct summands is also clear for the noncommutative Weil conjecture(s), the noncommutative standard conjecture of type D , and the noncommutative p -version of the Tate conjecture. In what regards the noncommutative strong form of the Tate conjecture, it follows from the equivalence $\text{ST}_{\text{nc}}(\text{NM}) \Leftrightarrow \text{T}_{\text{nc}}^p(\text{NM}) + \text{D}_{\text{nc}}(\text{NM})$. \square

Corollary 12.2. *Given noncommutative Chow motives $\text{NM}, \text{NM}' \in \text{NChow}(k)_\mathbb{Q}$, we have the equivalence of conjectures $\text{W}_{\text{nc}}(\text{NM} \oplus \text{NM}') \Leftrightarrow \text{W}_{\text{nc}}(\text{NM}) + \text{W}_{\text{nc}}(\text{NM}')$, the equivalence of conjectures $\text{W}_{\text{nc}}^l(\text{NM} \oplus \text{NM}') \Leftrightarrow \text{W}_{\text{nc}}^l(\text{NM}) + \text{W}_{\text{nc}}^l(\text{NM}')$, and the equivalence of conjectures $\text{ST}_{\text{nc}}(\text{NM} \oplus \text{NM}') \Leftrightarrow \text{ST}_{\text{nc}}(\text{NM}) + \text{ST}_{\text{nc}}(\text{NM}')$.*

Example 12.3 (Semi-orthogonal decompositions). Let $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ be smooth proper k -linear dg categories inducing a semi-orthogonal decomposition of triangulated categories $\text{H}^0(\mathcal{A}) = \langle \text{H}^0(\mathcal{B}), \text{H}^0(\mathcal{C}) \rangle$ in the sense of Bondal-Orlov [13]. As proved in [60, Prop. 2.2], the inclusions $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ give rise to an isomorphism of noncommutative Chow motives $U(\mathcal{A})_\mathbb{Q} \simeq U(\mathcal{B})_\mathbb{Q} \oplus U(\mathcal{C})_\mathbb{Q}$. Hence, Corollary 12.2 yields the equivalence $\text{W}_{\text{nc}}(\mathcal{A}) \Leftrightarrow \text{W}_{\text{nc}}(\mathcal{B}) + \text{W}_{\text{nc}}(\mathcal{C})$, the equivalence $\text{W}_{\text{nc}}^l(\mathcal{A}) \Leftrightarrow \text{W}_{\text{nc}}^l(\mathcal{B}) + \text{W}_{\text{nc}}^l(\mathcal{C})$, and also the equivalence $\text{ST}_{\text{nc}}(\mathcal{A}) \Leftrightarrow \text{ST}_{\text{nc}}(\mathcal{B}) + \text{ST}_{\text{nc}}(\mathcal{C})$.

Proof of Theorem 2.1. As proved in [63, Thm. 2.1], we have an isomorphism of noncommutative Chow motives $U(\mathrm{perf}_{\mathrm{dg}}(X))_{\mathbb{Q}} \simeq U(\mathrm{perf}_{\mathrm{dg}}(\mathcal{X}; \mathcal{F}))_{\mathbb{Q}}$. Hence, we have the equivalence of conjectures $W_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X)) \Leftrightarrow W_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X; \mathcal{F}))$, the equivalence of conjectures $W_{\mathrm{nc}}^l(\mathrm{perf}_{\mathrm{dg}}(X)) \Leftrightarrow W_{\mathrm{nc}}^l(\mathrm{perf}_{\mathrm{dg}}(X; \mathcal{F}))$, and the equivalence of conjectures $ST_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X)) \Leftrightarrow ST_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X; \mathcal{F}))$. Therefore, the proof follows from Theorems 1.5, 1.10 (plus Remark 10.3), and 1.17.

Proof of Theorem 2.4. The (noncommutative) Weil conjecture(s) as well as the (noncommutative) strong form of the Tate conjecture hold for $\mathrm{Spec}(k)$. Consequently, an iterated application of Example 12.3 to the semi-orthogonal decomposition (2.2) yields the equivalence of conjectures $W_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X)) \Leftrightarrow W_{\mathrm{nc}}(\mathcal{T}_{\mathrm{dg}}(X))$, the equivalence of conjectures $W_{\mathrm{nc}}^l(\mathrm{perf}_{\mathrm{dg}}(X)) \Leftrightarrow W_{\mathrm{nc}}^l(\mathcal{T}_{\mathrm{dg}}(X))$, and the equivalence of conjectures $ST_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X)) \Leftrightarrow ST_{\mathrm{nc}}(\mathcal{T}_{\mathrm{dg}}(X))$. Therefore, the proof follows from Theorems 1.5, 1.10 (plus Remark 10.3), and 1.17.

Proof of Theorem 2.5. Up to Morita equivalence, the dg category $X \ominus_{\mathrm{B}} Y$ admits a semi-orthogonal decomposition whose components are (Fourier-Mukai) equivalent to $\mathrm{perf}(X)$ and $\mathrm{perf}(Y)$. Consequently, the proof of Theorem 2.5 follows from the combination of Example 12.3 with Theorems 1.5, 1.10 (plus Remark 10.3), and 1.17.

Proof of Theorem 2.6. As proved by Ishii-Ueda in [26, Thm. 1.6], whenever the zero locus $D \hookrightarrow X$ of ς is smooth, we have a semi-orthogonal decomposition

$$(12.4) \quad \mathrm{perf}(\mathcal{X}) = \langle \mathrm{perf}(D)_{n-1}, \dots, \mathrm{perf}(D)_1, f^*(\mathrm{perf}(X)) \rangle,$$

where all the categories $\mathrm{perf}(D)_i$ are (Fourier-Mukai) equivalent to $\mathrm{perf}(D)$ and $f^*(\mathrm{perf}(X))$ is (Fourier-Mukai) equivalent to $\mathrm{perf}(X)$. Therefore, making use of Theorems 1.5, 1.10 (plus Remark 10.3), and 1.17, the proof follows from an iterated application of Example 12.3 to the semi-orthogonal decomposition (12.4).

Proof of Theorem 2.7. Let φ be the set of all cyclic subgroups of G and φ/\sim a set of representatives of conjugacy classes in φ . Since the category $\mathrm{NChow}(k)_{\mathbb{Q}}$ is \mathbb{Q} -linear, it follows from [62, Thm. 1.1 and Rk. 1.3(iii)] that the noncommutative Chow motive $U(\mathrm{perf}_{\mathrm{dg}}(\mathcal{X}))_{\mathbb{Q}}$ is a direct summand of the following direct sum:

$$(12.5) \quad \bigoplus_{\sigma \in \varphi/\sim} U(\mathrm{perf}_{\mathrm{dg}}(X^{\sigma} \times \mathrm{Spec}(k[\sigma])))_{\mathbb{Q}}.$$

Under the assumption $n|(q-1)$, the same holds with $X^{\sigma} \times \mathrm{Spec}(k[\sigma])$ replaced by X^{σ} ; consult [62, Cor. 1.5(ii)]. Recall from Proposition 12.1 that the noncommutative Weil conjecture(s) and the noncommutative strong form of the Tate conjecture are stable under direct sums and direct summands of noncommutative Chow motives. Therefore, the proof follows from the combination of (12.5) (under the assumption $n|(q-1)$, replace $X^{\sigma} \times \mathrm{Spec}(k[\sigma])$ by X^{σ}) with Theorems 1.5, 1.10 (plus Remark 10.3), and 1.17.

Proof of Theorem 2.11. The proof of Theorem 2.11 is similar to the proof of Theorem 2.7 (in the case where $n|(q-1)$). Simply, replace $\mathrm{perf}_{\mathrm{dg}}(\mathcal{X})$ by $\mathrm{perf}_{\mathrm{dg}}(\mathcal{X}; \mathcal{F})$, $\mathrm{perf}_{\mathrm{dg}}(X^{\sigma})$ by $\mathrm{perf}_{\mathrm{dg}}(Y_{\sigma})$, and [62, Cor. 1.5(ii)] by [62, Cor. 1.28(ii)].

Proof of Theorem 2.12. Consider the finite-dimensional k -algebra $H^0(A)$ and the associated semi-simple k -algebra $B := H^0(A)/\text{Jac}(H^0(A))$, where $\text{Jac}(H^0(A))$ stands for the Jacobson radical of $H^0(A)$. Let us write V_1, \dots, V_n for the simple (right) B -modules and $D_1 := \text{End}_B(V_1), \dots, D_n := \text{End}_B(V_n)$ for the associated division k -algebras. Note that since k is finite, the division k -algebras D_1, \dots, D_n are just finite field extensions $\kappa_1, \dots, \kappa_n$ of k . Under these notations, we have isomorphisms of noncommutative Chow motives

$$(12.6) \quad U(A)_{\mathbb{Q}} \simeq U(B)_{\mathbb{Q}} \simeq U(\kappa_1)_{\mathbb{Q}} \oplus \cdots \oplus U(\kappa_n)_{\mathbb{Q}},$$

where the first isomorphism was established by Raedschelders-Stevenson in [51, Thm. 3.5] and the second isomorphism follows from the fact that B is semi-simple. By combining the isomorphism (12.6) with Corollary 12.2, we obtain the equivalence $W_{\text{nc}}(A) \Leftrightarrow W_{\text{nc}}(\kappa_1) + \cdots + W_{\text{nc}}(\kappa_n)$, the equivalence $W_{\text{nc}}^l(A) \Leftrightarrow W_{\text{nc}}^l(\kappa_1) + \cdots + W_{\text{nc}}^l(\kappa_n)$, and the equivalence $\text{ST}_{\text{nc}}(A) \Leftrightarrow \text{ST}_{\text{nc}}(\kappa_1) + \cdots + \text{ST}_{\text{nc}}(\kappa_n)$. Consequently, since the conjectures $W_{\text{nc}}(\kappa_i)$, $W_{\text{nc}}^l(\kappa_i)$, and $\text{ST}_{\text{nc}}(\kappa_i)$ hold, the conjectures $W_{\text{nc}}(A)$, $W_{\text{nc}}^l(A)$, and $\text{ST}_{\text{nc}}(A)$, also hold.

Proof of Theorem 2.13. Following Orlov [50, §2.3], let us write $\text{Aus}(A)$ for the smooth proper Auslander dg k -algebra associated to A (Orlov used a different notation). As proved in [50, Thms. 2.18-2.19], we have a semi-orthogonal decomposition

$$\text{perf}(\text{Aus}(A)) = \langle \text{perf}(D_1), \dots, \text{perf}(D_n) \rangle,$$

where D_i is a division k -algebra; note that since k is finite, the division k -algebras D_1, \dots, D_n are just finite field extensions $\kappa_1, \dots, \kappa_n$ of k . Moreover, as also proved in *loc. cit.*, the category $\text{perf}(A)$ can be embedded (using a Fourier-Mukai functor) as an admissible triangulated subcategory of $\text{perf}(\text{Aus}(A))$. This implies that the noncommutative Chow motive $U(A)_{\mathbb{Q}}$ is a direct summand of the direct sum $\bigoplus_{i=1}^n U(\kappa_i)$. Consequently, since the conjectures $W_{\text{nc}}(\kappa_i)$, $W_{\text{nc}}^l(\kappa_i)$, and $\text{ST}_{\text{nc}}(\kappa_i)$ hold, we conclude from Proposition 12.1 that the conjectures $W_{\text{nc}}(A)$, $W_{\text{nc}}^l(A)$, and $\text{ST}_{\text{nc}}(A)$, also hold.

13. PROOF OF THEOREM 3.1

As proved in [41, Thm. 5.5] (see also [2, Thm. 2.3.7]), we have the following semi-orthogonal decomposition

$$\text{perf}(X) = \langle \text{perf}(\mathbb{P}^1; \mathcal{C}l_0(q)), \mathcal{O}_X(1), \dots, \mathcal{O}_X(n-4) \rangle,$$

where $\mathcal{C}l_0(q)$ stands for the sheaf of even parts of the Clifford algebra associated to the flat quadric fibration $f: Q \rightarrow \mathbb{P}^1$. Consequently, since the (noncommutative) Weil conjecture as well as the (noncommutative) strong form of the Tate conjecture hold for $\text{Spec}(k)$, an iterated application of Example 12.3 yields the following equivalences of conjectures:

$$(13.1) \quad W_{\text{nc}}(\text{perf}_{\text{dg}}(X)) \Leftrightarrow W_{\text{nc}}(\text{perf}_{\text{dg}}(\mathbb{P}^1; \mathcal{C}l_0(q)))$$

$$(13.2) \quad \text{ST}_{\text{nc}}(\text{perf}_{\text{dg}}(X)) \Leftrightarrow \text{ST}_{\text{nc}}(\text{perf}_{\text{dg}}(\mathbb{P}^1; \mathcal{C}l_0(q))).$$

We start by proving item (i). Following [41, §3.5] (see also [2, §1.6]), let \mathcal{Z} be the center of $\mathcal{C}l_0(q)$ and $\text{Spec}(\mathcal{Z}) =: \tilde{\mathbb{P}}^1 \rightarrow \mathbb{P}^1$ the discriminant cover of \mathbb{P}^1 . As explained in *loc. cit.*, $\tilde{\mathbb{P}}^1 \rightarrow \mathbb{P}^1$ is a 2-fold cover which is ramified over the (finite) set D of critical values of f . Moreover, since D is smooth, $\tilde{\mathbb{P}}^1$ is also smooth. Let us write \mathcal{F} for the sheaf $\mathcal{C}l_0(q)$ considered as a sheaf of noncommutative algebras over $\tilde{\mathbb{P}}^1$. As

proved in *loc. cit.*, since by assumption all the fibers of $f: Q \rightarrow \mathbb{P}^1$ have corank ≤ 1 , \mathcal{F} is a sheaf of Azumaya algebras over $\widehat{\mathbb{P}}^1$. Moreover, the category $\text{perf}(\mathbb{P}^1; \mathcal{C}l_0(q))$ is (Fourier-Mukai) equivalent to $\text{perf}(\widehat{\mathbb{P}}^1; \mathcal{F})$. Note that since the Brauer group of every smooth curve over a finite field k is trivial, the latter category reduces to $\text{perf}(\widehat{\mathbb{P}}^1)$. Therefore, thanks to Theorem 1.5, resp. Theorem 1.17, the equivalence (13.1), resp. (13.2), reduces to $W(X) \Leftrightarrow W(\widehat{\mathbb{P}}^1)$, resp. $ST(X) \Leftrightarrow ST(\widehat{\mathbb{P}}^1)$. Finally, since $\widehat{\mathbb{P}}^1$ is a curve, we hence conclude that the conjectures $W(X)$ and $ST(X)$ hold.

We now prove item (ii). Note first that $1/2 \in k$. Following [41, §3.6] (see also [2, §1.7]), let $\widehat{\mathbb{P}}^1$ the discriminant stack associated to the flat quadric fibration $f: Q \rightarrow \mathbb{P}^1$. As explained in *loc. cit.*, since $1/2 \in k$, $\widehat{\mathbb{P}}^1$ is a square root stack. Moreover, the underlying k -scheme of $\widehat{\mathbb{P}}^1$ is \mathbb{P}^1 . Therefore, it follows from Theorem 2.6 that the conjectures $W_{\text{nc}}(\text{perf}_{\text{dg}}(\widehat{\mathbb{P}}^1))$ and $ST_{\text{nc}}(\text{perf}_{\text{dg}}(\widehat{\mathbb{P}}^1))$ hold. Let us write \mathcal{F} for the sheaf $\mathcal{C}l_0(q)$ considered as a sheaf of noncommutative algebras over $\widehat{\mathbb{P}}^1$. As proved in *loc. cit.*, since by assumption all the fibers of $f: Q \rightarrow \mathbb{P}^1$ have corank ≤ 1 , \mathcal{F} is a sheaf of Azumaya algebras over $\widehat{\mathbb{P}}^1$. Moreover, the category $\text{perf}(\mathbb{P}^1; \mathcal{C}l_0(q))$ is (Fourier-Mukai) equivalent to $\text{perf}(\widehat{\mathbb{P}}^1; \mathcal{F})$. Note that since the Brauer group of every smooth curve over a finite field k is trivial, the latter category reduces to $\text{perf}(\widehat{\mathbb{P}}^1)$. Hence, thanks to Theorem 1.5, resp. Theorem 1.17, the equivalence (13.1), resp. (13.2), reduces to $W(X) \Leftrightarrow W_{\text{nc}}(\text{perf}_{\text{dg}}(\widehat{\mathbb{P}}^1))$, resp. $ST(X) \Leftrightarrow ST_{\text{nc}}(\text{perf}_{\text{dg}}(\widehat{\mathbb{P}}^1))$. Finally, since $W_{\text{nc}}(\text{perf}_{\text{dg}}(\widehat{\mathbb{P}}^1))$ and $ST_{\text{nc}}(\text{perf}_{\text{dg}}(\widehat{\mathbb{P}}^1))$ hold, we hence conclude that the conjectures $W(X)$ and $ST(X)$ also hold.

14. PROOF OF THEOREM 3.3

As proved in [6, Cor. 3.7], the following holds:

- (a) When $\dim(L) < d_2r$, the category $\text{perf}(X_L)$ admits a semi-orthogonal decomposition with one component (Fourier-Mukai) equivalent to $\text{perf}(Y_L)$ and with $(d_2r - \dim(L))\binom{d_1}{r}$ exceptional objects.
- (b) When $\dim(L) = d_2r$, the category $\text{perf}(X_L)$ is (Fourier-Mukai) equivalent to the category $\text{perf}(Y_L)$.
- (c) When $\dim(L) > d_2r$, the category $\text{perf}(Y_L)$ admits a semi-orthogonal decomposition with one component (Fourier-Mukai) equivalent to $\text{perf}(X_L)$ and with $(\dim(L) - d_2r)\binom{d_1}{r}$ exceptional objects.

Consequently, since the (noncommutative) Weil conjecture as well as the (noncommutative) strong form of the Tate conjecture hold for $\text{Spec}(k)$, by combining Theorems 1.5 and 1.17 with an iterated application of Example 12.3, we hence conclude from (a)-(c) that $W(X_L) \Leftrightarrow W(Y_L)$ and $ST(X_L) \Leftrightarrow ST(Y_L)$. Therefore, the proof of item (i), resp. item (ii), follows from the fact that $\dim(X_L) = r(d_1 + d_2 - r) - 1 - \dim(L)$, resp. $\dim(Y_L) = r(d_1 - d_2 - r) - 1 + \dim(L)$.

15. PROOF OF THEOREM 4.4

Given a prime number $p \neq p_1, \dots, p_m$, let us write $\chi_{(0,p)} := \dim_K TP_0(\mathfrak{A}_p)_{1/p}$ and $\chi_{(1,p)} := \dim_K TP_1(\mathfrak{A}_p)_{1/p}$. In the same vein, given $i = 1, \dots, m$, let us write $\chi_{(0,p_i)} := \dim_K TP_0(\mathfrak{A}_{p_i})_{1/p_i}$ and $\chi_{(1,p_i)} := \dim_K TP_1(\mathfrak{A}_{p_i})_{1/p_i}$. The next result, of independent interest, provides a uniform upper bound for these dimensions.

Proposition 15.1. *There exists an integer $C_0 \gg 0$, resp. $C_1 \gg 0$, such that $\chi_{(0,p)} \leq C_0$ and $\chi_{(0,p_i)} \leq C_0$, resp. $\chi_{(1,p)} \leq C_1$ and $\chi_{(1,p_i)} \leq C_1$, for every prime $p \neq p_1, \dots, p_m$ and $i = 1, \dots, m$.*

Proof. Following [60, §2.28], Hochschild homology gives rise to a symmetric monoidal functor $HH: \text{dgcatsp}(\mathbb{Z}[1/p_1, \dots, 1/p_m]) \rightarrow \mathcal{D}_c(\mathbb{Z}[1/p_1, \dots, 1/p_m])$. Since \mathfrak{A} is a smooth proper $\mathbb{Z}[1/p_1, \dots, 1/p_m]$ -linear dg category, this implies not only that the Hochschild homology modules $HH_n(\mathfrak{A}), n \in \mathbb{Z}$, are finitely generated but moreover that they are zero for $|n| \gg 0$. Let us choose a finite set of generators of $HH_n(\mathfrak{A})$ and write $\#_n$ for its cardinality. Under these choices, we set $C'_0 := \sum_{n \text{ even}} (\#_n + \#_{n-1})$ and $C'_1 := \sum_{n \text{ odd}} (\#_n + \#_{n-1})$.

Choose a prime $p \neq p_1, \dots, p_m$ and consider the associated symmetric monoidal functor between rigid symmetric monoidal categories:

$$(15.2) \quad - \otimes_{\mathbb{Z}[1/p_1, \dots, 1/p_m]}^{\mathbb{L}} \mathbb{F}_p: \text{Hmo}_{\text{sp}}(\mathbb{Z}[1/p_1, \dots, 1/p_m]) \longrightarrow \text{Hmo}_{\text{sp}}(\mathbb{F}_p).$$

As proved in [60, Prop. 2.24], the Hochschild homology $HH(\mathfrak{A})$, resp. $HH(\mathfrak{A}_p)$, may be understood as the Euler characteristic of \mathfrak{A} , resp. \mathfrak{A}_p , in the rigid symmetric monoidal category $\text{Hmo}_{\text{sp}}(\mathbb{Z}[1/p_1, \dots, 1/p_m])$, resp. $\text{Hmo}_{\text{sp}}(\mathbb{F}_p)$. Consequently, the above functor (15.2) yields an isomorphism $HH(\mathfrak{A}_p) \simeq HH(\mathfrak{A}) \otimes_{\mathbb{Z}[1/p_1, \dots, 1/p_m]}^{\mathbb{L}} \mathbb{F}_p$ in the derived category $\mathcal{D}_c(\mathbb{F}_p)$. Using the following free resolution

$$0 \longrightarrow \mathbb{Z}[1/p_1, \dots, 1/p_m] \xrightarrow{-p} \mathbb{Z}[1/p_1, \dots, 1/p_m] \longrightarrow \mathbb{F}_p \longrightarrow 0,$$

we hence obtain the Künneth (split) short exact sequence of \mathbb{F}_p -vector spaces

$$0 \longrightarrow HH_n(\mathfrak{A}) \otimes_{\mathbb{Z}[1/p_1, \dots, 1/p_m]} \mathbb{F}_p \longrightarrow HH_n(\mathfrak{A}_p) \longrightarrow \text{Tor}_p(HH_{n-1}(\mathfrak{A})) \longrightarrow 0,$$

where $\text{Tor}_p(HH_{n-1}(\mathfrak{A}))$ stands for the p -torsion subgroup of $HH_{n-1}(\mathfrak{A})$. This (split) short exact sequence naturally implies the following (in)equality:

$$\begin{aligned} \dim_{\mathbb{F}_p} HH_n(\mathfrak{A}_p) &= \dim_{\mathbb{F}_p} (HH_n(\mathfrak{A}) \otimes_{\mathbb{Z}[1/p_1, \dots, 1/p_m]} \mathbb{F}_p) + \dim_{\mathbb{F}_p} \text{Tor}_p(HH_{n-1}(\mathfrak{A})) \\ &\leq \#_n + \#_{n-1}. \end{aligned}$$

Similarly to §5.4, periodic cyclic homology (over a finite field) gives rise to a functor $HP_*(-): \text{dgcatsp}(\mathbb{F}_p) \rightarrow \text{mod}_{\mathbb{Z}}(\mathbb{F}_p[v^{\pm 1}])$ with values in the category of (degreewise finite-dimensional) \mathbb{Z} -graded $\mathbb{F}_p[v^{\pm 1}]$ -modules, where v is a variable of degree -2 . Thanks to the (convergent) Hodge-to-de Rham spectral sequence $HH_*(\mathfrak{A}_p)[u^{\pm 1}] \Rightarrow HP_*(\mathfrak{A}_p)$, where u is a formal variable of degree -2 , we have:

$$\dim_{\mathbb{F}_p} HP_0(\mathfrak{A}_p) \leq \sum_{n \text{ even}} \dim_{\mathbb{F}_p} HH_n(\mathfrak{A}_p) \quad \dim_{\mathbb{F}_p} HP_1(\mathfrak{A}_p) \leq \sum_{n \text{ odd}} \dim_{\mathbb{F}_p} HH_n(\mathfrak{A}_p).$$

By combining these inequalities with the fact that $\dim_{\mathbb{F}_p} HH_n(\mathfrak{A}_p) \leq \#_n + \#_{n-1}$, we hence conclude that $\dim_{\mathbb{F}_p} HP_0(\mathfrak{A}_p) \leq C'_0$ and $\dim_{\mathbb{F}_p} HP_1(\mathfrak{A}_p) \leq C'_1$.

Now, recall from §6 that $TP_*(\mathfrak{A}_p)$ is a (degreewise finitely-generated) \mathbb{Z} -graded module over $TP_*(k) \simeq \mathbb{Z}_p[v^{\pm 1}]$, where v is a variable of degree -2 . As proved in [1, Thm. 3.4] (see also [9]), in the same way that $\mathbb{Z}_p/p \simeq \mathbb{F}_p$, we have natural isomorphisms $\pi_*(TP(\mathfrak{A}_p)/p) \simeq HP_*(\mathfrak{A}_p)$. Via the inclusion $TP_*(\mathfrak{A}_p)/p \subseteq \pi_*(TP(\mathfrak{A}_p)/p)$, we hence conclude that $TP_0(\mathfrak{A}_p)/p$ and $TP_1(\mathfrak{A}_p)/p$ are finite-dimensional \mathbb{F}_p -vector spaces. Let $\chi'_{(0,p)}$ and $\chi'_{(1,p)}$ be their dimensions. It follows from [1, Lem. 2.12] that the \mathbb{Z}_p -module $TP_0(\mathfrak{A}_p)$, resp. $TP_1(\mathfrak{A}_p)$, is a quotient of the free \mathbb{Z}_p -module of rank $\chi'_{(0,p)}$, resp. $\chi'_{(1,p)}$. Therefore, after inverting p , we obtain the inequalities:

$$\chi_{(0,p)} \leq \chi'_{(0,p)} \leq \dim_{\mathbb{F}_p} HP_0(\mathfrak{A}_p) \leq C'_0 \quad \chi_{(1,p)} \leq \chi'_{(1,p)} \leq \dim_{\mathbb{F}_p} HP_1(\mathfrak{A}_p) \leq C'_1.$$

The proof follows now from the definitions $C_0 := \{C'_0, \chi_{(0,p_1)}, \dots, \chi_{(0,p_m)}\}$ and $C_1 := \max\{C'_1, \chi_{(1,p_1)}, \dots, \chi_{(1,p_m)}\}$. \square

Given a prime $p \neq p_1, \dots, p_m$ and an integer $n \geq 1$, consider the complex numbers

$$\begin{aligned} \#_{(0,p,n)} &:= \text{trace}(\mathbb{F}_0^{\circ n} \otimes_{K,\iota} \mathbb{C} | TP_0(\mathfrak{A}_p)_{1/p} \otimes_{K,\iota} \mathbb{C}) \\ \#_{(1,p,n)} &:= \text{trace}(\mathbb{F}_1^{\circ n} \otimes_{K,\iota} \mathbb{C} | TP_1(\mathfrak{A}_p)_{1/p} \otimes_{K,\iota} \mathbb{C}), \end{aligned}$$

where $\mathbb{F}_*^{\circ n}$ stands for the n -fold composition of the cyclotomic Frobenius \mathbb{F}_* (consult Notation 6.5). In the same vein, given $i = 1, \dots, m$, consider the complex numbers:

$$\begin{aligned} \#_{(0,p_i,n)} &:= \text{trace}(\mathbb{F}_0^{\circ n} \otimes_{K,\iota} \mathbb{C} | TP_0(\mathfrak{A}_{p_i})_{1/p_i} \otimes_{K,\iota} \mathbb{C}) \\ \#_{(1,p_i,n)} &:= \text{trace}(\mathbb{F}_1^{\circ n} \otimes_{K,\iota} \mathbb{C} | TP_1(\mathfrak{A}_{p_i})_{1/p_i} \otimes_{K,\iota} \mathbb{C}). \end{aligned}$$

Proposition 15.3. *Assume that the conjectures $W_{\text{nc}}(\mathfrak{A}_p)$, resp. $W_{\text{nc}}(\mathfrak{A}_{p_i})$, holds. Under this assumption, there exist integers $C_0, C_1 \gg 0$ such that $|\#_{(0,p,n)}| \leq C_0$ and $|\#_{(1,p,n)}| \leq C_1 p^{\frac{1}{2}n}$, resp. $|\#_{(0,p_i,n)}| \leq C_0$ and $|\#_{(1,p_i,n)}| \leq C_1 p_i^{\frac{1}{2}n}$, for every prime $p \neq p_1, \dots, p_m$ and $n \geq 1$, resp. for every $i = 1, \dots, m$ and $n \geq 1$.*

Proof. Given a prime $p \neq p_1, \dots, p_m$, let us write $\{\lambda_{(0,p,1)}, \dots, \lambda_{(0,p,\chi_{(0,p)})}\}$ for the eigenvalues (with multiplicity) of the automorphism $\mathbb{F}_0 \otimes_{K,\iota} \mathbb{C}$ of $TP_0(\mathfrak{A}_p)_{1/p} \otimes_{K,\iota} \mathbb{C}$ and $\{\lambda_{(1,p,1)}, \dots, \lambda_{(1,p,\chi_{(1,p)})}\}$ for the eigenvalues (with multiplicity) of the automorphism $\mathbb{F}_1 \otimes_{K,\iota} \mathbb{C}$ of $TP_1(\mathfrak{A}_p)_{1/p} \otimes_{K,\iota} \mathbb{C}$. In the same vein, given $i = 1, \dots, m$, let us write $\{\lambda_{(0,p_i,1)}, \dots, \lambda_{(0,p_i,\chi_{(0,p_i)})}\}$ for the eigenvalues (with multiplicity) of the automorphism $\mathbb{F}_0 \otimes_{K,\iota} \mathbb{C}$ of $TP_0(\mathfrak{A}_{p_i})_{1/p_i} \otimes_{K,\iota} \mathbb{C}$ and $\{\lambda_{(1,p_i,1)}, \dots, \lambda_{(1,p_i,\chi_{(1,p_i)})}\}$, for the eigenvalues of the automorphism $\mathbb{F}_1 \otimes_{K,\iota} \mathbb{C}$ of $TP_1(\mathfrak{A}_{p_i})_{1/p_i} \otimes_{K,\iota} \mathbb{C}$. Under these notations, we have the following (in)equalities

$$\begin{aligned} |\#_{(0,p,n)}| &= |\text{trace}(\mathbb{F}_0^{\circ n} \otimes_{K,\iota} \mathbb{C})| \\ &= |\lambda_{(0,p,1)}^n + \dots + \lambda_{(0,p,\chi_{(0,p)})}^n| \\ &\leq |\lambda_{(0,p,1)}|^n + \dots + |\lambda_{(0,p,\chi_{(0,p)})}|^n \\ (15.4) \quad &= \chi_{(0,p)} \\ (15.5) \quad &\leq C_0, \end{aligned}$$

where (15.4) follows from conjecture $W_{\text{nc}}(\mathfrak{A}_p)$ and (15.5) from Proposition 15.1; similarly with p replaced by p_i . In the same vein, we have the (in)equalities

$$\begin{aligned} |\#_{(1,p,n)}| &= |\text{trace}(\mathbb{F}_1^{\circ n} \otimes_{K,\iota} \mathbb{C})| \\ &= |\lambda_{(1,p,1)}^n + \dots + \lambda_{(1,p,\chi_{(1,p)})}^n| \\ &\leq |\lambda_{(1,p,1)}|^n + \dots + |\lambda_{(1,p,\chi_{(1,p)})}|^n \\ (15.6) \quad &= \chi_{(1,p)} p^{\frac{1}{2}n} \\ (15.7) \quad &\leq C_1 p^{\frac{1}{2}n}, \end{aligned}$$

where (15.6) follows from conjecture $W_{\text{nc}}(\mathfrak{A}_p)$ and (15.7) from Proposition 15.1; similarly with p replaced by p_i . \square

Recall the following general result, whose proof is a simple linear algebra exercise that we leave for the reader.

Lemma 15.8. *Given an endomorphism $f: V \rightarrow V$ of a finite-dimensional vector space, we have the following equality of formal power series*

$$\log\left(\frac{1}{\det(\text{id} - tf|V)}\right) = \sum_{n \geq 1} \text{trace}(f^{\circ n}) \frac{t^n}{n},$$

where $\log(t) := \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} (t-1)^n$.

Given a prime $p \neq p_1, \dots, p_m$, consider the following (auxiliar) formal power series $\phi_{(0,p)}(t) := \sum_{n \geq 1} \#_{(0,p,n)} \frac{t^n}{n}$ and $\phi_{(1,p)}(t) := \sum_{n \geq 1} \#_{(1,p,n)} \frac{t^n}{n}$ as well as their exponentiations $\varphi_{(0,p)}(t) := \sum_{n \geq 0} a_{(0,p,n)} t^n$ and $\varphi_{(1,p)}(t) := \sum_{n \geq 0} a_{(1,p,n)} t^n$. In the same vein, given $i = 1, \dots, m$, consider the following (auxiliar) formal power series $\phi_{(0,p_i)}(t) := \sum_{n \geq 1} \#_{(0,p_i,n)} \frac{t^n}{n}$ and $\phi_{(1,p_i)}(t) := \sum_{n \geq 1} \#_{(1,p_i,n)} \frac{t^n}{n}$ as well as their exponentiations $\varphi_{(0,p_i)}(t) := \sum_{n \geq 0} a_{(0,p_i,n)} t^n$ and $\varphi_{(1,p_i)}(t) := \sum_{n \geq 0} a_{(1,p_i,n)} t^n$. Note that thanks to the above Lemma 15.8, we have the following (formal) equalities:

$$(15.9) \quad \varphi_{(0,p)}(p^{-s}) = \zeta_{\text{even}}(\mathfrak{A}_p; s) \quad \varphi_{(0,p_i)}(p_i^{-s}) = \zeta_{\text{even}}(\mathfrak{A}_{p_i}; s)$$

$$(15.10) \quad \varphi_{(1,p)}(p^{-s}) = \zeta_{\text{odd}}(\mathfrak{A}_p; s) \quad \varphi_{(1,p_i)}(p_i^{-s}) = \zeta_{\text{odd}}(\mathfrak{A}_{p_i}; s).$$

Definition 15.11. Let $\varphi_0(s) := \sum_{n \geq 1} \frac{b_{(0,n)}}{n^s}$, resp. $\varphi_1(s) := \sum_{n \geq 1} \frac{b_{(1,n)}}{n^s}$, be the multiplicative Dirichlet series, where $b_{(0,n)} := a_{(0,p_{r_1}, v_{r_1})} \cdots a_{(0,p_{r_n}, v_{r_n})}$, resp. $b_{(1,n)} := a_{(1,p_{r_1}, v_{r_1})} \cdots a_{(1,p_{r_n}, v_{r_n})}$, is the product associated to the (unique) prime decomposition $p_{r_1}^{v_{r_1}} \cdots p_{r_n}^{v_{r_n}}$ of the integer $n \geq 1$.

Note that we have the following (formal) equalities

$$\begin{aligned} \varphi_0(s) &\stackrel{(a)}{=} \prod_{p \neq p_1, \dots, p_m} \sum_{n \geq 0} \frac{b_{(0,p^n)}}{p^{ns}} \cdot \prod_{1 \leq i \leq m} \left(\sum_{n \geq 0} \frac{b_{(0,p_i^n)}}{p_i^{ns}} \right) \\ &= \prod_{p \neq p_1, \dots, p_m} \varphi_{(0,p)}(p^{-s}) \cdot \prod_{1 \leq i \leq m} \varphi_{(0,p_i)}(p_i^{-s}) \\ &= L_{\text{even}}(\mathcal{A}; s) \end{aligned}$$

and

$$\begin{aligned} \varphi_1(s) &\stackrel{(b)}{=} \prod_{p \neq p_1, \dots, p_m} \sum_{n \geq 0} \frac{b_{(1,p^n)}}{p^{ns}} \cdot \prod_{1 \leq i \leq m} \left(\sum_{n \geq 0} \frac{b_{(1,p_i^n)}}{p_i^{ns}} \right) \\ &= \prod_{p \neq p_1, \dots, p_m} \varphi_{(1,p)}(p^{-s}) \cdot \prod_{1 \leq i \leq m} \varphi_{(1,p_i)}(p_i^{-s}) \\ &= L_{\text{odd}}(\mathcal{A}; s), \end{aligned}$$

where (a), resp. (b), follows from the Euler product decomposition of the (multiplicative) Dirichlet series $\varphi_0(s)$, resp. $\varphi_1(s)$. These (formal) equalities imply that the proof of Theorem 4.4 follows now from the next result:

Proposition 15.12. *Assume that the conjectures $W_{\text{nc}}(\mathfrak{A}_p)$ and $W_{\text{nc}}(\mathfrak{A}_{p_i})$ hold. Under these assumptions, the Dirichlet series $\varphi_0(s)$, resp. $\varphi_1(s)$, converges (absolutely) in the half-plane $\text{Re}(s) > 1$, resp. $\text{Re}(s) > \frac{3}{2}$. Moreover, $\varphi_0(s)$ and $\varphi_1(s)$ are non-zero in these half-plane regions.*

Proof. Let $z > 1$, resp. $z > \frac{3}{2}$, be a real number. Thanks to the classical properties of multiplicative Dirichlet series (see [52, Chap. VI §2]), it suffices to show that the infinite sum $\varphi_0(z)$, resp. $\varphi_1(z)$, converges (absolutely). Note that the

above (formal) equality $\varphi_0(s) = L_{\text{even}}(\mathcal{A}; s)$, resp. $\varphi_1(s) = L_{\text{odd}}(\mathcal{A}; s)$, implies that $\varphi_0(z)$, resp. $\varphi_1(z)$, converges (absolutely) if and only if the following infinite product $\prod_{p \neq p_1, \dots, p_m} \varphi_{(0,p)}(p^{-z}) \cdot \prod_{i=1}^m \varphi_{(0,p_i)}(p_i^{-z})$, resp. the following infinite product $\prod_{p \neq p_1, \dots, p_m} \varphi_{(1,p)}(p^{-z}) \cdot \prod_{i=1}^m \varphi_{(1,p_i)}(p_i^{-z})$, converges (absolutely). Using exponentiation, it is then enough to show that the sums $\sum_{p \neq p_1, \dots, p_m} \phi_{(0,p)}(p^{-z})$ and $\sum_{i=1}^m \phi_{(0,p_i)}(p_i^{-z})$, resp. $\sum_{p \neq p_1, \dots, p_m} \phi_{(1,p)}(p^{-z})$ and $\sum_{i=1}^m \phi_{(1,p_i)}(p_i^{-z})$, converge (absolutely). In what concerns the first sum, we have the following (in)equalities

$$(15.13) \quad \sum_{p \neq p_1, \dots, p_m} |\phi_{(0,p)}(p^{-z})| = \sum_{p \neq p_1, \dots, p_m} \sum_{n \geq 1} \frac{|\#_{(0,p,n)}|}{np^{nz}} \leq \sum_{p \neq p_1, \dots, p_m} \sum_{n \geq 1} \frac{C_0}{np^{nz}}$$

$$(15.14) \quad \leq C_0 \sum_{p \neq p_1, \dots, p_m} \sum_{n \geq 1} \frac{1}{p^{nz}} \leq C_0 \sum_{p \neq p_1, \dots, p_m} \frac{1}{p^z - 1},$$

where (15.13) follows from Proposition 15.3. Since the series $\sum_{p \neq p_1, \dots, p_m} \frac{1}{p^z - 1}$ is convergent (with $z > 1$), we hence conclude that $\sum_{p \neq p_1, \dots, p_m} \phi_{(0,p)}(p^{-z})$ converges (absolutely). Similarly, we have the following (in)equalities

$$(15.15) \quad \sum_{p \neq p_1, \dots, p_m} |\phi_{(1,p)}(p^{-z})| = \sum_{p \neq p_1, \dots, p_m} \sum_{n \geq 1} \frac{|\#_{(1,p,n)}|}{np^{nz}} \leq \sum_{p \neq p_1, \dots, p_m} \sum_{n \geq 1} \frac{C_1 p^{\frac{1}{2}n}}{np^{nz}}$$

$$(15.16) \quad \leq C_1 \sum_{p \neq p_1, \dots, p_m} \sum_{n \geq 1} \frac{1}{p^{n(z - \frac{1}{2})}} \leq \sum_{p \neq p_1, \dots, p_m} \frac{1}{p^{(z - \frac{1}{2})} - 1},$$

where (15.15) follows from Proposition (15.3). Since the series $\sum_{p \neq p_1, \dots, p_m} \frac{1}{p^{(z - \frac{1}{2})} - 1}$ is convergent (with $z > \frac{3}{2}$), we hence conclude that $\sum_{p \neq p_1, \dots, p_m} \phi_{(1,p)}(p^{-z})$ converges (absolutely). In what concerns the second sum, we have the (in)equalities

$$(15.17) \quad \sum_{1 \leq i \leq m} |\phi_{(0,p_i)}(p_i^{-z})| = \sum_{1 \leq i \leq m} \sum_{n \geq 1} \frac{|\#_{(0,p_i,n)}|}{np_i^{nz}} \leq \sum_{1 \leq i \leq m} \sum_{n \geq 1} \frac{C_0}{np_i^{nz}}$$

$$(15.18) \quad \leq C_0 \sum_{1 \leq i \leq m} \sum_{n \geq 1} \frac{1}{p_i^{nz}} \leq C_0 \sum_{1 \leq i \leq m} \frac{1}{p_i^z - 1},$$

where (15.17) follows from Proposition 15.3. This implies that $\sum_{i=1}^m \phi_{(0,p_i)}(p_i^{-z})$ converges (absolutely). Similarly, we have the following (in)equalities

$$(15.19) \quad \sum_{1 \leq i \leq m} |\phi_{(1,p_i)}(p_i^{-z})| = \sum_{1 \leq i \leq m} \sum_{n \geq 1} \frac{|\#_{(1,p_i,n)}|}{np_i^{nz}}$$

$$\leq \sum_{1 \leq i \leq m} \sum_{n \geq 1} \frac{C_1 p_i^{\frac{1}{2}n}}{np_i^{nz}}$$

$$(15.20) \quad \leq C_1 \sum_{1 \leq i \leq m} \sum_{n \geq 1} \frac{1}{p_i^{n(z-\frac{1}{2})}}$$

$$\leq C_1 \sum_{1 \leq i \leq m} \frac{1}{p_i^{(z-\frac{1}{2})} - 1},$$

where (15.19) follows from Proposition 15.3. This implies that $\sum_{i=1}^m \phi_{(1,p_i)}(p_i^{-z})$ converges (absolutely). Finally, note that the Dirichlet series $\varphi_0(s)$, resp. $\varphi_1(s)$, is non-zero in the half-plane $\operatorname{Re}(s) > 1$, resp. $\operatorname{Re}(s) > \frac{3}{2}$ because each one of its Euler factors (15.9), resp. (15.10), is non-zero in this half-plane region. \square

16. ALTERNATIVE PROOF OF SERRE'S CONVERGENCE RESULT

Let X be a smooth proper \mathbb{Q} -scheme of dimension d . Given an integer $0 \leq w \leq 2d$, consider the following L -function defined in §4:

$$(16.1) \quad L_w(X; s) := \prod_{p \neq p_1, \dots, p_m} \zeta_w(\mathfrak{X}_p; s) \cdot \prod_{1 \leq i \leq m} \zeta_w(\mathfrak{X}_{p_i}; s).$$

In the sixties, Serre [53, 54] proved the following result:

Theorem 16.2 (Serre). *The infinite product (16.1) converges (absolutely) in the half-plane $\operatorname{Re}(s) > \frac{w}{2} + 1$. Moreover, the L -function $L_w(X; s)$ is non-zero in this half-plane region.*

In this section, we present an alternative noncommutative proof of Theorem 16.2. Note that by combining Theorem 16.2 with the weight decomposition (4.1), we conclude that the L -function $L(X; s)$ of X converges (absolutely) in the half-plane $\operatorname{Re}(s) > d + 1$. Moreover, $L(X; s)$ is non-zero in this half-plane region.

Given a prime $p \neq p_1, \dots, p_m$, let us write $\beta_{(w,p)} := \dim_K H_{\text{crys}}^w(\mathfrak{X}_p)$. In the same vein, given $i = 1, \dots, m$, let us write $\beta_{(w,p_i)} := \dim_K H_{\text{crys}}^w(\mathfrak{X}_{p_i})$. The next result provides a uniform upper bound for these dimensions.

Proposition 16.3. *There exists an integer $C \gg 0$ such that $\beta_{(w,p)} \leq C$ and $\beta_{(w,p_i)} \leq C$ for every $p \neq p_1, \dots, p_m$ and $i = 1, \dots, m$.*

Proof. Consider the smooth proper \mathbb{Q} -linear dg category $\mathcal{A} := \operatorname{perf}_{\text{dg}}(X)$. In this case, similarly to Example 4.7, we can choose for \mathfrak{A} the dg category $\operatorname{perf}_{\text{dg}}(\mathfrak{X})$ and for \mathfrak{A}_i the dg category $\operatorname{perf}_{\text{dg}}(\mathfrak{X}_i)$. Consequently, since the dg categories \mathfrak{A}_p and \mathfrak{A}_{p_i} are Morita equivalent to $\operatorname{perf}_{\text{dg}}(\mathfrak{X}_p)$ and $\operatorname{perf}_{\text{dg}}(\mathfrak{X}_{p_i})$, respectively, we conclude from Proposition 15.1 that there exist integers $C_0 \gg 0$ and $C_1 \gg 0$ such that

$$\dim_K TP_0(\operatorname{perf}_{\text{dg}}(\mathfrak{X}_p)) \leq C_0 \quad \dim_K TP_1(\operatorname{perf}_{\text{dg}}(\mathfrak{X}_p)) \leq C_1;$$

similarly with p replaced by p_i . Thanks to (7.1)-(7.2), this implies that

$$\beta_{(w,p)} \leq \dim_K TP_0(\operatorname{perf}_{\text{dg}}(\mathfrak{X}_p)) + \dim_K TP_1(\operatorname{perf}_{\text{dg}}(\mathfrak{X}_p)) \leq C_0 + C_1;$$

similarly with p replaced by p_i . By setting $C := C_0 + C_1$, we hence conclude that $\beta_{(w,p)} \leq C$ and $\beta_{(w,p_i)} \leq C$ for every $p \neq p_1, \dots, p_m$ and $i = 1, \dots, m$. \square

Given a prime $p \neq p_1, \dots, p_m$ and an integer $n \geq 1$, consider the following integer $\#_{(w,p,n)} := \text{trace}((\text{Fr}^w)^{on} | H_{\text{crys}}^w(\mathfrak{X}_p))$. In the same vein, given $i = 1, \dots, m$, let us consider the integer $\#_{(w,p_i,n)} := \text{trace}((\text{Fr}^w)^{on} | H_{\text{crys}}^w(\mathfrak{X}_{p_i}))$.

Proposition 16.4. *There exists an integer $C \gg 0$ such that $|\#_{(w,p,n)}| \leq Cp^{\frac{w}{2}n}$, resp. $|\#_{(w,p_i,n)}| \leq Cp_i^{\frac{w}{2}n}$, for every $p \neq p_1, \dots, p_m$ and $n \geq 1$, resp. for every $i = 1, \dots, m$ and $n \geq 1$.*

Proof. Given a prime $p \neq p_1, \dots, p_m$, let us write $\{\lambda_{(w,p,1)}, \dots, \lambda_{(w,p,\beta_{(w,p)})}\}$ for the eigenvalues (with multiplicity) of the automorphism Fr^w of $H_{\text{crys}}^w(\mathfrak{X}_p)$. In the same vein, given $i = 1, \dots, m$, let us write $\{\lambda_{(w,p_i,1)}, \dots, \lambda_{(w,p_i,\beta_{(w,p_i)})}\}$ for the eigenvalues (with multiplicity) of the automorphism Fr^w of $H_{\text{crys}}^w(\mathfrak{X}_{p_i})$. Under these notations, we have the following (in)equalities

$$\begin{aligned} |\#_{(w,p,n)}| &= |\text{trace}(\text{Fr}^w)| \\ &= |\lambda_{(w,p,1)}^n + \dots + \lambda_{(w,p,\beta_{(w,p)})}^n| \\ &\leq |\lambda_{(w,p,1)}|^n + \dots + |\lambda_{(w,p,\beta_{(w,p)})}|^n \\ (16.5) \quad &\leq \beta_{(w,p)} p^{\frac{w}{2}n} \end{aligned}$$

$$(16.6) \quad \leq Cp^{\frac{w}{2}n},$$

where (16.5) follows from conjecture $W(\mathfrak{X}_p)$ (proved in [16]) and (16.6) from Proposition 16.3; similarly with p replaced by p_i . \square

Given a prime $p \neq p_1, \dots, p_m$, consider the following (formal) formal power series $\phi_{(w,p)}(t) := \sum_{n \geq 1} \#_{(w,p,n)} \frac{t^n}{n}$ and its exponentiation $\varphi_{(w,p)}(t) := \sum_{n \geq 0} a_{(w,p,n)} t^n$. In the same vein, given $i = 1, \dots, m$, consider the (auxiliar) formal power series $\phi_{(w,p_i)}(t) := \sum_{n \geq 1} \#_{(w,p_i,n)} \frac{t^n}{n}$ and its exponentiation $\varphi_{(w,p_i)}(t) := \sum_{n \geq 0} a_{(w,p_i,n)} t^n$. Note that thanks to the above Lemma 15.8, we have the (formal) equalities:

$$(16.7) \quad \varphi_{(w,p)}(p^{-s}) = \zeta_w(\mathfrak{X}_p; s) \quad \varphi_{(w,p_i)}(p_i^{-s}) = \zeta_w(\mathfrak{X}_{p_i}; s).$$

Definition 16.8. Let $\varphi_w(s) := \sum_{n \geq 1} \frac{b_{(w,n)}}{n^s}$ be the Dirichlet series, where $b_{(w,n)} := a_{(w,p_{r_1}, v_{r_1})} \cdots a_{(w,p_{r_n}, v_{r_n})}$ is the product associated to the (unique) prime decomposition $p_{r_1}^{v_{r_1}} \cdots p_{r_n}^{v_{r_n}}$ of the integer $n \geq 1$.

Note that we have the following (formal) equalities

$$\begin{aligned} \varphi_w(s) &\stackrel{(a)}{=} \prod_{p \neq p_1, \dots, p_m} \left(\sum_{n \geq 0} \frac{b_{(w,p^n)}}{p^{ns}} \right) \cdot \prod_{1 \leq i \leq m} \left(\sum_{n \geq 0} \frac{b_{(w,p_i^n)}}{p_i^{ns}} \right) \\ &= \prod_{p \neq p_1, \dots, p_m} \varphi_{(w,p)}(p^{-s}) \cdot \prod_{1 \leq i \leq m} \varphi_{(w,p_i)}(p_i^{-s}) \\ &= L_w(X; s), \end{aligned}$$

where (a) follows from the Euler product decomposition of the (multiplicative) Dirichlet series $\varphi_w(s)$. These (formal) equalities imply that the proof of Theorem 16.2 follows now from the next result:

Proposition 16.9. *The Dirichlet series $\varphi_w(s)$ converges (absolutely) in the half-plane $\text{Re}(s) > \frac{w}{2} + 1$. Moreover, $\varphi_w(s)$ is non-zero in this half-plane region.*

Proof. Let $z > \frac{w}{2} + 1$ be a real number. Similarly to the proof of Proposition 15.12, it is enough to show that the sums $\sum_{p \neq p_1, \dots, p_m} \phi_{(w,p)}(p^{-z})$ and $\sum_{i=1}^m \phi_{(w,p_i)}(p_i^{-z})$ converge (absolutely). Note that we have the following (in)equalities

$$(16.10) \quad \sum_{p \neq p_1, \dots, p_m} |\phi_{(w,p)}(p^{-z})| = \sum_{p \neq p_1, \dots, p_m} \sum_{n \geq 1} \frac{|\#_{(w,p,n)}|}{np^{nz}}$$

$$(16.11) \quad \leq \sum_{p \neq p_1, \dots, p_m} \sum_{n \geq 1} \frac{Cp^{\frac{w}{2}n}}{np^{nz}}$$

$$(16.11) \quad \leq C \sum_{p \neq p_1, \dots, p_m} \sum_{n \geq 1} \frac{1}{p^{n(z - \frac{w}{2})}}$$

$$\leq C \sum_{p \neq p_1, \dots, p_m} \frac{1}{p^{(z - \frac{w}{2})} - 1},$$

where (16.10) follows from Proposition 16.4. Since the series $\sum_{p \neq p_1, \dots, p_m} \frac{1}{p^{(z - \frac{w}{2})} - 1}$ is convergent (with $z > \frac{w}{2} + 1$), we hence conclude that $\sum_{p \neq p_1, \dots, p_m} \phi_{(w,p)}(p^{-z})$ converges (absolutely). Similarly, we have the following (in)equalities

$$(16.12) \quad \sum_{1 \leq i \leq m} |\phi_{(w,p_i)}(p_i^{-z})| = \sum_{1 \leq i \leq m} \sum_{n \geq 1} \frac{|\#_{(w,p_i,n)}|}{np_i^{nz}}$$

$$(16.13) \quad \leq \sum_{1 \leq i \leq m} \sum_{n \geq 1} \frac{Cp_i^{\frac{w}{2}n}}{np_i^{nz}}$$

$$(16.13) \quad \leq C \sum_{1 \leq i \leq m} \sum_{n \geq 1} \frac{1}{p_i^{n(z - \frac{w}{2})}}$$

$$\leq C \sum_{1 \leq i \leq m} \frac{1}{p_i^{(z - \frac{w}{2})} - 1},$$

where (16.12) follows from Proposition 16.4. This implies that $\sum_{i=1}^m \phi_{(w,p_i)}(p_i^{-z})$ converges (absolutely). Finally, note that $\varphi_w(s)$ is non-zero in the half-plane $\operatorname{Re}(s) > \frac{w}{2} + 1$ because its Euler factors (16.7) are non-zero in this region. \square

17. PROOF OF THEOREM 4.18

Recall first that, thanks to the Hochschild-Kostant-Rosenberg theorem (consult [19]), we have the following natural isomorphisms of finite-dimensional \mathbb{Q} -vector spaces $HP_0(\operatorname{perf}_{\operatorname{dg}}(X)) \simeq \bigoplus_{w \text{ even}} H_{dR}^w(X)$ and $HP_1(\operatorname{perf}_{\operatorname{dg}}(X)) \simeq \bigoplus_{w \text{ odd}} H_{dR}^w(X)$, where $H_{dR}^*(X)$ stands for the de Rham cohomology of X . Recall also that we have the following classical isomorphism

$$K_0(\operatorname{perf}_{\operatorname{dg}}(X))_{\mathbb{Q}} \xrightarrow{\simeq} \bigoplus_{0 \leq i \leq d} \mathcal{Z}^i(X)_{\mathbb{Q}} / \sim_{\operatorname{rat}} \quad [\mathcal{F}] \mapsto \operatorname{chern}(\mathcal{F}) \cdot \sqrt{\operatorname{td}}_X,$$

where $d := \dim(X)$, $\operatorname{chern}(\mathcal{F})$ stands for the Chern character of \mathcal{F} , and $\sqrt{\operatorname{td}}_X$ stands for the square root of the Todd class of X ; see [20, §18.3]. Under the above isomorphisms, the \mathbb{Q} -linear homomorphism $\operatorname{ch}: K_0(\operatorname{perf}_{\operatorname{dg}}(X))_{\mathbb{Q}} \rightarrow HP_0(\operatorname{perf}_{\operatorname{dg}}(X))$ defined in §5.4 corresponds to the classical cycle class map (with values in de Rham

cohomology). Consequently, we obtain an induced isomorphism

$$(17.1) \quad K_0(\mathrm{perf}_{\mathrm{dg}}(X))_{\mathbb{Q}/\sim_{\mathrm{hom}}} \simeq \bigoplus_{0 \leq i \leq d} \mathcal{Z}^i(X)_{\mathbb{Q}/\sim_{\mathrm{hom}}}$$

as well as an induced isomorphism:

$$(17.2) \quad K_0(\mathrm{perf}_{\mathrm{dg}}(X))_{\mathbb{Q}}^0 \simeq \bigoplus_{0 \leq i \leq d} \mathcal{Z}^i(X)_{\mathbb{Q}/\sim_{\mathrm{rat}}}^0 = \bigoplus_{1 \leq i \leq d} \mathcal{Z}^i(X)_{\mathbb{Q}/\sim_{\mathrm{rat}}}^0.$$

The first implication of Theorem 4.18 is a consequence of the equalities

$$(17.3) \quad \mathrm{ord}_{s=1} L_{\mathrm{even}}(\mathrm{perf}_{\mathrm{dg}}(X); s) = \sum_{w \text{ even}} \mathrm{ord}_{s=1} L_w(X; s + \frac{w}{2})$$

$$= \sum_{w \text{ even}} \mathrm{ord}_{s=\frac{w}{2}+1} L_w(X; s)$$

$$(17.4) \quad = - \sum_{w \text{ even}} \dim_{\mathbb{Q}} \mathcal{Z}^{\frac{w}{2}}(X)_{\mathbb{Q}/\sim_{\mathrm{hom}}}$$

$$(17.5) \quad = -\dim_{\mathbb{Q}} K_0(\mathrm{perf}_{\mathrm{dg}}(X))_{\mathbb{Q}/\sim_{\mathrm{hom}}},$$

where (17.3) follows from factorization (4.8) (and from Remark 4.12), (17.4) from Beilinson's conjecture $B_{\frac{w}{2}+1}^w(X)$, and (17.5) from isomorphism (17.1). Similarly, the second implication of Theorem 4.18 is a consequence of the equalities

$$(17.6) \quad \mathrm{ord}_{s=1} L_{\mathrm{odd}}(\mathrm{perf}_{\mathrm{dg}}(X); s) = \sum_{w \text{ odd}} \mathrm{ord}_{s=1} L_w(X; s + \frac{w-1}{2})$$

$$= \sum_{w \text{ odd}} \mathrm{ord}_{s=\frac{w+1}{2}} L_w(X; s)$$

$$(17.7) \quad = \sum_{w \text{ odd}} \dim_{\mathbb{Q}} \mathcal{Z}^{\frac{w+1}{2}}(X)_{\mathbb{Q}/\sim_{\mathrm{rat}}}^0$$

$$(17.8) \quad = \dim_{\mathbb{Q}} K_0(\mathrm{perf}_{\mathrm{dg}}(X))_{\mathbb{Q}}^0,$$

where (17.6) follows from factorization (4.9) (and from Remark 4.12), (17.7) from Beilinson's conjecture $B_{\frac{w+1}{2}}^w(X)$, and (17.8) from isomorphism (17.2).

Assume now that the Beilinson-Soulé vanishing conjecture holds. Under this assumption, the third implication of Theorem 4.18 is a consequence of the equalities

$$(17.9) \quad \mathrm{ord}_{s=0} L_{\mathrm{even}}(\mathrm{perf}_{\mathrm{dg}}(X); s) = \sum_{w \text{ even}} \mathrm{ord}_{s=0} L_w(X; s + \frac{w}{2})$$

$$= \sum_{w \text{ even}} \mathrm{ord}_{s=\frac{w}{2}} L_w(X; s)$$

$$(17.10) \quad = \sum_{w \text{ even}} \dim_{\mathbb{Q}} H_{\mathrm{mot}}^{w+1}(X; \mathbb{Q}(\frac{w}{2} + 1))$$

$$(17.11) \quad = \dim_{\mathbb{Q}} K_1(\mathrm{perf}_{\mathrm{dg}}(X))_{\mathbb{Q}},$$

where (17.9) follows from factorization (4.8) (and from Remark 4.12), (17.10) from Beilinson's conjecture $B_{\frac{w}{2}}^w(X)$, and (17.11) from Proposition 17.18 below. Similarly,

the fourth implication of Theorem 4.18 is a consequence of the equalities

$$(17.12) \quad \text{ord}_{s=-1} L_{\text{even}}(\text{perf}_{\text{dg}}(X); s) = \sum_{w \text{ even}} \text{ord}_{s=-1} L_w(X; s + \frac{w}{2})$$

$$= \sum_{w \text{ even}} \text{ord}_{s=\frac{w}{2}-1} L_w(X; s)$$

$$(17.13) \quad = \sum_{w \text{ even}} \dim_{\mathbb{Q}} H_{\text{mot}}^{w+1}(X; \mathbb{Q}(\frac{w}{2} + 2))$$

$$(17.14) \quad = \dim_{\mathbb{Q}} K_3(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}},$$

where (17.12) follows from factorization (4.8) (and from Remark 4.12), (17.13) from Beilinson's conjecture $B_w^{\frac{w}{2}-1}(X)$, and (17.14) from Proposition 17.18.

Finally, the last implication of Theorem 4.18 is a consequence of the equalities

$$(17.15) \quad \text{ord}_{s=0} L_{\text{odd}}(\text{perf}_{\text{dg}}(X); s) = \sum_{w \text{ odd}} \text{ord}_{s=0} L_w(X; s + \frac{w-1}{2})$$

$$= \sum_{w \text{ odd}} \text{ord}_{s=\frac{w-1}{2}} L_w(X; s)$$

$$(17.16) \quad = \sum_{w \text{ odd}} \dim_{\mathbb{Q}} H_{\text{mot}}^{w+1}(X; \mathbb{Q}(\frac{w+3}{2}))$$

$$(17.17) \quad = \dim_{\mathbb{Q}} K_2(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}},$$

where (17.15) follows from factorization (4.9) (and from Remark 4.12), (17.16) from Beilinson's conjecture $B_w^{\frac{w-1}{2}}(X)$, and (17.17) from Proposition 17.18.

Proposition 17.18. *Let X be a smooth proper \mathbb{Q} -scheme of dimension d and $n > 0$ an integer. Assuming the Beilinson-Soulé vanishing conjecture, we have the following direct sum decomposition(s) of rational algebraic K -theory:*

$$K_n(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}} \simeq \bigoplus_{\frac{n}{2} < r \leq d+n} H_{\text{mot}}^{2r-n}(X; \mathbb{Q}(r)) = \bigoplus_{\frac{n}{2} < r \leq d+n-1} H_{\text{mot}}^{2r-n}(X; \mathbb{Q}(r)).$$

Proof. Recall first that the algebraic K -theory groups $K_n(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}}$ of the dg category $\text{perf}_{\text{dg}}(X)$ are naturally isomorphic to the classical algebraic K -theory group $K_n(X)_{\mathbb{Q}}$ of X ; consult [60, §2.2.1]. As proved in [55, §2.8], we have the direct sum decomposition $K_n(X)_{\mathbb{Q}} \simeq \bigoplus_{r=0}^{d+n} K_n(X)_{\mathbb{Q}}^{(r)}$, where $K_n(X)_{\mathbb{Q}}^{(r)}$ stands for the r^{th} eigenspace with respect to the Adams operations on $K_n(X)_{\mathbb{Q}}$. Making use of the classical identification $H_{\text{mot}}^i(X; \mathbb{Q}(j)) \simeq K_{2j-i}(X)_{\mathbb{Q}}^{(j)}$, we hence conclude that

$$(17.19) \quad K_n(X)_{\mathbb{Q}} \simeq \bigoplus_{0 \leq r \leq d+n} H_{\text{mot}}^{2r-n}(X; \mathbb{Q}(r)).$$

It is well-known that $H_{\text{mot}}^{2d+n}(X; \mathbb{Q}(d+n))$ is isomorphic to the cokernel of the residue map $\partial: \bigoplus_{x \in X^{(d-1)}} K_{n+1}^M(\kappa(x))_{\mathbb{Q}} \rightarrow \bigoplus_{x \in X^{(d)}} K_n^M(\kappa(x))_{\mathbb{Q}}$, where $X^{(d)}$ stands for the set of points of codimension d on X and $K_n^M(\kappa(x))$ for the n^{th} Milnor K -theory group of the residue field $\kappa(x)$. Since $\kappa(x)$ is a number field, the Milnor K -theory groups $K_n^M(\kappa(x))$ are torsion. This implies that $H_{\text{mot}}^{2d+n}(X; \mathbb{Q}(d+n))$ is zero.

Assuming the Beilinson-Soulé vanishing conjecture, all the motivic cohomology groups $H_{\text{mot}}^{2r-n}(X; \mathbb{Q}(r))$, with $0 \leq r \leq \lfloor \frac{n}{2} \rfloor$, are zero. Consequently, since $H_{\text{mot}}^{2d+n}(X; \mathbb{Q}(d+n))$ is also zero, we obtain the sought decomposition(s). \square

Remark 17.20 (Potential generalization). Let X be a smooth proper \mathbb{Q} -scheme of dimension d . Assuming the Beilinson conjecture and the Beilinson-Soulé vanishing conjecture, note that the above proof of Theorem 4.18 (and of Proposition 17.18) shows that the following equalities hold

$$\begin{aligned} \text{ord}_{s=j} L_{\text{even}}(\text{perf}_{\text{dg}}(X); s) &= \dim_{\mathbb{Q}} K_{1-2j}(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}} & j \leq -2 \\ \text{ord}_{s=j} L_{\text{odd}}(\text{perf}_{\text{dg}}(X); s) &= \dim_{\mathbb{Q}} K_{2-2j}(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}} & j \leq -1 \end{aligned}$$

if and only if the groups $\{H_{\text{mot}}^{2r+2j-1}(X; \mathbb{Q}(r)) \mid d-j+1 < r \leq d-2j\}$, resp. $\{H_{\text{mot}}^{2r+2j-2}(X; \mathbb{Q}(r)) \mid d-j+1 < r \leq d-2j+1\}$, are zero. Unfortunately, to the best of the author's knowledge, nothing is known about these groups.

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GONÇALO TABUADA, DEPARTMENT OF MATHEMATICS, MIT, CAMBRIDGE, MA 02139, USA
E-mail address: `tabuada@math.mit.edu`
URL: <http://math.mit.edu/~tabuada>